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Yang-Baxterization and the BH algebra.

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Abstract

The BH algebra is defined by two sets of generators one of which satisfy the relations of the braid group and the other the relations of the Hecke algebra of projectors.Both sets also satisfy additional relations having the Birman-Wenzl generating relations as a particular case.In this paper we consider the problem of finding solutions of the Yang-Baxter equation from representations of the BH algebra by the method of Yang-Baxterization of these representations.

1 Introduction

The Yang-Baxter equation (YBE) plays a central role as the main criterium for integrability of lattice models in statistical mechanics. The equation bares a striking resemblance to the generating relations of the braid group algebra and were not for its dependence on a spectral parameter they would be the same thing. Many examples of solutions are known which in the limit when the parameter goes to zero or infinity the matrices satisfying the equation become a representation of the braid group. In [1-2] Jones considered the inverse problem of finding solutions of the Yang-Baxter equation from representations of the braid group. He named this procedure of constructing solutions of YBE,Yang-Baxterization.

There are several algebras which are associated to the braid group algebra like,for instance,the Temperley-Lieb algebra,the Hecke algebra,and the Birman-Wenzl algebra which are also closely related to integrable models in statistical mechanics.The Yang-Baxterization of the Birman-Wenzl (BW) algebra was considered by Jones himself in [1-2]. In [3] the problem of finding solutions of the YBE

$$R_1(x)R_2(xy)R_1 = R_2(y)R_1(xy)R_2(x) \tag{1}$$

from a given representation of the BW algebra was further considered.

The BW algebra [4] is defined by the generators σ_i and E_i,i=1,2,...,n-1,satisfying the relations

$$\sigma_i - \sigma_i^{-1} = m(1 - E_i) \tag{2.1}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| \geq 2, \tag{2.2}$$

$$E_iE_{i\pm1}E_i = E_i, E_iE_j = E_jE_i, \quad |i-j| \geq 2, \tag{2.3}$$

$$E_i^2 = (1 - \frac{l - l^{-1}}{m})E_i \tag{2.4}$$

$$\sigma_{i\pm1}\sigma_iE_{i\pm1} = E_i\sigma_{i\pm1}\sigma_i = E_iE_{i\pm1} \tag{2.5}$$

$$\sigma_{i\pm1}E_i\sigma_{i\pm1} = \sigma_i^{-1}E_{i\pm1}\sigma_i^{-1} \tag{2.6}$$

$$\sigma_{i\pm1}E_iE_{i\pm1} = \sigma_i^{-1}E_{i\pm1} \tag{2.7}$$

$$E_{i\pm1}E_i\sigma_{i\pm1} = E_{i\pm1}\sigma_i^{-1} \tag{2.8}$$

$$\sigma_iE_i = E_i\sigma_i = \frac{1}{l}E_i \tag{2.9}$$

$$E_i\sigma_{i\pm1}E_i = lE_i \tag{2.10}$$

$$\sigma_i^2 = m(\sigma_i - \frac{1}{l}E_i) + 1 \tag{2.11}$$

$$\sigma_i^3 = (m + \frac{1}{l})\sigma_i^2 - (1 - \frac{m}{l})\sigma_i - \frac{1}{l} \tag{2.12}$$

where m and l are the parameters of the algebra.

The starting point in the the analysis of [1-3] is to assume that the solution of the YBE is in the form

$$R_i(x) = A(x)\sigma_i + B(x)I + C(x)\sigma_i^{-1} \tag{3}$$

where the σ_i are given by representations of the BW algebra and satisfy the following cubic relation:

$$(\sigma_i - \lambda_1)(\sigma_i - \lambda_2)(\sigma_i - \lambda_3) = 0 \tag{4}$$

For $m = q - q^{-1}$ and $\lambda_1 = q$,$\lambda_2 = -q^{-1}$, $\lambda_3 = l^{-1}$,this equation is precisely the cubic relation (2.12). The solution R is then required to satisfy the following three conditions:

