DIAS-STP-94-38

GENERALISED MOONSHINE AND ABELIAN ORBIFOLD CONSTRUCTIONS

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0. Introduction. We consider the application of Abelian orbifold constructions in Meromorphic Conformal Field Theory (MCFT) [Go,DGM] towards an understanding of various aspects of Monstrous Moonshine [CN] and Generalised Moonshine [N]. We review some of the basic concepts in MCFT and Abelian orbifold constructions [DHVW] of MCFTs and summarise some of the relevant physics lore surrounding such constructions including aspects of the modular group, the fusion algebra and the notion of a self-dual MCFT. The FLM Moonshine Module, $\mathcal{V}^\natural$, [FLM1,FLM2] is historically the first example of such a construction being a $\mathbb{Z}_2$ orbifolding of the Leech lattice MCFT, $\mathcal{V}^\Lambda$. We review the usefulness of these ideas in understanding Monstrous Moonshine, the genus zero property for Thompson series [CN] which we have shown is equivalent to the property that the only meromorphic $\mathbb{Z}_n$ orbifoldings of $\mathcal{V}^\natural$ are $\mathcal{V}^\Lambda$ and $\mathcal{V}^\natural$ itself (assuming that $\mathcal{V}^\natural$ is uniquely determined by its characteristic function $J(\tau)$) [T1,T2]. We show that these constraints on the possible $\mathbb{Z}_n$ orbifoldings of $\mathcal{V}^\natural$ are also sufficient to demonstrate the genus zero property for Generalised Moonshine functions in the simplest non-trivial prime cases by considering $\mathbb{Z}_p\times\mathbb{Z}_p$ orbifoldings of $\mathcal{V}^\natural$. Thus Monstrous Moonshine implies Generalised Moonshine in these cases.

1. Meromorphic Conformal Field Theory. In this section, we will review some of the basic properties of Meromorphic Conformal Field Theory (MCFT) (or chiral algebras) as described by Goddard [Go]. This is a physically motivated approach to Vertex Operator Algebras [B1,FLM2,FHL] containing the same essential ideas. Let $\mathcal{H}$ denote some Hilbert space with a dense subspace of states $\{\phi\}$ including a unique 'vacuum state' $|0\rangle$ with properties described below. In a MCFT we define a set of conformal fields or vertex operators $\mathcal{V}$ such that corresponding to each state ϕ there exists an operator $V(\phi,z)\in\mathcal{V}$ acting on $\mathcal{H}$ with

$$\lim_{z\to 0} V(\phi,z)|0\rangle=\phi \tag{1.1}$$

It is assumed that there exists Virasoro operators L_n which form the modes of $V(\omega,z)$ (see below) for a Virasoro state ω where

$$[L_n,V(\phi,z)] =z^n[z\frac{d}{dz}+(n+1)h_\phi]V(\phi,z) \tag{1.2a}$$

$$[L_m,L_n] =(m-n)L_{m+n}+\frac{C}{12}m(m^2-1)\delta_{m,-n} \tag{1.2b}$$

Maths Subject Classification 81T40,17B69,20D08,11F11

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

where h_ϕ is called the conformal weight of ϕ and C is the central charge for the representation of the Virasoro algebra (1.2b). The Virasoro state ω has conformal weight 2. From (1.2a), L_0 defines a discrete grading on $\mathcal{H}$ with $L_0|\phi\rangle = h_\phi|\phi\rangle$. We assume that $\mathcal{V}$ is unitary so that $h_\phi \geq 0$. By a *Meromorphic CFT*, we will mean a CFT for which the conformal weights are integral and where the operators $\mathcal{V}$ obey the (bosonic) Locality Property :

$$V(\phi, z)V(\psi, w) = V(\psi, w)V(\phi, z) \tag{1.3}$$

with $|z| > |w|$ on the LHS analytically continued to $|z| < |w|$ on the RHS. (These assumptions ensure that all correlation functions are meromorphic). These operators can then be shown to satisfy the Duality Property [Go,FHL]

