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On holomorphic factorization of two-dimensional gravity action

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Abstract

We solve the integrability conditions for the local covariant formulation of the induced action of 2d-gravity and propose gauge conditions under which the chiral fermion action is an expansion of the Polyakov action in the case when both functions $f^{\pm}$ are retained.

Polyakov's famous work on the quantization of 2d-gravity [1] is restricted to the case of a metric with Minkovskian signature. On the other hand, the classification of closed two-dimensional surfaces exists only in the case with euclidean signature.

These approaches are consistent if the theory of two-dimensional gravity can be represented as the sum of independent holomorphic and antiholomorphic parts as happens in 2d-conformal field theory [2].

This problem was stated first by R. Stora. There are different approaches to this problem. One of these is developed by Stora's group: they integrate the diffeomorphism anomaly on a Riemann surface of arbitrary genius $g \geq 2$ [3]. The holomorphic factorization of the effective action

$$\Gamma[\mu,\bar{\mu}] = \Gamma[\mu] + \Gamma^*[\bar{\mu}] \tag{1}$$

is postulated in this work.

Because the Einstein-Hilbert action in two dimensions is trivial and reduces to the Euler characteristic of the surface it should be more interesting to consider the theory of induced 2d-gravity. This theory appears in particular in the context of string theory.

There is a hypothesis that all induced theories are universal [4]; in particular theories of 2d gravitation induced by scalars, fermions, etc., are equivalent.

The 2d-gravity induced by chiral fermions is a remarkable theory - in this case almost all known anomalies (conformal, gauge and gravitational) are present.

In our opinion the most convenient one to analyse is the theory obtained by considering three-dimensional fermions interacting with a surface embedded in R^3 , i.e. the induced Dirac theory.

This approach is the one closest to string theory and allows us to investigate the dynamics of the interaction of the embedded surface with the ambient space.

The embedding of a surface in R^3 is described by

$$\Sigma : \quad X^\mu = X^\mu(\xi^\alpha) \tag{2}$$

where X^μ are coordinates in R^3 and ξ^α is ones on Σ. The embedding induces a metric on Σ :

$$g_{\alpha\beta}(\xi) = \partial_\alpha X^\mu \cdot \partial_\beta X^\mu \tag{3}$$

The induced Dirac action is described by projection of the usual flat Dirac action in R^3 on Σ :

$$S_{ID} = \frac{i}{2} \int_\Sigma d^2\xi \sqrt{g}\, g^{\alpha\beta} \partial_\alpha X^\mu (\bar{\psi} \gamma_\mu \overleftrightarrow{\partial_\beta} \psi) \tag{4}$$

where $\psi(\xi)$ and $\bar{\psi}(\xi)$ are spinors in R^3 .

To translate the interaction of a surface with R^3 , induced by embedding, to the more familiar interaction with gravitational and gauge fields, we introduce the zweibeins of the induced metric

$$e^a_\alpha(\xi) \cdot e^a_\beta(\xi) = g_{\alpha\beta}(\xi) \tag{5}$$

and the vector field n_μ orthogonal to surface:

$$n_\mu(\xi) \cdot \partial_\alpha X^\mu(\xi) = 0 \quad , (n_\mu)^2 = 1 \tag{6}$$

Then we obtain an orthonormal basis of R^3 at each point of Σ :

$$\{n^\mu(\xi),\ X^\mu_a(\xi)\}$$

$$X^\mu_a(\xi) = e^\alpha_a(\xi) \cdot \partial_\alpha X^\mu \ , \quad X^\mu_a \cdot X^\mu_b = \delta_{ab} \tag{7}$$

We introduce also an SO(3)-matrix $\Omega(\xi)$ which rotates this basis to a fixed one:

$$\gamma^a(\xi) = X^a_\mu(\xi)\cdot\gamma^\mu = \Omega^{-1}(\xi)\sigma^a\Omega(\xi)$$
$$n(\xi) = n_\mu(\xi)\cdot\gamma^\mu = \Omega^{-1}(\xi)\sigma^3\Omega(\xi) \tag{8}$$

here $\sigma^- = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$, $\sigma^+ = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli-matrices and the γ^μ differ from them by a global rotation.

