DIAS-STP-92-29

MONSTROUS MOONSHINE AND THE UNIQUENESS OF THE MOONSHINE MODULE *

Michael P. Tuite †
Department of Mathematical Physics
University College
Galway, Ireland

and

Dublin Institute for Advanced Studies
10 Burlington Road
Dublin 4, Ireland

ABSTRACT

In this talk we consider the relationship between the conjectured uniqueness of the Moonshine module $\mathcal{V}^\natural$ of Frenkel, Lepowsky and Meurman and Monstrous Moonshine, the genus zero property for Thompson series discovered by Conway and Norton. We discuss some evidence to support the uniqueness of $\mathcal{V}^\natural$ by considering possible alternative orbifold constructions of $\mathcal{V}^\natural$ from a Leech lattice compactified string. Within these constructions we find a new relationship between the centralisers of the Monster group and the Conway group generalising an observation made by Conway and Norton. We also relate the uniqueness of $\mathcal{V}^\natural$ to Monstrous Moonshine and argue that given this uniqueness, then the genus zero properties hold if and only if orbifolding $\mathcal{V}^\natural$ with respect to a monster element reproduces $\mathcal{V}^\natural$ or the Leech theory.

---

* Talk presented at the Nato Advanced Research Workshop on 'Low dimensional topology and quantum field theory', Cambridge, 6-13 Sept 1992.

† EMAIL: mphtuite@bodkin.ucg.ie

The Moonshine Module. The Moonshine module [1] of Frenkel, Lepowsky and Meurman (FLM) is the first example of an orbifold CFT [2] and is constructed from a string compactified to R^{24}/Λ where Λ is the Leech lattice, the unique 24 dimensional even self-dual lattice without roots i.e. $\lambda^2 \neq 2$ cf. [3]. The orbifolding is then based on the Z_2 reflection automorphism of Λ.

Let $\mathcal{V}^\Lambda$ denote the set of vertex operators $\{\phi(z)\}$ for the Leech lattice CFT which forms a closed meromorphic operator product algebra (OPA) with central charge 24 [1,4]

$$\phi_i(z)\phi_j(w) \sim \sum_k C_{ijk}(z-w)^{h_k - h_i - h_j}\phi_k(w) + \ldots \tag{1}$$

We will represent such an OPA schematically by $\phi\phi \sim \phi$. The 1-loop partition function $Z(\tau) = \mathrm{Tr}_{\mathcal{V}^\Lambda}(q^{L_0})$ is a modular invariant and meromorphic function of τ with a unique simple pole at $q = e^{2\pi i\tau} = 0$ and is given by the unique (up to an additive constant) modular invariant function $J(\tau)$

$$\begin{aligned} Z(\tau) &= J(\tau) + 24 \\ J(\tau) &= \frac{E_2^3}{\eta^{24}} - 744 = \frac{1}{q} + 0 + 196884q + \ldots \end{aligned} \quad . \tag{2}$$

The constant 24 reflects the existence of 24 massless (conformal weight 1) operators in this theory. $\eta(\tau) = q^{1/24}\prod_n(1-q^n)$ and $E_2(\tau)$ is the Eisenstein modular form of weight 4 [5].

The FLM Moonshine module [1] is an orbifold CFT based on the Z_2 lattice reflection automorphism $\bar{r} : \lambda \to -\lambda$ for $\lambda \in \Lambda$. $\bar{r}$ lifts to a family of Z_2 automorphisms of $\mathcal{V}^\Lambda$ preserving (1) from which family one automorphism r is chosen. Defining the projection $\mathcal{P}_r = (1+r)/2$, the set of operators $\mathcal{P}_r\mathcal{V}^\Lambda$ then also form a closed meromorphic OPA. However, the corresponding partition function $\mathrm{Tr}_{\mathcal{P}_r\mathcal{V}^\Lambda}(q^{L_0}) = \frac{1}{2}({}^{1}\underset{1}{\square} + {}^{r}\underset{1}{\square})$ is not modular invariant, employing standard notation for the world-sheet torus boundary conditions e.g. [6]. Thus, under a modular transformation $\tau \to -1/\tau$, ${}^{r}\underset{1}{\square} = 1/\eta_{\bar{r}}(\tau) \to {}^{1}\underset{r}{\square} = 2^{12}\eta_{\bar{r}}(\tau/2) = 2^{12}q^{1/2} + \ldots$ where $\eta_{\bar{r}}(\tau) = [\eta(2\tau)/\eta(\tau)]^{24}$. Therefore the introduction of a 'twisted' sector with vacuum energy $1/2$ and degeneracy 2^{12} is necessary to form a modular invariant theory [1,2]. The states of this sector are constructed from twisted vertex operators $\mathcal{V}_r = \{\psi(z)\}$ acting on the untwisted vacuum. Thus $\mathcal{V}^\Lambda$ is enlarged by $\mathcal{V}_r$ to $\mathcal{V}' = \mathcal{V}^\Lambda \oplus \mathcal{V}_r$ which forms a closed non-meromorphic OPA [1,7,8,9] where (schematically)

