

Large deviations and overflow probabilities for the general single-server queue, with applications

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Abstract

We consider from a thermodynamic viewpoint queueing systems where the workload process is assumed to have an associated large deviation principle with arbitrary scaling: there exist increasing scaling functions $(a_t, v_t, t \in \mathbb{R}_+)$ and a *rate function* I such that if $(W_t, t \in \mathbb{R}_+)$ denotes the workload process, then

$$\lim_{t \rightarrow \infty} v_t^{-1} \log P(W_t/a_t > w) = -I(w)$$

on the continuity set of I . In the case that $a_t = v_t = t$ it has been argued heuristically, and recently proved in a fairly general context (for discrete time models) by Glynn and Whitt [8], that the queue-length distribution (that is, the distribution of supremum of the workload process $Q = \sup_{t \geq 0} W_t$) decays exponentially:

$$P(Q > b) \sim e^{-\delta b}$$

and the decay rate δ is directly related to the rate function I . We establish conditions for a more general result to hold, where the scaling functions are not necessarily linear in t : we find that the queue-length distribution has an exponential tail only if $\lim_{t \rightarrow \infty} a_t/v_t$ is finite and strictly positive; otherwise, provided our conditions are satisfied, the tail probabilities decay like

$$P(Q > b) \sim e^{-\delta v(a^{-1}(b))}.$$

We apply our results to a range of workload processes, including fractional Brownian motion (a model that has been proposed in the literature (see, for example, Leland *et al* [10] and Norros [13]) to account for self-similarity and long range dependence) and, more generally, Gaussian processes with stationary increments. We also show that the martingale upper bound estimates obtained by Daley and Duffield [5], when the workload is modelled as an Ornstein-Uhlenbeck position process, are asymptotically correct.

1 Introduction

Consider a general single-server queue. For $s, t \in T$ ($T = \mathbb{Z}_+$ or \mathbb{R}_+), we denote by $A_{s,t}$ the amount of work that arrives to be processed in the time interval $[s, t)$, and by $S_{s,t}$ the amount of work that can be processed in the same time interval. (If more work arrives than can be processed, the surplus waits in the queue.) The *workload process* W is defined by

$$W_t := A_{-t,0} - S_{-t,0}, \quad (1)$$

and the *queue-length at time zero* is given by

$$Q = \sup_{t \geq 0} W_t. \quad (2)$$

There has been a recent flood of literature and discussion on the tail behaviour of the queue-length distribution, motivated by potential applications to the design and control of high-speed telecommunication networks. In this paper we generalise results of Glynn and Whitt [8] (see also references therein) on the problem of characterising the tails of the queue-length distribution in terms of the large deviation properties of the workload process. Our results are quite general and can, for example, be applied to processes that exhibit long range dependence, as have been proposed by Leland *et al* [10]. They can also be applied in other areas of probability theory.

We begin our discussion with the following easy inequality: for $b > 0$,

$$P(\sup_{t \geq 0} W_t > b) \geq \sup_{t \geq 0} P(W_t > b). \quad (3)$$

At first sight, this may seem a rather crude estimate. However, there are circumstances in which for large b we have

$$P(\sup_{t \geq 0} W_t > b) \simeq \sup_{t \geq 0} P(W_t > b). \quad (4)$$

The heuristics behind this claim are:

- *The principle of the largest term, or Laplace's method:* Suppose that $T = \mathbb{Z}_+$, and consider the following calculation.

$$P(\sup_n W_n > b) = P \bigcup_n \{W_n > b\} \quad (5)$$

$$\leq \sum_n P(W_n > b). \quad (6)$$

Roughly speaking, if the probabilities $P(W_n > b)$ decay sufficiently fast in b , then the principle of the largest term applies, yielding (4).

- *Rare events occur in the most likely way:* this well known principle is the probabilistic counterpart of the principle of the largest term. In this case it translates as “suppose that the process W is unlikely to ever reach a given level b ; if W is conditioned to reach b , then it will do so at the time when this is most likely to occur”.

