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Towards Time - Dynamics for Bosonic Systems in Quantum Statistical Mechanics, 2

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Abstract: This is a continuation of a study of time-evolution of a state of an infinitely-extended one-dimensional boson lattice system, which was initiated in the paper [1] under the same title We consider here finite range, pair interaction potentials and prove that the family of the time-evolved states provides a solution to the corresponding Liouville equation and BBGKY hierarchy.

Key words and phrases: one-dimensional lattice boson system, diagonal state, finite-range interaction potential, time-evolution Liouville's equation, BBGKY hierarchy

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1. Introduction

This paper is a continuation of [1]. We studied in the paper [1] the non-equilibrium time-evolution problem for a "locally-unbounded" phase space quantum model where the "traditional" approach [2] based on a limiting $*$ - automorphism group of an underlying C*-algebra does not work. The model was on the one-dimensional lattice $\mathbb{Z}^1$ and was given by its formal Hamiltonian

$$H = -\frac{1}{2} \sum_{j \in \mathbb{Z}^1} a_j^+ (\Delta a)_j + \sum_{j,j' \in \mathbb{Z}^1} \Phi(|j-j'|) n_j n_{j'} , \qquad (1.1)$$

where a_j^+, a_j and $n_j = a_j^+ a_j$ are, respectively, the (bosonic) creation, annihilation and particle number operators in the Hilbert space $\mathcal{H}_j$ which is realized as $\ell_2(\mathbb{Z}^1_+)$ with the standard orthonormal basis $\{ e_s^{(j)}, s \in \mathbb{Z}^1_+ \}$. Let $\mathcal{B}_j$ denote the C*-algebra of bounded operators in $\mathcal{H}_j$ and $\mathcal{B} = \bigotimes_{j \in \mathbb{Z}^1} \mathcal{B}_j$ be the corresponding quasilocal C*-algebra. We have constructed in [1] a family $\{ \varphi_t , t \in \mathbb{R}^1 \}$ of states of $\mathcal{B}$ which describes the time-evolution of an initial state $\varphi = \varphi_0$. The states φ_t are given by

$$\varphi_t(A) = \lim_{\Lambda \nearrow \mathbb{Z}^1} \varphi_{\Lambda,t}(A), \quad A \in \mathcal{B}, \quad t \in \mathbb{R}^1, \qquad (1.2)$$

where

$$\varphi_{\Lambda,t}(A) = \varphi(e^{itH_\Lambda} A e^{-itH_\Lambda}) \qquad (1.3$$

and H_Λ is the Hamiltonian (1.1) confined to the finite volu $\Lambda \subset \mathbb{Z}^1$ (bounded interval on the lattice).

In the paper [1] we assumed that the (locally normal) initial state φ is diagonal. This condition means that the densit matrix of the state φ in any bounded interval $\Lambda \subset \mathbb{Z}^1$

is diagonal in the reference basis $\{ e_x^{(\Lambda)} \}$ in the space $\mathcal{H}_\Lambda = \bigotimes_{j \in \Lambda} \mathcal{H}_j$. Here x is an arbitrary occupation number configuration in Λ, i.e., a function $\Lambda \to \mathbb{Z}^1_+$ (as in [1], we shall use the notation $x \dot{\subset} \Lambda$). The diagonality of a state φ means in physical terms that it describes a classical particle system on the lattice. Of course, this property is destroyed by the time-evolution, but it makes possible to formulate further assumptions on the initial state in natural probabilistic terms. For example, the basic assumption that the state φ has long space "holes" which are free of particles is related to Borel - Cantelli type assertions (see [1]).