$$\phi\phi \sim \phi, \qquad \phi\psi \sim \psi, \qquad \psi\psi \sim \phi \tag{3}$$

$\bar{r}$ can also be lifted to an automorphism r of (3) where the operators of $\mathcal{P}_r\mathcal{V}_r$ have integral conformal weight. Then $\mathcal{V}^\natural = \mathcal{P}_r\mathcal{V}'$ forms a closed meromorphic OPA, the FLM Moonshine module [1]. The r projection ensures the absence of untwisted massless operators whereas the twisted sector operators are all massive since the twisted vacuum energy is 1/2. Thus the orbifold partition function is

$$\mathrm{Tr}_{\mathcal{V}^\natural}(q^{L_0}) = \mathcal{P}_r \underset{1}{\square} + \mathcal{P}_r \underset{r}{\square} = J(\tau) \tag{4}$$

The absence of massless operators in $\mathcal{V}^\natural$ sets the Moonshine module apart from other CFTs. Usually such operators are present and form a closed Kac-Moody algebra. However, the 196884 conformal weight 2 operators in $\mathcal{V}^\natural$, including the energy-momentum tensor $T(z)$ can be used to define a closed non-associative commutative algebra. FLM demonstrated [1] that this algebra is an affine version of the 196883 dimensional Griess algebra [10] together with $T(z)$. The automorphism group of the Griess algebra is the Monster M. FLM showed that M is the automorphism group for the OPA of $\mathcal{V}^\natural$ where the operators of $\mathcal{V}^\natural$ of a given conformal weight form a (reducible) representation of M. This demonstrates an observation of McKay and Thompson [11] that the coefficients of $J(\tau)$ are positive sums of dimensions of irreducible representations of M e.g. the coefficient of q is $196884 = 1 + 196883$, the sum of the trivial and adjoint representation.

We may identify an involution $i \in M$, defined like a 'fermion number', under which all untwisted (twisted) operators have eigenvalue $+1(-1)$ where i also respects (3). The centraliser of i can be found [1] to give $C(i|M) = \{g \in M | ig = gi\} = 2_+^{1+24}.\mathrm{Co}_1$ where Co_1 is the Conway simple group (the automorphism group Co_0 of Λ modulo the reflection automorphism $\bar{r}$), 2_+^{1+24} is an extra-special group and $A.B$ denotes a group with normal subgroup A with $B = A.B/A$. This result is an essential part of the FLM construction since M is generated by $2_+^{1+24}.\mathrm{Co}_1$ and a second involution σ [10]. FLM constructed σ, which mixes the untwisted and twisted sectors, from a hidden triality symmetry [1,12] and hence showed that the automorphism group of $\mathcal{V}^\natural$ is M.

The automorphisms i and r can be said to be 'dual' to each other in the sense that they are both automorphisms of $\mathcal{V}'$ and that the subsets invariant under i and r, $\mathcal{V}^\Lambda$ and $\mathcal{V}^\natural$ repectively, form meromorphic OPAs. In addition, we may 'reorbifold' $\mathcal{V}^\natural$ with respect to i to reproduce $\mathcal{V}^\Lambda$. Thus

$$\begin{tikzcd} & \mathcal{V}' \arrow[dl, "\mathcal{P}_i"'] \arrow[dr, "\mathcal{P}_r"] & \\ \mathcal{V}^\Lambda \arrow[rr, shift left, "r"] & & \mathcal{V}^\natural \arrow[ll, shift left, "i"] \end{tikzcd} \tag{5}$$

where the horizontal arrows denote an orbifolding and the diagonal arrows a projection [13].