This argument (namely that (4) can provide a good estimate) has been exploited and made rigorous in a variety of contexts: we refer the reader to the books of Dembo and Zeitouni [6] and Aldous [2] for details and references. (For a general introduction to large deviation theory, the paper of Lewis and Pfister [11] is an excellent source.) However, more relevant to our discussion here is the acute observation that using (4), the tail behaviour of the queue-length distribution can be derived from the large deviation behaviour of the workload process: this was originally proposed by Kesidis *et al* [9] and later made rigorous by Glynn and Whitt [8] (see also references therein and Chang [4] for related work).

Roughly speaking, Glynn and Whitt proved the following result. If $T = \mathbb{Z}_+$ (discrete time) and the pair $(W_n/n, n)$ satisfy a large deviation principle with some well-behaved rate function, then

$$\lim_{b \rightarrow \infty} b^{-1} \log P(Q > b) = -\delta, \quad (7)$$

where

$$\delta = \sup\{\theta : \lambda(\theta) \leq 0\}, \quad (8)$$

and λ is the cumulant generating function defined by

$$\lambda(\theta) := \lim_{n \rightarrow \infty} n^{-1} \log E e^{\theta W_n}. \quad (9)$$

This is a very general result, and extremely useful for applications. In particular, it can be applied to any stable system with stationary arrivals and deterministic service rate, provided the arrivals process does not possess long range dependence. However, there has been some suggestion recently that real traffic can exhibit long range dependence, most notably by Leland *et al* [10], who propose *fractional Brownian motion* (with negative drift) as a canonical model for the workload process. This has motivated us to generalise Glynn and Whitt's result. Roughly speaking, we show that if there exist increasing scaling functions (a_t, v_t) such that the pair $(W_t/a_t, v_t)$ satisfy a large deviation principle with rate function I , and if there exists a scaling function (h_t) such that the limit

$$g(c) = \lim_{t \rightarrow \infty} \frac{v(a^{-1}(c/t))}{h_t} \quad (10)$$

exists for each $c > 0$, then (under suitable hypotheses)

$$\lim_{b \rightarrow \infty} h_b^{-1} \log P(Q > b) = - \inf_{c > 0} g(c) I(c). \quad (11)$$

It is not hard to check that this agrees with Glynn and Whitt's result in the case $a_t = v_t = t$ (see Lemma 2.1 below). For fractional Brownian motion (with negative drift), $a_t \equiv t$ and v_t is polynomial in t ; in this case the appropriate scaling function is $h_b := v_b$. A natural generalisation of fractional Brownian motion as a model for the workload is to consider more general Gaussian processes with stationary increments: we demonstrate, as one might expect, that the tail behaviour of the queue-length distribution is governed by the asymptotic variance of the workload process.

We include one more example. Daley and Duffield [5] have justified the Ornstein-Uhlenbeck position process (with negative drift) as an approximation for the workload in a queue with a large number of independent bursty sources; we show that the upper bound estimates which they obtain via martingale methods are asymptotically correct.

We present our results in §2, and the applications in §3.

2 Large deviations and overflow probabilities

Consider a stochastic process $(W_t, t \in T)$, where $T = \mathbb{Z}_+$ or \mathbb{R}_+ , and set

$$Q = \sup_{t \geq 0} W_t. \quad (12)$$

Recall that for the general single-server queue, we are thinking of W as the workload and Q as the queue-length at time zero. In this section we characterise the tail behaviour of Q in terms of the large deviation properties of W , based on the heuristics described in the introduction. We use the terminology of Dembo and Zeitouni [6]. Our basic assumption is the following.

Hypothesis 2.1 (i) *There exists functions $a, v : T \rightarrow T$ that increase to infinity, such that for each $\theta \in \mathbb{R}$, the cumulant generating function defined as the limit*

$$\lambda(\theta) := \lim_{t \rightarrow \infty} v_t^{-1} \log E e^{\theta v_t W_t / a_t} \quad (13)$$

exists as an extended real number.

