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The Spin-Statistics Connection from Homology Groups of Configuration Space and an Anyon Wess-Zumino Term

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Abstract: The first and second homology groups H_i for configuration spaces of framed two-dimensional particles and antiparticles, with annihilation included, are computed when up to two particles and an antiparticle are present. The set of 'frames' considered are S^2, SO(2) and SO(3). It is found that the H_1 groups are those of the 'frames' and are generated by a cycle corresponding to a 2π frame rotation. This same cycle is homologous to the exchange path - the spin - statistics theorem. Furthermore for the frame space SO(2), H_2 contains a Z subgroup which implies the existence of a nontrivial Wess-Zumino term. A rotationally and translationally invariant, topologically nontrivial Wess-Zumino term for a pair of anyons and an antianyon is exhibited for this case.

1. Introduction

Although the axioms of local relativistic quantum field theory are sufficient to guarantee a spin -statistics theorem[1], results concerning spin-statistics correlations for extended objects such as solitons, monopoles and vortices have shown that they are by no means necessary.[2] This has raised the question as to what general assumptions are needed for the spin- statistics theorem and has led to the investigation [3,4] of the general topological properties of systems of particles and extended objects and their physical consequences. One of the results that has emerged from this work is the importance of pair- creation and annihilation [4] for this topology and in particular for the spin - statistics connection, but many questions are still unresolved.

One way to appreciate the role of topology in quantum theory is as follows: The quantum mechanical Hilbert space of square integrable vector valued "functions " $\psi(q)$ over a classical configuration space C is made up of sections of vector bundles over C. For a variety of physically interesting configuration spaces C, these vector bundles incorporate the spin-type and the statistics, as well as other topological properties. Furthermore many of the relevant topological properties can be described by the homotopy and homology groups of the classical configuration spaces which are associated with these vector bundles. For example, the set of U(1) (and hence line) bundles over C is characterized by $H^2(C,Z)$ which is isomorphic to $H_2(C, Z)^* \oplus \mathrm{Tor}\, H_1(C, Z)$ where $H_2(C, Z)^*$ can be thought of as the non-torsion part of $H_2(C, Z)$ and Tor $H_1(C, Z)$ is the torsion subgroup of $H_1(C, Z)$.[5] [The torsion subgroup of an abelian group is its maximal finite subgroup.] Futhermore, in the case of flat bundles,the spin- statistics theorem can be expressed as the statement that the exchange of two particles and a 2π rotation of one particle correspond to the same nontrivial element of the fundamental group $\pi_1(C)$. For more general bundles it asserts (in three dimensions) a homotopy, not between loops but between a certain pair of mappings of RP^2 into C.[4] Similarly, the condition for the existence of a nontrivial Wess-Zumino term is that the second homology group $H_2(C, Z)$ contains a Z group.

A particular sort of system that has been investigated[6,7] using this type of approach is that of identical particles and anti-particles on R^d or two-dimensional surfaces with handles[7], each carrying a 'frame' F, the frame having been introduced in order to describe intrinsic spin. In this case, the classical configuration space C is (as a set) of the form C =

$\coprod_{m,n} Q_{m,n}$ where $Q_{m,n}$ are the spaces containing m particles and n anti-particles, all with distinct locations (but see Ref. 8), and the sum runs over all possibilities. The topology of each subspace $Q_{m,n}$ (or just Q_n if the particle is its own antiparticle) is the topology of an appropriate frame bundle modified by the fact that the particles are assumed to be indistinguishable. The basic problem (solved in the second paper of ref. 6) is to construct a (Hausdorff) topology for the full space C such that pair creation and annihilation can proceed smoothly. Now there remains the technical problem of finding the precise topological properties (in particular the homotopy and homology groups) that are introduced by the construction and of analyzing these properties.

In the present paper, we consider a limited version of this problem wherein the individual units are point particles and they move in two - dimensional Euclidean space R^2, carrying 'frames' which embody the notion of spin. By 'carrying frames' is meant that a single particle (or anti-particle) is represented by a bundle over R^2, three possible fibres, namely SO(2), S^2, and SO(3), being considered. (The S^2 and SO(3) bundles are not so natural for R^2, but are considered with future work on R^3 generalizations in mind.) The restriction to point particles and to two dimensions is for simplicity and because of the present interest in two-dimensional systems, particularly in the theory of anyons. But our intention is to generalize the results to three dimensions and to apply them to extended objects such as monopoles and strings later.

Our solution [6] to the problem of finding an appropriate topology for the configuration space is reviewed in Section 2. The essential idea is to introduce open neighbourhoods of the vacuum (and corresponding neighborhoods of non-vacuum configurations) that allow a particle and a antiparticle to annihilate provided their positions and frames are suitably aligned. (If a second particle is nearby in space, we require in particular that the two particles be on opposite sides of the antiparticle, a situation which we call 'syzygy' from analogy with planetary alignments). The complete topology is then obtained from these neighborhoods. There remains the problem of determining some of the detailed properties of the resulting topological space, including its homology and homotopy groups, especially insofar as they help answer the question of how many inequivalent vector bundles the space admits.

We will concentrate on the homology groups in this work, these being sufficient if we restrict ourselves to line bundles. One may hope to solve for these groups by using the Mayer-Vietoris exact sequence[9], which expresses the homology of a union of two spaces in terms of the homologies of the separate spaces and of their intersection (which for our spaces is often relatively easy to compute). This process is carried out here explicitly for the first and second homology groups in the case of the subspaces $X_{1,1} = Q_{1,1} \cup Q_{0,0}$ and $X_{2,1} = Q_{2,1} \cup Q_{1,0}$. The generalization to higher homology groups and to subspaces with more particles and thence to the full space C should proceed along the same lines, although for the full space, certain notions of limits are expected to be involved.

A result of particular importance we find is that, for the frames SO(2) and S^2, the exchange loop of two particles is homologous to a loop in which one particles undergoes a 2π rotation. This entails the spin-statistics correlation for these cases. But the most important result is that, for SO(2)-frames, the exchange loop is not homologous to zero. This nontriviality (insofar as it persists for the full space C) confirms that our framework is broad enough to admit Fermi statistics and spinorial angular momentum. Had this not been the case, the spin-statistics proof of Ref. 6 would have been essentially vacuous, since it would have been based on assumptions which excluded precisely the phenomena it was aiming to illuminate.

The layout of the paper is as follows. In Section 2, we describe the topology of the spaces considered. In Section 3, we discuss how the Mayer -Vietoris theorem can be used to determine the homology groups H_1 and H_2 of these spaces while in Section 4, the method is illustrated by the simplest non-trivial example, namely the space $X_{1,1} = Q_{1,1} \cup Q_{0,0}$. In Section 5, we outline the calculation of H_1 and H_2 of $X_{2,1} = Q_{2,1} \cup Q_{1,0}$, which is the simplest subspace in which two particles can exchange. To find the homology groups of this subspace, it is necessary to subdivide it into further subspaces and apply the theorem four times, which is done in Sections 6, 7, and 8. In order to keep the train of operations as clear as possible we have inserted a flow diagram, and have presented the results in a series of tables in these Sections. The Appendix contains a technical discussion establishing that a certain space retracts to another space.

We find that, for SO(2) frames, $H_2(X_{2,1})$ contains a Z group. Hence, as alluded to above, there should exist a nontrivial (closed but nonexact) Wess-Zumino two form on $X_{2,1}$. By the nature of $X_{2,1}$, it will be compatible with creation-annihilation processes. We find such a two form which is rotationally and translationally invariant as well. It vanishes as a particle and antiparticle approach annihilation as required by creation and annihilation processes. This term is exhibited in Section 9.

2. Topology of the Space of Framed Particles and Antiparticles

The topology of the space of framed particles and antiparticles is described in detail in ref. 6. For completeness, we give a brief description of the space including the 'reflection' and 'syzygy' conditions for annihilation. Let $X = (x, F^{(x)})$, $(\overline{X} = (\overline{x}, \overline{F}^{(\overline{x})}))$ denote the position and 'frame' orientation of a particle (antiparticle). By $F^{(x)}$ we mean a generic 'frame' $\in SO(2), S^2$ or $SO(3)$ attached to the particle located at position x. Then

$$\{[X^1, X^2 \ldots\ldots, X^m; \overline{X}^1, \overline{X}^2 \ldots\ldots \overline{X}^n] \mid x^i, \overline{x}^j \in R^d;\ x^i \neq x^j, \overline{x}^i \neq \overline{x}^j \text{ if } i \neq j;\ x^i \neq \overline{x}^j\} = Q_{m,n},$$

$$[X^1, ..X^i...X^j..,X^m; \overline{X}^1, ..\overline{X}^l..\overline{X}^k..\overline{X}^n] = [X^1, ..X^j...X^i..,X^m; \overline{X}^1, ..\overline{X}^k..\overline{X}^l...\overline{X}^n] \tag{1}$$

where $Q_{m,n}$ denotes the sector of our configuration space consisting of m particles and n antiparticles. Here we also introduce the vacuum ("VAC") by setting

$$Q_{0,0} = \{VAC\}. \tag{2}$$

Now the concept " ε - close" is defined as follows:

(i) Particles X and Y are ε close iff $|x - y| < \varepsilon$ and $d(F^{(x)}, F^{(y)}) < \varepsilon/L$ (where L is some length), and similiarly for antiparticles $\overline{X}$ and $\overline{Y}$.

