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A non-Abelian Higgs model with instantons and Sphaleron

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Abstract :

We present a SU(2) X SU(2) X U(1) model in (3+1) dimensions, which has instanton solutions in $\mathbb{R}_4$, and a sphaleron on $\mathbb{R}_3$ in the static limit.

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Sometime ago, Manton[1)] found a very interesting property of the Dashen, Hasslacher and Neveu[2)] (DHN) solution to the SU(2) Yang-Mills-Higgs (complex doublet) system in 3 dimensions. He showed that the DHN solution occured at the maximal energy position on a non-contractible-loop (NCL) in field space, and hence argued that it should be unstable. This instability, against a one-parameter family of fluctuations, was later demontrated by Burzlaff[3)] explicitely.

This solution, which is unstable against decay into the vacuum, occurs at the maximum of a minimum energy path connecting topologically inequivalent vacuua of the Eclidean Theory, and was named a sphaleron by Klinkhamer and Manton [4)]. The physical interest of the sphaleron derives from its relevence to the fermion-number violation in electroweak theory. In the electroweak theory however there are no instanton solutions, so that the topologically distinct vacuua referred to above are characterised by the instantons of the YangMills (YM) model, which does not feature a Higgs field. Futhermone, the DHN sphaleron has SU(2) gauge group, and not SU (2)xU(1) as the electroweak theory. The latter point was largely settled in ref [4], where it was shown numerically that it could be justified to include the U(1) field. Also, more vecently, Kleihaus, Kunz and Brihaye [6)] have included the U(1) fied in their construction of the electroweak sphaleron, by departing from a 3 dimensional radial field configuration which would suppress the U(1) field, and working with an axial field configuration, which being radial in 2 dimensions,still features a non-trivial U(1) field.

While the latter question of employing the DHN solution as the sphaleron of the electroweak theory is to same extent justified [4,6)] the previous matter, namely that strictly speaking there is no instanton solution to the Weinberg-Salam model remains open. This leads us to the question whether we can have a model, which has instanton solutions, and, a sphaleron solution in its static limit ? This is the main question we address ourselves to in the present work.

Before analysing it more closely, we note that this question has been answered positively in the case of lower dimensions, to different degrees of completeness. First, Forgacs and Horvath [7)] classified models in (1+1) and (2+1) dimensions which have this propertly, namely the Abelian Higgs model with vortex[8)] solutions as instantons, and the (adjoint-representation) SU(2) Higgs model whith the 't Hooft-Polykov monopole as instanton[9)], respectivily.

In the case of the Abelian Higgs model in (1+1) dimensions, Bochkarev and

Shaposhnikov[10] and Grigoriev and Rubakov[11] showed that the static limit of this system had a sphaleron solution on $\mathbb{R}_1$. In the static limit, this system reduces to the ϕ^4 model in $\mathbb{R}_1$.To construct unstable solutions to it is rather more subtle than to the SU(2) YM-Higgs (doublet) case in $\mathbb{R}_3$,. and involves requiring the fields to be periodic, resulting in an instability [10] against a parameter τ ($\infty \geqslant \tau \geqslant -\infty$), the sphaleron ocurring for $\tau = \frac{1}{2}$, in contrast with the DHN case where the instability is parametrised by an angular parameter μ ($0 \leqslant \mu \leqslant \pi$). Manton and Samols[12] showed that this was equivalent to putting the field on a circle of finite circumference L, where the sphaleron occuurs for $L \rightarrow \infty$.

The case of the (adjoint) SU(2) Higgs model in (2+1) dimensions is simpler and more akin to the (3+1) case, and hence deserves further study, but we defer this because even as a toy model, it does not feature any explicit solutions. As it is, we have recently proposed[13] a (non-gauge) toy model in (2+1) dimensions satisfying these criteria, and with an explicit sphaleron solution.

In addition to the above mentioned lower dimensional models[7][13], Mottola and Wipf[14] have employed a (1+1) dimensional toy model based on the O(3) sigma-model, which has technical advantages over the corresponding Abelian Higgs model of Ref [7, 10, 11]. This model[14] however, shares with the electroweak model[1] the feature that the instantons and sphaleron are supported by different dynamics.

Finally we come to our above stated aim of constructing a model in (3+1) dimensions, which features both instantons on $\mathbb{R}_4$, and a sphaleron on $\mathbb{R}_3$, in the static limit. This task was previously addressed by Ratra and Yaffe[15], who considered the field configurations of the Weinberg-Salam model, that correspond to a non-zero Chern-Pontryagin (C-P) charge. These however are not solutions, since the (scaling) Virial theorem[8] in this case contradicts the existence of finite action solutions with non-trivial Higgs field.