$$V(\phi, z)V(\psi, w) = V(V(\phi, z-w)\psi, w) \tag{1.4}$$

with $|z| > |w|$ and $|z-w| < |w|$ respectively and where $V(\phi, z)$ is extended by linearity to any state in $\mathcal{H}$. These are essentially the defining properties of a vertex (operator) algebra as defined in [B1] and developed in [FLM2]. All the conformal fields in a MCFT also obey the Monodromy condition :

$$V(\phi, e^{2\pi i}z) = V(\phi, z) \tag{1.5}$$

so that the mode expansion for each operator is $V(\phi, z) = \sum_{k\in\mathbb{Z}} \phi_k z^{-k-h_\phi}$ with $\phi_k|0\rangle = 0$ for all $k > -h_\phi$ and $\phi_{-h_\phi}|0\rangle = \phi$ from (1.1) e.g. the modes for the Virasoro (energy-momentum) operator $V(\omega, z)$ are $\{L_n\}$ as above with $\omega = L_{-2}|0\rangle$. Then (1.4) leads to an exact form of the usual operator product expansion of CFT [BPZ]

$$V(\phi, z)V(\psi, w) = \sum_{k=0}^{\infty} (z-w)^{k-h_\phi-h_\psi} V(\chi, w) \tag{1.6}$$

where $\chi = \phi_{-k+h_\psi}(\psi)$ is a state of conformal weight k. We will sometimes schematically write such an expansion as $\mathcal{V}\mathcal{V} \sim \mathcal{V}$.

2. The Modular Group and Self-Dual MCFTs. Let $\mathcal{V}$ be a MCFT and define the characteristic function (or partition function) for $\mathcal{V}$ by the following trace

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}}(q^{L_0 - C/24}) \tag{2.1}$$

where $\tau \in H$, the upper half complex plane. In string theory models, $Z(\tau)$ arises when finding the probability for a closed string to form a 2-torus parameterised by τ. The simplest example, is the one-dimensional $C = 1/24$ bosonic string which has characteristic function $1/\eta(\tau)$ where $\eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n)$. For a d dimensional $C = d/24$ string compactified by an even lattice Λ, we obtain a MCFT denoted by $\mathcal{V}^\Lambda$, with $Z(\tau) = \Theta_\Lambda(\tau)/[\eta(\tau)]^d$ where $\Theta_\Lambda = \sum_{\lambda\in\Lambda} q^{\lambda^2/2}$.

The behaviour of $Z(\tau)$ under the action of the modular group $\Gamma = SL(2,\mathbb{Z})$, generated by $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$, is related to the meromorphic properties of $\mathcal{V}$ and to properties of the meromorphic irreducible representations of

$\mathcal{V}$. For a MCFT we clearly have $Z(\tau+1)=e^{2\pi i C/24}Z(\tau)$ and, in particular, $Z(\tau)$ is T invariant for $C=24$.

Let us now discuss the meaning of the S transformation. Let $\tilde{\mathcal{V}}$ denote an irreducible meromorphic representation for $\mathcal{V}$ acting on a Hilbert space $\mathcal{H}^K$ and let $\mathcal{V}^K$ be the corresponding set of intertwiners acting on $\mathcal{H}$ that create the states of $\mathcal{H}^K$ from the original vacuum vector $|0\rangle \in \mathcal{H}$ [FHL,DGM]. Then as in (1.3) and (1.4) we have

$$\begin{aligned}
\tilde{V}(\phi,z)\tilde{V}(\psi,w) &= \tilde{V}(\psi,w)\tilde{V}(\phi,z) && (2.2a)\\
\tilde{V}(\phi,z)\tilde{V}(\psi,w) &= \tilde{V}(V(\phi,z-w)\psi,w) && (2.2b)\\
\tilde{V}(\phi,z)V^K(\chi,w) &= V^K(\chi,w)V(\phi,z) && (2.2c)\\
\tilde{V}(\phi,z)V^K(\chi,w) &= V^K(\tilde{V}(\phi,z-w)\chi,w) && (2.2d)
\end{aligned}$$

(up to suitable analytic continuations) for $\tilde{V}(\phi,z)\in\tilde{\mathcal{V}}$, $V^K(\chi,z)\in\mathcal{V}^K$ with $\phi,\psi\in\mathcal{H}$ and $\chi\in\mathcal{H}^K$. Given such a representation, we thus naturally extend $\mathcal{V}$ to act on $\mathcal{H}\oplus\mathcal{H}^K$ and henceforth we drop the tilde notation distinguishing the space on which $\mathcal{V}$ acts. We also define the characteristic function $Z^K(\tau)=\mathrm{Tr}_{\mathcal{H}^K}(q^{L_0-1})$ for $\mathcal{V}^K$. In general, the conformal weights of $\mathcal{H}^K$ are not integral but are equal $\mathit{mod}\ \mathbb{Z}$ and hence $Z^K(\tau)$ is T invariant up to a phase.