(ii) The particle X and antiparticle $\overline{Y}$ are ε-close iff $|x - \overline{y}| < \varepsilon$ and $d(F^{(x)}, R(x - \overline{x})\overline{F}^{(\overline{y})}) < |x - \overline{y}|/L$ where $|x - \overline{y}|$ is the Euclidian distance between points x and $\overline{y}$, $d(F^{(x)}, F^{(y)})$ is the geodesic distance between $F^{(x)}$ and $F^{(y)}$ in the space of frames and $R(x - \overline{x})\overline{F}^{(\overline{y})}$ is the frame which results when the anti-frame $\overline{F}^{(\overline{y})}$ is reflected in the plane perpendicular to the vector $x - \overline{y}$.

Reflections are here defined imagining R^2 to be a subspace of R^3. This is convenient, although not necessary for the frame space $\mathcal{F} = SO(2)$.

This concept of ε – close is used to define an ε – neighborhood in $Q_{m,n}$ of a point in $Q_{m,n}$ in the obvious way. We further need to define when a point $Y = \{Y^1,..,Y^{m+p}; \overline{Y}^1,..,\overline{Y}^{n+p}\} \in Q_{m+p,n+p}$, $p \geq 0$, is in a ε–neighborhood of a point $X = \{X^1,..,X^m, \overline{X}^1,..\overline{X}^n\} \in Q_{m,n}$, that is when p particle - antiparticle pairs are 'close' to annihilation . To this end we define a viable labelling of Y with respect to X as one that satisfies the following:

(i) Y^i is ε – close to X^i for i = 1,...,m ;

(ii) $\overline{Y}^i$ is ε – close to $\overline{X}^i$ for i = 1,...,n ;

(iii) Y^{i+m} is ε – close to Y^{i+n} for i = 1,..p.

Also,we say that *a triplet X,Y, $\overline{Z}$ is in syzygy if (and only if)*

$$\left.\begin{array}{l} |\overline{x - \overline{z}} + \overline{y - \overline{z}}| \\ d(F^{(x)}, R(x - \overline{z})\overline{F}^{(\overline{z})}) \\ d(F^{(y)}, R(y - \overline{z})\overline{F}^{(\overline{z})}) \end{array}\right\} < \frac{|x - \overline{z}| + |y - \overline{z}|}{L} \qquad (3)$$

A similiar definition of syzygy applies to the triplet X, Y, Z. $\overline{x - \overline{z}}$ here indicates the unit vector in the direction x - $\overline{z}$.

Finally we say that $Y \in N_\varepsilon[X]$, that is, Y is an element of an ε- neighborhood of X, if there exists a viable labelling of Y with respect to X, and for all viable labellings all suitable triplets are in syzygy. By suitable triplets we mean that at least one member of the triplet comes from the set $\{ Y^{m+1},...Y^{m+p}; \overline{Y}^{n+1},...\overline{Y}^{n+p}\}$, that is one 'new' particle or one 'new' antiparticle must be a member of the triplet.

In words, the condition that a particle-antiparticle pair be close to annihilation is first that their frames nearly satisfy the reflection condition. In addition, if another particle is nearby in space, the three must be in syzygy. This means that the two particles are on opposite sides of the antiparticle and both particles nearly satisfy the reflection condition with the antiparticle.

3. The Mayer - Vietoris Sequence.

The tool we will use to calculate the homology groups of our spaces will be the Mayer-Vietoris Sequence (MVS). It is an exact sequence relating the homology groups of subspaces to the homology groups of the whole space.

Given a topological space $C = A \cup B$ where A and B are open subspaces, let $Y=A\cap B$. Then the MVS is the exact sequence

$$..\rightarrow H_n(Y) \rightarrow H_n(A)\oplus H_n(B) \rightarrow H_n(C) \rightarrow H_{n-1}(Y) \rightarrow H_{n-1}(A)\oplus H_{n-1}(B) \rightarrow H_{n-1}(C)\rightarrow.. \qquad (4)$$

Here we regard the homology groups as having coefficients in Z. The homomorphisms in the sequence for fixed n are (up to a sign) those induced by the inclusion maps of A and B in C. In some of our applications, A will in fact be closed in C. But we will be able to find an open $\tilde{A}$ which is a 'thickened' version of A and retractable to A. $\tilde{A}$ then has the same topology as A and the MVS still obtains with $Y = \tilde{A} \cap B$. Figure 1 indicates the mappings involved. The signs in the figure indicate the signs attached to the induced homomorphisms. Note that an element of $H_n(Y)$ is mapped with opposite signs to $H_n A)$ and $H_n(B)$.

We plan to investigate the homology groups of $X_{m,n} = X_{m-1,n-1} \cup Q_{m,n}$ for small values of m,n. We do not thus allow arbitrary numbers of particles and antiparticles. Now $Q_{m,n}$ is open in $X_{m,n}$ whereas $X_{m-1,n-1}$ is closed in $X_{m,n}$. However, if we can define a thickened $X_{m-1,n-1}$, call it $\tilde{X}_{m-1,n-1}$, such that $\tilde{X}_{m-1,n-1}$ is open in $X_{m,n}$ and retracts to $X_{m,n}$, then we can apply MVS with $A=X_{m-1,n-1}$, $B = Q_{m,n}$, $Y = \tilde{X}_{m-1,n-1} \cap Q_{m,n}$ and $C = X_{m,n}$.

It is readily seen that the content of MVS can be summarized in terms of the short exact sequences

$$0 \rightarrow \frac{H_p(A) \oplus H_p(B)}{\text{image of } m_p} \rightarrow H_p(A \cup B) \rightarrow \text{Kernel of } m_{p-1} \rightarrow 0. \quad (5)$$

Here m_p is the map

$$m_p : H_p(\tilde{A} \cap B) \rightarrow H_p(A) \oplus H_p(B). \quad (6)$$

These short sequences will be our main calculational tool and will be referred to as SES.

The Kunneth formula [9]

$$H_p(X \otimes Y) = \bigoplus_{k+q=p} H_k(X) \otimes H_q(Y) + \sum_{j=0}^{p-1} \text{Tor}(H_{p-j-1}(X), H_j(Y)) \quad (7)$$

which relates the homology group of a product space to the properties of the homology groups of the factors, will also prove useful. Here Tor(A, B) is the so called torsion product of the abelian groups A and B. Since Tor (Z, Z) = 0 and Tor $(Z, Z_n) = 0$,[9] the Tor terms vanish in all applications of the Kunneth formula in this paper. This is because we calculate only H_2 using this formula and H_0 has no torsion factors. We also note the following formulae for the tensor product of abelian groups (regarded as Z modules): 1) $Z \otimes Z_p = Z_p$, and 2) $Z_m \otimes Z_n = Z_p$ where p = greatest common divisor of m and n.

H_1 and H_2 for the three spaces of frames used in this paper are listed in Table 1.

4. H_1 and H_2 of $X_{1,1}$

As a simple application of the procedure, we will use MVS in the form of SES to calculate H_1 and H_2 of $X_{1,1} = Q_{0,0} \cup Q_{1,1}$.

To apply SES directly, we need to decide what is $\tilde{Q}_{0,0}$ and to know H_i for $\tilde{Q}_{0,0} \cap Q_{1,1}$, $Q_{1,1}$ and $Q_{0,0}$. We take the space $\tilde{Q}_{0,0}$ to be $Q_{0,0} \cup$ {all pairs within an $\bar{\varepsilon}$ neighbourhood}, $\bar{\varepsilon}$ being very small. It clearly retracts to $Q_{0,0}$ as it should. Since $Q_{0,0}$ is a single point, $H_i(Q_{0,0})$ = { Identity}, that is it is the trivial group as shown in Table 2. In the spaces $\tilde{Q}_{0,0} \cap Q_{1,1}$ and $Q_{1,1}$, we can retract the position of the antiparticle to the origin. $Q_{1,1}$ can be further retracted so as to place the particle on the unit circle. In $\tilde{Q}_{0,0} \cap Q_{1,1}$, the pair can be retracted to a distance ε smaller than $\bar{\varepsilon}$ and with their frames satisfying the reflection condition exactly.

It follows that $Q_{1,1} \approx S^1 \times \mathcal{F} \times \bar{\mathcal{F}}$ where S^1 is the closed circle around the antiparticle and $\mathcal{F}$ and $\bar{\mathcal{F}}$ are the frame spaces for particles and antiparticles. Now $H_1(S^1) = Z$ and is generated by the cycle γ of the particle circling the antiparticle with its frame fixed. We denote generators by Greek letters and the generators of $H_1(\mathcal{F})$ by ϕ, $H_2(\mathcal{F})$ by σ, $H_1(\bar{\mathcal{F}})$ by $\bar{\phi}$ and $H_2(\bar{\mathcal{F}})$ by $\bar{\sigma}$. We thus obtain the entries for $Q_{1,1}$ in Table 2. (The generators of a group is written in paranthesis following its symbol.) Of course the Kunneth formula was used to compute $H_2(Q_{1,1})$.

