A (2+1)-dimensional model with instanton and sphaleron solutions

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Abstract: We present a model with instantons in $\mathbb{R}_3$, and in its static limit a sphaleron in $\mathbb{R}_2$.

The solution found for the SU(2) Yang–Mills (YM) field interacting with an isospin–half Higgs field found by Dashen, Hasslacher and Neveu[1] (DHN) was interpreted by Manton[2] and Klinkhamer[3], as the unstable field configuration situated at a point of a noncontractible loop (NCL) in configuration space, whose energy is maximal. They named this field the sphaleron[3], and suggested that it could be important as a mechanism for classical transitions responsible for baryon–number non–conservation, because the sphaleron can be cast into the form of a path connecting topologically distinct vacua. There has been considerable activity recently in this area[4], and because of the considerable complexity involved in estimating the baryon–number violation in electroweak theory, much attention has also been devoted to carrying out this programme employing toy models in lower (than physical) dimensions.

In the most intensively studied models, namely the electroweak model in (3+1)–dimensions and the (modified) Sigma model employed by Mottola and Wipf[6] in (1+1)–dimensions, the topologically distinct vacua are defined by the YM and Sigma models in $\mathbb{R}_4$ and $\mathbb{R}_2$ respectively, while the sphalerons are the fields of a YM–Higgs, (the electroweak) model, and a special[6] scale–breaking Sigma model, in $\mathbb{R}_3$ and $\mathbb{R}_1$ respectively. In fact, in these cases, the instantons and the sphalerons are the results of distinct dynamical models.

It is our aim in the present Letter, to propose a new model in (2+1)–dimensions, for which both the instanton fields characterising the distinct topological vacua, and the sphaleron field, are solutions of the one model, respectively on $\mathbb{R}_3$ and $\mathbb{R}_2$.

To put our task in perspective, we should note that ours is not the first such model proposed. Forgács and Horváth[7] proposed the Abelian Higgs model in (1+1)–dimensions, whose instantons are the Ginzburg–Landau vortices[7] in $\mathbb{R}_2$, and in the static limit, the model is nothing but the ϕ^4 model in $\mathbb{R}_1$, which however has only stable solutions. To find the (unstable) sphaleron in this model, Manton and Samols[5] considered the ϕ^4 model on S^1 instead of $\mathbb{R}_1$. Next in one dimension higher, they proposed[7] the YM–Higgs model, with the Higgs field in the adjoint representation of SU(2) in (2+1)–dimensions. Clearly, the instanton[9] of this model is the 't Hooft–Polyakov monopole in $\mathbb{R}_3$. In the static case on $\mathbb{R}_2$, there is every reason to expect that the same model should have a sphaleron as argued in Ref. [7], which can be systematically constructed, but it is not possible to find this sphaleron explicitly, like in the cases of Refs. [5,6]. We plan to consider this case in detail elsewhere. By contrast to this latter (2+1)–dimensional case, the sphaleron in the model we shall propose below can be explicitly evaluated, and by contrast to the former (1+1) dimensional case, our sphaleron field is defined on $\mathbb{R}_2$ and not S^2.

The model: Our (2+1)–dimensional model is extracted from a class of models[10] on $\mathbb{R}_3$, which have soliton solutions. These solitons are taken as the instantons of our model, just like the 't Hooft–Polyakov monopole was taken to be the instanton in (2+1)–dimensions, mentioned above[9]. The systematics of constructing such models were previously[11] considered, but here we restrict ourselves to one specific model with the required properties.

Our dynamical field is an SU(2) scalar field $\Phi = \vec{\phi}\cdot\vec{\sigma}$, and the dynamics is controlled by a symmetry breaking potential, a quartic and a sextic kinetic term, but no quadratic kinetic term. Denoting $\Phi_\mu = \partial_\mu \Phi$, $\Phi_{\mu\nu} = [\Phi_\mu, \Phi_\nu]$ and $\Phi_{\mu\nu\rho} = (\{\Phi_{\mu\nu}, \Phi_\rho\} + \text{cycl. } \mu\nu\rho)$, with $\mu = i,3$; $i = 1,2$, it is given by the Lagrangian

$$\mathcal{L} = \mathrm{tr}\left[\Phi^2_{\mu\nu\rho} + \frac{1}{8}\Phi^2_{\mu\nu} - (1-\Phi^2)^2\right]. \tag{1}$$

That the kinetic terms are totally antisymmetic in the free indices assures that only velocity–square terms occur in (1), and for simplicity we have suppressed all dimensional constants in (1).