By a *Rational MCFT*, we will mean a MCFT which has a finite number M of such irreducible representations $\{\mathcal{V}^K\}$, $K=0,...,M-1$ (with $\mathcal{V}\equiv\mathcal{V}^0$) and where every representation of $\mathcal{V}$ is reducible. For a Rational MCFT, Zhu has shown that each characteristic function $Z^K(\tau)$ is holomorphic on the upper half plane H (given a certain growth condition which is conjectured to follow from rationality) and the functions $\{Z^K\}$ transform amongst themselves under the modular group Γ [Z].

These properties can also be understood if $\mathcal{V}$ together with (possibly multiple copies of) its intertwiners form a non-meromorphic CFT which we call the *Dual CFT* to $\mathcal{V}$ and denote by $\mathcal{V}^*$. We can think of $\mathcal{V}^*$ as comprising the maximal (in some sense!) set of vertex operators of central charge C that are local with respect to $\mathcal{V}$. $\mathcal{V}^*$ is expected to satisfy an operator product algebra given by some generalised version of (1.6) where schematically

$$\mathcal{V}^I\mathcal{V}^J \sim \sum_{K=0}^{M-1} N^{IJK}\mathcal{V}^K \tag{2.3}$$

where N^{IJK} are non-negative integers determining the decomposition in terms of irreducible representations of $\mathcal{V}$ of the non-meromorphic algebra - these are the *Fusion Rules* for $\mathcal{V}^*$ [Ve]. The coefficients N^{IJK} satisfy a commutative associative algebra which is diagonalised by $S:Z^I\to S^{IJ}Z^J$ where S^{IJ} is a unitary symmetric matrix [Ve]. In addition, we assume $\mathcal{V}^*$ is a unitary CFT, so that $S^{I0}/S^{00}\geq 1$ with equality iff we have *Abelian Fusion Rules* i.e. for every given $I,J=0,...M-1$, $N^{IJK}=1$ for some unique K so that every pair of intertwiners fuses to form a unique intertwiner. Assuming Abelian Fusion Rules we then find, since S^I_J is symmetric and unitary, that

$$S: Z(\tau)\to \epsilon_S\frac{1}{\sqrt{M}}\sum_K Z^K(\tau)=\epsilon_S|\mathcal{H}^*/\mathcal{H}|^{-1/2}\mathrm{Tr}_{\mathcal{H}^*}(q^{L_0-1}) \tag{2.4}$$

where $|\epsilon_S| = 1$ and $\mathcal{H}^*$ denotes the Hilbert space $\oplus_{K=0}^{M-1}\mathcal{H}^K$ for $\mathcal{V}^* \equiv \oplus_K \mathcal{V}^K$ and $M = |\mathcal{H}^*/\mathcal{H}|$. If furthermore $\{Z^K\}$ is charge conjugation invariant then S^{IJ} is real so that $\epsilon_S = 1$. This formula can be verified for an even lattice Λ MCFT where the irreducible representations for $\mathcal{V}^\Lambda$ are indexed by Λ^*/Λ where Λ^* is the dual lattice [D1]. In this case, $\mathcal{V}^\Lambda$ is naturally embedded in the non-meromorphic CFT $\mathcal{V}^{\Lambda^*}$ so that $(\mathcal{V}^\Lambda)^* = \mathcal{V}^{\Lambda^*}$. Furthermore, the fusion rules are abelian from the abelian structure of Λ^* [DL]. Then, under the action of S, $Z_\Lambda = \Theta_\Lambda/\eta^d \to |\Lambda^*/\Lambda|^{-1/2}\Theta^*_\Lambda/\eta^d$ in the usual way in agreement with (2.4) with $\epsilon_S = 1$. Similarly, (2.4) holds with $\epsilon_S = 1$ for the Abelian orbifold constructions discussed below.