Similiarly, from the preceding discussion of $\tilde{Q}_{0,0} \cap Q_{1,1}$, we can see that $\tilde{Q}_{0,0} \cap Q_{1,1} \approx S^1 \times \bar{\mathcal{F}}$. However, here, since the frame of the particle is 'locked' to that of the antiparticle when the particle traverses the circle, $H_1(S^1)$ is generated by the cycle $\gamma + 2\phi$. The entries for $\tilde{Q}_{0,0} \cap Q_{1,1}$ in Table 2 are determined remembering this fact.

The mapping m_i, i = 1,2, of the generators of $H_i(\tilde{Q}_{0,0} \cap Q_{1,1})$ to $H_i(Q_{0,0}) \oplus H_i(Q_{1,1})$ induced by the mappings of the spaces is clear and the results are indicated in Table 2. One sees that Ker $m_i = 0$, i =1,2, and these entries are made in Table 2. We must now compute $\text{Coker}\, m_i := \dfrac{H_i(Q_{0,0}) \oplus H_i(Q_{1,1})}{\text{Image}\, m_i}$. The independent generators of Coker m_i can be chosen to be those of $H_i(Q_{0,0}) \oplus H_i(Q_{1,1})$ subject to the conditions that the images of the generators of $H_i(\tilde{Q}_{0,0} \cap Q_{1,1})$ under m_i be set to zero. Thus we arrive at the entries for Coker m_i in Table 2.

Finally, to calculate $H_i(X_{1,1})$, we apply the SES

$$0 \to \text{Coker}\, m_i \to H_i(X_{1,1}) \to \text{Ker}\, m_{i-1} \to 0.$$

We will work with reduced homology [8], the MVS being valid for it as well. Its zeroeth homology group is distinguished by a tilde and $\tilde{H}_0(M) = \{0\}$ for a connected space M.[8] Since Ker m_1 = Ker $m_0 = 0$ (the latter following from the fact that the spaces are connected), we obtain Coker $m_i = H_i(X_{1,1})$, and thus the final entries in Table 2.

The procedure used to determine the entries in Table 2 will be repeated for the Tables corresponding to other spaces of concern to us.

5. On the Calculation of $H_1(X_{2,1})$ and $H_2(X_{2,1})$

We propose to use MVS to calculate H_1 and H_2 of $X_{2,1}$. Thus we must define $\tilde{Q}_{1,0}$ which is the thickening of $Q_{1,0}$ and which retracts to $Q_{1,0}$. We must also know H_1 and H_2 of $Q_{2,1}$, $X_{1,0} = Q_{1,0}$ and $\tilde{Q}_{1,0} \cap Q_{2,1}$. As for $\tilde{Q}_{1,0}$, it consists of $Q_{1,0}$ and all points of $Q_{2,1}$ within an ε-neighbourhood of any point of $Q_{1,0}$. We show in the Appendix that $\tilde{Q}_{1,0}$ retracts to $Q_{1,0}$. Since $Q_{2,1}$ and $\tilde{Q}_{1,0} \cap Q_{2,1}$ are rather complicated spaces themselves, we use MVS in turn to calculate their H's. Fig 2 illustrates a 'flow diagram' of four sets of mappings which we perform to complete the calculation of $H_i(X_{2,1})$.

As before, all generators of homology groups will be indicated by Greek letters. It is useful to define some particular generating cycles. In the space of two particles and one antiparticle, there are two types of non-trivial closed cycles that the particles can undergo: i) a cycle γ in which the closed path of the particle encircles the antiparticle anti-clockwise, but not the other particle, and ii) a cycle β, in which the closed path of the particle encircles both the antiparticle and the other particle anti-clockwise. One can argue that $\gamma = (\bar{\varepsilon} - \varepsilon)$ and $\beta = (\bar{\varepsilon} + \varepsilon)$ where a) $\bar{\varepsilon}$ is the exchange of the two particles where the exchange path described by both particles encircles the anti-particle, and b) ε is the exchange where the exchange path does not encircle the the anti-particle. We note that these generators do not affect the frames of the particles and antiparticle. We denote the generators of $H_1(\mathcal{F}_i)$ and $H_1(\overline{\mathcal{F}})$ by ϕ_i and $\overline{\phi}$ and those of $H_2(\mathcal{F}_i)$ and $H_2(\overline{\mathcal{F}})$ by σ_i and $\overline{\sigma}$.

6. H_1 and H_2 of $Q_{2,1}$

We start with the calculation of H_1 and H_2 of $Q_{2,1}$ which as indicated in Fig. 2 (a) and Fig. 2(b) will entail the application of MVS twice. We first note that $Q_{2,1}$ can be considered as the space $X \otimes \overline{\mathcal{F}}$. We then split off the frame space $\overline{\mathcal{F}}$ of the antiparticle and consider X ($\overline{\mathcal{F}}$ will be restored later). We then retract X to a simpler space by first moving the antiparticle to the origin and then sending the innermost particle to the circle of radius of 1 (in some suitable units). Of course, this last may cause both particles to move to the circle. X is topologically equivalent to $A \cup B$, where A is the space with two particles on the unit circle with the antiparticle at the center and B is the space with one particle on the unit circle and one particle at a distance greater than 1 from the antiparticle. These spaces are depicted in Fig. 3. In that Figure, $\tilde{A}$ is the thickening of A. It is the union of A and $\tilde{A} \cap B$. The space $\tilde{A} \cap B$ here is the space with one particle outside the unit circle within a ring of some small ε, but with a line connecting the particle on the unit circle excluded. It is also depicted in Fig. 3. It is clear that $\tilde{A}$ is retractable to A. One may thus use the MVS involving these spaces depicted in Fig 2 to compute H_1 and H_2 of $A \cup B$. Of course to use the MVS for X we need to know the homology groups of A, B, and $\tilde{A} \cap B$. That is, we need to argue the contents of Table 3 for $\tilde{A} \cap B$, A and B. We can finally apply the Kunneth formula to find the homology groups of $Q_{2,1}$.

It is clear that B is retractable to the space $S^1 \times S^1 \times \mathcal{F} \times \mathcal{F}$ and that the knowledge of H_1 and H_2 of S^1 and $\mathcal{F}$, together with the use of the Kunneth formula implies that of B. From Table 1, $H_1(S^1) = Z$ and $H_2(S^1) = 0$. Since one S^1 of space B corresponds to particle 1 circling the antiparticle but not particle 2 while the other corresponds to particle 2 circling both the antiparticle and particle 1, we write the first H_1 as $Z(\overline{\varepsilon} - \varepsilon)$ and the second as $Z(\overline{\varepsilon} + \varepsilon)$. The entries in Table 3 for $H_1(B)$ and $H_2(B)$ follow.

Since the particle in $\tilde{A} \cap B$ which is outside the unit circle can be retracted to be diametrically opposite the particle on the unit circle, $\tilde{A} \cap B \approx S^1 \times \mathcal{F} \times \mathcal{F}$. It should be clear that the generator of $H_1(S^1)$ corresponds to both particles circling the antiparticle and thus to $2\overline{\varepsilon}$, as entered in Table 3. An application of the Kunneth formula yields $H_2(\tilde{A} \cap B)$.

However the space A is not a product of simple spaces. (A is a twisted fibre bundle with base RP^1 and fibre ($\mathcal{F} \times \mathcal{F}$).) We will first use MVS to compute its homology before continuing with the calculation for $A \cup B$. Towards this end, we note that A can be contracted so that the particles on the unit circle are antipodal.

We will compute H_i (A) using the decomposition of A as $A' \cup B'$ depicted in Fig.4. A' is the space with one of the antipodal particles at the point marked with a vertical line. B' is the space with no particle at this marked point. We define particle 1 to be the particle reached first moving anticlockwise from the marked point . $\tilde{A}'$ is the thickening of A'. It is the union of A' and $\tilde{A}' \cap B'$. Clearly $\tilde{A}'$ is retractable to A' and $\tilde{A}' \cap B'$ is the disjoint union of the spaces indicated in Fig. 4. Thus $A' \approx \mathcal{F} \times \mathcal{F} \approx B'$ and $\tilde{A}' \cap B' \approx$ $(\mathcal{F}_1 \times \mathcal{F}_2) \cup (\mathcal{F}_2' \times \mathcal{F}_1')$ (The subscript on $\mathcal{F}$ distinguishes the particle associated with it. The prime is for distinguishing the two subspaces.) We can now fill in the entries for A', B' and $\tilde{A}' \cap B'$ in Table 4.

In order to use (3) to compute H_1 of A, we need to know the kernel Ker m_0 of m_0 where $m_0{:}H_0(\tilde{A}' \cap B') \to H_0(A') \oplus H_0(B')$. It is perhaps simplest to use reduced homology as before for which $\tilde{H}_0(A') = \tilde{H}_0(B') = 0$ (since both A' and B' are connected) and for which $\tilde{H}_0(\tilde{A}' \cap B') = Z$ (since $\tilde{A}' \cap B'$ has two disconnected components)[8]. Thus m_0: $Z \to 0 \oplus 0$ which implies that Ker $m_0 = Z$. It is also reasonably clear that the generator of this Z , which interchanges the two disconnected pieces of $\tilde{A}' \cap B'$ corresponds to the 'exchange' of the two reference points and hence can be written as $Z(\varepsilon)$. One can see that the maps $m_i{:}H_i(\tilde{A}' \cap B') \to H_i(A') \oplus H_i(B')$ transform the generators as entered in Table 4, and from these transformations deduce the entries for Ker m_i and Coker m_i.

