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INFINITE PARTICLE HAMILTONIAN DYNAMICS OF CHERN - SIMONS TYPE.

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In this paper we give an example of the d-dimensional integrable infinite particle Hamiltonian system, originating from the Topological Quantum Field Theory.

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INTRODUCTION.

Statistical mechanics has to describe the infinite particle dynamics. Since that, a natural question arises: is it possible to find a Hamiltonian mechanical system, whose infinite set of equations of motion is completely integrable? It is natural to consider at first the systems, whose equations of motion are integrable for any number of particles.

One-dimensional systems of particles, interacting through four types of potentials have the integrable equations of motion for finite number of particles[1]. Other symmetric integrable systems of particles, interacting through many-body potentials are not known.

It is impossible to answer the question for these one-dimensional systems since the method of Lax pair does not work for infinite number of particles.

In this paper we give an example of an integrable infinite particle Hamiltonian d-dimensinal dynamnics that is related to Chern-Simons dynamics of charged particles [2].

Our n-particle system is described by the Hamiltonian

(1) $$H(P_n,X_n) = (2m)^{-1} \sum_{j=1}^{n} (p_j - a_j(X_n))^2,$$

$$P_n = (p_1,p_2,\dots,p_n),\ X_n = (x_1,x_2,\dots,x_n),\ x_j \in \mathbb{R}^d,\ p_j \in \mathbb{R}^d,$$

(2) $$a_j(X_n) = \nabla_j U(X_n),$$

where ∇ is the gradient,and U is the potential energy, corresponding

to the pair potential ϕ. The Chern-Simons systems corresponds to the case d=2 and n=2N, $U = U_{CS}$

$$U_{CS}(X_n) = \sum_{1\le k<j\le n} e_k e_j \, \phi_{CS}(x_j - x_k), \quad \phi_{CS}(x-y) = \mathrm{arctg}(x^2-y^2)(x^1-y^1)^{-1}.$$

It is natural to expect that the S-C dynamics is completely integrable since it is derived from the Topological Field Theory.

And it is quite surprising to find out that the Hamiltonian system defined above is integrable in d dimension if the condition (1) is satisfied. We shall show this in the first paragraph. It is not really difficult to do.

The integration of the infinite-particle system is not trivial.

Only in the case the function U is expressed through the pair short range smooth potential ϕ we are able to establish that the dynamics is integrable and there exist an infinite sequense of integrals of motion . They are defined on such the space of initial configurations that permit to solve the Cauchy problem for the equation of motion.

In the second paragragh we establish that the evolution operator of flow is congruent to the evolution operator of the free evolution.

We end the paper with discussion.

1.Dynamics of finite particle system.

Let us show that the Poisson bracket $\{V_j,V_k\}$ vanishes,where

$$V_j = p_j - a_j(X_n), \quad \text{if} \quad \nabla_J a_k = \nabla_k a_j.$$

Let by ∇^*_j the gradient in p_j be denoted. Then

$$\text{(1.1)} \qquad \{V_j,V_k\} = \sum_{l=1}^{n} \{ ((\nabla_l V_j),(\nabla^*_l V_k)) -((\nabla^*_l V_j),(\nabla_l V_k)) \} =$$

$$= \sum_{l=1}^{n} \{((-\nabla_l a_j), \delta_{k,l}) + (\delta_{j,l}, (\nabla_l a_k)) \} = 0.$$

Sinse ∇ is the vector valued operation we use brackets (,) for the scalar product in d-dimensional space.But it is convinient to omit them.

PROPOSITION 1.1.

If $a_j(X_n) = (\nabla_j U)(X_n)$ and $U \in C^2(\mathbb{R}^{dn})$, then $V_j(P_n,X_n)$ are the integrals of motion for the system with the Hamiltonian

$$(1.2) \qquad H(P_n,X_n) = (2m)^{-1} \sum_{j=1}^{n} (p_j - a_j(X_n))^2 = (2m)^{-1} \sum_{j=1}^{n} V_j^2 .$$

PROOF. It follows immediatly from (1.1) and the fact that H depends on V_j. In other words we have to check that

$$\{ V_j^2, V_k \} = 0, \sum_{j=1}^{n} \{ V_j^2 , V_k \} = 0.$$

Now let us write the equation of motion the Hamiltonian H, given by (1.2).

$$(1.3) \qquad \dot{x}_j = \nabla_j H = m^{-1} (p_j - a_j(X_n))$$

$$\dot{p}_j = m^{-1} \sum_{k=1}^{n} (p_k - a_k(X_n)) (\nabla_j a_k)(X_n),$$

where the dot over variables means the time derivative
From the first equation of motion and proposition (1.1) it follows that

$$x_j(t) = x_j + m^{-1} t(p_j - a_j(X_n))$$

(p_j,x_j), $j = 1,\dots,n$ are the initial momenta and position vectores.
Then substituting the first equation (1.3) into the second we obtain

$$\dot{p}_j = \sum_{k=1}^{n} \dot{x}_k (\nabla_j a_k)(X_n(t)) = \sum_{k=1}^{n} \dot{x}_k (\nabla_k a_j)(X_n(t)) = \dot{a}_j(X_n(t))$$

This equality holds if the conditions of the proposition (1.1) hold
Hence, the second equation of motion (1.3) is a consequence of the

integrals of motion. Thus we have proved the proposition.

