DIAS-STP 91-11

GIBBS SYSTEM OF INTERACTING SCALAR FIELDS AND PARTICLES AS AN ORIGIN OF THE SINE-GORDON TRANSFORMATION

W.I.SKRYPNIK*

The Gibbs system, defined by the formal measure on the Cartesian product of spaces of particle and field variables, is introduced. The Sine-Gordon transformation for the Gibbs systems of particles interacting via many-body potentials is derived with the help of the reduction of the introduced system.

* On leave of absence from the Institute of mathematics of the Ukrainian Academy of Science. Permanent address: 252601,Kiev 4 ,Rep in sir.3, USSR,Ukraine

INTRODUCTION

The system of charged particles is the most fundamental object of the statistical mechanics and the only tool of rigorous dealing with it is the Sine-Gordon transformation. This explains the interest in this transformation.

The Sine-Gordon (S-G) trasformation relates Gibbs systems of particles, interacting through a pair positive-definite potential, and Gibbs field systems [1] . The former are defined by Gibbs (multiplicative) perturbations of the Poisson measure and the latter are defined by the similar perturbations of a gaussian measure.

This relation is exploited in statistical nechanics [2-4].

In this paper we establish that the Sine-Gordon transformation can be generalized to the class of Gibbs systems of particles, interacting through many-body potentials. Initially we tried to obtain the generalization for the case of a special three-body potential, which is connected with a system of diffusing particles [5]. The interest for the generalization isbased on the idea, that quantum mechnical interaction between nuclei and electrons generates an effective many-body potential between molecules,i.e.classical particles, which regu-regularize the divergencies, created by the Coulomb pair potential.

The proposed generalization is based on the introduction of the Gibbs systems of interacting fields and particles.

The Sine-Gordon transformation of a particle Gibbs system looks like transition to an effective Gibbs scalar field system in the system of interacting fields and particles after integrating out the particle variables. And in the case of absence of 2k+1 - body potentials the effective Lagrangian contains the usual S-G Lagrangian and the logarithm of the Fourier-Wiener transform.

Our paper is organized as follows.

In the first paragraph we introduce Gibbs systems of separate particles and fields and derive the usual Sine-Gordon transformation.

In the second paragraph we introduce the Gibbs system of interacting fields and particles and present the reduction procedure that yields the generalized Sine-Gordon transformation for Gibbs systems on a formal level.

In the third paragraph we give a rigorous interpretation of the the obtained formulas.

1.Gibbs systems of independent particles and fields.

Let us consider the d-dimensional system of particles, interacting through the pair potential $c_o(x)$, $x \in \mathbb{R}^d$

$$C_o(x) = (2\pi)^{-d} \int \exp\{i(k,x)\} Q^{-1}(k^2) dk$$

where Q is a positive polynomial or an entire function.

We assume that the Fourier transform is considered in the sense of the generalized functions. This assumption restricts the class of Q considered. For example the case

$$Q(k^2) = (k^2)^l \, , \; l \geq \frac{d}{2}$$

is excluded.

The grand partition function Ξ_P^Λ given by

$$\Xi_P^\Lambda = \sum_{n \geq o} \frac{1}{n!} z^n \int_{\Lambda^n} \exp\{ - \beta U(X) \} \, dX_n$$

where $X_n = (x_1, x_2, \ldots, x_n)$, $dX_n = dx_1 \bullet dx_2 \bullet\bullet\bullet dx_n$,

$$U_o(X_n) = \sum_{i<j=1} C_o(x_i - x_j),$$

$z = \exp\{\beta\mu\}$ is the activity, β is the inverse temperature, μ is the chemical potential.

Let us denote by $d_P\rho$ the formal measure defined on the space Ω_P^o of the integer valued measures $\rho(x) = \sum_{j\geq o} \delta(x - x_j)$, such that $X = (x_1, x_2, \ldots.)$ is a finite sequence, by the equality

$$d_P\rho = \sum_{n\geq o} \frac{1}{n!} dX_n.$$

Properly normalized $d_P\rho$ generates the Poisson measure $\pi_o(d\rho)$ on the space Ω_P of integer valued measures such that X is an infinite locally finite sequence.If X is restricted to Λ, then $\pi_o(d^\Lambda\rho) = \exp\{-V(\Lambda)\} d_P^\Lambda\rho$, where $V(\Lambda)$ is the volume of Λ.