In the situation under consideration we are able to prove that the states φ_t, $t \in R^1$, given by (1.2), (1.3) provide a solution to the infinite-volume Liouville equation

$$\frac{d}{dt} p_t^{(\Lambda)} = i \, tr_{\Lambda_+ \setminus \Lambda} [p_t^{(\Lambda_+)}, \bar{H}^0_{\Lambda_+}] + \qquad (1.4)$$

$$+ i \sum_{\substack{j, j' \in \mathbb{Z}^1: \\ \Lambda \cap \{j, j'\} \neq \emptyset}} tr_{\{j,j'\} \setminus \Lambda} [p_t^{(\Lambda \cup \{j,j'\})}, \Phi(|j-j'|) n_j n_{j'}] .$$

Equation (1.4) is on a family $\{ p_t^{(\Lambda)} \}$ of density matrices in finite volumes $\Lambda \subset \mathbb{Z}^1$, which obey the standard consistency condition $tr_{\Lambda' \setminus \Lambda} p_t^{(\Lambda')} = p_t^{(\Lambda)}$, $\Lambda' \supset \Lambda$. The term $[p_t^{(\Lambda_+)}, \bar{H}^0_{\Lambda_+}]$ in the RHS of (1.4) corresponds to the kinetic energy: here Λ_+ is $[v_1 - 1, v_2 + 1]$ for $\Lambda = [v_1, v_2]$, for a general case Λ_+ is defined as $\{ j \in \mathbb{Z}^1 : dist(j, \Lambda) \leq 1 \}$. Namely,

$$\bar{H}^{0}_{\Lambda_+} = -\frac{1}{2}\sum_{\substack{j\in\Lambda_+:\\ j-1\in\Lambda_+}} a_j^+ a_{j-1} - \frac{1}{2}\sum_{\substack{j\in\Lambda_+:\\ j+1\in\Lambda_+}} a_j^+ a_{j+1} + \sum_{j\in\Lambda} n_j \tag{1.5}$$

(the notation $\bar{H}^{0}_{\Lambda_+}$ is used to stress that no boundary condition is involved in the RHS of (1.5)). The partial trace $tr_{\Lambda^\sim}$, $\Lambda^\sim \subset \mathbb{Z}^1$, is taken in $\mathcal{H}_{\Lambda^\sim}$; in the case $\Lambda^\sim = \emptyset$ (which occurs when $\{j, j'\} \subset \Lambda$ in the second-term sum in the RHS of (1.4) no trace is taken.

Equivalently, equation (1.4) may be rewritten in the following form

$$\frac{d}{dt}\rho_t^{(\Lambda)} = i\, tr_{\Lambda_{+r}\setminus\Lambda}\left[\rho_t^{(\Lambda_{+r})}, \bar{H}_{\Lambda_{+r}}\right]. \tag{1.6}$$

Here and below $\Lambda_{+r} = \{j\in\mathbb{Z}^1 : dist(j,\Lambda)\le r\}$, $r = \frac{1}{2}\mathrm{diam\,supp}\,\Phi$. The precise formulation of the result is given in the next section.

Another object related to the time-evolution in statistical mechanics is the BBGKY hierarchy:

$$\frac{d}{dt}\rho_t^{(n)} = i\left[\rho_t^{(n)}, H^{(n)}\right] + \mathbb{A}\,\rho_t^{(n+1)}, \quad n\geqslant 0,\ t\in\mathbb{R}^1. \tag{1.7}$$

This is a system of equations for a sequence $\{\rho_t^{(n)}, n\geqslant 0\}$ of operators $\rho_t^{(n)} : \mathcal{H}^{(n)} \longrightarrow \mathcal{H}^{(n)}$ where $\mathcal{H}^{(n)} = (\mathcal{H}^{(1)})^{\otimes n}_{symm}$ is the symmetrized tensor product of n copies of the underlying Hilbert space $\mathcal{H}^{(1)} = \ell_2(\mathbb{Z}^1)$ ($\mathcal{H}^{(0)}$ is set to be the one-dimensional complex space). Thereby the matrix elements of $\rho_t^{(n)}$ are labelled by pairs (x, x') of occupation number configurations $x, x' \dot{\subset} \mathbb{Z}^1$ with $|x| = |x'| = n$