Monstrous Moonshine. The operators of $\mathcal{V}^\natural$ of a given conformal weight form reducible representations of M. The Thompson series $T_g(\tau)$ for $g \in M$ is defined by the trace

$$T_g(\tau) = \mathrm{Tr}_{\mathcal{V}^\natural}(g q^{L_0}) = \frac{1}{q} + 0 + [1 + \chi(g)]q + ... \tag{6}$$

which depends only on the conjugacy class of g where $\chi(g)$ is the character in the 196883 dimensional irreducible representation. Thus for i defined above, it is easy to show $T_i(\tau) = [\eta_{\mathcal{T}}(\tau)]^{-1} + 24$.

The Thompson series for the identity element is $J(\tau)$ which is unique (up to a constant) for the following reasons. Let $\mathcal{F} = H/\Gamma$ be the fundamental region where $\Gamma = \mathrm{SL}(2, Z)$ is the full modular group and H is the upper half complex plane. Adding the point at infinity, the compactification $\overline{\mathcal{F}}$ is isomorphic to the Riemann sphere of genus zero where the function $J(\tau)$ realises this isomorphism. Such a function is called a *hauptmodul for the genus zero modular group* Γ. A modular invariant meromorphic function is a hauptmodul if and only if it possesses a unique simple pole. Once the location of this pole is specified, this function is itself unique up to a constant cf. [5,14].

Based on 'experimental' evidence, Conway and Norton [15] conjectured that each $T_g(\tau)$ is a hauptmodul for a genus zero modular group Γ_g. This has recently been rigorously demonstrated by Borcherds although the origin of the genus zero property remains obscure [16]. In general, for g of order n, $T_g(\tau)$ is found to be invariant up to phases of order (at most) h under $\Gamma_0(n) = \{\begin{pmatrix} a & b \\ nc & d \end{pmatrix} | \det = 1\}$ where $h|n$ and $h|24$. $T_g(\tau)$ is fixed by Γ_g with $\Gamma_0(N) \subseteq \Gamma_g \subseteq \mathcal{N}(N) = \{\rho \in \mathrm{SL}(2, R) | \rho\Gamma_0(N) = \Gamma_0(N)\rho\}$, the normaliser of $\Gamma_0(N)$ in $\mathrm{SL}(2, R)$ where $N = nh$. Furthermore, Γ_g is a genus zero modular group and $T_g(\tau)$ is the corresponding hauptmodul with a simple pole at $q = 0$. Consider the elements of prime order $n = p$. Apart from one class of order 3 with $h = 3$, we have $h = 1$ in each case. Thus either $\Gamma_g = \Gamma_0(p)$ or $\Gamma_0(p)+$, generated by $\Gamma_0(p)$ and the Fricke involution $W_p : \tau \to -1/p\tau$ with $W_p^2 = 1$, the only non-trivial element of $\mathcal{N}(p)$. $\Gamma_0(p)$ is of genus zero when $(p-1)|24$ $(p = 2, 3, 5, 7, 13)$ where the hauptmodul is $[\eta(\tau)/\eta(p\tau)]^{2d} + 2d$ with $2d = 24/(p-1)$. There is a class of M denoted by $p-$ for each such prime with this Thompson series e.g. the involution i belongs to the class $2-$. $\Gamma_0(p)+$ is of genus zero for $2 \leq p \leq 31$ or $p = 41, 47, 59, 71$, which constitute all the prime divisors of the order of M

[17]. Similarly, there is a class of M, denoted by $p+$, for each such prime with Thompson series fixed by $\Gamma_0(p)+$.

It is natural to interprete the Thompson series $T_g(\tau)$ as a contribution to the partition function for a further orbifolding of $\mathcal{V}^\natural$ with respect to g [18,14]. In particular, we expect that under $\tau \to -1/\tau$, $T_g(\tau)$ transforms to the partition function for a g twisted sector $\mathcal{V}_g$ as follows:

$$T_g(\tau) = g \underset{1}{\square}^{\natural} \to 1 \underset{g}{\square}^{\natural} = N_g q^{E_0^g} + \ldots \tag{7}$$

where $\natural$ denotes a trace contribution to the orbifolding of $\mathcal{V}^\natural$ and $\mathcal{V}_g$ has vacuum energy E_0^g and degeneracy N_g. For many classes of M, the method of construction of $\mathcal{V}_g$ is not known. However, for certain elements discussed below and some others, a construction can be given [14,13].