For (1) defined on $\mathbb{R}_d$, the virial theorem[8] states

$$(d-6)\ \|\Phi_{\mu\nu\rho}\|^2 + (d-4)\ \|\Phi_{\mu\nu}\|^2 + (d-0)\ \|S\|^2 = 0, \tag{2}$$

where $S = 1-\Phi^2$, and $\|S\|^2$ denotes the volume integral of S^2, etc.

It is therefore obvious that finite–action solutions on $\mathbb{R}_3$, and finite–energy solutions on $\mathbb{R}_2$ are viable for the model (1). Note also that in the static limit on $\mathbb{R}_2$, (1) is nothing else than

$$\mathcal{L}_{\text{static}} = \Phi_{ij}^2 - S^2, \tag{3}$$

which resembles the complex scalar field ($\varphi = \phi_1 + i\phi_2$) model considered in Refs. [12]

$$\mathcal{L}_0 = (i\partial_{[i}\varphi\partial_{j]}\varphi^*)^2 + (1-|\varphi|^2)^2, \tag{4}$$

which has soliton solutions[12]. By contrast, (3) does not have soliton solutions on $\mathbb{R}_2$. We shall explain this below.

Instantons: That the field equations of (1) are endowed with instanton solutions can be deduced from the work of Ref. [10], so that we shall not give a complete construction here. We shall restrict ourselves to the demonstration of relevant points here, such as the fact that these instantons are not absolutely minimal field configurations satisfying a selfdual equation, and hence cannot be evaluated explicitly. We shall also demonstrate the topological stability of these solutions, not least by way of showing that the solutions in the static limit (3) are not topologically stable.

Consider the inequality

$$\text{tr}(\Phi_{\mu\nu\rho} - \frac{1}{3!}\,\epsilon_{\mu\nu\rho}S)^2 \geq 0. \tag{5}$$

When we add that positive definite term tr $\Phi_{\mu\nu}^2$ to the left hand side of (5) we have a lower bound for $\mathcal{L}$ in (1)

$$\int_{\mathbb{R}_3} \mathcal{L} \geq \int 2\,\epsilon_{\mu\nu\rho}\,\text{tr}\,S\,\Phi_\mu\,\Phi_\nu\,\Phi_\rho = \int \partial_\mu \Omega_\mu \quad , \tag{6}$$

where we have denoted the cross term in the right hand side as a total divergence, which is easy to show, c.f. Ref [10]. The action therefore is bounded from below by the surface integral on the right hand side of (6), which is non–vanishing provided that the asymptotic property

$$\mathrm{tr}\ \Phi^2 = \vec{\phi}^2 \xrightarrow[|x_\mu| \to \infty]{} 1 \tag{7}$$

is satisfied. This establishes the topological stability.

Were it not for the quartic kinetic term $\Phi^2_{\mu\nu}$ in (1), saturating (5) would have given the exact solution[10]. We must however retain $\Phi^2_{\mu\nu}$ in (1), otherwise in the static limit, the dynamics of (3) would be trivial and there would be no sphaleron. It is important to note that for the static limit (3), the right hand side of the inequality corresponding to (6), namely

$$\mathcal{L}_{\mathrm{static}} \geq 2\ \epsilon_{ij}\ \mathrm{tr}\ S\ \Phi_i\ \Phi_j \equiv 0, \tag{8}$$

does not supply a nontrivial topological bound. (The corresponding quantity for (4) is $i\epsilon_{ij}(1-|\varphi|^2)\partial_i\varphi\partial_j\varphi^*$, which is a total divergence[12].)