The SES

$$0 \to \text{Coker } m_1 \to H_1(A) \to \text{Ker } m_0 \to 0$$

becomes

$$0 \to H_1(\phi) \to H_1(A) \to Z(\varepsilon) \to 0$$

which implies

$$H_1(A) \approx H_1(\phi) + Z(\varepsilon) .$$

To deduce $H_2(A)$ from the SES

$$0 \to \text{Coker } m_2 \to H_2(A) \to \text{Ker } m_1 \to 0 ,$$

we have to consider the cases with different framings separately. We have, for $\mathcal{F} = SO(2)$ and $\mathcal{F} = SO(3)$,

$$0 \rightarrow Z_2 \rightarrow H_2(A) \rightarrow H_1(\phi_1 + \phi_2 - \phi_1' - \phi_2') \rightarrow 0 \quad ,$$

and for $\mathcal{F} = S^2$,

$$0 \rightarrow H_2(\sigma) \rightarrow H_2(A) \rightarrow 0 \ .$$

These SES in turn imply the entries for H_2 in Table 4.

We are now in a position to complete the calculation of the entries in Table 3. One can argue that the transformation of the generators under the map $m_i : H_1(\tilde{A} \cap B) \rightarrow H_1(A) \oplus H_1(B)$ is as entered in Table 3. From these, Ker m_i and Coker m_i, which are entered in that Table, follow. First note that Ker $m_0 = 0$. [$\tilde{A} \cap B$, A and B are all connected and thus $\tilde{H}_0 = 0$ for all.] Since Ker m_0 = Ker $m_1 = 0$, the SES give $H_i(\tilde{A} \cup B)$ = Coker m_i.

7. H_1 and H_2 of $\tilde{Q}_{1.0} \cap Q_{2,1}$

We turn to the calculation of $H_1(\tilde{Q}_{1,0} \cap Q_{2,1})$ and $H_2(\tilde{Q}_{1,0} \cap Q_{2,1})$. The topology of $\tilde{Q}_{1,0} \cap Q_{2,1}$ is complicated in that it contains on the one hand the region in which both particles are close to the antiparticle, and thus must satisfy the 'syzygy' condition , and the region in which only one particle is close to the antiparticle. We will choose $\tilde{Q}_{1,0}$ to be $N_\varepsilon(Q_{1,0})$ with $\varepsilon << L$, where L is the length used in the definition of neighborhoods in Sec II. [More precisely, $\tilde{Q}_{1,0}$ is the intersection of $N_\varepsilon(Q_{1,0})$ with $\tilde{Q}_{1,0} \cup Q_{2,1}$.] We define $\tilde{I}$ to be the space such that one particle is within a distance ε of the antiparticle while the second particle is within a distance 2ε of the antiparticle. Since $\varepsilon << L$, these particles are nearly in 'syzygy' and their frames are nearly aligned by the reflection rule to the antiparticle frame. Similarly we define $\tilde{O}$ to be the space such that one particle is within a distance ε of the antiparticle while the second particle is at a distance larger then ε from the antiparticle. Clearly $\tilde{Q}_{1,0} \cap Q_{2,1} = \tilde{I} \cup \tilde{O}$ where I is the space with both particles at the same distance $d < \varepsilon$ from the antiparticle, in perfect 'syzygy', and have their frames perfectly aligned by the reflection rule with the antiparticle frame. I is a retraction of $\tilde{I}$. We define O to be the space wherein one particle is within a distance ε of the antiparticle with its frame perfectly aligned by the reflection rule with the antiparticle frame while the second particle is at a distance larger then 2ε from the antiparticle. O is a retraction of $\tilde{O}$.

The spaces $\tilde{I} \cap \tilde{O}$, I and O are depicted in Fig.5. We will use the MVS of these spaces depicted in Fig. 2(c) to compute $H_i(\tilde{Q}_{1,0} \cap Q_{2,1})$.

Note that $\tilde{I} \cap \tilde{O} = S^1 \times \mathcal{F}$, $I = RP^1 \times \mathcal{F}$ and $O = S^1 \times \bar{\mathcal{F}} \times \mathcal{F}$. It can now be seen that the entries for H_1 and H_2 for these spaces and for the images of the generators under the maps $m_i{:}H_i(\tilde{I} \cap \tilde{O}) \rightarrow H_i(I) \oplus H_i(O)$ are as indicated in Table 5. Hence we can determine Coker m_i and show that Ker m_1 = Ker $m_2 = 0$. Since $\tilde{I} \cap \tilde{O}$, I and O are connected, $\tilde{H}_0(\tilde{I} \cap \tilde{O}) = \tilde{H}_0(I) = \tilde{H}_0(O) = 0$, and thus Ker $m_0 = 0$. Thus the SES gives $H_i(I \cup O)$ = Coker m_i as entered in Table 5.

8. H_1 and H_2 of $X_{2,1} = \tilde{Q}_{1.0} \cup Q_{2,1}$

Finally we can apply the MVS to calculate $H_1(X_{2,1})$ and $H_2(X_{2,1})$. We know H_1 and H_2 of $Q_{2,1} = \tilde{A} \cup B \otimes \bar{\mathcal{F}}$ (Table 1), of $I \cup O = \tilde{Q}_{1,0} \cap Q_{2,1}$ (Table 4) and of $Q_{1,0} = \mathcal{F}$ (Table 1). These H's are entered in Table 6 and we proceed to calculate $H_1(X_{2,1})$ and $H_2(X_{2,1})$. In the Appendix, we argue that $\tilde{Q}_{1,0}$ retracts to $Q_{1,0}$ and hence that we can use the MVS corresponding to Fig. 2(d). We can show that the map $m_i{:}H_i(\tilde{Q}_{1,0} \cap Q_{2,1}) \rightarrow H_1(Q_{1,0}) \oplus H_1(Q_{2,1})$ transforms the generators as entered in Table 6. In deducing the transformations of the generators by m_2 for $\mathcal{F} = SO(3)$, we have used the conditions $\phi_i \otimes (\bar{\varepsilon}+\varepsilon) = -\phi_i \otimes (\bar{\varepsilon}-\varepsilon)$ in the cokernel of the map $m_2{:}H_2(\tilde{A} \cap B) \rightarrow H_2(A) \oplus H_2(B)$. From these maps, we see that Ker m_1 = Ker $m_2 = 0$. Again Ker $m_0 = 0$ since the spaces involved are connected. Coker m_i are now easily determined. Note that the SES gives $H_i(X_{2,1})$ = Coker m_i since Ker m_0 = Ker $m_1 = 0$.

We can use ϕ as the single independent generator of Coker $m_1 = H_1(X_{2,1})$. Thus $H_1(X_{2,1}) = H_1(\phi)$. Note the following: This is i) Z for $\mathcal{F} = SO(2)$, ii) {0} for $\mathcal{F} = S^2$ and iii) Z_2 for $\mathcal{F} = SO(3)$. It is of course the first case that is of interest for the two-dimensional spin-statistics theorem.

Now, vector bundles on any topological space associated with the abelian representations of π_1 are classified(as bundles with flat connection) by representations of H_1. For these bundles, the spin-statistics theorem is the statement that the exchange and 2π rotation loops are homologous. This result is implicit in the preceding considerations and is also explicitly shown in ref. 6. However, in ref. 6, we did not establish that $H_1(X_{2,1}) =$ Z for $\mathcal{F} = SO(2)$ (or indeed that it is non-zero for any choice of frames). This result is necessary to have the possibility of all spins, fractional or otherwise. We have shown this result here.

Now in two spatial dimensions, rotation and exchange refer essentially to motion restricted to a circle, and only the first homology group of a circle is nontrivial. Therefore our result that the exchange and rotation loops are homologous would appear to be a rather general spin-statistics relation in two dimensions. For the abelian (i.e. line) bundles specifically, there is no topological distinction among the various bundles over a circle, and the concepts of spin and statistics can refer only to some distinguished connection (unlike in higher dimensions, where the boson and fermion bundles differ topologically). If that connection is flat, as is often assumed, then we see explicitly that $H_1(C)$ determines everything, since its characters classify the flat U(1) connections on C.

On the other hand, there are in some cases topologically nontrivial line bundles over C, corresponding to the introduction of Wess-Zumino terms in the curvature, as described in Sec. 9. For these, the physical meaning of our homological spin-statistics equivalence is less clear, since the phase associated with a specifice exchange (or rotation) would depend on the details of the exchange path. Whether the assignment of a definite fractional statistics or spin to the particles would continue to make sense for these curved bundles appears to us as an open question, hinging on whether some suitably simple analog of flatness could be used to single out a preferred class of connections associated with the representations of $H_1(C)$. (For example one might try the criterion that the curvature be "harmonic" in some generalized sense appropriate to the non-manifold C.)