Consider now this orbifold picture of $T_g(\tau)$ for the prime classes $p+$ and $p-$, although the analysis given can be generalised to all classes [14,19,13]. Under a modular transformation $\gamma : \tau \to \frac{a\tau+b}{c\tau+d}$ we find $g \underset{1}{\square}^{\natural} \to g^{-d} \underset{g^c}{\square}^{\natural}$ assuming that no extra global phase occurs [20] (such a phase corresponds to $h \neq 1$ in the original Moonshine conjectures [14,13]). For $\gamma \in \Gamma_0(p)$ with $c = 0 \bmod p$ we find $\gamma : T_g(\tau) \to T_{g^{-d}}(\tau) = T_g(\tau)$ since d and p are relatively prime and $T_g(\tau)$ is $\Gamma_0(p)$ invariant.

The genus zero property can be also understood as follows. $T_g(\tau)$ always has a simple pole at $q = 0$ ($\tau = \infty$). The only other possible pole for $T_g(\tau)$ is at $\tau = 0$ since the fundamental region $\mathcal{F}_p = H/\Gamma_0(p)$ for $\Gamma_0(p)$ has only these two cusp points [21]. From (7), $T_g(\tau)$ has a pole at $\tau = 0$ if and only if $E_g^0 < 0$. Thus $T_g(\tau)$ is a hauptmodul for $\Gamma_0(p)$ if and only if $E_g^0 \geq 0$. Also from (7), $T_g(W_p(\tau)) = 1 \underset{g}{\square}^{\natural}(p\tau)$, so that $T_g(\tau)$ is a hauptmodul for $\Gamma_0(p)+$ if and only if $E_g^0 = -1/p$ and $N_g = 1$.

For classes of type $p+$, $T_g(\tau) = 1 \underset{g}{\square}^{\natural}(p\tau)$ is a series in q with non-negative coefficients since the RHS of (7) is the $\mathcal{V}_g$ partition function. For classes of type $p-$, $T_g(\tau)$ has coefficients of mixed sign. In general, all classes of M can be divided into two such types i.e. classes with Thompson series invariant (or not invariant) under a Fricke involution $W_N : \tau \to -1/N\tau$ which are called Fricke (or non-Fricke) classes. There are a total of

121 Fricke classes all of which have non-negative coefficient Thompson series and 51 non-Fricke classes with mixed sign coefficients for similar reasons to the prime ordered classes described. This division of the classes of M will be important below.

The FLM Uniqueness Conjecture. FLM have conjectured that $\mathcal{V}^\natural$ is characterised (up to isomorphism) as follows [1]: *$\mathcal{V}^\natural$ is the unique meromorphic conformal field theory with modular invariant partition function $J(\tau)$ and central charge 24.* This is analogous to the uniqueness property of the Leech lattice as being the only even self-dual lattice in 24 dimensions without roots.

Let us now consider orbifold models based on other automorphisms a of the untwisted Leech lattice theory $\mathcal{V}^\Lambda$ lifted from automorphisms $\overline{a} \in \mathrm{Co}_0$ [19,22]. $\overline{a}$ will be chosen so that each model contains no massless operators, has a meromorphic OPA and is modular invariant with partition function $J(\tau)$ and hence, should reproduce $\mathcal{V}^\natural$. Each $\overline{a} \in \mathrm{Co}_0$ can be parameterised as follows

$$\det(x - \overline{a}) = \prod_{k|n} (x^k - 1)^{a_k} \tag{8a}$$

$$\sum_{k|n} a_k = 0 \tag{8b}$$

with $\sum_{k|n} k a_k = 24$ where $k|n$ denotes that k divides n, the order of $\overline{a}$ and $\{a_k\}$ are integers. (8b) is imposed to ensure the absence of fixed points for $\overline{a}$ so that no massless operators in $\mathcal{V}^\Lambda$ survive the $\mathcal{P}_a$ projection. For $n = p$ prime, we have $a_p = -a_1 = 2d$ where $(p-1)2d = 24$.