<u>Sphaleron:</u> To construct the sphaleron field on $\mathbb{R}_2$ for $\mathcal{L}_{\mathrm{static}}$ given by (3), we follow Manton's procedure of Ref. [2]. We introduce the unit vector

$$P(\mu,\theta) = (\sin\mu\,\sin\theta,\ \sin^2\mu\,\cos\theta + \cos^2\mu,\ \sin\mu\,\cos\mu\,(\cos\theta-1)), \tag{9}$$

which features one polar angle, μ, and one azymuthal angle θ, analogous to Manton's case, where the unit vector was parametrized by two polar angles and one azymuthal angle. The unit vector (9) was also employed in Ref. [6] in a somewhat different framework. Following Manton, we identify the asymptotic field $\vec{\phi}^\infty$ with the unit vector $\vec{P}$ given by

(9). We see that for $\mu = 0$ and $\mu = \pi$ $\vec{\phi}^{\infty} = (0,1,0) = \vec{\phi}_{vac}$, which we identify with the vacuum field.

Next we seek the sphaleron field, for which $\vec{\phi}_{sph}(\mu = \frac{\pi}{2})$ is the maximal energy configuration, and for which $\vec{\phi}_{sph}(\mu = 0) = \vec{\phi}_{sph}(\mu = \pi) = \vec{\phi}_{vac}$, such that as μ increases from 0 to π, the vacuum field is connected to itself. Such a $\vec{\phi}_{sph}$ would be an NCL. After some thought, we see that

$$\vec{\phi}_{sph}(r,\mu,\theta) = h(r)\,\vec{\phi}^{\infty} + (1-h(r)) \begin{bmatrix} 0 \\ \cos^2\mu \\ -\sin\mu\,\cos\mu \end{bmatrix}, \tag{10}$$

where $\vec{\phi}^{\infty} = P(\mu,\theta)$ given by (9) and $h(\infty) = 1$.

Calculating the energy functional $E = \int \mathcal{L}_{statrc}\, r\, dr\, d\theta$ for $\vec{\phi} = \vec{\phi}_{sph}$ we find

$$E = 2\pi \sin^4\mu \int \left[\left[\frac{hh'}{r}\right]^2 + (1-h^2)^2 \right] r dr\ . \tag{11}$$

That E takes on its maximal value for $\mu = \frac{\pi}{2}$ is manifest, as also is the fact that E vanishes for both $\mu = 0$ and $\mu = \pi$, for the vacuum field. Moreover, we can now find the function h(r) explicitly by varying $L[h,h']$ defined by $E = \int L[h,h']dr$, with respect to h(r) we find

$$\frac{1}{r}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}h^2\right] + 4\,(1-h^2) = 0, \tag{12}$$

which, in terms of $\chi = 1 - h^2$, and $\rho = r^2$ can be expressed as

$$\frac{d^2\chi}{d\rho^2} - \chi = 0 \tag{13}$$

which can be integrated, and setting the arbitrary constant of this first integration equal to zero we end up with

$$\frac{d\chi}{d\rho} = \mp \chi. \tag{14}$$

Choosing the upper sign, we find the solution with the desired asymptotic behaviour to be

$$h^2 = 1 - e^{-r^2}, \tag{15}$$

explicitly. In performing the second integral, we have chosen the constant of integration so as to satisfy $h(0) = 0$.

Topological charge: Finally, we evaluate the topological charge density, which is defined by the integrand on the right hand side of (6), for the sphaleron field (10), with the assumption[13] that the polar parameter μ now depends on $t = x_3$, with $\mu(t = -\infty) = 0$ and $\mu(t = +\infty) = \pi$.

After performing the summation over the indices $\mu = (i,3)$, this density is proportional to

$$\rho = i\epsilon_{ij} \, \mathrm{tr}\, S(\Phi_t \Phi_i \Phi_j + \Phi_j \Phi_t \Phi_i + \Phi_i \Phi_j \Phi_t) \tag{16}$$

Substituting (10) into (16), denoting $\dot{\mu} = \frac{d\mu}{dt}$, we see that

$$\rho = 12\,(1-h^2)\,(1-h\cos\theta)\,\frac{hh'}{r}\,\dot{\mu}\,\sin^5\mu, \tag{17}$$

Finally, integrating over t and θ in $q = \int \rho \, r \, dr \, d\theta \, dt$, we find

$$q = \frac{64\pi}{5} \int_0^\infty \frac{d}{dr}\left(h^2 - \tfrac{1}{2} h^4\right) dr = \frac{32\pi}{5}$$

which could have been evaluated directly from our knowledge of $h(0) = 0$ and $h(\infty) = 1$, even if we did not have the explicit form (15) for h.

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