If $Z(\tau)$ is S invariant and hence $\mathcal{V}$ is the unique irreducible representation for itself, we define $\mathcal{V}$ to be a *Self-Dual MCFT*. This is only possible for $C = 0$ *mod* 8 [Go]. For $C = 24$, then $Z(\tau)$ is modular invariant with a unique simple pole at $q = 0$ on H/Γ which is equivalent to the Riemann sphere of genus zero. Hence $Z(\tau)$ is given by $J(\tau)$, the *hauptmodul* for Γ [Se]

$$Z(\tau) = J(\tau) + N_0 \tag{2.5a}$$

$$J(\tau) = \frac{E_2^3}{\eta^{24}} - 744 = \frac{1}{q} + 0 + 196884q + 21493760q^2 + \ldots \tag{2.5b}$$

with $E_2(\tau)$ the Eisenstein modular form of weight 4 [Se] and where N_0 is the number of conformal weight 1 operators in $\mathcal{V}$. Examples of such theories are lattice models where Λ is a Niemeier even self-dual lattice. Then $\mathcal{V}^\Lambda$ is meromorphic self-dual because Λ is even self-dual. In particular, for the Leech lattice which contains no roots, $N_0 = 24$. Other examples of self-dual C=24 MCFTs are the Moonshine Module $\mathcal{V}^\natural$ with $Z(\tau) = J(\tau)$ and other orbifold constructions as we now describe. In general, there are thought to be just 71 such independent self-dual MCFTs [Sch].

3. Abelian Orbifolding of a Self-Dual MCFT. Let $\mathcal{V}$ be a self-dual MCFT and let $Aut(\mathcal{V})$ denote the automorphism group preserving the operator algebra for $\mathcal{V}$ with

$$gV(\phi, z)g^{-1} = V(g\phi, z) \tag{3.1}$$

for each $g \in Aut(\mathcal{V})$. Consider G any finite abelian subgroup of $Aut(\mathcal{V})$ generated by m commuting elements $\{g_1, \ldots, g_m\}$ of order $n_1, \ldots, n_m$. Let $\mathcal{P}_G\mathcal{V}$ denote the operators invariant under G with projection operator $\mathcal{P}_G = \frac{1}{|G|}\sum_{g\in G} g$. $\mathcal{P}_G\mathcal{V}$ is a MCFT but is not self-dual as can be seen by studying the corresponding characteristic function $\mathrm{Tr}_{\mathcal{P}_G\mathcal{H}}(q^{L_0-1})$ which is not S invariant. In particular, consider the trace for each $g \in G$

$$Z(1, g) \equiv \mathrm{Tr}_{\mathcal{H}}(gq^{L_0-1}) \tag{3.2}$$

where we introduce standard notation indicating boundary conditions on the 2-torus where the first label 1 refers to the monodromy condition (1.5). Using path integral methods in string theory [DHVW] one can argue that under $S : \tau \to -1/\tau$ the boundary conditions are interchanged for $Z(1, g)$ charge conjugation invariant and $\mathcal{V}$ a self-dual theory so that

$$S : Z(1, g) \to Z(g, 1) = \mathrm{Tr}_{\mathcal{H}_g}(q^{L_0-1}) = D_g q^{E_0^g} + \ldots \tag{3.3}$$

the characteristic function for $\mathcal{H}_g$, the 'g-twisted' Hilbert space. We assume that $\mathcal{H}_g$ is uniquely defined (up to isomorphism) for each $g \in G$. The parameters E_0^g and D_g are called the g-twisted vacuum energy and degeneracy. Note also that the remaining coefficients of the powers are necessarily all non-negative integers. We assume that each twisted state $\psi \in \mathcal{H}_g$ of conformal weight h_ψ is created from $|0\rangle$ by the action of a twisted operator with the following g-twisted Monodromy property:

$$V(\psi, e^{2\pi i} z) = gV(\psi, z)g^{-1} = e^{-2\pi i h_\psi} V(\psi, z) \tag{3.4}$$