Since a is an OPA automorphism for $\mathcal{V}^\Lambda$, the a invariant subspace $\mathcal{P}_a\mathcal{V}^\Lambda$ also forms a closed meromorphic OPA. The partition function $\mathrm{Tr}_{\mathcal{P}_a\mathcal{V}^\Lambda}(q^{L_0})$ is not modular invariant, as before, necessitating the introduction of sectors $\mathcal{V}_a$ twisted by a. Thus under $\tau \to -1/\tau$

$$a\underset{1}{\square} = \frac{1}{\eta_{\overline{a}}(\tau)} \to 1\underset{a}{\square} = D_a^{1/2} \prod_{k|n} \eta(\tau/k)^{-a_k} = D_a^{1/2} q^{E_0^a}(1 + O(q^{1/n})) \tag{9}$$

with $\eta_{\overline{a}}(\tau) = \prod_k \eta(k\tau)^{a_k}$ and $D_a = \det(1 - \overline{a})$ where $D_a^{1/2}$ and $E_0^a = -\frac{1}{24}\sum_k \frac{a_k}{k}$ are the degeneracy and energy of the a twisted vacuum. Under $\tau \to \tau + n$, the a twisted partition function is invariant up to a phase $\exp(2\pi i n E_0^a)$. For modular consistency of the orbifold partition function we must have $nE_0^a = 0 \bmod 1$ i.e. there is no global phase anomaly [20].

In addition, if $E_0^a > 0$, then the a twisted sector does not reintroduce massless states. We therefore restrict ourselves to $\overline{a} \in \mathrm{Co}_0$ obeying [19]

$$\sum_{k|n} a_k = 0 \tag{10a}$$

$$E_0^a > 0 \tag{10b}$$

$$nE_0^a = 0 \bmod 1 \tag{10c}$$

There are a total of 38 classes of Co_0 [23] that obey these constraints [19]. If we relax condition (10c) then a further 13 classes of Co_0 obey only (10a-b) [24,13]. Each of these 13 classes is characterised by some $h \neq 1$ where $h|24$ with $h|k$ for all $a_k \neq 0$. In all 51 cases the parameters $\{a_k\}$ obey $a_k = -a_{nh/k}$ and so $E_0^a = 1/nh$ which violates (10c) for $h \neq 1$. $a \underset{1}{\square}$ is invariant up to phases of order h under $\Gamma_0(n)$ and is a hauptmodul for Γ_a with $\Gamma_0(N) \subseteq \Gamma_a \subset \mathcal{N}(N)$, $N = nh$, where Γ_a is one of the genus zero modular groups considered by Conway and Norton. Furthermore, since $E_0^a > 0$, $a \underset{1}{\square}$ cannot be Fricke invariant and hence these 51 hauptmoduls are the 51 non-Fricke Monster group hauptmoduls. Thus there is a correspondence between 51 classes $\{\overline{a}\}$ of Co_0 and the 51 non-Fricke classes of M. We will explicitly identify an element $g_n \in M$ of each such class below.

$\mathcal{V}_a$ with the partition function $1 \underset{a}{\square}$ of (9) has a standard construction [25]. Likewise, $\mathcal{V}_{a^k}$ twisted sectors must be introduced for modular invariance and OPA closure. Then the following intertwining non-meromorphic OPA should hold (schematically)

$$\psi_{a^j}\psi_{a^k} \sim \psi_{a^{j+k}} \tag{11}$$

with $\psi_{a^k}(z) \in \mathcal{V}_{a^k}$. Apart from the original Z_2 case, this OPA has only been rigorously constructed in the prime ordered cases [22]. We will assume that it is true in general. We therefore enlarge $\mathcal{V}^\Lambda$ by the introduction of $\mathcal{V}_{a^k}$ to $\mathcal{V}' = \mathcal{V}^\Lambda \oplus \mathcal{V}_a \oplus \ldots \oplus \mathcal{V}_{a^{n-1}}$ which forms a closed non-meromorphic OPA. The projection $\mathcal{V}^a_{\mathrm{orb}} = \mathcal{P}_a\mathcal{V}'$ then forms a meromorphic OPA. (10c) is sufficient to guarantee the modular invariance of the corresponding partition function. (10b) can be also shown to be sufficient to ensure no massless operators appear in $\mathcal{P}_a\mathcal{V}_{a^k}$ [19,13]. Thus, for the 38 automorphisms obeying (10a-c), the partition function is modular invariant and is given by $Z_{\mathrm{orb}}(\tau) = J(\tau)$. Therefore $\mathcal{V}^a_{\mathrm{orb}} \equiv \mathcal{V}^\natural$ according to the FLM uniqueness conjecture. Let us now consider some evidence to support this.