PROPOSITION 2.1.

If the conditions of the proposition hold then the solution of (1.3) is given by

$$x_j(t) = x_j + m^{-1}t(p_j - a_j(X_n)) \tag{1.4}$$

$$p_j(t) = p_j + a_j(X_n(t)) - a_j(X_n)$$

where p_j, x_j are the initial data.

Moreover the evolution operator U_n^t of the dynamical flow is given by

$$U_n^t = Z_n^{-1} U_{on}^t Z_n \tag{1.5}$$

where U_{on}^t is the evolution operator of the free dynamics and

$$Z_n(P_n,X_n) = (p_j - a_j(X_n), x_j)_{j=1}^n \tag{1.6}$$

2.INFINITE PARTICLE DYNAMICS

The equation of motion for the infinite particle system of S-C type with the formal Hamiltonian H(P,X) is written as follows

$$\dot{x}_j(t) = m^{-1}(p_j(t) - a_j(X(t))), \quad j\geq 1, \tag{2.1}$$

$$\dot{p}_j(t) = m^{-1} \sum_{k\geq 1} (p_k(t) - a_k(X(t)).)(\nabla_j a)_k(X(t)) ,$$

where (P,X) is an infinite sequence and $a_j(X) = (\nabla_j U)(X)$.

Now we assume that the "potential energy" U is of two-body character, i.e.

$$(2.2) \qquad a_j(X) = \sum_{k\geq 1} (\nabla\phi)(x_j - x_k) \ , \ \phi \in C_0^{\infty}(\mathbb{R}^d)$$

that is, ϕ is a smooth function with compact support.
It is a usual situation in the Statistical Mechanics that the Hamiltonian is divergent while equation of motion make sense for infinite particle systems. The same happens in our case.

Formally (2.1) is integrated easilly. Really let us substitute the fist equation of motion into the second. As a result

$$(2.3) \qquad \dot{p}_j(t) = \sum_{k\geq 1} \dot{x}_k(t) \ (\nabla_j \ a_k)(X(t)) = \dot{a}_j(X(t))$$

If the sum is convergent than the functions $V_j = p_j(t) - a_j(X(t))$ are the integrals of motion and the analog of (1.4) for the solution of (2.1) has to hold.

Let the eq.(2.1) hold and put

$$x_j^*(t) = x_j + p_j^* t.$$

It is clear that $a_j(X^*(t))$ are well defined functions if (2.2) holds and in a bounded domain there is a finite number of particles.

Let $\mathbb{R}^{\infty d}$ be the infinite Cartesian product of $\mathbb{R}^d$ and $\mathbb{R}_0^{\infty d}$ be its subset of locally finite configurations (in a bounded domain there is a finite number of particles).

Let now U_0^t be the evolution operator a free dynamics on $\mathcal{X}$, $\mathcal{X}^{2d} = \mathbb{R}^{\infty d} \times \mathbb{R}_0^{\infty d}$ and Z is given on $\mathcal{X}^{2d}$ by

$$Z(P,X) = (\ P - A(X),\ X\ ), \quad A(X) = \{a_j(X)\},$$

where $a_j(X)$ satisfies (2.2). The infinite sum in (2.3) is finite and $a_j(X(t))$ is a smooth function in t. Hence the following theorem is true

THEOREM 2.1.

If the conditions (2.2) are satisfied then the evolution operator U^t

$$U^t = Z^{-1} U_o^{\ t} Z$$

solves the Cauchy problem for (2.1) in χ^{2d}.

3 DISCUSSION

In spite of the fact that the Theorem 2.1 is simple it gives the first example of the integrable infinite particle mechanical system. In this case there exist a sequence of the integrals of motion and in the same time the dynamics is reduced by a simple transform to the free dynamics.

The simplicity of the dynamics can be expected since the grand partition function of our system coincides with the grand partition function of the free gas. The Gibbs correlation functions looks also simple but they do not coincide with the correlation functions of the free gas. They induce the probability measure on χ^{2d}, which defines the equilibrium state of the infinite particle system of C-S type.

It is an interesting question what kind of infinite dimensional Lie group does appear in the infinite particle system and how is it connected with equilibrium states.

The other interesting question concerns the procedure of quantization : what is the direct procedure of quantization of the evolution U^t on χ^{2d} ?(By the simple shift of phase the n-particle quantum Hamiltonian of C-S type is reduced to the free Hamiltonian but this relation is lost in the infinite particle case).

And let us return to the true Chern-Simons system.

The C-S is long range and this makes nontrivial even the computation of the Gibbs correlation functions. The problem concerning describing the set of configurations which allow to solve the Cauchy problem seems nontrivial.

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