As a result, we obtain

$$S_{ID} = \int_\Sigma d^2\xi\sqrt{g}\,\bar\psi \not{D}\psi = \int_\Sigma d^2\xi\sqrt{g}(\bar\psi_L D_L\psi_L + \bar\psi_R D_R\psi_R)$$
$$\not{D} = i\Omega^{-1}\sigma^a e^\alpha_a(\partial_\alpha - \tfrac{1}{4}\Gamma_\alpha^{\ bc}\sigma_{bc})\Omega \tag{9}$$

where $\psi_{L \atop R} = \frac{1}{2}(1 \pm n(\xi))\psi$ and $\Gamma_\alpha^{\ bc} = -e^b_\beta\nabla_\alpha e^{\beta c}$ is the spin-connection related to Σ . Here we have used the relation

$$(X^a_\mu dX^a_\nu + n_\mu dn_\nu)\gamma^\mu\cdot\gamma^\nu = 4\Omega^{-1}d\Omega = \Gamma^{ab}\gamma_a\gamma_b + 2h^a\cdot\gamma_a\cdot n \tag{10}$$

which relates the matrix Ω to the expansion coefficients:

$$dX^a_\mu = \Gamma^{ab}\cdot X^b_\mu + h^a\cdot n_\mu$$
$$dn_\mu = -h^a\cdot X^a_\mu \tag{11}$$

The induced Dirac operator coincides with the usual one except for the presence of the matrix $\Omega(\xi)$. When the surface is embedded in R^d , the induced Dirac operator contains also an $SO(d-2)$ gauge field. However the fields parametrizing operator (9) are not completely independent. The equations (11) have the integrability conditions:

$$d\Gamma^{ab} = -h^a \wedge h^b$$
$$dh^a = \Gamma^{ab} \wedge h^b \tag{12}$$

besides these, there is one more constraint:

$$h^a \wedge e^a = 0 \tag{13}$$

this expresses the symmetry of the second quadratic form of the surface:

$$h_{\alpha\beta} = n_\mu \cdot \partial_\alpha \partial_\beta X^\mu = h_{\beta\alpha} = e^a_\alpha \cdot h^a_\beta \tag{14}$$

Computing the determinant of the chiral Dirac operator in the usual reparametrisation - invariant manner, we obtain the effective action for chiral fermions which, in our interpretation, is an action of 2d gravity:

$$S_{eff} = \frac{1}{2}\int_\Sigma d^2\xi \sqrt{g}\, g^{\alpha\beta}\, \Gamma^{ab}_\alpha \Gamma^{ba}_\beta \pm \frac{1}{2}\int_B \Gamma^{ab} d\Gamma^{ba} \tag{15}$$

here B is any three-manifold whose boundary is Σ. The first term in the action (15) is one ordinary Liouville action plus the kinetic term related to the angle of Lorentz-rotations. The sign $\pm$ corresponds to right and left fermions through the projection operators $1/2\,(1 \pm n(\xi))$. By using the integrability condition (12), it is easy to show that the Chern-Simons term in (15) becomes the integral

$$\int_B tr(d\Omega \cdot \Omega^{-1})^3 \tag{16}$$

of the density of the Hopf-invariant of the mapping $S^3 \to S^2$. (When the integrand is a closed 3-manifold, it is equal to an integral multiple of $1/8\pi^2$).

To represent the 3-form (16) in more convenient form, we parametrize the fields $h^\pm$ as

$$h^\pm = exp(\phi \pm i\alpha_e) \cdot df^\pm \tag{17}$$

The light-cone coordinates are defined by $f^{\pm} = (f^1 \pm i f^2)/\sqrt{2}$ The second constrain (12) then gives us

$$dh^{\pm} = (d\phi \pm i d\alpha_e)\wedge h^{\pm} = \pm 2\Gamma \wedge h^{\pm}$$

or

$$\begin{aligned} d\phi + i d\alpha_e - 2\Gamma &= 2a_+ \cdot df^+ \\ d\phi - i d\alpha_e + 2\Gamma &= 2a_- \cdot df^- \end{aligned} \qquad (18)$$