(ii) *(Note that λ is automatically convex.) $\lambda(\cdot)$ is essentially smooth, lower semi-continuous and there exists $\theta > 0$ for which $\lambda(\theta) < 0$.*

(iii) *There exists an increasing function $h : T \rightarrow T$ such that the limit*

$$g(c) := \lim_{t \rightarrow \infty} \frac{v(a^{-1}(t/c))}{h_t} \quad (14)$$

exists for each $c > 0$, where

$$a^{-1}(x) := \sup\{s \in T : a(s) \leq x\}. \quad (15)$$

If Hypothesis 2.1 is satisfied, then by the Gärtner-Ellis theorem, the pair $(W_t/a_t, v_t)$ satisfy a large deviation principle with good rate function given by the Fenchel-Legendre transform of λ , which we denote by λ^* . In other words, for any Borel set Γ ,

$$\limsup_{t \rightarrow \infty} v_t^{-1} \log P(W_t/a_t \in \Gamma) \leq - \inf_{x \in \Gamma} \lambda^*(x), \quad (16)$$

and

$$\liminf_{t \rightarrow \infty} v_t^{-1} \log P(W_t/a_t \in \Gamma) \geq - \inf_{x \in \Gamma^o} \lambda^*(x), \quad (17)$$

where

$$\lambda^*(x) := \sup_{\theta \in \mathbb{R}} \{\theta x - \lambda(\theta)\}. \quad (18)$$

For $x > 0$, it follows that

$$\limsup_{t \rightarrow \infty} v_t^{-1} \log P(W_t/a_t > x) \leq -\lambda^*(x), \quad (19)$$

and

$$\liminf_{t \rightarrow \infty} v_t^{-1} \log P(W_t/a_t > x) \geq -\lambda^*(x^+). \quad (20)$$

The following lower bound result is essentially an immediate consequence of the basic inequality (3).

Theorem 2.1 *If Hypothesis 2.1 is satisfied, then*

$$\liminf_{b \rightarrow \infty} h_b^{-1} \log P(Q > b) \geq - \inf_{c > 0} g(c) \lambda^*(c^+).$$

Proof. For each $c > 0$,

$$\liminf_{b \rightarrow \infty} h_b^{-1} \log P(Q > b) \geq \liminf_{b \rightarrow \infty} h_b^{-1} \log P(W_{a^{-1}(b/c)} > b) \quad (21)$$

$$= g(c) \liminf_{t \rightarrow \infty} v_t^{-1} \log P(W_t/a_t > c) \quad (22)$$

$$\geq -g(c) \lambda^*(c^+). \quad (23)$$

The result follows. □

To state a complimentary upper bound, we first record some hypotheses.

Hypothesis 2.2 *There exists $d > 0$ such that*

(i)

$$\inf_{c > 0} g(c) \lambda^*(c) = \inf_{c > d} g(c) \lambda^*(c) < \infty;$$

(ii)

$$\liminf_{t \rightarrow \infty} \inf_{c > d} \frac{\lambda^*(c)v_t}{h(ca_t)} = \inf_{c > d} \lambda^*(c)g(c);$$

(iii) for each $\gamma > 0$,

$$\limsup_{b \rightarrow \infty} h_b^{-1} \log \sum_{k=\lceil a^{-1}(b/d) \rceil}^{\infty} e^{-\gamma v_k} \leq -\inf_{c > 0} g(c)\lambda^*(c);$$

(iv)

$$\limsup_{b \rightarrow \infty} h_b^{-1} \log a^{-1}(b/d) = 0.$$

The next hypothesis is not required if $T = \mathbb{Z}_+$; in the case $T = \mathbb{R}_+$ define, for $n \in \mathbb{Z}_+$,

$$W_n^* := \sup_{0 \leq r < 1} W_{n+r}. \quad (24)$$

Hypothesis 2.3 ($T = \mathbb{R}_+$.) *Either*

$$\limsup_{n \rightarrow \infty} v_n^{-1} \log E e^{\theta v_n (W_n^* - W_n)/a_n} = 0 \quad (25)$$

for all $\theta > 0$; or (25) holds for some $\theta > 0$, and

$$\limsup_{n \rightarrow \infty} v_n^{-1} \log P(W_n^* - W_n > xa_n) \leq -\lambda^*(x) \quad (26)$$

for all $x > 0$.