(by $|y|$ we denote, as in [1], the number of particles in an occupation number configuration y : $|y| = \sum_j y(j)$). The first term in the RHS of (1.7) is the commutator with the n-particle Hamiltonian $H^{(n)}$ corresponding to (1.1). Like $\rho_t^{(n)}$, the operator $H^{(n)}$ acts in $\mathcal{H}^{(n)}$ and its matrix elements are again labeled by pairs (x, x') where $x, x' \in \mathbb{Z}^1$, $|x| = |x'|$. They are given by

$$(H^{(n)})_{x',x} = -\frac{1}{4} \sum_{j,j' \in \mathbb{Z}^1} \Phi(|j-j'|)\, x(j)\, x(j'),$$

if occupation number configurations x and x' coinside,

$$(H^{(n)})_{x',x} = -\frac{1}{4}\, x(j)^{1/2} \left(x(j \mp 1) + 1 \right)^{1/2},$$

$$\text{if} \qquad x' = x - \delta_j + \delta_{j \mp 1},$$

and

$$(H^{(n)})_{x',x} = 0, \qquad \text{otherwise}.$$

Here and below δ_j denotes the one-particle configuration concentrated at a site $j \in \mathbb{Z}^1$.

As to the second term in the RHS of (1.7), this is the following operator in $\mathcal{H}^{(n)}$:

$$\mathbb{A} \rho_t^{(n+1)} = i \sum_{j \in \mathbb{Z}^1} [U(j), \rho_t^{(n+1)}(j)]. \tag{1.8}$$

Here $\rho_t^{(n+1)}(j)$ is an operator

$$\left(\rho_t^{(n+1)}(j)\right)_{x,x'} = \left(\rho_t^{(n+1)}\right)_{x+\delta_j,\, x'+\delta_j}, \quad x, x' \dot{\subset} \mathbb{Z}^1, \; |x| = |x'| = n,$$

(recall that $\rho_t^{(n+1)}$ acts in $\mathcal{H}^{(n+1)}$ and therefore its matrix elements are labeled by pairs (y, y') with $|y| = |y'| = n+1$. Likewise, $U(j)$ is an operator in $\mathcal{H}^{(n)}$ with

$$\left(U(j)\right)_{x,x'} = \sum_{k \in \mathbb{Z}^1} \Phi(|j-k|)\, x(k),$$

if occupation number configurations x and x' coinside,

$$\left(U(j)\right)_{x,x'} = 0, \quad \text{otherwise.}$$

In our situation we prove that a solution to (1.7) is given by the family of operators with

$$\left(\rho_t^{(n)}\right)_{x,x'} = \mathcal{S}_t\left(\prod_{j \in \mathbb{Z}^1} a_j^{+\,x'(j)} \prod_{k \in \mathbb{Z}^1} a_k^{x(k)}\right), \tag{1.9}$$

$$x, x' \dot{\subset} \mathbb{Z}^1, \; |x| = |x'| = n, \quad n \geq 0, \quad t \in \mathbb{R}^1.$$

Of course, the existence of the expectations in the RHS of (1.9) should be guaranteed. For the precise formulation of this result, see Section 3.

Notice that we do not state that our solutions to the Liouville and BBGKY equations are unique: one must be able for this to construct the time evolution for a much larger family of initial states $\mathcal{S}$.

2. The Liouville equation

Theorem 1. Suppose that a diagonal state φ has properties (d^*) and (d^{**}) from [1]. Then, for any bounded interval Λ^0 and any operator $A \in \mathcal{B}$ localized in Λ^0 $(A \in \mathcal{B}_{\Lambda^0})$, the function $t \mapsto \varphi_t(A)$ is smooth and the following equation takes place

$$\frac{d}{dt}\varphi_t(A) = i\,\varphi_t([\bar{H}_{\Lambda^0_{+r}}, A]), \quad t \in \mathbb{R}^1, \tag{2.1}$$

where the RHS of the equality is defined as the limit

$$\lim_{s\to\infty} \varphi_t(\Pi^{(\Lambda^0_{+r})}_{<s}[\bar{H}_{\Lambda^0_{+r}}, A]\,\Pi^{(\Lambda^0_{+r})}_{<s}), \tag{2.2}$$

$\Pi^{(\Lambda)}_{<s}$ is the orthogonal projection in $\mathcal{H}_\Lambda$ onto the subspace generated by the occupation number configurations $x \,\dot{\subset}\, \Lambda$ with $|x| < s$. $\triangleleft$

Equation (2.1) may be considered as a weak form of the infinite-volume Liouville equation. From this point of view, a family of states φ_t, $t \in \mathbb{R}^1$, is called sometimes a weak solution of the Liouville equation.