The Gibbs particle system is described by the Gibbs measure μ_P^Λ

$$(1.1) \qquad \mu_P^\Lambda = (\Xi_P^\Lambda)^{-1} \exp\{-\beta U_\Lambda(\rho)\} d_P\rho$$

in this formula $d_P^\Lambda\rho$ can be written, since

$$U_\Lambda(\rho) = \sum_{k\geq o} \int_{\Lambda^k} \Phi_k(X_k)\rho(x_1)\cdots\rho(x_k)dX_k,$$

where $\Phi_1 = -\mu$, $\Phi_2 = C_o$, Φ_k is a k-body potential, that can depend on Λ.

Now let us consider the Gibbs field system. The corresponding Gibbs measure is formally given on Ω_F^o by μ_F^Λ

$$(1.2) \qquad \mu_P^\Lambda = (\Xi_F^\Lambda)^{-1} \exp\{-L_\Lambda(\varphi)\} d_F\varphi ,$$

where $d_F\varphi = \prod_{x\in\mathbb{R}^d} d\varphi(x)$, $\Omega_P^o = \mathbb{R}^{\mathbb{R}^d}$, $\Xi_F^\Lambda = \int_{\Omega_F^o} \exp\{-L_\Lambda(\varphi)\} d_F\varphi$

It is possible to give a rigorous meaning for (1.2) if the

Lagrangian L_Λ is given by

$$L_\Lambda(\varphi) = \int_{\mathbb{R}^d} (Q(\Delta)\varphi)(x)\varphi(x)dx + V_\Lambda(\varphi)$$

where Δ is the laplacian, and $C_o(o) < \infty$.

Then the Gibbs measure of the field system is given by

$$(1.3) \qquad \mu_F^\Lambda(d\varphi) = (\Xi_F^\Lambda)^{-1} \exp\{ - V_\Lambda(\varphi)\} \mu_o(d\varphi)$$

where μ_o is the Gaussian measure with the covariance $C_o(x-y)$,and

$$(1.4) \qquad \Xi_F^\Lambda = \int_{\Omega_F} \exp\{ - V(\varphi)\}\mu_o(d\varphi)$$

Ω_F is the probability space which is usually a subset of Ω_F^o. We shall omit it in all the integrals in what follows.
(1.3) is derived from (1.2) by multiplying the numerator and the denominator by

$$(\mathrm{Det}\ \pi^{-1}Q(\Delta))^{\frac{1}{2}}$$

and using the equality

$$(1.5) \qquad \mu_o(d\varphi) = (\mathrm{Det}Q(\Delta))^{\frac{1}{2}}\exp\{ - (Q(\Delta)\varphi,\varphi)\}d_F\frac{\varphi}{\sqrt{\pi}}$$

PROPOSITION 1.1(S-G transformation)

If $V_\Lambda(\varphi) = - \hat{z} \int_\Lambda \exp\{ i\sqrt{\beta}\varphi(x)\}dx$, $z=\hat{z}\exp\{2^{-1}\beta C_o(0)\}$, $\Phi_k=0,k>3$,then

$$(1.6) \qquad \Xi_P^\Lambda = \Xi_F^\Lambda$$

PROOF is simple and it is based on the expansion of $\exp\{ - V_\Lambda(\varphi)\}$ into series and application of the formula

$$\int\mu_o(d\varphi) \exp\{ \sum_{j=1}^n i\sqrt{\beta}\ \varphi(x) \} = \exp \{ - \beta \sum_{i\le j=1}^n C_o(x_i-x_j) \}$$

2. GIBBS SYSTEMS OF INTERACTING PARTICLES AND SCALAR FIELDS

The phase space of the system of interacting particles and fields is $\Omega^{o} = \Omega^{o}_{F} \times \Omega^{o}_{P}$. The potential energy is given by

$$U_{\Lambda}(\rho,\varphi) = U_{\Lambda}(\varphi) + \int_{\Lambda} V_{x}(\varphi)\rho(x)dx + V_{\Lambda}(\rho,\varphi) \tag{2.1}$$

$$U_{\Lambda}(\varphi) > -\infty,\ V_{x}(\varphi) > -\infty,\ V_{\Lambda}(\rho,\varphi) > -\infty .$$

where $V_{\Lambda}(\rho,\varphi)$ is a nonlinear function of ρ .It is sufficient to consider the case when it is zero.