1)the boundary condition :

$$R_i(0) \sim \sigma_i \tag{4.1}$$

2)the initial condition:

$$R_i(1) \sim I \tag{4.2}$$

3)the unitarity condition:

$$R_i(x)R_i(x^{-1}) = f(x)I, \tag{4.3}$$

for some function f(x).It was found in reference [3] that the coefficients A,B,and C can then be chosen of the following form:

$$A(x) = -\lambda_3^{-1}(x-1), C(x) = \lambda_1 x(x-1)$$

$$B(x) = (1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_2}{\lambda_3})x \tag{5.1}$$

$$A(x) = -\lambda_3^{-1}(x-1), C(x) = \lambda_2 x(x-1)$$

$$B(x) = (1 + \frac{\lambda_2}{\lambda_1} + \frac{\lambda_2}{\lambda_3} + \frac{\lambda_1}{\lambda_3})x \tag{5.2}$$

$$A(x) = -\lambda_2^{-1}(x-1), C(x) = \lambda_1 x(x-1)$$

$$B(x) = (1 + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_3}{\lambda_2})x \tag{5.3}$$

Under the permutations $(\lambda_1, \lambda_2, \lambda_3) \to (\lambda_2, \lambda_1, \lambda_3)$ and $(\lambda_1, \lambda_2, \lambda_3) \to (\lambda_1, \lambda_3, \lambda_2)$ the functions in (5.2) and (5.3) can be obtained from those in (5.1). R given by (3) and (5.1-3) solves the YBE if and only if the following identity is satisfied:

$$f_3^+\theta_3^+ + f_3^-\theta_3^- + f_2\theta_2 + f_1^+\theta_1^+ + f_1^-\theta_1^- = 0 \tag{6}$$

where

$$\theta_3^{\pm} = \sigma_1^{\pm}\sigma_2^{\mp}\sigma_1^{\pm} - \sigma_2^{\pm}\sigma_1^{\mp}\sigma_2^{\pm}$$

$$\theta_2 = \sigma_1\sigma_2^{-1} - \sigma_2\sigma_1^{-1} + \sigma_2^{-1}\sigma_1 - \sigma_1^{-1}\sigma_2$$

$$\theta_1^{\pm} = \sigma_1^{\pm} - \sigma_2^{\pm} \tag{6.1}$$

and $f_3^{\pm}, f_2, f_1^{\pm}$ are given by

$$f_3^+ = \frac{\lambda_1}{\lambda_3^2}, f_3^- = -\frac{\lambda_1^2}{\lambda_3}$$

$$f_2 = -\frac{\lambda_1}{\lambda_3}(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_3} + \frac{\lambda_1}{\lambda_3})$$

$$f_1^{\pm} = \mp\lambda_2^{\mp} f_2 \tag{6.2}$$

According to [3] a sufficient condition for σ to satisfy the identity (6) is that it admits the BW algebraic structure,that is, assuming $l^{-1} = \lambda_3, m = \lambda_1 + \lambda_2$ and $\lambda_1\lambda_2 = -1$, σ and E defined by (2.1) satisfy the BW relations.One just has to use the relations of the algebra to check that for coefficients A,B,C given by (5.1) and (5.2) equation (6) is satisfied.Thus,two solutions of YBE of the form (3) are found. We will show in the next section that when σ admits the $B\mathcal{H}$ algebraic structure given below equation (6) is also satisfied.

2 The BH algebra

In our previous paper [5] the following algebra which we called the $B\mathcal{H}$ algebra was found. It depends on two parameters l and m and has generators H_i and σ_i,i=1,...,n-1,satisfying the relations (2.2) and the following

$$\sigma_i - \sigma_i^{-1} = m(1 - H_i) \tag{7.1}$$

$$H_i H_{i\pm1} H_i - H_i = H_{i\pm1} H_i H_{i\pm1} - H_{i\pm1} \tag{7.2}$$

$$H_i^2 = (1 - \frac{l - l^{-1}}{m}) H_i \tag{7.3}$$

$$H_i H_j = H_j H_i, \ \ |i - j| \geq 2, \tag{7.4}$$

$$\sigma_{i\pm1} \sigma_i H_{i\pm1} = H_i \sigma_{i\pm1} \sigma_i \tag{7.5}$$

$$\sigma_{i\pm1} \sigma_i H_{i\pm1} - H_i H_{i\pm1} = \sigma_i \sigma_{i\pm1} H_i - H_{i\pm1} H_i \tag{7.6}$$