The model we propose is the SO(4) x U(1) Higgs model introduced in ref [16], which has topologically stable finite action solutions on $\mathbb{R}_4$. It differs from the standard electroweak theory in that the gauge group is $SU_+(2) \times SU_-(2) \times U(1) \approx SO(4) \times U(1)$, instead of SU(2) x U(1), and consequently the Higgs multiplet consists of two independent Higgs doublets (one for each $SU_\pm(2)$), instead of a single doublet. We denote the SO(4) x U(1) connection $\widetilde{A}_\mu$ as

$$\widetilde{A}_\mu = A_\mu + i\,\gamma_5\, a_\mu \tag{1a}$$

$$A_\mu = \begin{bmatrix} A_\mu^{(+)} & \\ & A_\mu^{(-)} \end{bmatrix} \qquad (1b)$$

where $A_\mu^{(\pm)}$ are respectively the $SU_\pm$ (2) (antihermitian) connections, and a_μ the U(1) field. The Higgs multiplet is the (antiselfdual) 4x4 matrix field

$$\Phi = \begin{bmatrix} & \psi \\ -\psi^+ & \end{bmatrix} \qquad (2)$$

where the 2x2 consituent matrix field ψ consists of the two independent doublets, and is therefore subject to no constraints.

The special property of the Higgs field(2),Φ, is that it can be subjected to spherical summetry contraints with respect to the gauge group SO(4), without trivialising, in contrast with the complex Higgs doublet in electroweak theory[1]. (It was remarked also by Manton[1], that the radial symmetry of the energy functional of the DHN sphaleron can be understood by considering these fields as an SO(4) gauge theory).

The most important consequence of this property is that we can set the U(1) field in (1a) equal to zero by imposing spherical symmetry in $\mathbb{R}_4$, or as the case may be, axial symmetry, corresponding to spherical symmetry in $\mathbb{R}_3$ for the static case. This last configuration pertains to the sphaleron solution to be discussed below. Accordingly, we do not have to resort to a perturbation method[4], as in the case of the DHN sphaleron corresponding to $\theta_W = 0$ electroweak theory. In our case, we can restrict our considerations to the SO(4) gauge field (2b), A_μ, just by imposition of symmetry, and not by setting $\theta_W = 0$ by hand.

Our SO(4) x U (1) model, which was derived[16] from the 8-dimensional generalised YM (GYM) system[17] by dimensional reduction, is given by the Lagrangian.

$$L = \mathrm{tr}\,[S^2_{\mu\nu\rho\sigma} + 2^2\,\lambda_1\,S^2_{\mu\nu\rho} + 3^2.2\,\lambda_2\,S^2_{\mu\nu} + 3^3.2\,\lambda_3\,S^2_\mu + 3^3.2\lambda_4\,(S^2)^2], \qquad (3)$$

where the dimensionless numbers $\lambda_1,..,\lambda_4$ may take any real positive values except $\lambda_1=\lambda_2=\lambda_3=\lambda_4 = 1$ as explained in ref [16], and

$$S_{\mu\nu\rho\sigma} = \tilde{F}_{\mu\nu\rho\sigma} = \left\{\tilde{F}_{\mu\nu}, \tilde{F}_{\rho\sigma}\right\} + \text{cycl.}\ (\nu,\rho,\sigma) \tag{4a}$$

$$S_{\mu\nu\rho} = \left\{\tilde{F}_{\mu\nu}, \tilde{D}_{\rho}\Phi\right\} + \text{cycl.}\ (\mu,\nu,\rho) \tag{4b}$$

$$S_{\mu\nu} = i\left\{S, \tilde{F}_{\mu\nu}\right\} + [\tilde{D}_{\mu}\Phi, \tilde{D}_{\nu}\Phi]) \tag{4c}$$

$$S_{\mu} = i\left\{S, \tilde{D}_{\mu}\Phi\right\}, \quad S = -(\Phi^2 + \eta^2). \tag{4d,e}$$

The dimensional constant η in (4e) is the inverse of the radius of S^4, the four-sphere, over which the dimensional reduction from $\mathbb{R}_4 \times S^4$ was performed to derive (3). Naturally $\tilde{F}_{\mu\nu}$ and $\tilde{D}_{\mu}\Phi$ denote the curvature and the covariant derivatrive, respectively, of the connection (1a), $\tilde{A}_{\mu}$. For the actual instanton fields given in ref [16], we replace $(\tilde{A},\tilde{F}, \tilde{D}\,\Phi)$, by $(A,F,D\,\Phi)$ pertaining to (1b), as the former is a spherically symmetric system.