The Gibbs system is characterized by the formal measure $\mu^{\Lambda}_{F,P}$

$$\mu^{\Lambda}_{F,P}(d\varphi,d\rho) = (\Xi^{\Lambda}_{F,P})^{-1}\delta(Q(\Delta)\varphi - \rho)\exp\{ - \beta U_{\Lambda}(\rho,\varphi)\}d_{F}\varphi\ d_{P}\rho \tag{2.2}$$

where

$$\delta(Q(\Delta) - \rho) = \prod_{x\in\mathbb{R}^{d}} \delta((Q(\Delta)\varphi)(x) - \rho(x))$$

$$\Xi^{\Lambda}_{F,P} = \int_{\Omega^{o}} \exp\{ - \beta U_{\Lambda}(\rho,\varphi)\}\delta(Q(\Delta)\varphi - \rho)d_{F}\varphi\ d_{P}\rho \tag{2.3}$$

PROPOSITION 2.1(Formal S-G transformation)

If the potential energy $U_{\Lambda}(\rho)$ of the Gibbs particle system depends on the potential energy $U_{\Lambda}(\rho,\varphi)$ of the system of interacting particles and scalar field as follows

$$U_{\Lambda}(\rho) = U_{\Lambda}(\rho,Q^{-1}(\Delta)\rho) \tag{2.4}$$

then

$$\Xi^{\Lambda}_{p} = \hat{\Xi}^{\Lambda}_{F,P} = (\mathrm{Det}Q(\Delta))\ \Xi^{\Lambda}_{F,P} \tag{2.5}$$

PROOF: With the help of the equality

$$(\mathrm{Det}Q(\Delta))\delta(Q(\Delta)\varphi - \rho) = \delta(\varphi - Q^{-1}(\Delta)\rho)$$

and (2.4) we obtain

$$\hat{\Xi}^{\Lambda}_{F,P} = \int d_P\rho \int d_F\varphi \exp\{ - \beta U_\Lambda(\rho,\varphi)\}\delta(\varphi - Q(\Delta)^{-1}\rho) =$$

$$\int d_P\rho \exp\{ - \beta U_\Lambda(\rho)\} = \Xi^{\Lambda}_P$$

PROPOSITION 2.2 (FORMAL S-G TRANSFORMATION)
For the grand partition function $\hat{\Xi}^{\Lambda}_{F,P}$ of the Gibbs system of interacting particles and scalar field the following representation holds

(2.6)

$$\hat{\Xi}^{\Lambda}_{F,P} = \int \mathrm{Det}\sqrt{Q(\Delta)}\, d_F\frac{\varphi^*}{\sqrt{\pi}} \int \mathrm{Det}\sqrt{Q(\Delta)}\, d_F\frac{\varphi}{\sqrt{\pi}} \exp\{ - L^*_\Lambda(\varphi,\varphi^*)\}\exp\{i(Q(\Delta)\varphi,\varphi^*)\}$$

where $$L^*_\Lambda(\varphi,\varphi^*) = - \beta U_\Lambda(\varphi) + \int_\Lambda \exp\{i\varphi^*(x) - \beta V_x(\varphi)\}dx.$$

PROOF:Let us use the formula

$$\delta(\varphi) = \int \exp\{i(\varphi^*,\varphi)\}\, d\frac{\varphi}{\pi}$$

Then substitute this equality into (2.3) we obtain

$$\hat{\Xi}^{\Lambda}_{F,P} = \int \mathrm{Det}\sqrt{Q(\Delta)}\, d_F\frac{\varphi^*}{\sqrt{\pi}} \int \mathrm{Det}\sqrt{Q(\Delta)}\, d_F\frac{\varphi}{\sqrt{\pi}} \exp\{i(Q(\Delta)\varphi,\varphi^*) - \beta U_\Lambda(\varphi)\}\times$$

$$\times [\int \exp\{i(\varphi^*,\rho) - \beta\int_\Lambda V_x(\varphi)\rho(x)dx\; d_P\rho =$$

$$= \exp\{- L_\Lambda(\varphi^*,\varphi) + \beta U_\Lambda(\varphi)\}.$$

Let us show now that the S-G transformation, i.e. eq. (1.6) follows from eq.(2.6).