$$\sigma_{i\pm1} H_i \sigma_{i\pm1} - \sigma_i^{-1} H_{i\pm1} \sigma_i^{-1} = \sigma_i H_{i\pm1} \sigma_i - \sigma_{i\pm1}^{-1} H_i \sigma_{i\pm1}^{-1} \tag{7.7}$$

$$\sigma_{i\pm1} H_i H_{i\pm1} - \sigma_i^{-1} H_{i\pm1} = \sigma_i H_{i\pm1} H_i - \sigma_{i\pm1}^{-1} H_i \tag{7.8}$$

$$H_{i\pm1} H_i \sigma_{i\pm1} - H_{i\pm1} \sigma_i^{-1} = H_i H_{i\pm1} \sigma_i - H_i \sigma_{i\pm1}^{-1} \tag{7.9}$$

$$\sigma_i H_i = H_i \sigma_i = \frac{1}{l} H_i \tag{7.10}$$

$$H_i \sigma_{i\pm1} H_i - l H_i = H_{i\pm1} \sigma_i H_{i\pm1} - l H_{i\pm1} \tag{7.11}$$

$$\sigma_i^2 = m(\sigma_i - \frac{1}{l} H_i) + 1 \tag{7.12}$$

$$\sigma_i^3 = (m + \frac{1}{l}) \sigma_i^2 - (1 - \frac{m}{l}) \sigma_i - \frac{1}{l} \tag{7.13}$$

This algebra was obtained from the general relations satisfied by the algebraic weights defined on the Hecke algebra at the point where the partition function becomes dual.See reference [5] for details. Observe that when both sides of relations (7.2,7.6-9,7.11) are equal to zero relations (7.1-13) become the generating relations of the Birman-Wenzl algebra (2). Thus,every representation of BW algebra is also a representation of the $B\mathcal{H}$ algebra.

We now prove the following

Theorem:Suppose σ and H defined by (7.1) satisfy the BH algebra (7).Then σ satisfies equation (6) for A,B,C given by (5.1) and (5.2).

Proof:

Using the relations of the algebra we rewrite relation (6.1) as follows:

$$\theta_3^+ = m^2 \{ \sigma_2 - \frac{1}{l} H_2 - \sigma_1 + \frac{1}{l} H_1 - \sigma_2 H_1$$

$$+\sigma_1 H_2 - H_1\sigma_2 + mH_1 + H_2\sigma_1 - mH_2\} \tag{8.1}$$

$$\theta_3^- = m^2\{\sigma_2 - \sigma_2 H_1 - H_1\sigma_2 + lH_1$$

$$-lH_2 - \sigma_1 + \sigma_1 H_2 + H_2\sigma_1\} \tag{8.2}$$

$$\theta_2 = m\{-2\sigma_1 + 2\sigma_2 + \sigma_1 H_2 - \sigma_2 H_1$$

$$+H_2\sigma_1 - H_1\sigma_2\} \tag{8.3}$$

$$\theta_1^+ = \sigma_1 - \sigma_2 \tag{8.4}$$

$$\theta_1^- = \sigma_1 - \sigma_2 + mH_1 - mH_2 \tag{8.5}$$

Without loss of generality,we take $\lambda_3 = 1/l$,$m = \lambda_1 + \lambda_2$ and $\lambda_1.\lambda_2 = -1$. Upon substitution of relations (8) in the left hand side of (6) it is observed that all terms cancel out and the identity is satisfied for for $(\lambda_1, \lambda_2, \lambda_3)$ and $(\lambda_2, \lambda_1, \lambda_3)$ but not for $(\lambda_1, \lambda_3, \lambda_2)$.

In terms of the generators σ and H of the $B\mathcal{H}$ algebra (7),the solutions in (3) can be expressed as

$$R(x) = (A + C)\sigma + (B - mC)I + mCH \tag{9}$$

for A,B,C given by (5.1) and (5.2). Thus,two solutions of YBE are associated to each representation of the braid group satisfying the $B\mathcal{H}$ algebra (7). Because every representation of the BW algebra is also a representation of $B\mathcal{H}$ the solutions found in [3] are automatically given by (9),too.

To conclude,we mention that in [3] a reduced form of the BW algebra and its Yang-Baxterization is considered.The same arguments used there can be carried on to a reduced form of the $B\mathcal{H}$ algebra.

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