Proof of Theorem 1. At the first step we shall check equation (2.1) for $A = E^{(\Lambda^0)}_{x,y}$, $|x| = |y|$.

Lemma 1.1. Given bounded $\Lambda \supset \Lambda^0$, the convergence

$$\varphi_{\Lambda,t}([\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}]) = \lim_{s\to\infty} \varphi_{\Lambda,t}(\Pi^{(\Lambda^0_{+r})}_{<s}[\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}]\,\Pi^{(\Lambda^0_{+r})}_{<s}) \tag{2.3}$$

is uniform for $\Lambda \supset \Lambda^0_{+r}$ and t in compacts, and the function

$$t \mapsto \varphi_{\Lambda,t}([\bar{H}_\Lambda, E^{(\Lambda^0)}_{x,y}]) = \varphi_{\Lambda,t}([\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}])$$

is bounded on compacts uniformly for $\Lambda \supset \Lambda^0_{+r}$. Moreover,

$$\varphi_{\Lambda,t}([\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}]) =$$

$$= \lim_{s\to\infty} \varphi_{\Lambda,t}(\Pi^{(\Lambda)}_{<s}[\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}]\Pi^{(\Lambda)}_{<s}), \tag{2.4}$$

and the convergence (2.4) is also uniform for t in compacts.

Proof. Writing

$$\varphi_{\Lambda,t}(\Pi^{(\Lambda^0_{+r})}_{<s}[\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}]\Pi^{(\Lambda^0_{+r})}_{<s}) =$$

$$= \sum_{j\in\Lambda^0_{+r}} \varphi_{\Lambda,t}(\Pi^{(\Lambda^0_{+r})}_{<s}[a^+_j(\Delta a)_j, E^{(\Lambda^0)}_{x,y}]\Pi^{(\Lambda^0_{+r})}_{<s}) + \tag{2.5}$$

$$+ \sum_{j,j'\in\Lambda^0_{+r}} \Phi(|j-j'|)\varphi_{\Lambda,t}(\Pi^{(\Lambda^0_{+r})}_{<s}[n_j n_{j'}, E^{(\Lambda^0)}_{x,y}]\Pi^{(\Lambda^0_{+r})}_{<s}),$$

we see that it suffices to check that every term in the RHS of (2.5) converges uniformly to the corresponding quantity as $s \to \infty$.

We restrict ourselves to the term

$$\varphi_{\Lambda,t}(\Pi^{(\Lambda^0_{+r})}_{<s} n_j n_{j'} E^{(\Lambda^0)}_{x,y} \Pi^{(\Lambda^0_{+r})}_{<s}), \quad j\in\Lambda^0,\ j'\notin\Lambda^0. \tag{2.6}$$

To prove the convergence of (2.6) it suffices to check that

$$|\varphi_{\Lambda,t}(\Pi^{(\Lambda^0_{+r})}_{s,s'} n_{j'} \Pi^{(\Lambda^0_{+r})}_{s,s'})| \xrightarrow{s\to\infty} 0, \tag{2.7}$$

where $s' > s$, $\Pi^{(\Lambda^0_{+r})}_{s,s'} = \Pi^{(\Lambda^0_{+r})}_{>s} \Pi^{(\Lambda^0_{+r})}_{<s'}$. The LHS of (2.7) is estimated by

$$|\mathcal{S}_{\Lambda,t}(\Pi^{(\Lambda^0_{+r})}_{s,s'} n_{j'} \Pi^{(j')}_{<\ln s} \Pi^{(\Lambda^0_{+r})}_{s,s'})| + |\mathcal{S}_{\Lambda,t}(\Pi^{(\Lambda^0_{+r})}_{s,s'} n_{j'} \Pi^{(j')}_{\geqslant \ln s} \Pi^{(\Lambda^0_{+r})}_{s,s'})| \leqslant$$

$$\leq \| \Pi^{(j')}_{<\ln s} n_{j'} \Pi^{(j')}_{<\ln s} \| \cdot |\mathcal{S}_{\Lambda,t}(\Pi^{(\Lambda^0_{+r})}_{s,s'})| +$$