To do it we have to assume that

$$U_\Lambda(\rho,\varphi) = -(\mu + C_o(0)\int_\Lambda \rho(x)dx + (Q(\Delta)\varphi,\varphi)$$

In this case $U_\Lambda(\varphi)$ is the quadratic form and

$$V_x(\varphi) = - \mu - C_o(0), \qquad z\int \exp\{i\varphi^*(x) - \beta V_x(\varphi)\} = V_\Lambda(\tfrac{\varphi}{\sqrt{\beta}}).$$

Then (2.6) yields

$$\Xi^\Lambda_{F,P} = \int \mathrm{Det}\sqrt{Q(\Delta)}\, d_F\tfrac{\varphi^*}{\sqrt{\pi}} \exp\{ - V_\Lambda(\tfrac{\varphi}{\sqrt{\beta}})\} \int \mathrm{Det}\sqrt{Q(\Delta)}\; d_F\tfrac{\varphi}{\sqrt{\pi}} \times$$

$$\times \exp\{ - \beta(Q(\Delta)\varphi,\varphi) + i(Q(\Delta)\varphi,\varphi^*)\}.$$

Now let us rescale the fields:$\varphi \Rightarrow \sqrt{\beta}\varphi$, $\varphi^* \Rightarrow \frac{1}{\sqrt{\beta}}\varphi^*$. So β disappears in the right side of the last equality. Taking into account (1.5) we derive the following equality

$$\hat{\Xi}^\Lambda_{F,P} = \int \mu_o(d\varphi^*)\exp\{- V_\Lambda(\varphi^*) + \tfrac{1}{2}(Q(\Delta)\varphi^*,\varphi^*)\}\int\mu_o(d\varphi)\exp\{i(Q(\Delta)\varphi,\varphi)\}.$$

The second integral is easilly computed

$$\int \mu_o(d\varphi)\exp\{i(Q(\Delta)\varphi,\varphi^*)\} = \exp\{ - \tfrac{1}{2}(Q(\Delta)\varphi^*,\varphi^*)\}$$

Hence we obtain the usual S-G transformation.

3.The generalized S-G transformation.

There are cases when the formula (2.6) can be written in a rigorous form (MAIN THEOREM)

THEOREM 3.1.

Let potentuial energy $U_\Lambda(\rho,\varphi)$ in (2.1) is such that

$$U_\Lambda(\varphi) = (Q(\Delta)\varphi,\varphi) + U^*_\Lambda(\varphi), \quad U^*_\Lambda(\varphi)>-\infty .$$

then the following equality holds

$$(3.1) \quad \hat{\Xi}^{\Lambda}_{F,P} = \int \mu_o(d\varphi^*) \int \mu_o(d\varphi)\exp\{i(\varphi,\varphi^*)_Q + \tfrac{1}{2}(\varphi^*,\varphi^*) - \overset{*}{L}_\Lambda(\varphi,\varphi^*)\}$$

where

$$\overset{*}{L}_\Lambda(\varphi,\varphi^*) = \beta\overset{*}{U}_\Lambda(\varphi) - \int_\Lambda \exp\{i\varphi^*(x) - \beta V_x(\varphi)\}dx, \quad (\varphi,\varphi^*)=(Q(\Delta)\varphi,\varphi^*).$$

If the potential energy $U_\Lambda(\rho)$ of the particle Gibbs system satisfies then Ξ^Λ_P equals the right side of (3.1) also.

PROOF follows from props. (2.1), (2.2), and eq. (1.5).

It follows from the main theorem that

$$(3.2) \qquad \Xi^\Lambda_P = \int \mu_o(d\varphi^*)\exp\{- \overset{*}{L}_\Lambda(\varphi^*)\}$$

where $\overset{*}{L}_\Lambda(\varphi^*)$ can be interpreted as an effective Lagrangian.