Although we state the above hypothesis in terms of what happens over unit intervals, the length of the intervals can be arbitrarily small; in words, the hypothesis ensures that the continuous-time process is *locally* well-behaved.

Theorem 2.2 *Suppose that Hypotheses 2.1 and 2.2 are satisfied and, if $T = \mathbb{R}_+$, that Hypothesis 2.3 is also satisfied. Then*

$$\limsup_{b \rightarrow \infty} h_b^{-1} \log P(Q > b) \leq -\inf_{c > 0} g(c)\lambda^*(c).$$

In most applications the scaling functions are polynomial: we present this case as a corollary. Note that in this case there is no need to check Hypothesis 2.2.

Corollary 2.3 *Suppose that Hypotheses 2.1 is satisfied and, if $T = \mathbb{R}_+$, that Hypothesis 2.3 is also satisfied. If $a_t = t^a$ and $v_t = t^v$ for some $a, v > 0$, then*

$$\limsup_{b \rightarrow \infty} b^{-v/a} \log P(Q > b) \leq - \inf_{c > 0} c^{-v/a} \lambda^*(c).$$

If λ^ is continuous we can combine this with Theorem 2.1 to get*

$$\lim_{b \rightarrow \infty} b^{-v/a} \log P(Q > b) = - \inf_{c > 0} c^{-v/a} \lambda^*(c).$$

The following lemma, due to Kesidis *et al* [9] and Duffield [7], reconciles our result with that of Glynn and Whitt [8] for the linear case.

Lemma 2.1 *In the above context, if Hypothesis 2.1 is satisfied,*

$$\inf_{c > 0} c^{-1} \lambda^*(c) = \sup\{\theta : \lambda(\theta) \leq 0\}.$$

Proof of Theorem 2.2. First suppose that $T = \mathbb{Z}_+$. Let $d > 0$ be such that Hypothesis 2.2 is satisfied.

$$P(Q > b) \leq P\left(\sup_{n < a^{-1}(b/d)} W_n > b\right) + P\left(\sup_{n \geq a^{-1}(b/d)} W_n > b\right) \quad (27)$$

$$\leq a^{-1}(b/d) \sup_{c > d} P(W_{a^{-1}(b/c)} > b) + \sum_{n \geq a^{-1}(b/d)} P(W_n > b). \quad (28)$$

By Hypothesis 2.1 we know that there exist $\theta, \epsilon > 0$ for which $\lambda(\theta) + \epsilon < 0$. Thus, by Chernoff's inequality and the definition of λ , we have for k sufficiently large,

$$P(W_k > b) \leq e^{-\theta v_k b / a_k} E e^{-\theta v_k W_k / a_k} \quad (29)$$

$$\leq e^{[\lambda(\theta) + \epsilon] v_k}. \quad (30)$$

Combining this with Hypothesis 2.2(iii) we get

$$\limsup_{b \rightarrow \infty} h_b^{-1} \log \sum_{n \geq a^{-1}(b/d)} P(W_n > b) \leq - \inf_{c > 0} g(c) \lambda^*(c). \quad (31)$$

To treat the first term in (28) we appeal to the large deviation upper bound (19) and Hypothesis 2.2(ii, iv): for any $\delta > 0$,

$$\begin{aligned} & \limsup_{b \rightarrow \infty} h_b^{-1} \log \left[a^{-1}(b/d) \sup_{c > d} P(W_{a^{-1}(b/c)} > b) \right] \\ &= \limsup_{n \rightarrow \infty} \sup_{c > d} h(ca_n)^{-1} \log P(W_n/a_n > c) \end{aligned} \quad (32)$$

$$\leq \limsup_{n \rightarrow \infty} \sup_{c > d} h(ca_n)^{-1} v_n [\delta - \lambda^*(c)] \quad (33)$$

$$\leq - \lim_{n \rightarrow \infty} \inf_{c > d} \frac{\lambda^*(c) v_n}{h(ca_n)} + \delta g(d) \quad (34)$$

$$= - \inf_{c > d} g(c) \lambda^*(c) + \delta g(d) \quad (35)$$

Since δ is arbitrary, we can combine the inequalities (28), (31) and (35), and Hypothesis 2.2(i), to obtain the result.