$$+ \sum_{k \geqslant \ln s} |\mathcal{S}_{\Lambda,t}(\Pi^{(\Lambda^0_{+r})}_{s,s'} n_{j'} \Pi^{(j')}_{=k} \Pi^{(\Lambda^0_{+r})}_{s,s'})| \leqslant$$

$$\leq \ln s \, |\mathcal{S}_{\Lambda,t}(\Pi^{(\Lambda^0_{+r})}_{>s})| + \sum_{k \geqslant \ln s} k \, |\mathcal{S}_{\Lambda,t}(\Pi^{(j')}_{\geqslant k})| <$$

$$< c_1 \ln s \cdot s^{-(2+\alpha)} + c_2 \sum_{k \geqslant \ln s} k \cdot k^{-(2+\alpha)} \qquad (2.8$$

where c_1, c_2, α are some positive constants. The last bound in (2.8) is the consequence of bounds obtained in the proof of Theorem 2 from [1].

The proof of equality (2.4) proceeds along the same scheme This finishes the proof of Lemma 1.1. □

Note that the simple modification of these arguments gives that for any $j_1, \dots, j_\ell \in \mathbb{Z}^1$ and $\alpha_1, \dots, \alpha_\ell \in \mathbb{Z}^1_+$

$$\mathcal{S}_{\Lambda,t}(n^{\alpha_1}_{j_1} \cdot \ldots \cdot n^{\alpha_\ell}_{j_\ell}) < \infty \qquad (2.9)$$

uniformly for t in compacts.

From Lemma 1.1 one obtains that, for fixed Λ, $\mathcal{S}_{\Lambda,t}([\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}])$ is a continuous function of the variable and the following equality is valid

$$\mathcal{S}_{\Lambda,t}(E^{(\Lambda^0)}_{x,y}) = \mathcal{S}(E^{(\Lambda^0)}_{x,y}) + i \int_0^t \mathcal{S}_{\Lambda,t'}([\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}]) \, dt' . \qquad (2.1($$

Lemma 1.2. The following limits exist uniformly for t in compacts

$$\varphi_t([\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}]) = $$
$$= \lim_{s\to\infty} \varphi_t(\Pi^{(\Lambda^0_{+r})}_{<s} [\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}] \Pi^{(\Lambda^0_{+r})}_{<s})$$

and

$$\lim_{\Lambda \uparrow \mathbb{Z}^1} \varphi_{\Lambda,t}([\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}]). \tag{2.11}$$

Moreover, the both limits are equal. ◁

Proof. From Lemma 1.1 one has that the convergence in (2.3) is uniform for $\Lambda \supset \Lambda^0_{+r}$. From [1] (see Theorem 3) one can obtain the convergence

$$\lim_{\Lambda \uparrow \mathbb{Z}^1} \varphi_{\Lambda,t}(\Pi^{(\Lambda^0_{+r})}_{<s} [\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}] \Pi^{(\Lambda^0_{+r})}_{<s}) =$$
$$= \varphi_t(\Pi^{(\Lambda^0_{+r})}_{<s} [\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}] \Pi^{(\Lambda^0_{+r})}_{<s}).$$

Hence, the both limits exist and are equal. Lemma 1.2 is proven

From (2.10) and Lemma 1.2 one obtains that $\varphi_t([\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}])$ is a continuous function of t and

$$\varphi_t(E^{(\Lambda^0)}_{x,y}) = \varphi(E^{(\Lambda^0)}_{x,y}) + i\int_0^t \varphi_{t'}([\bar{H}_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}])dt'.$$

Lemma 1.3. Let $A \in \mathcal{B}$ be an operator localized in Λ^0. Then, for any $u \in \mathbb{Z}^1_+$

$$t \mapsto \varphi_{\Lambda,t}([\bar{H}_{\Lambda^0_{+r}}, \Pi^{(\Lambda^0)}_{>u} A \Pi^{(\Lambda^0)}_{>u}]) \tag{2.12}$$

is a continuous function which is bounded uniformly for t in compacts. The value (2.12) converges to 0 as $u \to \infty$ uniformly in Λ. ◁