As the SO(4) instanton of the model (3) was discussed in quite some detail in ref [16], we suffice here by only recording the surface integral, which provides the lower bound on the volume integral of (3),

$$\int \Omega_{\mu}\, d\Sigma_{\mu} = \int (\Omega_{\mu}^{(2)} + \Omega_{\mu}^{(1)} + \Omega_{\mu}^{(0)})\ d\Sigma_{\mu} \tag{5}$$

with

$$\Omega_{\mu}^{(2)} = -\frac{1}{2}\eta^4\, \varepsilon_{\mu\nu\rho\sigma}\, \mathrm{tr}\, \gamma_5\, A_{\nu}\, (F_{\rho\sigma} - \frac{2}{3} A_{\rho}A_{\sigma}) \tag{6a}$$

$$\Omega_{\mu}^{(1)} = \frac{1}{4}\eta^2\, \varepsilon_{\mu\nu\rho\sigma}\, \mathrm{tr}\, \gamma_5\, \Phi \left\{F_{\rho\sigma}, D_{\nu}\,\Phi\right\} \tag{6b}$$

$$\Omega_{\mu}^{(0)} = \frac{i}{12}\, \varepsilon_{\mu\nu\rho\sigma}\, \mathrm{tr}\, \gamma_5\, \Phi\, (\left\{S,F_{\rho\sigma}\right\} + [D_{\rho}\,\Phi, D_{\sigma}\,\Phi])\, D_{\nu}\,\Phi. \tag{6c}$$

We have parametrised (6) in terms of $(F,D\Phi)$, even though more generally, this surface integral is given by (6) in terms of $(\tilde{F},\tilde{D}\,\Phi)$. This is done in anticipation of the fact that the instantons under consideration are spherically symmetric[16] or, as the case may be, axially symmetric.

Together with the lower bound (5), the existence of instantons can be assured by requiring suitable boundary conditions, most important amongst which is

$$\mathrm{tr}\,\Phi^2 \xrightarrow[|x_\mu| \to \infty]{} -\eta^2. \tag{7}$$

We now address ourselves to the question, whether in the static limit, the model (3) has a sphaleron solution. To discuss this static limit, we first recast the important quantities (3) and (6) in axially symmetric configuration. Following the procedures used in refs [15,18,19], we express the SO(4) axially symmetric field $A_\mu = (A_i, A_4)$, i = 1,2,3 and Φ, as function of $r = \sqrt{x_i x_i}$ and $t = x_4$,

$$A_i = \left(\frac{\chi_2+1}{r}\right)\gamma_{ij}\, n_j + \left[\frac{\chi_1}{r}\,\delta_{ij} - \left(\frac{\chi_1}{r} - a_2\right) n_i\, n_j\right]\gamma_{j4} \tag{8a}$$

$$A_4 = a_1\, \gamma_{j4}\, n_j \tag{8b}$$

$$\Phi = \phi_1\, \gamma_5\, \gamma_4 + \phi_2\, \gamma_5\, \gamma_j\, n_j, \tag{8c}$$

where $a_\alpha = (a_1, a_2)$, $\chi_\alpha = \chi_1, \chi_2)$ and $\phi_\alpha = (\phi_1, \phi_2)$ are all real functions of the two variables x_α= (r,t), with the unit vector $\vec{n}$ defined by $x_i = r\, n_i$. Introducing furthur the notation $\chi = \chi_1 + i\chi_2$, $\varphi = \phi_1 + i\,\phi_2$, $u = (|\chi|^2 - 1)$, $v = (|\varphi|^2 - \eta^2)$ and $w = (\chi\varphi^* + \chi^*\varphi)$, (3) and (6) are expressed as,

$$L = \frac{3}{r^4}\,[u\, f_{\alpha\beta} - i\, D_{[\alpha}\, \chi\, D_{\beta]}\, \chi^*]^2 + \frac{3.2^3}{r^2}\,[\, w\, f_{\alpha\beta} + i\,(D_{[\alpha}\, \chi^*\, D_{\beta]}\, \varphi + D_{[\alpha}\varphi^* D_{\beta]}\, \chi)]^2$$

$$+3^2.2^3\, [v\, f_{\alpha\beta} - i\, D_{[\alpha}\, \varphi\, D_{\beta]}\varphi^*]^2 + 2^5.3^5\, v^2\, |D_i\varphi|^2$$

$$+ \frac{3.2^5}{r^2}\,(v\, D_\alpha\, \chi + w\, D_\alpha\, \varphi)\,(v\, D_\alpha\, \chi^* + w\, D_\alpha\, \varphi^*)$$