Let $M^a_{\rm orb}$ be the automorphism group of the OPA for $\mathcal{V}^a_{\rm orb}$ where we expect $M = M^a_{\rm orb}$. We define $i_a \in M^a_{\rm orb}$ of order n (which generalises the involution i in the original FLM construction) under which all the operators of $\mathcal{V}_{a^k}$ are eigenstates with eigenvalue $e^{2\pi ik/n}$. i_a is also an automorphism of $\mathcal{V}'$ and is 'dual' to the automorphism a where $\mathcal{P}_a\mathcal{V}' = \mathcal{V}^a_{\rm orb}$ and $\mathcal{P}_{i_a}\mathcal{V}' = \mathcal{V}^\Lambda$. Furthermore, we may reorbifold $\mathcal{V}^a_{\rm orb}$ with respect to i_a to reproduce $\mathcal{V}^\Lambda$ as before [13]

$$\begin{array}{ccc} & \mathcal{V}' & \\ {}^{\mathcal{P}_{i_a}}\swarrow & & \searrow^{\mathcal{P}_a} \\ \mathcal{V}^\Lambda & \overset{a}{\underset{i_a}{\rightleftarrows}} & \mathcal{V}^a_{\rm orb} \end{array} \tag{12}$$

Thus if $\mathcal{V}^a_{\rm orb} \equiv \mathcal{V}^\natural$, we can explicitly construct the twisted sectors $\mathcal{V}_{i_a^k}$ assumed earlier for $i_a \in M$. We may also compute the Thompson series for $i_a \in M^a_{\rm orb}$ by taking the trace over $\mathcal{V}^a_{\rm orb}$ to obtain

$$T^{\rm orb}_{i_a}(\tau) = {\rm Tr}_{\mathcal{V}^a_{\rm orb}}(i_a q^{L_0}) = \frac{1}{\eta_{\overline{a}}(\tau)} - a_1 \tag{13}$$

which is the hauptmodul for the genus zero modular group Γ_a introduced earlier [19]. Thus each $i_a \in M^a_{\rm orb}$ dual to a has the same Thompson series as a corresponding non-Fricke element of M e.g. for $\overline{a}$ of prime order p, $T^{\rm orb}_{i_a}(\tau) = [\eta(\tau)/\eta(p\tau)]^{2d} + 2d = T_{p-}(\tau)$. Note also, from (7), that $\mathcal{V}_{i_a}$ has vacuum energy $E_0^{i_a} = 0$ and degeneracy $-a_1 > 0$. (13) may be generalised to the other 13 classes violating (10c) where $\overline{a}^h$, of order $n' = n/h$, can be employed to construct an orbifold with partition function $J(\tau)$. Let g_n denote the lifting of $\overline{a}$ where $g_n^h = i_{a^h}$ is dual to a^h, a lifting of $\overline{a}^h$ (for $h = 1$, $g_n = i_a$). We may then compute the Thompson series for g_n as a trace over $\mathcal{V}^{a^h}_{\rm orb}$ to show that (13) again holds so that g_n has the same Thompson series as a non-Fricke element of M [13].

We may also compute the centraliser $C(g_n|M^{a^h}_{\rm orb}) = \{g \in M^{a^h}_{\rm orb}|g_n^{-1}gg_n = g\}$. For the 38 classes with $h = 1$ this consists of automorphisms that do not mix the sectors $\mathcal{P}_a\mathcal{V}_{a^k}$. For the other 13 automorphisms g_n, $C(g_n|M^{a^h}_{\rm orb}) \subset C(i_{a^h}|M^{a^h}_{\rm orb})$. In general, $c \in C(i_a|M^a_{\rm orb})$ must commute with a and therefore c is lifted from the automorphism $\overline{c} \in G_n = C(\overline{a}|{\rm Co}_0)/ < \overline{a} >$. One can then show that [24,13]

$$C(g_n|M^{a^h}_{\rm orb}) = \hat{L}_{\overline{a}}.G_n \tag{14}$$

where $\hat{L}_{\overline{a}} = n.L_{\overline{a}}$, an extension of $L_{\overline{a}} = \Lambda/(1-\overline{a})\Lambda$ by a cyclic group of order n. $\hat{L}_{\overline{a}}$ arises from the vaccum structure of $\mathcal{V}_a$ where $D_a = |L_{\overline{a}}|$. With $M^a_{\rm orb} = M$, (14) generalises a a well-known observation of Conway and Norton concerning the 5 prime classes where

$C(p-|M) = p_+^{1+2d}.G_p$ and $i_a = p-$ [15]. For the other 46 classes, there are 11 cases for which (14) can be checked using the available information about these centralisers [15,26]. In general, the order of these groups agrees with (14) in each case supporting the very likely validity of the result.