We denote the set of such operators for each $g \in G$ by $\mathcal{V}_g$. We also assume that $\oplus_{g\in G}\mathcal{V}_g$ satisfies a non-meromorphic version of the Locality property (1.3) and a G-invariant operator product expansion generalising (1.6) (up to suitable analytic continuation) where

$$\begin{aligned} V(\psi,z)V(\chi,w) =& \epsilon_{\psi,\chi} V(\chi,w)V(\psi,z) && (3.5a)\\ V(\psi,z)V(\chi,w) =& \sum_\rho (z-w)^{h_\rho - h_\psi - h_\chi} V(\rho, w) && (3.5b)\end{aligned}$$

for $\psi \in \mathcal{H}_g$, $\chi \in \mathcal{H}_h$ and $\rho \in \mathcal{H}_{gh}$ for $g, h \in G$. The Locality phase $\epsilon_{\psi,\chi}$ is of order dividing $|G|$ and is unity for $\phi \in \mathcal{P}_G\mathcal{H}$ and any $\psi \in \mathcal{H}_g$. This latter property implies that each twisted sector $\mathcal{V}_g$ is the intertwiner for a meromorphic representation of $\mathcal{P}_G\mathcal{V}$ as in (2.2c,d). This representation can be then further decomposed into $|G|$ irreducible representations $\mathcal{V}_g = \oplus_{j_k}\mathcal{V}_g^{\{j_k\}}$ labelled by the eigenvalues $\{\exp 2\pi i j_k/n_k\}$ of the generators $\{g_k\}$ for G. Thus, in the language of the last section, we have a set of $|G|^2$ irreducible representations for the Rational MCFT $\mathcal{P}_G\mathcal{V}$ which together form the Dual CFT given by $(\mathcal{P}_G\mathcal{V})^* = \oplus_{g,j_k}\mathcal{V}_g^{\{j_k\}}$ with Abelian Fusion Rules : $\mathcal{V}_g^{\{i_k\}}\mathcal{V}_h^{\{j_k\}} \sim \mathcal{V}_{gh}^{\{i_k+j_k\}}$ from (3.5b). Then (2.4) is recovered with $\epsilon_S = 1$ using (3.3) where

$$S : Z(1, \mathcal{P}_G) = \mathrm{Tr}_{\mathcal{P}_G\mathcal{H}}(q^{L_0-1}) \to \frac{1}{|G|}\sum_{g\in G} Z(g,1) = \frac{1}{|G|}\mathrm{Tr}_{(\mathcal{P}_G\mathcal{H})^*}(q^{L_0-1}) \tag{3.6}$$

where $(\mathcal{P}_G\mathcal{H})^* \equiv \oplus_{g\in G}\mathcal{H}_g$.

Since the operators of $\mathcal{V}_g$ are eigenvalues of g, the centraliser of $C(g|Aut(\mathcal{V}))$ has a natural extension as the automorphism group, which we denote by C_g, of the non-meromorphic algebra $\mathcal{V}\mathcal{V}_g \sim \mathcal{V}_g$. This extension depends on the g-twisted vacuum degeneracy D_g. Defining $G_n = C(g|Aut(\mathcal{V}))/\langle g\rangle$, in general one finds that $C_g = \hat{L}.G_n$ for some extension $\hat{L} = \langle g\rangle.L$ determined by the automorphism group acting on the twisted vacuum of $\mathcal{H}_g$. (Here $A.B$ denotes a group with normal subgroup A where $B = A.B/A$). If the twisted vacuum is unique ($D_g = 1$), then $C_g = \langle g\rangle \times G_n$. For each $h \in C_g$ we can then generalise (3.2) to define

$$\begin{aligned} &Z(g,h) = \mathrm{Tr}_{\mathcal{H}_g}(hq^{L_0-1}) && (3.7a)\\ &T: Z(g,h) \to Z(g, g^{-1}h), \quad S: Z(g,h)\to Z(h,g^{-1}) && (3.7b)\end{aligned}$$

which transform under T as given in (3.4) and under S by an interchange of g and h boundary conditions assuming (3.3) in general. Then for any $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ in Γ,