The proof of Lemma 1.3 is given by a combination of arguments from the proofs of Lemma 1.1 and Lemma 1.2. Here one uses the following bound

$$|\varphi_{\Lambda,t}(n_j n_{j'}, \Pi^{(\Lambda^0)}_{>u} A \Pi^{(\Lambda^0)}_{>u})| \leq$$

$$\leq [\varphi_{\Lambda,t}(n_j^2 n_{j'}^2)]^{1/2} \cdot \|A\| \, [\varphi_{\Lambda,t}(\Pi^{(\Lambda^0)}_{>u})]^{1/2}.$$

The details are omitted. □

From Lemma 1.3 one obtains

$$\varphi_t(A) = \varphi(A) + i\int_0^t \varphi_{t'}([\bar{H}_{\Lambda^0_{+r}}, A])\,dt' \qquad (2.13)$$

for any $A \in \mathcal{B}$ localized in Λ^0. The last remark is that (2.13) is equivalent to (2.1). Theorem 1 is proven. □

<u>Corollary</u> 1. Assume that a diagonal state φ satisfies the conditions of Theorem 1. Let $\rho_t^{(\Lambda^0)}$ be the density matrix of the state φ_t in a volume Λ^0. The family of matrices $\rho_t^{(\Lambda)}$, $t \in \mathbb{R}^1$, provides a solution to the infinite-volume Liouville equation written in the operator form

$$\frac{d}{dt}(\rho_t^{(\Lambda^0)}) = i\, \mathrm{tr}_{\Lambda^0_{+r}\setminus\Lambda^0}[\rho_t^{(\Lambda^0_{+r})}, \bar{H}_{\Lambda^0_{+r}}], \quad t \in \mathbb{R}^1.$$

The derivative in the LHS is understood in the weak operator topology. ◁

Equivalently, this equation may be rewritten in the form (1.4).

Proof. From Lemma 1.2 and bounds (2.8) we have

$$\vartheta_t([\bar H_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}]) = \lim_{s\to\infty} \vartheta_t\left(\Pi^{(\Lambda^0_{+r})}_{<s}[\bar H_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}]\Pi^{(\Lambda^0_{+r})}_{<s}\right) =$$

$$= \sum_{z,w\in\Lambda^0_{+r}} \left(p_t^{(\Lambda^0_{+r})}\right)_{z,w} \left([\bar H_{\Lambda^0_{+r}}, E^{(\Lambda^0)}_{x,y}]\right)_{w,z} \qquad (2.14$$

and the series in the RHS of (2.14) is absolutely convergent. Hence, (2.14) may be rewritten in the following form

$$\sum_{z,w\in\Lambda^0_{+r}} \left([p_t^{(\Lambda^0_{+r})}, \bar H_{\Lambda^0_{+r}}]\right)_{z,w} \left(E^{(\Lambda^0)}_{x,y} \otimes 1\!\!1^{(\Lambda^0_{+r}\setminus\Lambda^0)}\right)_{w,z} =$$

$$= \sum_{z\in\Lambda^0_{+r}\setminus\Lambda^0} \left([p_t^{(\Lambda^0_{+r})}, \bar H_{\Lambda^0_{+r}}]\right)_{z\vee x,\, z\vee y}$$

and from (2.1) one obtains

$$\frac{d}{dt}\left(\left(p_t^{(\Lambda^0)}\right)_{x,y}\right) = i\left(\mathrm{tr}_{\Lambda^0_{+r}\setminus\Lambda^0}[p_t^{(\Lambda^0_{+r})}, \bar H_{\Lambda^0_{+r}}]\right)_{x,y} .$$

From the last equation and Lemma 1.3 it follows that for any $f, g \in \mathcal{H}_{\Lambda^0}$

$$\frac{d}{dt}\left(\langle p_t^{(\Lambda^0)} f, g\rangle_{\Lambda^0}\right) = i\left\langle\left(\mathrm{tr}_{\Lambda^0_{+r}\setminus\Lambda^0}[p_t^{(\Lambda^0_{+r})}, \bar H_{\Lambda^0_{+r}}]\right) f, g\right\rangle_{\Lambda^0}, \qquad (2.1$$

where $\langle\cdot,\cdot\rangle_{\Lambda^0}$ is the scalar product in $\mathcal{H}_{\Lambda^0}$. □