Now suppose $T = \mathbb{R}_+$. If (25) holds for $\theta \leq \theta^* \leq \infty$, then by Hölder's inequality we have

$$\limsup_{n \rightarrow \infty} v_n^{-1} \log E e^{\theta v_n W_n^*/a_n} \leq \lambda(\theta/p), \quad (36)$$

for $0 < p < 1$, $\theta \leq (1-p)\theta^*$. If (25) holds for all $\theta > 0$ we can let $p \nearrow 1$ and combine this with the trivial lower bound

$$\liminf_{n \rightarrow \infty} v_n^{-1} \log E e^{\theta v_n W_n^*/a_n} \geq \lambda(\theta), \quad (37)$$

to get

$$\lim_{n \rightarrow \infty} v_n^{-1} \log E e^{\theta v_n W_n^*/a_n} = \lambda(\theta). \quad (38)$$

We have thus shown that in this case Hypothesis 2.1 is satisfied by the (discrete time) process $(W_n^*, n \in \mathbb{Z}_+)$ and the result follows.

Finally, if (25) holds only for some $\theta > 0$ we modify the proof of the discrete-time case: step (30) is justified by (36) and step (33) by the hypothesis (26); the rest of the proof is identical. \square

Proof of Corollary 2.3. In this case we have $h_b = b^{v/a}$, $g(c) = c^{-v/a}$ and the statement follows from Theorem 2.2 provided Hypothesis 2.2 is satisfied. To check (i) we observe the

following (given Hypothesis 2.1, this is standard convex analysis):

$$0 < \lambda^*(0) \equiv -\inf_{\theta \in \mathbb{R}} \lambda(\theta) < \infty; \quad (39)$$

the interior of the effective domain of λ^* contains the origin, so we have

$$c^{-v/a} \lambda^*(c) < \infty \quad (40)$$

for $c > 0$ sufficiently close to zero. It follows that

$$\lim_{c \rightarrow 0^+} c^{-v/a} \lambda^*(c) = +\infty, \quad (41)$$

which, together with (40), implies (i).

Condition (ii) is trivial in this case.

We check (iii) by appealing to the asymptotic properties of incomplete gamma functions (see, for example, [1, p260]):

$$\limsup_{b \rightarrow \infty} b^{-v/a} \log \sum_{k=\lceil (b/d)^{1/a} \rceil}^{\infty} e^{-\gamma k^v} = \limsup_{b \rightarrow \infty} b^{-v/a} \log \int_{(b/d)^{1/a}}^{\infty} e^{-\gamma t^v} dt \quad (42)$$

$$= \limsup_{b \rightarrow \infty} b^{-v/a} \log \Gamma(v^{-1}, c(b/d)^{v/a}) \quad (43)$$

$$= -\gamma d^{-v/a}, \quad (44)$$

where

$$\Gamma(w, x) := \int_x^{\infty} r^{w-1} e^{-r} dr. \quad (45)$$

Now we can choose d sufficiently small to ensure that (iii) is satisfied.

Condition (iv) is trivial. □

3 Applications

3.1 Gaussian processes with stationary increments

Let $(Z_t, t \in \mathbb{R}_+)$ be a zero-mean Gaussian process with stationary increments and covariance function

$$\Gamma(s, t) = EZ_s Z_t, \quad (46)$$

and set

$$W_t := Z_t - \mu t. \quad (47)$$

This is quite a general model for the workload process, and includes fractional Brownian motion; the practical generality is that we allow “different levels of burstiness at different time-scales”.