$\gamma : Z(g,h) \to Z(h^{-c}g^{a}, h^{d}g^{-b})$. In particular, for all $g, h \in G$, these characters form a basis for the characters of the irreducible representations $\mathcal{V}_g^{\{j_k\}}$ of the Rational MCFT $\mathcal{P}_G\mathcal{V}$ [DVVV]. Thus, each $Z(g,h)$ is expected to be holomorphic on H [Z]. Other important properties of $Z(g,h)$ are that given charge conjugation invariance then $Z(g,h) = Z(g^{-1}, h^{-1})$ so that γ and $-\gamma$ act equally for each $\gamma \in \Gamma$. Finally, given the uniqueness of the twisted sectors, it also clear that under conjugation by any element $x \in Aut(\mathcal{V})$, with $g \to g^x = xgx^{-1}$, then $x(\mathcal{V}_g)x^{-1}$ is isomorphic to $\mathcal{V}_{g^x}$ so that $Z(g,h) = Z(g^x, h^x)$ for all $x \in Aut(\mathcal{V})$.

The construction of operators obeying (3.4) is only known in string theory-like models [DHVW,L,DGM,DM1] where the automorphism g is lifted from an automorphism of the embedding space of the string, typically a lattice automorphism. The properties of (3.5) are assumed in the physics literature [DFMS,DVVV] and are only so far understood in limited settings for vertex operator algebras [H]. The modular transformation properties (3.7b) for $Z(g,h)$ can be explicitly demonstrated in many cases [Va,DM1].

The G orbifold MCFT is now constructed from the projection $\mathcal{V}^G_{orb} = \mathcal{P}_G((\mathcal{P}_G\mathcal{V})^*)$ which has characteristic function $Z_{orb} = \sum_{g,h\in G} Z(g,h)$. In general, g may act projectively on $\mathcal{V}_g$ in (3.4) for a given $g \in G$ of order n so that g^n is a global phase. Then $\mathcal{V}^G_{orb}$ is not meromorphic and Z_{orb} is not T invariant. Such a 'global phase anomaly' is absent whenever $nE_0^g = 0 \ mod\ 1$ [Va] so that the operators of $\mathcal{P}_{\langle g\rangle}\mathcal{V}_g$ are of integer conformal weight. Assuming no such anomalies arise then $\mathcal{V}^G_{orb}$ is a self-dual MCFT and so $Z_{orb}(\tau) = J(\tau) + N_0^{orb}$ as in (2.5) where N_0^{orb} is the number of conformal weight 1 operators in $\mathcal{V}^G_{orb}$.

The OPA (3.5) is also preserved by the action of the dual automorphism group G^*, defined as follows. Recalling that $G = \langle g_1, ..., g_m\rangle$ with g_k of order n_k, we define g_k^* by

$$g_k^* V(\psi, z) g_k^{*-1} = e^{2\pi i r_k/n_k} V(\psi, z) \tag{3.8}$$

for each $\psi \in \mathcal{V}_g$ where $g = g_1^{r_1}...g_m^{r_m}$. Then $G^* = \langle g_1^*, ..., g_m^*\rangle$ is clearly an automorphism group for (3.5) and is isomorphic to G. We may then consider the orbifolding of $\mathcal{V}^G_{orb}$ with respect to G^*. The G^* invariant operators of $\mathcal{V}^G_{orb}$ are $\mathcal{P}_{G^*}\mathcal{V}^G_{orb} = \mathcal{P}_G\mathcal{V}$ as before. Therefore the projection of the dual is $P_{G^*}(\mathcal{P}_G\mathcal{V})^* = \mathcal{V}$ i.e. orbifolding $\mathcal{V}^G_{orb}$ with respect to G^* reproduces $\mathcal{V}$. Thus the two self-dual MCFTs $\mathcal{V}$ and $\mathcal{V}^G_{orb}$ are placed on an equal footing with each an Abelian orbifolding of the other. Thus we have :

$$\begin{tikzcd}
 & (\mathcal{P}_G\mathcal{V})^* \arrow[dl, "\mathcal{P}_{G^*}"'] \arrow[dr, "\mathcal{P}_G"] & \\
\mathcal{V} \arrow[rr, shift left, "G"] \arrow[dr, "\mathcal{P}_G"'] & & \mathcal{V}^G_{orb} \arrow[ll, shift left, "G^*"] \arrow[dl, "\mathcal{P}_{G^*}"] \\
 & \mathcal{P}_G\mathcal{V} &
\end{tikzcd} \tag{3.9}$$

where the horizontal arrows represent an orbifolding with respect to the indicated automorphism group and the diagonal arrows are projections.