From (2.15) one can directly deduce the following

<u>Corollary</u> 2. The equation (2.15) may be rewritten in the following vector form: for any $s \in \mathbb{Z}^1$ and $f \in \Pi^{(\Lambda^0)}_{<s}\mathcal{H}_{\Lambda}$

$$\frac{d}{dt} p_t^{(\Lambda^0)} f = i\, \mathrm{tr}_{\Lambda^0_{+r}\setminus\Lambda^0}[p_t^{(\Lambda^0_{+r})}, \bar H_{\Lambda^0_{+r}}] f \quad . \triangleleft$$

3. The BBGKY hierarchy.

Theorem 2. Let a diagonal state φ have properties (d^*) and (d^{**}) from [1]. Then, for any $n \geqslant 0$ and any pair x, x' of occupation number configurations with $|x| = |x'| = n$ the function

$$t \longmapsto R_t(x, x')$$

is smooth and the following equation takes place:

$$\frac{d}{dt} R_t^{(n)}(x,x') = i(\mathbb{H}^{(n)} R_t^{(n)})(x,x') + (\mathbb{A} R_t^{(n+1)})(x,x') . \tag{3.1}$$

Here $R_t(x, x')$ is defined by

$$R_t(x,x') = \lim_{s\to\infty} \varphi_t \Big(\Pi_{<s}^{(\Lambda^0)} \prod_{j\in\mathbb{Z}^1} a_j^{+\,x'(j)} \prod_{k\in\mathbb{Z}^1} a_k^{x(k)} \Pi_{<s}^{(\Lambda^0)} \Big), \tag{3.2}$$

$\Lambda^0 = \Lambda^0(x, x') = \{ \ell \in \mathbb{Z}^1 : x(\ell) + x'(\ell) \geqslant 1 \}$ is the support of $(x + x')$, the projectors $\Pi_{<s}^{(\Lambda^0)}$ were introduced before. $\mathbb{H}^{(n)} R_t^{(n)}$ is given by

$$\mathbb{H}^{(n)} R_t^{(n)}(x,x') = \sum_y \left(R_t^{(n)}(x,y)(H^{(n)})_{y,x'} - (H^{(n)})_{x,y} R_t^{(n)}(y,x') \right) \tag{3.3}$$

and $\mathbb{A} R_t^{(n+1)}$ is given by

$$(\mathbb{A} R_t^{(n+1)})_{x,x'} =$$

$$= i \sum_{j,k\in\mathbb{Z}^1} \Phi(|j-k|)(x(k) - x'(k)) R_t^{(n+1)}(x+\delta_j, x'+\delta_j) . \tag{3.4} \triangleleft$$

Equation (3.1) may be considered as a weak form of the infinite-volume BBGKY hierarchy for the time-evolution under consideration.

Proof of Theorem 2. The proof of Theorem 2 follows the sam line of arguments as that of Theorem 1. To avoid repetitions, we shall omit technical details.

Let $x, y \dot{\subset} \Lambda^0$ be occupation number configuratio in Λ^0. We denote

$$a^+(x) = \prod_{j\in\Lambda^0} a_j^{+\,x(j)}, \quad a(y) = \prod_{j\in\Lambda^0} a_j^{y(j)} .$$

Lemma 3.1. Given bounded $\Lambda \supset \Lambda^0$, the limits

$$\begin{aligned} &\mathcal{S}_{\Lambda,t}(a^+(x)a(y)) = R_{\Lambda,t}(y,x) = \\ &= \lim_{s\to\infty} \mathcal{S}_{\Lambda,t}(\Pi_{<s}^{(\Lambda^0)} a^+(x)a(y)\,\Pi_{<s}^{(\Lambda^0)}), \\ &\mathcal{S}_{\Lambda,t}([\bar{H}_{\Lambda^0_{+r}}, a^+(x)a(y)]) = \\ &= \lim_{s\to\infty} \mathcal{S}_{\Lambda,t}(\Pi_{<s}^{(\Lambda^0_{+r})} [\bar{H}_{\Lambda^0_{+r}}, a^+(x)a(y)]\,\Pi_{<s}^{(\Lambda^0_{+r})}) \end{aligned} \qquad (3.5$$