Setting

$$\sigma_t^2 := \Gamma(t, t), \quad (48)$$

we have

$$\lambda(\theta) := \lim_{t \rightarrow \infty} \frac{\sigma_t^2}{t^2} \log E e^{\theta t W_t / \sigma_t^2} \quad (49)$$

$$= \frac{1}{2} \theta^2 - \theta \mu, \quad (50)$$

and we see that Hypothesis 2.1 is satisfied with scaling functions $a_t = t$ and $v_t = t^2 / \sigma_t^2$, provided the limit

$$g(c) := \lim_{t \rightarrow \infty} \frac{\sigma_t^2}{c^2 \sigma_{t/c}^2} \quad (51)$$

exists for each $c > 0$. If σ_t^2 is such that Hypothesis 2.2 is satisfied, and if Hypothesis 2.3 is satisfied, then by Theorems 2.1 and 2.2 we have

$$\lim_{b \rightarrow \infty} \frac{\sigma_b^2}{b^2} \log P(\sup_t W_t > b) = - \inf_{c > 0} g(c) (c + \mu)^2 / 2. \quad (52)$$

In particular, if the variance σ_t^2 is asymptotically linear in t :

$$\sigma_t^2 / t \rightarrow \sigma^2 > 0 \quad (53)$$

say, as $t \rightarrow \infty$; then

$$\lim_{b \rightarrow \infty} b^{-1} \log P(Q > b) = - \inf_{c > 0} c^{-1} (c + \mu)^2 / 2\sigma^2 \quad (54)$$

$$= -2\mu / \sigma^2. \quad (55)$$

Finally we remark that to check Hypothesis 2.3 it is sufficient to check that

$$E \sup_{0 \leq t \leq 1} |Z_t| < \infty. \quad (56)$$

This follows from Borell’s inequality [3] for Gaussian processes (see also [6, Exercise 5.2.14]).

3.2 Fractional Brownian motion

A special case of the above is where

$$2\Gamma(s, t) = s^{2H} + t^{2H} - |s - t|^{2H}, \quad (57)$$

for some $0 < H < 1$. In this case the process Z is called *fractional Brownian motion*. The parameter H is called the *Hurst parameter*, and when $H > 1/2$ the process exhibits long range dependence. This has been proposed as a model for the workload by Leland *et al* [10], based on observations of Ethernet traffic data. A lower bound for the tail of the queue-length distribution in this case was obtained by Norros [13], using the inequality (3).

Since the scaling functions are polynomial in this case, we need only check (56) and apply Corollary 2.3 to get

$$\lim_{b \rightarrow \infty} b^{-2(1-H)} \log P(Q > b) = - \inf_{c > 0} c^{-2(1-H)} (c + \mu)^2 / 2. \quad (58)$$

This is consistent with the lower bound estimate of Norros, and moreover demonstrates that it is asymptotically correct.

We now check (56). By symmetry it is sufficient to show that

$$E \sup_{0 \leq t \leq 1} Z_t < \infty. \quad (59)$$

To do this we use the stochastic integral representation of fractional Brownian motion (see, for example, [12]): if B_1 and B_2 are two independent, one-dimensional Brownian motions started at zero, and if we set

$$Z_t := \int_0^\infty \left[(t+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right] dB_1(s) + \int_0^t (t-s)^{H-\frac{1}{2}} dB_2(s), \quad (60)$$

then Z is fractional Brownian motion with Hurst parameter H . Now by Itô's formula—or more specifically, by [15, p139, Remark 5°]—we have

$$Z_t = \left(H - \frac{1}{2}\right) \int_0^\infty \left[s^{H-\frac{3}{2}} - (t+s)^{H-\frac{3}{2}} \right] B_1(s) ds + \left(H - \frac{1}{2}\right) \int_0^t (t-s)^{H-\frac{3}{2}} dB_2(s) \quad (61)$$

$$\leq \int_0^\infty \left[s^{H-\frac{3}{2}} - (1+s)^{H-\frac{3}{2}} \right] B_1^+(s) ds + \left(H - \frac{1}{2}\right) \int_0^t (t-s)^{H-\frac{3}{2}} dB_2^+(s), \quad (62)$$

and so

$$E \sup_{0 \leq t \leq 1} Z_t \leq \int_0^\infty [s^{H-\frac{3}{2}} - (1+s)^{H-\frac{3}{2}}] EB_1^+(s) ds + E \sup_{0 \leq t \leq 1} B_2^+(t) \quad (63)$$

$$\leq \int_0^\infty [s^{H-\frac{3}{2}} - (1+s)^{H-\frac{3}{2}}] s^{1/2} ds + 2 \quad (64)$$

$$< \infty, \quad (65)$$

as required.