Inserting this in the first constrain, we see that it yields the Liouville equation for ϕ :

$$\frac{\partial^2 \phi}{\partial f^+ \partial f^-} = -\exp(2\phi) \qquad (19)$$

To solve this, we use the method described in [5]. Namely, we make the substitution $\exp(-\phi) = \rho$ then (19) becomes

$$\rho \partial_+ \partial_- \rho - \partial_+ \rho \cdot \partial_- \rho = 1 \qquad (20)$$

Differentiating this equation with respect to f^+, we obtain

$$\rho^2 \partial_- \left(\frac{\partial_+^2 \rho}{\rho} \right) = 0 \quad , \quad i.e. \quad \partial_+^2 \rho = \rho \cdot M(f^+) \qquad (21)$$

Analogously, we obtain $\rho^2 \partial_+ \left(\partial_-^2 \rho / \rho \right) = 0$ and $\partial_-^2 \rho = \rho \cdot N(f^-)$
For a given function $N(f^-)$, this equation has two independent solutions $b(f^-)$ and $B(f^-)$ so that ρ can be written as

$$\rho(f^+, f^-) = a(f^+) b(f^-) + A(f^+) \cdot B(f^-) \qquad (22)$$

Inserting this in (21), and taking account of the independence of $b(f^-)$ and $B(f^-)$, we obtain

$$\frac{a''(f^+)}{a(f^+)} = \frac{A''(f^+)}{A(f^+)} = M(f^+)$$

It follows that Wronskian of a and A is zero:

$$a''\cdot A - a\cdot A'' = (a'\cdot A - a\cdot A')' = 0 \text{ so that } a'A - aA' = C = const \tag{23}$$

Introducing more convenient functions

$$p(f^-) = B(f^-)/b(f^-) \quad \text{and} \quad q(f^+) = A(f^+)/a(f^+) \tag{24}$$

we have

$$q'(f^+) = C/a^2(f^+) \tag{25}$$

and analogously $\quad p'(f^-) = \tilde{C}/b^2(f^-)$
so that (22) now reads

$$\rho(f^+, f^-) = (1 + p(f^-)\cdot q(f^+))\sqrt{\frac{C\tilde{C}}{p'(f^-)q'(f^+)}} \tag{26}$$

Bringing this into (20), we find that $\sqrt{C\tilde{C}} = 1$ (27)

Introducing a new variable, $\quad i\alpha = i\alpha_e + \frac{1}{2}\ln\frac{p'(f^-)}{q'(f^+)}$ (28)

we obtain expressions for Γ and $h^{\pm}$:

$$\begin{aligned} 2\Gamma &= i d\alpha + \frac{p dq - q dp}{1 + pq} \\ h^+ &= \frac{\exp(i\alpha)}{1 + pq} dq \\ h^- &= \frac{\exp(-i\alpha)}{1 + pq} dp \end{aligned} \tag{29}$$

the matrix $\Omega(\xi)$ is expressed in terms α, p and q in a local manner by

$$\Omega(\xi) = \frac{1}{\sqrt{1 + pq}} \begin{pmatrix} \exp(i\alpha/2) & q\cdot\exp(i\alpha/2) \\ -p\cdot\exp(-i\alpha/2) & \exp(-i\alpha/2) \end{pmatrix} \tag{30}$$

Comparing $$\Omega^{-1} = \frac{1}{\sqrt{1+pq}} \begin{pmatrix} \exp(-i\alpha/2) & -q\cdot\exp(i\alpha/2) \\ p\cdot\exp(-i\alpha/2) & \exp(i\alpha/2) \end{pmatrix}$$

with $$\Omega^{\dagger} = \frac{1}{\sqrt{1+p^*q^*}} \begin{pmatrix} \exp(-i\alpha^*/2) & -p^*\exp(i\alpha^*/2) \\ q^*\exp(-i\alpha^*/2) & \exp(i\alpha^*/2) \end{pmatrix}$$

we find that the unitarity condition $\Omega^{\dagger} = \Omega^{-1}$ is equivalent to

$$p(f^-) = (q(f^+))^* \qquad \text{and} \qquad \alpha = \alpha^* \tag{31}$$

in other wor ds, α is real and p is the complex conjugate of q
The 3-form (16) is now written as

$$\int_{\Sigma} d^2f \cdot \alpha(f) \cdot R\sqrt{g}(f) \tag{32}$$

We parametrize the zweibeins of the induced metric as

$$e^{\pm} = \exp(\theta \pm i\alpha_e) \cdot dF^{\pm} \tag{33}$$

and introduce

$$\begin{aligned} \lambda &= \partial F^+/\partial f^+ \quad , \quad \lambda\cdot\mu = \partial F^+/\partial f^- \\ \bar\lambda &= \partial F^-/\partial f^- \quad , \quad \bar\lambda\cdot\bar\mu = \partial F^-/\partial f^+ \end{aligned} \tag{34}$$