$$+\frac{3.2^5}{r^2}\,(u\, D_\alpha\, \varphi + w\, D_\alpha\, \chi)\,(u\, D_\alpha\, \varphi^* + w\, D_\alpha\, \chi^*)$$

$$+ 2^3.\, 3^3\, v^4 + \frac{2^4 3^3}{r^2}\, v^2\, w^2 + \frac{2^2 3^2}{r^4}\,(w^2 + 2uv)^2, \tag{9}$$

and

$$\Omega^{(2)}_\alpha = \frac{3^5}{r^2}\,\eta^4\, \varepsilon_{\alpha\beta}\,[2a_\beta + i\,(\chi^*\, D_\beta\, \chi - \chi\, D_\beta\, \chi^*)] \tag{10a}$$

$$\Omega^{(1)}_\alpha = \frac{3^5 i}{r^2}\,\eta^2\, \varepsilon_{\alpha\beta}\,[\, 2\,|\varphi|^2\,(\chi^*\, D_\beta\, \chi - \chi\, D_\beta\, \chi^*) + 2\,(\,|\chi|^2 - 1)\,(\varphi^* D_\beta\, \varphi - \varphi\, D_\beta\, \varphi^*)$$
$$+ \chi^*\, \varphi\,(\chi^*\, D_\beta\, \varphi - \varphi\, D_\beta\, \chi^*) + \chi\, \varphi^*\,(\varphi^*\, D_\beta\, \chi - \chi\, D_\beta\, \varphi^*) \tag{10b}$$

$$\Omega_\alpha^{(0)} = \frac{3^5 i}{r^2}\, \varepsilon_{\alpha\beta} [\, |\varphi|^4 (\chi^* D_\beta\ \chi - \chi\, D_\beta\, \chi^*) + |\varphi|^2\ (2\,|\chi|^2 - 1)\, (\varphi^* D_\beta\, \varphi - \varphi\, D_\beta\, \varphi^*)$$
$$+ 2\,|\varphi|^2 (\varphi\chi^{*2}\, D_\beta\, \varphi - \varphi^*\, \chi^2\, D_\beta\, \chi^*)] \qquad (10c)$$

where $f_{\alpha\beta}$ is the (Abelian) curvature of the connection a_α, and $D_\alpha = (\partial_\alpha + i\, a_\alpha)$ its covariant derivatrive.

The form (9) of the Lagrangian is very useful in analysing the Ansatz for the NCL in the field space of the static system, which following Manton's prescription[1), is found by identifying the asymptotic Higgs field with $\Phi^\infty = \gamma_5\, \gamma_\mu\, p_\mu$, in terms of the unit four-vector

$$p = (\sin\mu\, \sin\theta\, \cos\phi,\ \sin\mu\, \sin\theta\, \sin\phi,\ \sin^2\mu\, \cos\theta + \cos^2\mu,$$
$$\sin\mu\, \cos\mu\, (\cos\theta - 1)) \qquad (11)$$

Here, we shall defer this detailed analysis, and concentrate instead on a preliminary search for the sphaleron, following rather Burzlaff's prescription [3)], generally.

We take the static limit[15)] by requiring that $f_{\alpha\beta}$ vanish and φ and χ be time independent up to a U(1) gauge transformation with parameter $\Lambda = \Lambda\, (r,t)$,

$$a_\alpha = \partial_\alpha\, \Lambda\, ,\ \chi = e^{-i\Lambda}\, \hat{\chi}\, (r)\, ,\ \varphi = e^{-i\Lambda}\, \hat{\varphi}\, (r), \qquad (12)$$

and by further restricting $\hat{\varphi} = ih\,(r)$ and $\hat{\chi}\,(r) = (f(r) + 1)$, where h and f are real functions of r.

The resulting energy density descending from (9), is the SO(3) spherically symmetric field configuration on $\mathbb{R}_3$, of the following Lagrangian,

$$L_3 = \mathrm{tr}\, [\, 2^2\, \lambda_1\, S_{ijk}^2 + 3^2\, \lambda_2\, S_{ij}^2 + 3^3\, 2\lambda_3\, S_i^2 + 3^2\, \lambda_4\, (S^2)^2], \qquad (13)$$

where we have employed the notation given by (4b-e) with the spherically symmetric fields on $\mathbb{R}_3$

$$A_i = \frac{f(r)}{r}\, \gamma_{ij}\, n_j\, ,\ \Phi = h(r)\, \gamma_5\, \gamma_j\, n_j \qquad (14a,b)$$

We have not yet expressd (13) explicitly in terms of f(r) and h(r), because we shall arrive at that expression presently in a more instructive way.