In the case when 2k+1-body potentials are absent,i.e.

$$V_x(\varphi) = -\mu - C_o(0)$$

then this Lagrangian is the sum of two terms

(3.3)

$$L^*_\Lambda(\varphi^*) = V_\Lambda(\varphi^*) - \ln[\exp\{\tfrac{1}{2}(\varphi^*,\varphi^*)_Q\}\int \mu_o(d\varphi)\exp\{i(\varphi,\varphi^*)_Q - \beta U^*_\Lambda(\varphi)\}]$$

There is the problem how to make the measure $\mu^\Lambda_{F,P}$ rigorous. It is cleare that we have to make meaningful the "probability" measure

$$\mu^{*\Lambda}_{F,P} = (\Xi^{*\Lambda}_{F,P})^{-1}\exp\{-\beta U^*_\Lambda(\varphi) - \int_\Lambda V_x(\varphi)\rho(x)dx\}\ \mu_o(d\varphi)\pi_o(d\rho)\delta(\varphi - Q(\Delta)^{-1}\rho)$$

We shall make it somewhere else.

Let us consider some examples.

$$.V_x(\varphi) = V^o_x(\varphi) + V^*_x(\varphi),\quad V^o_x(\varphi) = -\mu - C_o(0).$$

1 If $V_x(\varphi) = \varphi^{21}(x)$, then

$$U_\Lambda(\rho) = U(X) = \sum_{j>0}\ (\sum_{i>0}\ C_o(x_j - x_i))^{21}.$$

2.If $\qquad V^*_x(\varphi) = 0,\quad U^*_\Lambda(\varphi) = (\int_\Lambda \varphi^2(x)dx^2)^1,$ then

$$(3.4)\ U_\Lambda(\rho) = U(X) = (\sum_{i,j} C_\Lambda(x_j,x_i)\ )^{21}, C_\Lambda(x,y) = \int_\Lambda C_o(x-z)C_o(y-z)dz.$$

The effective Lagrangian can be derived in the latter case with the help of the theorem

THEOREM 3.2

If the potential energy of the particle Gibbs system satisfies (3.4) then

$$(3.5)\ \Xi_P^\Lambda = \int \mu_o(d\varphi)\exp\{ - V_\Lambda(\varphi)\} \int m_l(q)\{\mathrm{Det}(I + i\beta^{(2l)^{-1}} q\hat{\chi}_\Lambda Q(\Delta)^{-1})\}^{\frac{1}{2}}\times$$

$$\times\exp\{i(C^*_{\Lambda,q}\varphi,\varphi)\}dq, \quad C^*_{\Lambda,q} = \beta^{(2l)^{-1}} q\hat{\chi}_\Lambda(Q(\Delta) + iq\beta^{(2l)^{-1}}\hat{\chi}_\Lambda)^{-1}Q\ (\Delta)$$

where $\hat{\chi}_\Lambda$ is the operator of the multiplication by the characteristic function of the compact domain Λ .

PROOF is simple and it can be found in [6].

4.DISCUSSION

The Gibbs system described by the measure (2.2)reflects the fact that in order to derive the generalized S-G transformation we have to introduce the field φ. It has the meaning of the electric field created by the distribution of charges.

There is a suspicion that the introduction of the Gibbs system of interacting scalar field and particles is not really needed for the derivation of the generalized S-G transformation, since the introduction of the measure $\delta(Q(\Delta)\varphi-\rho)$ means meerly the change of the variable. This view is wrong in general since it is impossible to represent in terms of the bounded below function of the field variable a 2k+1-body potential.

There is also the question: why have we to omitt the last term in (2.1)? The only answer is that we don't have examples that it contributes to the potential energy of the particle system. If this term is left then the particle variable is not easilly integrated out.

In conclusion the author expresses his sincere gratitude to professor John Lewis for inviting him to the Dublin Institute of Advanced Study and kindness and thanks Nick Duffield for help with printing.

REFERANCES

1.Siegert A.J.F. Physica,1960, v.26,p.30.

2.Kac M. Probability and related topics in physics,AMS, 1966.

3.Skrypnik W.I. Teor.mat.fiz.,1976,v.29,p.323.

4.Brydges D. Commun.math.phys.,1978, v.53,p.313.

5.Skrypnik W.I. Preprint of the Dublin Institute of Advanced Study DIAS-STP 89-33, 1989.

6.Skrypnik W.I. Teor.mat.fiz,1991,in print.