Notice that the Lorentz-angle α_e in (33) is the same as in (17) because the dependence of $h^{\pm}$ on α_e comes from the zweibeins (the second quadratic form (14) is invariant with respect to Lorentz-rotations).

In fact, (34) relates the conformal structure defined by the induced metric with the one defined by the second quadratic form.

We propose that the effective action is computed when the Weil - and Lorentz-symmetries are fixed by the conditions:

$$2\theta = -\ln \lambda\cdot\bar\lambda \tag{35a}$$

and

$$2\alpha_\ell = -i \ln \lambda/\bar{\lambda} \tag{35b}$$

which are equivalent to $e^+_+ = 1 \;,\; e^+_- = \mu$
$e^-_+ = \bar{\mu} \;,\; e^-_- = 1$

As a result, we obtain that in the gauge (35), the effective action of the left fermions is

$$S_{left} = \int_\Sigma \frac{d^2f}{1-\mu\bar{\mu}} (\partial_+ - \mu\partial_-)\ln\lambda \cdot (\partial_- - \bar{\mu}\partial_+)\ln\lambda + \frac{i}{4}\int_\Sigma \ln\frac{p'}{q'} \cdot R\sqrt{g}\, d^2f \tag{36}$$

and $S_{right} = S^*_{left}$

The energy-momentum tensor corresponding to (36),

$$T_{\alpha\beta} = \partial_\alpha \ln\lambda \cdot \partial_\beta \ln\lambda - \frac{1}{2} g_{\alpha\beta}(g^{\gamma\delta}\partial_\gamma \ln\lambda \cdot \partial_\delta \ln\lambda) \tag{37}$$

is traceless. (The second term gives no contribution to the stress-tensor because the variation of $R\sqrt{g}$ on the metric is $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \cdot R = 0$ in two dimensions).
Integrating by parts in (36), we have

$$S_{left} = S_P(F^+) + \frac{1}{2}\int_\Sigma \alpha_{rep} \cdot R\sqrt{g} \cdot d^2f$$
$$S_{right} = S_P(F^-) + \frac{1}{2}\int_\Sigma \alpha_{rep} \cdot R\sqrt{g} \cdot d^2f \tag{38}$$

where $\alpha_{rep} = i \ln p'/p'^*$
the expression

$$S_P(F^+) = \int \frac{d^2f}{1-\mu\bar{\mu}} \cdot \partial_+\mu \cdot (\partial_+ - \bar{\mu}\cdot\partial_-)\ln\lambda \tag{39}$$

coincides with Polyakov's action when $\mu = 0$ and $\lambda = 1$ i.e. when $F^-(f) = f^-$

If we try to recognize this answer starting from Polyakov's action and make an opposite reparametrization to that which yields the light-cone gauge.

$$f^{-\prime} = F^{-}(f^{+}, f^{-}) \tag{40}$$

we see that $d^2f'(\partial_-' - \mu'\partial_+')\ln\lambda' \cdot \partial_+' \ln\lambda' \rightarrow$

$$\rightarrow \frac{d^2f}{1-\mu\bar{\mu}}(\partial_- - \mu\partial_+)\ln\lambda\cdot(1-\mu\bar{\mu})\cdot(\partial_+ - \bar{\mu}\partial_-)\ln\lambda\cdot(1-\mu\bar{\mu})$$

and to obtain S_{left}, we should make additional Weil-

$$\theta \rightarrow \theta - \frac{1}{2}\ln(1-\mu\bar{\mu}) \tag{41a}$$

and Lorentz

$$\alpha \rightarrow \alpha + \frac{1}{2}\ln p'(f^-) \tag{41b}$$

transformations which are needed to compensate (37) so that the gauge conditions remain unchanged.

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