9. A Wess - Zumino Term for $\mathcal{F} = SO(2)$

We have shown that $H_2(X_{2,1}) = H_1(\bar{\phi}) \otimes Z(\bar{\varepsilon}+\varepsilon) = Z$ for $\mathcal{F} = SO(2)$. Hence, as noted in the Introduction, $X_{2,1}$ has non-trivial closed two forms which when integrated over the two surface defined by the two cycle $\bar{\phi} \otimes (\bar{\varepsilon}+\varepsilon)$ is not zero. Such a two form approaches zero in the region of $X_{2,1}$ where one of the particles is near annihilation with the antiparticle and vanishes idenically in $Q_{1,0}$. It is a candidate for writing a nontrivial Wess - Zumino term in the Lagrangian describing the interaction of two such particles and an antiparticle. We now exhibit such a closed two form which is rotationally and translationally invariant as well.

Define $x^{(i)} - \bar{x} = |x^{(i)} - \bar{x}| (\cos\theta_i, \sin\theta_i)$, $F_i = e^{i\phi_i}$, $\bar{F} = e^{i\bar{\phi}}$, $x_1 - x_2 = |x_1 - x_2| (\cos\theta_{12}, \sin\theta_{12})$. Here the angles are measured from some arbitrary direction. One can easily see that when particle 1 is near annihilation $\phi_1 - \theta_1 \approx \theta_1 - \bar{\phi} + \pi$ and $\theta_{12} = \theta_2 + \pi$. If both particles are close to annihilation, we have in addition $\theta_{12} = \theta_1$.

Define the closed two form

$$w = d(\phi_1 - \theta_1) \wedge d(\theta_1 - \bar{\phi}) + d(\phi_2 - \theta_2) \wedge d(\theta_2 - \bar{\phi}) + d(\theta_1 + \theta_2 - 2\bar{\phi}) \wedge d(\theta_{12} - \bar{\phi}) - [d(\phi_1 - \theta_1) + d(\phi_2 - \theta_2)] \wedge d(\theta_{12} - \bar{\phi}) \quad (8)$$

One easily checks that $w \approx 0$ in the annihilation region. Furthermore

$$\int_{\bar{\phi}\otimes(\bar{\varepsilon}+\varepsilon)} w = \int_{\bar{\phi}\otimes(\bar{\varepsilon}+\varepsilon)} d\theta_2 \wedge d\bar{\phi} = -(2\pi)^2. \quad (9)$$

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Appendix

In this Appendix, we argue that $\tilde{Q}_{1,0}$ retracts to $Q_{1,0}$ so that both have the same homotopy and homology groups. $\tilde{Q}_{1,0}$ is $Q_{1,0}$ thickened into $Q_{2,1}$ and characterized by some ε thickening. By this we mean the following: $\tilde{Q}_{1,0}$ contains $Q_{1,0}$ and in addition every point in $Q_{2,1}$ which is in an ε - neighbourhood of any point of $Q_{1,0}$. Loosely stated,we must show that there exists a continuous deformation (or more precisely, a retraction) of $\tilde{Q}_{1,0}$ to $Q_{1,0}$. We will pick our ε to be $<< L$ where L is the length used to define syzygy for a triplet in Sec.II. Thus if $p \in \tilde{Q}_{1,0} \cap Q_{2,1}$, then $p=\{[x^1 F^1, x^2 F^2]\, \bar{x}\, \bar{F}\}$ with $|x^i - \bar{x}| < \varepsilon$ and $d(F^i, R(x^i - \bar{x})\bar{F}) < \frac{\varepsilon}{L}$. [Unlike in Section 2, we omit superscripts on frames to indicate where they are attached. We do so for simplicity.] Further the triplet is in syzygy. (See Sec. II).

Our procedure for retraction will treat different regions of $\tilde{Q}_{1,0} \cap Q_{2,1}$ differently. To exhibit the retraction, we first define a continous transition function $f_\varepsilon(\delta)$ which vanishes for $\delta < \varepsilon$ and is equal to 1 for $\delta > 2\varepsilon$. Such a function is indicated in Fig.9.

The retraction consists of three steps. In Step I, the frames of the particles are rotated by different amounts in different regions; in Step II, the positions of the particles are brought into syzygy to a different degree in different regions; and finally in Step III, the positions of the particles are brought near to the antiparticle again to a different degree in different regions. All of this must be done in a continous manner, that is in a way so that the space $\tilde{Q}_{1,0} \cap Q_{2,1}$ does not 'tear'.

Step I: If $d(F^i, R(x^i - \bar{x})\bar{F}) = \Delta/L$, partially align by rotating $F^{(i)}$ along the geodesic path so that $d(F^i, R(x^i - \bar{x})\bar{F}) = f_\varepsilon(\Delta)\Delta/L$. Note that if $d(F^i, R(x^i - \bar{x})\bar{F}) \geq 2\varepsilon/L$, no rotation of F^i takes place whereas if $d(F^i, R(x^i - \bar{x})\bar{F}) \leq \varepsilon/L$, perfect alignment of $F^{(i)}$ and $\bar{F}$ is effected.

Step II: Note there is always at least one x^i such that $|x^i - \bar{x}| < \varepsilon$. By Step I, any such particle has had its frame perfectly aligned with the antiparticle. Now we move x^i and x^j, $j \neq i$, into a more approximate syzygy (for retraction)with respect to the

the antiparticle in the ɔllowing manner. Define the unit vectors $\hat{n}^i = \frac{x^i - \bar{x}}{|x^i - \bar{x}|}$,

and with these the uıt vector $\hat{s} = \frac{\hat{n}^i - \hat{n}^j}{|\hat{n}^i - \hat{n}^j|}$. Let $\hat{n}^i$ make an angle θ with $\hat{s}$

then $\hat{n}^j$ makes the ame angle θ with $-\hat{s}$. Let $\delta_{max} = \text{Max}\, |x^i - \bar{x}|$. Now move x^i so that $\hat{n}^i$ rotate towards $\hat{s}$ until it makes an angle $f_\varepsilon(\delta_{max})\theta$ with $\hat{s}$, and similiarly move x ꞉ that it makes an angle $f_\varepsilon(\delta_{max})\theta$ with $-\hat{s}$. Also rotate the frames so that thelegree of frame alignment, $d(F^i, R(x^i - \bar{x})\,\bar{F}) = f_\varepsilon(\Delta)\Delta/L$, obtained in Step I ; maintained. This will require a frame rotation of twice the angle that the univectors rotate. If $\delta_{max} > 2\varepsilon$, the positions of the particles have not been changedn this step,whereas if $\delta_{max} < \varepsilon$, the two particles end in perfect syzygy wn their frames perfectly aligned.

Step III: If $|x^i - \bar{x}| = \delta_i$ move x_i towards $\bar{x}$ so that $|x^i - \bar{x}|$ approaches $f_\varepsilon(\delta_i)\,\delta_i$. This last step effectan annihilation of a particle - anti-particle pair leaving a particle, and the retractn is complete.

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Figure Captions

Fig. 1: Mappings involved in a Mayer-Vietoris Sequence.

Fig. 2: Flow diagram of MVS's used to calculate $H_1(X_{2,1})$ and $H_2(X_{2,1})$.

Fig. 3 Spaces involved in the MVS used to calculate $H_1(Q_{2,1})$ and $H_2(Q_{2,1})$.

Fig. 4: Spaces involved in the MVS used to calculate $H_1(A)$ and $H_2(A)$. $\tilde{A}' \cap B'$ is the disjoint union of the two spaces vertically above "$\tilde{A}' \cap B'$" .

Fig. 5: Spaces involved in the MVS used to calculate $\tilde{Q}_{1,0} \cap Q_{2,1}$.

Fig. 6: The transition function $f_\varepsilon(\delta)$ used in the retraction procedure.

	SO(3)	S^2	SO(2)
H_1	Z_2	0	Z
H_2	0	Z	0

Table 1

	H_1	H_2
$\tilde{Q}_{0,0} \cap Q_{1,1}$	$H_1(\bar{\phi}) + Z(\gamma + 2\phi)$	$H_2(\bar{\sigma}) + H_1(\bar{\phi}) \otimes Z(\gamma + 2\phi)$
$Q_{0,0}$	0	0
$Q_{1,1}$	$H_1(\bar{\phi}) + H_1(\phi) + Z(\gamma)$	$H_2(\sigma) + H_2(\bar{\sigma})) + H_1(\phi) \otimes H_1(\bar{\phi})$ $+H_1(\phi) \otimes Z(\gamma) + H_1(\bar{\phi}) \otimes Z(\gamma)$
$H(\tilde{Q}_{0,0} \cap Q_{1,1}) \rightarrow$ $H(Q_{0,0}) \oplus H(Q_{1,1})$	$\gamma + 2\phi \rightarrow 0 \oplus \gamma + 2\phi$ $\bar{\phi} \rightarrow 0 \oplus \bar{\phi}$	$\bar{\sigma} \rightarrow 0 \oplus \bar{\sigma}$ $\bar{\phi} \otimes (\gamma + 2\phi) \rightarrow 0 \oplus [\bar{\phi} \otimes \gamma + 2(\bar{\phi} \otimes \phi)]$
Ker m	0	0
Coker m	$H_1(\phi)$	$H_2(\sigma) + H_1(\phi) \otimes Z(\gamma)$ $+ H_1(\bar{\phi}) \otimes Z(\gamma)$
$\tilde{Q}_{0,0} \cup Q_{1,1}$	$H_1(\phi)$	$H_2(\sigma) + H_1(\phi) \otimes Z(\gamma)$ $+ H_1(\bar{\phi}) \otimes Z(\gamma)$