Both (13) and (14) support the conjecture that $\mathcal{V}^a_{\mathrm{orb}} \equiv \mathcal{V}^\natural$. This can only be proved by finding a generalised version of σ in the FLM construction which mixes the untwisted and twisted sectors [1,12] i.e. there should exist some permutation group Σ_n which mixes the sectors of $\mathcal{V}^a_{\mathrm{orb}}$ where $C(g_n|M)$ and Σ_n generate M. In the prime cases $p \neq 2$, Σ_p has been recently constructed and it has been rigorously shown that $M^a_{\mathrm{orb}} = M$ for $p = 3$ and almost so for $p = 5, 7, 13$ [22].

Monstrous Moonshine from the Uniqueness of $\mathcal{V}^\natural$. Let us now assume that the FLM Uniqueness conjecture is correct. We can then argue that *Thompson series are hauptmoduls if and only if orbifolding $\mathcal{V}^\natural$ with respect to elements of M reproduces $\mathcal{V}^\natural$ or $\mathcal{V}^\Lambda$*. Thus Monstrous Moonshine is intimately linked to the uniqueness of $\mathcal{V}^\natural$.

From (12), orbifolding $\mathcal{V}^\natural$ with respect to the 38 non-Fricke elements i_a dual to a reproduces $\mathcal{V}^\Lambda$. We may similarly consider the orbifolding of $\mathcal{V}^\natural$ with respect to the Fricke elements $\{f\}$ with $h = 1$ which lead to a modular invariant theory $\mathcal{V}^f_{\mathrm{orb}}$ [14,13], given that the operators $\mathcal{V}_{f^k}$ can be constructed. Assuming that the Thompson series are hauptmoduls we find that $\mathcal{V}^f_{\mathrm{orb}} \equiv \mathcal{V}^\natural$ i.e. orbifolding $\mathcal{V}^\natural$ with respect to a Fricke automorphism reproduces $\mathcal{V}^\natural$ again. Thus we have [13]

$$\mathcal{V}^\Lambda \underset{\underset{i_a}{\longleftarrow}}{\overset{a}{\longrightarrow}} \mathcal{V}^\natural \overset{f}{\longleftrightarrow} \mathcal{V}^\natural \tag{15}$$

For example, consider f an element of a prime class $p+$. Fricke invariance implies $1\ \square_{f^k} = T_f(\tau/p) = q^{-1/p} + 0 + O(q^{1/p})$ so that there is a 'gap' in the spectrum of $\mathcal{V}_{f^k}$ and no massless operators are reintroduced in orbifolding $\mathcal{V}^\natural$. Thus the modular invariant partition function for $\mathcal{V}^f_{\mathrm{orb}}$ is $J(\tau)$ and hence $\mathcal{V}^f_{\mathrm{orb}} \equiv \mathcal{V}^\natural$. A similar argument can be made in the general case [13].

The converse to the above also holds i.e. assuming that (15) is true for all automorphisms of M that define a modular consistent theory, then the Thompson series are hauptmoduls. To see this, firstly consider an orbifolding with respect to $i_a \in M$ which reproduces $\mathcal{V}^\Lambda$. i_a must be dual to one of the 38 automorphisms obeying (10a-c) and has non-Fricke invariant Thompson series (13) which is the hauptmodul for a genus zero group.