3.3 Ornstein-Uhlenbeck position process

Another example where the workload is modelled by a Gaussian process with stationary increments is the following. Consider a queue with constant service rate, for which the workload W_t is the position component of a stationary Ornstein-Uhlenbeck process with negative drift. Such an arrival process has been proposed by Norros *et al* [14] as a model of continuous correlated arrivals. It has been shown by Daley and Duffield [5] that this arrival process occurs as the heavy traffic limit of superposed 2-state markov fluid sources under suitable rescaling of time and mean activity. Moreover, using martingale methods they obtain exponential upper bounds for the tail of the corresponding queue length distribution. Here we show that the exponential decay constant is equal to that obtained by the foregoing large deviation arguments, so that the upper bound estimate is asymptotically correct.

To be precise we consider the stationary Ornstein-Uhlenbeck process $(V_t, t \in \mathbb{R}_+)$, defined to be the solution of the stochastic differential equation

$$dV_t = -V_t dt + \sqrt{2}(\mu/\nu) dB(t) \quad (66)$$

where V_0 is normally distributed with zero mean and variance $(\mu/\nu)^2$. Here B is standard Brownian motion, $\nu > 0$ is a load parameter (the case $\nu = 0$ corresponding to unit load), and μ is the service rate. The corresponding position process (with zero initial condition) is

$$Z_t = \int_0^t V_s ds, \quad (67)$$

and the workload is

$$W_t = Z_t - \mu t. \quad (68)$$

In [5] it is shown that

$$P(Q > b) \leq e^{-\nu^2/2 - \nu^2 b/\mu}. \quad (69)$$

In fact, as we now confirm,

$$\lim_{b \rightarrow \infty} b^{-1} \log P(Q > b) = -\nu^2/\mu. \quad (70)$$

To do this we note that Z is a centered Gaussian process with stationary increments, so we can use the arguments of §3.1. We begin by calculating the variance σ_t^2 of Z_t . The solution of (66) is

$$V_t = e^{-t}V_0 + e^{-t}\sqrt{2}(\mu/\nu) \int_0^t e^s dB(s), \quad (71)$$

and so the covariance function of V is (for $t \geq s \geq 0$)

$$\Xi(s, t) := EV_t V_s \quad (72)$$

$$= e^{-(t+s)} \left(\text{Var}(V_0) + 2(\mu/\nu)^2 E \left(\int_0^t e^{t'} dB(t') \int_0^s e^{s'} dB(s') \right) \right) \quad (73)$$

$$= (\mu/\nu)^2 e^{-(t+s)} \left(1 + 2 \int_0^s e^{2s'} ds' \right) \quad (74)$$

$$= (\mu/\nu)^2 e^{s-t}; \quad (75)$$

hence

$$\sigma_t^2 = 2 \int_{0 \leq s' \leq s \leq t} \Xi(s, s') ds ds' \quad (76)$$

$$= 2(\mu/\nu)^2 (t + e^{-t} - 1). \quad (77)$$

The result now follows from (55), provided (56) is satisfied. Again, by symmetry, it is sufficient to check that

$$E \sup_{0 \leq t \leq 1} Z_t < \infty. \quad (78)$$

But

$$E \sup_{0 \leq t \leq 1} Z_t = E \sup_{0 \leq t < 1} \int_0^t V_{t'} dt' \quad (79)$$

$$\leq E \sup_{0 \leq t < 1} V_t \quad (80)$$

$$\leq \sqrt{2}(\mu/\nu) E \sup_{0 \leq t < 1} B(t) \quad (81)$$

$$= 2\sqrt{2}\mu/\nu, \quad (82)$$

and so we are done.

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