Table 2

	H_1	H_2
$\tilde{A} \cap B$	$H_1(\phi_1)+H_1(\phi_2)$ $+ Z(2\bar{\varepsilon})$	$H_2(\sigma_1)+H_2(\sigma_2)$ $+ H_1(\phi_1)\otimes H_1(\phi_2) + \Sigma\, H_1(\phi_i)\otimes Z(2\bar{\varepsilon})$
A	$H_1(\phi) + Z(\bar{\varepsilon})$	F = SO(2): $H_1(\phi_1)\otimes H_1(\phi_2) + Z_2[\bar{\varepsilon}\otimes(\phi_1+\phi_2)]$ S^2: $H_2(\sigma)$ SO(3): $H_1(\phi_1)\otimes H_1(\phi_2) + Z_2[\bar{\varepsilon}\otimes(\phi_1+\phi_2)]$
B	$H_1(\phi_1)+H_1(\phi_2)$ $+ Z(\bar{\varepsilon} - \varepsilon) + Z(\bar{\varepsilon} + \varepsilon)$	$H_2(\sigma_1)+H_2(\sigma_2) + H_1(\phi_1)\otimes H_1(\phi_2)$ $+ Z(\bar{\varepsilon} - \varepsilon) \otimes Z(\bar{\varepsilon} + \varepsilon)$ $+ [H_1(\phi_1)+H_1(\phi_2)] \otimes [Z(\bar{\varepsilon} - \varepsilon)+Z(\bar{\varepsilon} + \varepsilon)]$
$H(\tilde{A} \cap B) \rightarrow$ $H(A) \oplus$ $H(B)$	$\phi_1 \rightarrow -\phi \oplus \phi_1$ $\phi_2 \rightarrow -\phi \oplus \phi_2$ $2\bar{\varepsilon} \rightarrow -\bar{\varepsilon} \oplus [(\bar{\varepsilon} - \varepsilon)+(\bar{\varepsilon} + \varepsilon)]$	$F = S^2$: $\sigma_1 \rightarrow -\sigma \oplus \sigma_1$ $\sigma_2 \rightarrow -\sigma \oplus \sigma_2$ F = SO(2) and SO(3): $\phi_1\otimes\phi_2 \rightarrow -(\phi_1\otimes\phi_2) \oplus (\phi_1\otimes\phi_2)$ $\phi_1\otimes 2\bar{\varepsilon} \rightarrow 0 \oplus (\phi_1 \otimes [(\bar{\varepsilon} - \varepsilon)+(\bar{\varepsilon} + \varepsilon)])$ $\phi_2\otimes 2\bar{\varepsilon} \rightarrow 0 \oplus (\phi_2 \otimes [(\bar{\varepsilon} - \varepsilon)+(\bar{\varepsilon} + \varepsilon)])$
Ker m	0	0
Coker m	$H_1(\phi) + Z(\bar{\varepsilon}) + Z(\bar{\varepsilon} + \varepsilon)$	$H_2(A) +$ $[H_1(\phi_1)+H_1(\phi_2)+Z(\bar{\varepsilon} - \varepsilon)] \otimes Z(\bar{\varepsilon} + \varepsilon)$
$\tilde{A} \cup B$	$H_1(\phi) + Z(\bar{\varepsilon}) + Z(\bar{\varepsilon} + \varepsilon)$	$H_2(A) +$ $[H_1(\phi_1)+H_1(\phi_2)+Z(\bar{\varepsilon} - \varepsilon)] \otimes Z(\bar{\varepsilon} + \varepsilon)$

Note: In Coker m_2 for F = SO(2) and SO(3), $H_1(\phi_i) \otimes Z(\bar{\varepsilon} + \varepsilon) = H_1(\phi_i) \otimes Z(\bar{\varepsilon} - \varepsilon)$.

Table 3

	H_1	H_2
$\tilde{A}' \cap B'$	$H_1(\phi_1) + H_1(\phi_2) +$ $H_1(\phi_1') + H_1(\phi_2')$	$H_2(\sigma_1) + H_2(\sigma_2) + H_2(\sigma_1') + H_2(\sigma_2') +$ $H_1(\phi_1) \otimes H_1(\phi_2) + H_1(\phi_1') \otimes H_1(\phi_2')$
A'	$H_1(\phi_1) + H_1(\phi_2)$	$H_2(\sigma_1) + H_2(\sigma_2)$ $+ H_1(\phi_1) \otimes H_1(\phi_2)$
B'	$H_1(\phi_1) + H_1(\phi_2)$	$H_2(\sigma_1) + H_2(\sigma_2)$ $+ H_1(\phi_1) \otimes H_1(\phi_2)$
$H(\tilde{A}' \cap B')$ $\rightarrow$ $H(A') \oplus H(B')$	$\phi_1 \rightarrow -\phi_1 \oplus \phi_1$ $\phi_2 \rightarrow -\phi_2 \oplus \phi_2$ $\phi_1' \rightarrow -\phi_2 \oplus \phi_1$ $\phi_2' \rightarrow -\phi_1 \oplus \phi_2$	$\sigma_1 \rightarrow -\sigma_1 \oplus \sigma_1$ $\sigma_2 \rightarrow -\sigma_2 \oplus \sigma_2$ $\sigma_2' \rightarrow -\sigma_2 \oplus \sigma_1$ $\sigma_1' \rightarrow -\sigma_1 \oplus \sigma_2$ ** $\phi_1 \otimes \phi_2 \rightarrow \phi_1 \otimes \phi_2 \oplus \phi_1 \otimes \phi_2$ $\phi_1' \otimes \phi_2' \rightarrow \phi_1 \otimes \phi_2 \oplus \phi_2 \otimes \phi_1$
Ker m	$H_1(\phi_1 + \phi_2 - \phi_1' - \phi_2')$	0
Coker m	$H_1(\phi_1 + \phi_2)$	F = SO(2) S^2 SO(3) Z_2 $H_2(\sigma)$ Z_2
A $\Rightarrow$	* $H_1(\phi) + Z(\bar{\varepsilon})$	*** F = SO(2) S^2 SO(3) $H_1(\phi_1) \otimes H_1(\phi_2) + Z_2[\bar{\varepsilon} \otimes (\phi_1 + \phi_2)]$ $H_2(\sigma)$ $H_1(\phi_1) \otimes H_1(\phi_2) + Z_2[\bar{\varepsilon} \otimes (\phi_1 + \phi_2)]$

* Ker $m_0 = Z(\bar{\varepsilon})$. See text.

**Setting $\phi_1 \otimes \phi_2 \oplus \phi_1 \otimes \phi_2 = 0$ and $\phi_1 \otimes \phi_2 \oplus \phi_2 \otimes \phi_1 = 0$ implies relations such that Coker $m_2 = Z_2$ for SO(2). These relations are automatically satisfied for SO(3).

*** For SO(2), $H_1(\phi_1) \otimes H_1(\phi_2) = Z$. For SO(3), $H_1(\phi_1) \otimes H_1(\phi_2) = Z_2$. The generating two cycle $\bar{\varepsilon} \otimes \phi$, of the extra Z_2, is written assuming the two particles in the cycle $\bar{\varepsilon}$ have the same frame orientation. In A , this cycle generates a Z_2 of an RP^2.