4. The Moonshine Module and Monstrous Moonshine. The FLM Moonshine module $\mathcal{V}^\natural$ is historically the first example of a self-dual orbifold MCFT [FLM1] and is constructed as a $\mathbb{Z}_2$ orbifolding of $\mathcal{V}^\Lambda$, which will denote the Leech lattice MCFT from now on. The $\mathbb{Z}_2$ automorphism r of $\mathcal{V}^\Lambda$ chosen is lifted from the lattice reflection $\overline{r}$ so that $\mathcal{P}_{\langle r\rangle}\mathcal{V}^\Lambda$ contains no conformal weight 1 operators. The r-twisted space $\mathcal{H}_r$ on the other hand has vacuum energy $E_0^r = 1/2$ (and is hence global phase anomaly free) but likewise contains no conformal weight 1 operators since $E_0^r > 0$. The resulting orbifold MCFT, $\mathcal{V}^\natural = \mathcal{P}_{\langle r\rangle}(\mathcal{V}^\Lambda \oplus \mathcal{V}_r)$, therefore has characteristic function $J(\tau)$. As shown by FLM, a symmetrisation of the vertex algebra of the 196884 conformal weight 2 operators (including the Virasoro operator $V(\omega, z)$) forms an affine version of the 196883 dimensional Griess algebra [Gr] whose automorphism group is the Monster $\mathbb{M}$. FLM went on to show that $\mathbb{M} = Aut(\mathcal{V}^\natural)$ [FLM1,FLM2]. Note that we can identify as in (3.8), the automorphism group $\langle r^*\rangle$ dual to $\langle r\rangle$. By considering $Aut(\mathcal{P}_{\langle r\rangle}\mathcal{V}^\Lambda)$ and $Aut(\mathcal{P}_{\langle r\rangle}\mathcal{V}_r)$, the centraliser $C(r^*|\mathbb{M})$ can be found to be $C(r^*|\mathbb{M}) = 2_+^{1+24}.Co_1$ where Co_1 denotes the Conway simple group (i.e. the automorphism group Co_0 of Λ modulo $\overline{r}$), 2_+^{1+24} is an extra-special 2-group. Then $\mathbb{M}$ is generated by C and another involution that mixes the twisted and untwisted sectors [Gr,FLM1,FLM2]. Furthermore, $\mathcal{V}^\natural$ can be orbifolded with respect to $\langle r^*\rangle$ as in (3.9) to recover $\mathcal{V}^\Lambda$.

FLM have conjectured that $\mathcal{V}^\natural$ is characterised (up to isomorphism) as follows [FLM2]:

$\mathcal{V}^\natural$ Uniqueness Conjecture. *$\mathcal{V}^\natural$ is the unique CFT with characteristic function $J(\tau)$.*

This is stated in the context of the assumptions of Sections 1 and 2 where $Z(\tau) = J(\tau)$ is modular invariant and hence $\mathcal{V}^\natural$ is a self-dual C=24 MCFT. Furthermore, $\mathcal{V}^\natural$ forms the unique irreducible representation for itself [D2]. We will now consider briefly some evidence for this conjecture.

We may consider other possible $\mathbb{Z}_n$ orbifoldings of $\mathcal{V}^\Lambda$ with characteristic function $J(\tau)$ which should reproduce $\mathcal{V}^\natural$ according to this conjecture. In general, we can classify all automorphisms a of $\mathcal{V}^\Lambda$ lifted from automorphisms $\overline{a} \in \mathrm{Co}_0$, (for which $\mathcal{V}_a$ can be explicitly constructed) so that [T2]

(i) $\mathcal{P}_{\langle a\rangle}\mathcal{V}^\Lambda$ contains no conformal dimension 1 operators i.e. $\overline{a}$ is fixed point free.
(ii) $E_0^a > 0$ i.e. $\mathcal{V}_a$ contains no conformal dimension 1 operators.
(iii) $\mathcal{V}_a$ is global phase anomaly free i.e. $nE_0^a = 0 \ mod\ 1$ for $\overline{a}$ of order n.