are uniform for $\Lambda \supset \Lambda^0$ and t in compacts. The functions

$$t \mapsto \mathcal{S}_{\Lambda,t}([\bar{H}_{\Lambda^0_{+r}}, a^+(x)a(y)]), \quad t \mapsto R_{\Lambda,t}(y,x)$$

are continuous and bounded on compacts uniformly for $\Lambda \supset \Lambda^0$. Moreover, the following limits exist

$$\begin{aligned} &R_{\Lambda,t}(y,x) = \lim_{s\to\infty} \mathcal{S}_{\Lambda,t}(\Pi_{<s}^{(\Lambda)} a^+(x)a(y)\,\Pi_{<s}^{(\Lambda)}), \\ &\mathcal{S}_{\Lambda,t}([\bar{H}_{\Lambda^0_{+r}}, a^+(x)a(y)]) = \\ &= \lim_{s\to\infty} \mathcal{S}_{\Lambda,t}(\Pi_{<s}^{(\Lambda)} [\bar{H}_{\Lambda^0_{+r}}, a^+(x)a(y)]\,\Pi_{<s}^{(\Lambda)}) \end{aligned} \qquad (3.6$$

and the convergence in (3.6) is uniform for t in compacts. $\triangleleft$

The proof of Lemma 3.1 proceeds in the same way as Lemma 2.1.

From Lemma 3.1 one obtains that for fixed $\Lambda \supset \Lambda^0_{+r}$ the following equality is valid

$$\varsigma_{\Lambda,t}\left(a^+(x)a(y)\right) = $$
$$= \varsigma\left(a^+(x)a(y)\right) + i\int_0^t \varsigma_{\Lambda,t'}\left(\left[\bar{H}_{\Lambda^0_{+r}}, a^+(x)a(y)\right]\right)dt'. \tag{3.7}$$

Lemma 3.2. Uniformly for t in compacts the following limits exist:

$$R_t(y,x) = \varsigma_t\left(a^+(x)a(y)\right) =$$
$$= \lim_{s\to\infty} \varsigma_t\left(\Pi^{(\Lambda^0)}_{<s}\, a^+(x)a(y)\,\Pi^{(\Lambda^0)}_{<s}\right) =$$
$$= \lim_{\Lambda \nearrow \mathbb{Z}^1} R_{\Lambda,t}(y,x)$$

and

$$\varsigma_t\left(\left[\bar{H}_{\Lambda^0_{+r}}, a^+(x)a(y)\right]\right) =$$
$$= \lim_{s\to\infty} \varsigma_t\left(\Pi^{(\Lambda^0_{+r})}_{<s}\left[\bar{H}_{\Lambda^0_{+r}}, a^+(x)a(y)\right]\Pi^{(\Lambda^0_{+r})}_{<s}\right) =$$
$$= \lim_{\Lambda \nearrow \mathbb{Z}^1} \varsigma_{\Lambda,t}\left(\left[\bar{H}_{\Lambda^0_{+r}}, a^+(x)a(y)\right]\right). \triangleleft$$

The proof of Lemma 3.2 repeats that of Lemma 2.2.

As in Section 2, from (3.7) and Lemma 3.2 one obtains

$$R_t(y,x) = \varsigma\left(a^+(x)a(y)\right) + i\int_0^t \varsigma_{t'}\left(\left[\bar{H}_{\Lambda^0_{+r}}, a^+(x)a(y)\right]\right)dt' \tag{3.8}$$

Hence, the function $t \mapsto R_t(y,x)$ is smooth, and direc computation gives that the equality (3.8) may rewritten in the form (3.1). Theorem 2 is proven. $\square$

References

1. A.G.Shuhov, Yu.M.Suhov, A.V.Teslenko. Towards time-dynamics for bosonic systems in quantum statistical mechanics. Preprint DIAS-STP-89-29.

2. Bratteli O., Robinson D.W. Operator algebras and quantum statistical mechanics, Vols 1,2. Berlin et al.: Springer-Verlag, 1979, 1981.