Table 4

	H_1	H_2
$\tilde{I} \cap \tilde{O}$	$H_1(\phi_1 + \phi_2 - \bar{\phi})$ $+ Z[2(\bar{\varepsilon} + \phi_1 + \phi_2)]$	$H_2(\sigma_1 + \sigma_2 - \bar{\sigma})$ $+ H_1(\phi_1 + \phi_2 - \bar{\phi}) \otimes Z[2(\bar{\varepsilon} + \phi_1 + \phi_1)]$
I	$H_1(\phi_1 + \phi_2 - \bar{\phi})$ $+ Z(\bar{\varepsilon} + \phi_1 + \phi_2)$	$H_2(\sigma_1 + \sigma_2 - \bar{\sigma}) +$ $H_1(\phi_1 + \phi_2 - \bar{\phi}) \otimes Z(\bar{\varepsilon} + \phi_1 + \phi_2)$
O	$H_1(\phi_1 - \bar{\phi}) + H_1(\phi_2)$ $Z(\bar{\varepsilon} - \varepsilon + 2\phi_1) + Z(\bar{\varepsilon} + \varepsilon)$	$H_2(\sigma_1 - \bar{\sigma}) + H_2(\sigma_2) +$ $H_1(\phi_1 - \bar{\phi}) \otimes H_1(\phi_2) +$ $[H_1(\phi_1 - \bar{\phi}) + H_1(\phi_2)] \otimes [Z(\bar{\varepsilon} - \varepsilon + 2\phi_1) +$ $Z(\bar{\varepsilon} + \varepsilon)]$ $+ Z(\bar{\varepsilon} - \varepsilon + 2\phi_1) \otimes Z(\bar{\varepsilon} + \varepsilon)$
$H(\tilde{I} \cap \tilde{O})$ $\to$ $H(I) \oplus H(O)$	$\phi_1 + \phi_2 - \bar{\phi} \to$ $- (\phi_1 + \phi_2 - \bar{\phi}) \oplus$ $[(\phi_1 - \bar{\phi}) + \phi_2]$ $2(\bar{\varepsilon} + \phi_1 + \phi_2) \to$ $- 2(\bar{\varepsilon} + \phi_1 + \phi_2) \oplus [(\bar{\varepsilon} - \varepsilon +$ $2\phi_1) + (\bar{\varepsilon} + \varepsilon) + 2\phi_2]$	$(\sigma_1 + \sigma_2 - \bar{\sigma}) \to (\sigma_1 + \sigma_2 - \bar{\sigma}) \oplus [(\sigma_1 - \bar{\sigma})$ $+ \sigma_2]$ $(\phi_1 + \phi_2 - \bar{\phi}) \otimes 2(\bar{\varepsilon} + \phi_1 + \phi_2) \to$ $(\phi_1 + \phi_2 - \bar{\phi}) \otimes 2(\bar{\varepsilon} + \phi_1 + \phi_2)]$ $\oplus [(\phi_1 - \bar{\phi}) + \phi_2] \otimes [(\bar{\varepsilon} - \varepsilon + 2\phi_1) + (\bar{\varepsilon} + \varepsilon)$ $+ 2\phi_2]$
Ker m	0	0
Coker m	$H_1(\phi_1 + \phi_2 - \bar{\phi}) +$ $H_1(\phi_1 - \bar{\phi}) + Z(\phi_1 + \phi_2 + \bar{\varepsilon})$ $+ Z(2\phi_1 + \bar{\varepsilon} - \varepsilon)$	$H_2(\sigma_1 + \sigma_2 - \bar{\sigma}) + H_2(\sigma_1 - \bar{\sigma})$ $+ H_1(\phi_1 - \bar{\phi}) \otimes H_1(\phi_2)$ $+ [H_1(\phi_1 - \bar{\phi}) + H_1(\phi_2)] \otimes$ $[Z(\bar{\varepsilon} + \varepsilon) + Z(\bar{\varepsilon} - \varepsilon + 2\phi_1)]$ $+ Z(\bar{\varepsilon} - \varepsilon + 2\phi_1) \otimes Z(\bar{\varepsilon} + \varepsilon)$
$I \cup O$ (= $\tilde{Q}_{1,0} \cap Q_{2,1}$)	$H_1(\phi_1 + \phi_2 - \bar{\phi}) +$ $H_1(\phi_1 - \bar{\phi}) + Z(\phi_1 + \phi_2 + \bar{\varepsilon})$ $+ Z(2\phi_1 + \bar{\varepsilon} - \varepsilon)$	$H_2(\sigma_1 + \sigma_2 - \bar{\sigma}) + H_2(\sigma_1 - \bar{\sigma})$ $+ H_1(\phi_1 - \bar{\phi}) \otimes H_1(\phi_2)$ $+ [H_1(\phi_1 - \bar{\phi}) + H_1(\phi_1 + \phi_2 - \bar{\phi})] \otimes$ $[Z(\bar{\varepsilon} - \varepsilon) + Z(\bar{\varepsilon} - \varepsilon + 2\phi_1)]$ $+ Z(\bar{\varepsilon} - \varepsilon + 2\phi_1) \otimes Z(\bar{\varepsilon} - \varepsilon)$

Table 5

	H_1	H_2
$I \cup O = \tilde{Q}_{1,0} \cap Q_{2,1}$	$H_1(\phi_1+\phi_2-\bar{\phi}) + H_1(\phi_1-\bar{\phi}) + Z(\phi_1+\phi_2+\bar{\varepsilon}) + Z(2\phi_1+\bar{\varepsilon}-\varepsilon)$	$H_2(\sigma_1+\sigma_2-\bar{\sigma}) + H_2(\sigma_1-\bar{\sigma}) + H_1(\phi_1-\bar{\phi}) \otimes H_1(\phi_2) + [H_1(\phi_1-\bar{\phi})+H_1(\phi_2)] \otimes [Z(\bar{\varepsilon}+\varepsilon) + Z(\bar{\varepsilon}-\varepsilon+2\phi_1)] + Z(\bar{\varepsilon}-\varepsilon+2\phi_1) \otimes Z(\bar{\varepsilon}+\varepsilon)$
$Q_{1,0}$	$H_1(\phi)$	$H_2(\sigma)$
$(\tilde{A} \cup B) \otimes \bar{F} = Q_{2,1}$	$H_1(\phi) + H_1(\bar{\phi}) + Z(\bar{\varepsilon}) + Z(\bar{\varepsilon}+\varepsilon)$	$H_2(A) + H_2(\bar{\sigma}) + H_1(\bar{\phi}) \otimes [H_1(\phi) + Z(\bar{\varepsilon}) + Z(\bar{\varepsilon}+\varepsilon)] + [H_1(\phi_1)+H_1(\phi_2)+Z(\bar{\varepsilon}-\varepsilon)] \otimes Z(\bar{\varepsilon}+\varepsilon)$
$\tilde{Q}_{1,0} \cap Q_{2,1} \to Q_{1,0} \oplus Q_{2,1}$	$\phi_1+\phi_2-\bar{\phi} \to -\phi \oplus (2\phi-\bar{\phi})$ $\phi_1-\bar{\phi} \to 0 \oplus (\phi-\bar{\phi})$ $\phi_1+\phi_2+\bar{\varepsilon} \to -\phi \oplus (2\phi+\bar{\varepsilon})$ $2\phi_1+\bar{\varepsilon}-\varepsilon \to 0 \oplus [2\phi+(\bar{\varepsilon}-\varepsilon)]$	$F = S^2$: $\sigma_1+\sigma_2-\bar{\sigma} \to -\sigma \oplus (\sigma-\bar{\sigma})$ $\sigma_1-\bar{\sigma} \to 0 \oplus (\sigma-\bar{\sigma})$ $Z(\bar{\varepsilon}-\varepsilon) \otimes Z(\bar{\varepsilon}-\varepsilon) \to 0 \oplus [Z(\bar{\varepsilon}+\varepsilon) \otimes Z(\bar{\varepsilon}-\varepsilon)]$ $F = SO(2)$ and $SO(3)$: $(\phi_1-\bar{\phi}) \otimes \phi_2 \to 0 \oplus [(\phi_1 \otimes \phi_2) - (\bar{\phi} \otimes \phi)]$ $(\phi_1-\bar{\phi}) \otimes (\bar{\varepsilon}+\varepsilon) \to 0 \oplus [(\phi)-(\bar{\phi})] \otimes (\bar{\varepsilon}+\varepsilon)$ $\phi_2 \otimes (\bar{\varepsilon}+\varepsilon) \to 0 \oplus [\phi_2 \otimes (\bar{\varepsilon}+\varepsilon)]$ $(\phi_1-\bar{\phi}) \otimes (\bar{\varepsilon}-\varepsilon+2\phi_1) \to 0 \oplus \{[-(\phi_1) \otimes (\bar{\varepsilon}+\varepsilon)] - \bar{\phi} \otimes [2\bar{\varepsilon}-(\bar{\varepsilon}+\varepsilon)] - 2\bar{\phi} \otimes \phi\}$ $\phi_2 \otimes (\bar{\varepsilon}-\varepsilon+2\phi_1) \to 0 \oplus \{[\phi_2 \otimes (\bar{\varepsilon}+\varepsilon)] - (\phi_1 \otimes \phi_2)\}$ $(\bar{\varepsilon}-\varepsilon+2\phi_1) \otimes (\bar{\varepsilon}+\varepsilon) \to 0 \oplus [(\bar{\varepsilon}-\varepsilon)+2\phi_1] \otimes (\bar{\varepsilon}+\varepsilon)$
Ker m	0	0
Coker m	$H_1(\phi)$	$F = SO(3)$ and $F = SO(2)$: $Z_2[\bar{\varepsilon} \otimes (\phi_1+\phi_2)] + H_1(\bar{\phi}) \otimes Z(\bar{\varepsilon}+\varepsilon)$ $F = S^2$: 0
$X_{2,1}$	$H_1(\phi)$	$F = SO(3)$ and $F = SO(2)$: $Z_2[\bar{\varepsilon} \otimes (\phi_1+\phi_2)] + H_1(\bar{\phi}) \otimes Z(\bar{\varepsilon}+\varepsilon)$ $F = S^2$; 0

Note: In deducing the transformations of the generators by m_2 for $F = SO(2)$ and $SO(3)$ we have used the conditions $\phi_i \otimes (\bar{\varepsilon}+\varepsilon) = -\phi_i \otimes (\bar{\varepsilon}-\varepsilon)$ in Coker of the map $H(\tilde{A} \cap B) \to H(A) \oplus H(B)$. See Table 3 .