That a spherically symmetric solution to the Euler-Lagrange equations of the energy functional of (13) exists, is not in question. That is evident from a consideration of the corresponding (scaling) Virial theorem[8], and the definite sign of (13), L_3. It is even easy to find a surface integral[20] in this case, giving the energy integral topological stability. But for this solution to be a sphaleron, we must demonstrate that it is unstable against, for example[3], perturbations in some parameter, and that the energy functional, namely the 3-dimensional volume integral of (13) for the field configuration (14), be the maximal point of this one parameter family.

In analogy with the one parameter Ansatz of Burzlaff[3] for the DHN Sphaleron, we make here the following Ansatz

$$\Phi = h(r)\,\Phi^{\infty}, \quad A_i = -\frac{1}{4} f(r)\,[\Phi^{\infty}, \partial_i \Phi^{\infty}] \qquad (15a,b)$$

with

$$\Phi^{\infty} = \gamma_5\, \gamma_\mu\, q_\mu \qquad (16a)$$

$$q_\mu = (\sin\mu \sin\theta \cos\phi,\ \sin\mu \sin\theta \sin\phi,\ \sin\mu \cos\theta,\ \cos\mu) \qquad (16b)$$

Notice that (Φ, A_i) given by (14), coincide with the fields (15-16) for $\mu = \frac{\pi}{2}$.

It is left to calculate (13) for the fields (15)-(16), and inquire as to whether the maximal value of the former occurs at $\frac{\pi}{2}$. If so, and if the energy functional vanishes for $\mu = 0, \pi$, then (14) is unstable against the fluctuations of μ, and is a sphaleron solution at $\mu = \frac{\pi}{2}$.

We give the result, term by term,

$$\mathrm{tr}\, S_{ijk}^2 = -\frac{24}{r4}\, [f\,(f+2)\, h' + h\,(f+1)\, f\,']^2 \sin^4\mu \qquad (17a)$$

$$\mathrm{tr}\, S_{ij}^2 = -\frac{2}{r^2}\, [2\, ([h^2-\eta^2)\,(f+1)]')^2 + \frac{1}{r^2}\,(f(f+2)\,(h^2-\eta^2) + 2h^2\,(f+1)^2)^2 \sin^2\mu]\, \sin^2\mu \qquad (17b)$$

$$\mathrm{tr}\, S_i^2 = -16\,(h^2-\eta^2)^2\, [h'^2 + \frac{2h^2}{r^2}\,(f+1)^2 \sin^2\mu] \qquad (17c)$$

$$\mathrm{tr}\, (S^2)^2 = -(h^2-\eta^2)^4 \qquad (17d)$$

For $\mu = \frac{\pi}{2}$, substitution of (17) into (13) coincides with the static limit of (9). It is clear from (17) that L_3 for the fields (14)-(15) is (negative) definite, as it is expected to be. So we would expect that the nonzero topological lower bound, (c.f. ref [20]) not explicitely constructed here, would ensure a nontrivial solution. It is not however easy to give a strict proof of existence, for example by adapting the Tyupkin, Fat e'ev, Schwarz (TFS)[21)] existence proof of the monopole, as was done in ref[3]. This is because of the unusual kinetic terms in (17), which are not simply the velocity terms f'^2 or h'^2, but some functions of f and/or h times these velocity terms.(This is in principle feasible, and for a simpler example of this kind, namely for the seven dimensional selfdual GYM-Higgs 'monopole', the TFS proof was successfully adapted[22)23)].

There remains finally the question as to whether this solution, is indeed a sphaleron, by testing to find out if the energy functional

$$E = 4\pi \int (r^2 L_3 [f, f'; h, h']) dr$$

is maximal for $\mu = \frac{\pi}{2}$. This is manifestly so, because each term in (17) has definite sign, and depends on the parameter μ only through the function $\sin^2\mu$. This is exactly analogous to the situation for the DHN sphaleron[3)].

We have presented a (3+1) dimensional SO(4) x U(1) Higgs model which has instantons in $\mathbb{R}_4$, and in the static limit a finite energy solution in $\mathbb{R}_3$. We have explicitly demonstated that this finite energy solution lies at the maximal energy point of a one parameter family and is therefore a sphaleron in $\mathbb{R}_3$. This sphaleron field configuration is not obtained by putting the U(1) field of the model equal to zero by hand, but rather by the imposition of spherical symmetry (in $\mathbb{R}_3$).

The explicit construction of the NCL in field space, and the analysis of its properties on the lines of ref [1], which is a straighforward task, is deferred elsewhere. We also defer the less straightforward task of attempting to analyse the t-dependence[24)] of the instability parameter μ. Both these questions are at present under active consideration.

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