Similarly, as discussed above, the other non-Fricke automorphisms can also be found with a corresponding genus zero Thompson series. For the remaining Fricke classes of M we provide an argument for f an element of prime order. We wish to show that $\mathcal{V}_f$ has the correct vacuum structure so that $T_f(\tau)$ is a hauptmodul for $\Gamma_0(p)+$. In the orbifolding of $\mathcal{V}^\natural$ with respect to f which reproduces $\mathcal{V}^\natural$, let $i_f \in M$ be dual to f with eigenvectors $\mathcal{V}_{f^k}$ for eigenvalue $e^{2\pi ik/n}$. Then it can be shown that $T_{i_f}(\tau) = T_f(\tau)$ so that i_f is in the same class as f. Furthermore, the centralisers obey $C(f|M) \subseteq C(i_f|M)$ with the necessary equality only when the $\mathcal{V}_f$ vacuum is unique i.e. $N_f = 1$. Since the twisted sector $\mathcal{V}_f$ does not reintroduce massless operators, the vacuum energy obeys either (a) $E_f^0 = -1/p$ or (b) $E_f^0 > 0$. (a) is possible because the absence of massless operators in $\mathcal{V}^\natural$ allows for a similar 'gap' in the spectrum of $\mathcal{V}_g$. If (b) holds, then $T_f(\tau)$ has a unique simple pole at $q = 0$ and must be a hauptmodul for $\Gamma_0(p)$ with $(p-1)|24$ and $T_f(\tau) = [\eta(\tau)/\eta(p\tau)]^{2d} + 2d$. However, this is impossible since then $E_f^0 = 0$ with $N_f = 2d$ from (7). Thus (15) implies that $\mathcal{V}_f$ has vacuum structure $E_f^0 = -1/f$ with $N_f = 1$ and hence, as described before, $T_f(\tau)$ is a hauptmodul for the genus zero group $\Gamma_0(p)+$ and f is of class $p+$. A similar argument can be given for the other Fricke classes [13].

References

[1] Frenkel, I., Lepowsky, J. and Meurman, A., Proc.Natl.Acad.Sci.USA **81** (1984) 3256; J.Lepowsky et al. (eds.), Vertex operators in mathematics and physics, (Springer Verlag, New York, 1985); Vertex operator algebras and the monster, (Academic Press, New York, 1988).

[2] Dixon, L., Harvey, J.A., Vafa, C., and Witten, E., Nucl.Phys. **B261** (1985) 678; Nucl.Phys. **B274** (1986) 285.

[3] Conway, J.H. and Sloane, N.J.A., Sphere packings, lattices and groups, (Springer Verlag, New York, 1988).

[4] Goddard, P., Proceedings of the CIRM Luminy conference, 1988, (World Scientific, Singapore, 1989).

[5] Serre, J-P., A course in arithmetic, (Springer Verlag, New York, 1970).

[6] Ginsparg, P., Les Houches, Session XLIX, 1988, "Fields, strings and critical phenomena", ed. E. Brezin and J. Zinn-Justin, Elsevier Science Publishers (1989).

[7] Dixon, L., Friedan, D., Martinec, E. and Shenker, S., Nucl.Phys. **B282** (1987) 13.

[8] Corrigan, E. and Hollowood, T.J., Nucl.Phys. **B304** (1988) 77.

[9] Dolan, L., Goddard, P. and Montague, P., Nucl.Phys. **B338** (1990) 529.
[10] Griess, R., Inv.Math. **68** (1982) 1.
[11] Thompson, J.G., Bull.London Math.Soc.**11** (1979) 347.
[12] Dolan, L., Goddard, P. and Montague, P., Phys.Lett. **B236** (1990) 165.
[13] Tuite, M.P., DIAS preprint 1992, in preparation.
[14] Tuite, M.P., Commun.Math.Phys. **146** (1992) 277.
[15] Conway, J.H. and Norton, S.P., Bull.London.Math.Soc. **11** (1979) 308.
[16] Borcherds, R., Univ. Cambridge DPMMS preprint 1989.
[17] Ogg, A., Bull.Soc.Math.France **102** (1974) 449.
[18] Dixon, L. Ginsparg, P. and Harvey, J.A., Comm.Math.Phys. **119** (1988) 285.
[19] Tuite, M.P., DIAS-STP-90-30, To appear in Commun.Math.Phys..
[20] Vafa, C., Nucl.Phys. **B273** (1986) 592.
[21] Gunning, R.C., Lectures on modular forms, (Princeton University Press, Princeton, 1962).
[22] Dong, C. and Mason, G., U.C.Santa Cruz Preprint 1992.
[23] Kondo, T., J.Math.Soc.Japan **37** (1985) 337.
[24] Tuite, M.P., DIAS-STP-91-25 Aug 1991.
[25] Lepowsky, J., Proc.Natl.Acad.Sci.USA **82** (1985) 8295; Kac, V. and Peterson, D., Proceedings of the Argonne symposium on anomolies, geometry, topology, 1985 (World Scientific, Singapore, 1985).; Corrigan, E. and Hollowood, T.J., Nucl.Phys. **B303** (1988) 135.
[26] Wilson, R., J.Alg. **85** (1983) 144.