Table 6

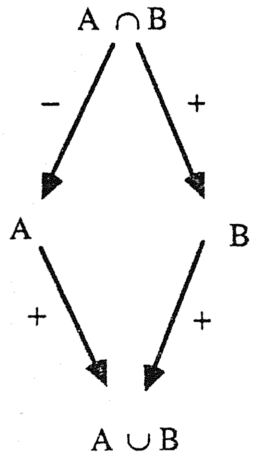
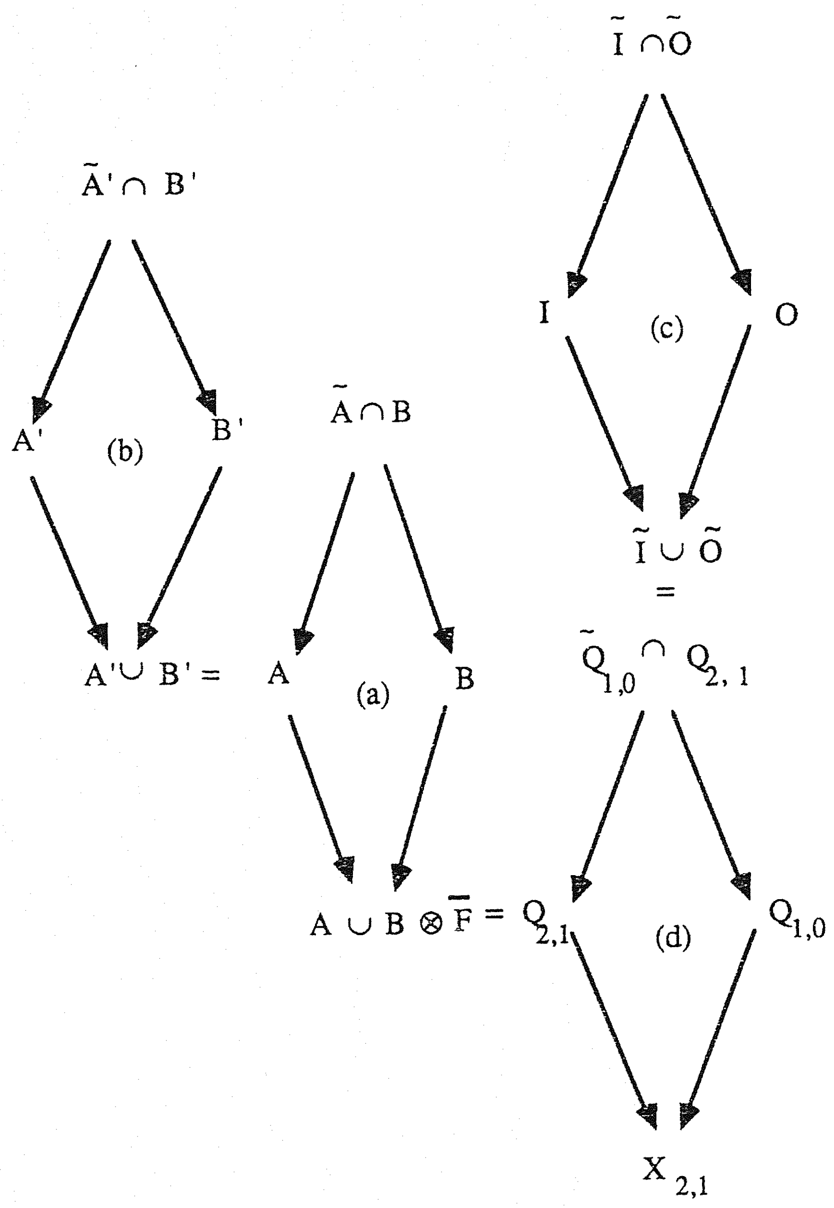


Fig. 1



$\tilde{I} \cap \tilde{O}$
$\tilde{A}' \cap B'$
I
(c)
O
A'
(b)
B'
$\tilde{A} \cap B$
$\tilde{I} \cup \tilde{O}$
=
$\tilde{Q}_{1,0} \cap Q_{2,1}$
$A' \cup B' =$
A
(a)
B
$A \cup B \otimes \overline{F} = Q_{2,1}$
(d)
$Q_{1,0}$
$X_{2,1}$

Fig. 2

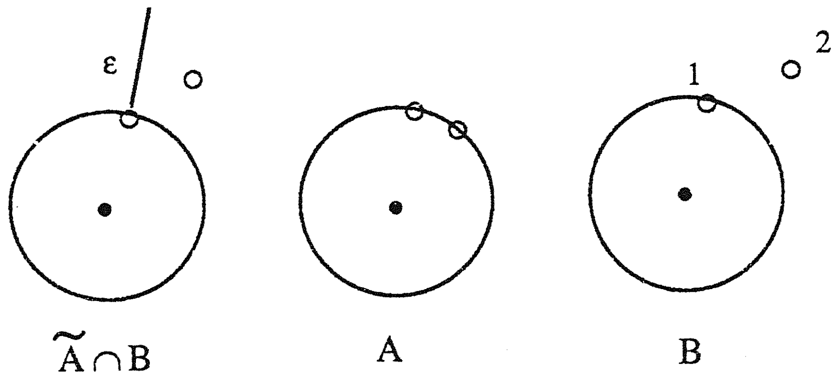


Fig. 3

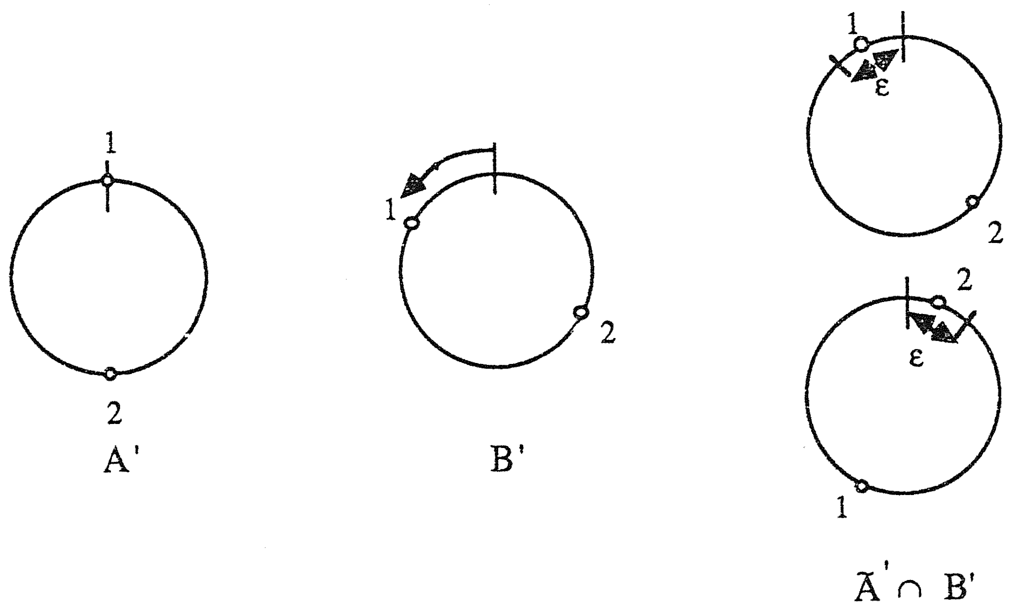


Fig. 4

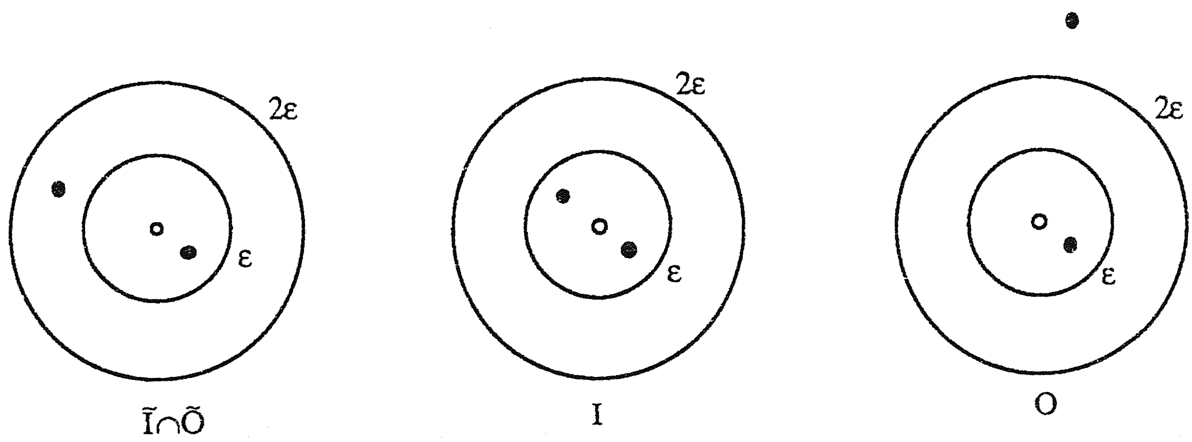


Fig. 5

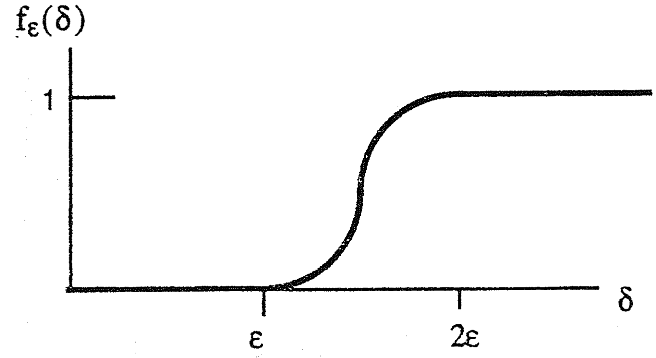


Fig. 6