There are 51 classes of Co_0 obeying (i) and (ii) only and 38 classes satisfying (i),(ii) and (iii). These 38 classes include 5 prime ordered cases for which $(p-1)|24$. These have been considered in much greater detail by Dong and Mason [DM2] who reconstructed $\mathcal{V}^\natural$ exactly for $p = 3$ and by Montague who also analysed the $p = 3$ case [M]. For each of these 38 classes, we expect that a self-dual MCFT $\mathcal{V}_{orb}^{\langle a\rangle}$ with characteristic function $J(\tau)$ exists. Furthermore, orbifolding $\mathcal{V}_{orb}^{\langle a\rangle}$ with respect to the dual group $\langle a^*\rangle$ defined as in (3.8) reproduces $\mathcal{V}^\Lambda$ with $\mathcal{V} = \mathcal{V}^\Lambda$, $G = \langle a\rangle$ and $\mathcal{V}_{orb}^G = \mathcal{V}^\natural$ in (3.9). By analysing $Aut(\mathcal{P}_{\langle a\rangle}\mathcal{V}_{a^k})$ for $k = 0, ..., n-1$ we can calculate explicitly the centraliser [T2]

$$C(a^*|Aut(\mathcal{V}_{orb}^{\langle a\rangle})) = \hat{L}.G_n \tag{4.1}$$

where $G_n = C(\overline{a}|\mathrm{Co}_0)/\langle\overline{a}\rangle$ and $\hat{L} = n.L$ is a cyclic extension of $L = \Lambda/(1-\overline{a})\Lambda$. For the prime ordered cases, this reduces to a well-known centraliser formula for $\mathbb{M}$ [CN]. (4.1) can also be shown to hold for all 51 classes obeying (i) and (ii) once a^* is appropriately defined and is verified for $Aut(\mathcal{V}_{orb}^{\langle a\rangle}) = \mathbb{M}$ in many cases [T2]. All of this provides evidence that $\mathcal{V}_{orb}^{\langle a\rangle} = \mathcal{V}^\natural$ in each construction lending weight to the uniqueness conjecture. Further evidence is given below.

Let us now define the Thompson-McKay series $T_g(\tau)$ for each $g \in \mathbb{M}$

$$T_g(\tau) = \mathrm{Tr}_{\mathcal{H}^\natural}(gq^{L_0-1}) = \frac{1}{q} + 0 + [1+\chi_A(g)]q + \dots \tag{4.2}$$

where $\chi_A(g)$ is the character of the 196883 dimensional adjoint representation for $\mathbb{M}$. This trace is obviously reminscent of (3.2) and this interpretation will be further explored below. The Thompson series for the identity element is $J(\tau)$, which is the hauptmodul for the genus zero modular group $\Gamma = SL(2,\mathbb{Z})$ as already stated. By calculating the first ten terms of $T_g(\tau)$ for each conjugacy class of $\mathbb{M}$, Conway and Norton [CN] conjectured

Monstrous Moonshine. *For each $g \in \mathbb{M}$, $T_g(\tau)$ is the hauptmodul for a genus zero fixing modular group Γ_g.*

Borcherds has now demonstrated this rigorously although the origin of the genus zero property remains obscure [B2]. In general, for g of order n, $T_g(\tau)$ is found to be invariant under $\Gamma_0(n) = \{\begin{pmatrix} a & b \\ nc & d \end{pmatrix} |det = 1\}$ up to h^{th} roots of unity where $h|n$ and $h|24$. $T_g(\tau)$ is fixed by Γ_g with $\Gamma_0(N) \subseteq \Gamma_g$ and contained in the normaliser of $\Gamma_0(N)$ in $SL(2,\mathbb{R})$ where $N = nh$ [CN]. This normaliser always contains the Fricke involution $W_N : \tau \to -1/N\tau$ where $W_N^2 = 1 \bmod \Gamma_0(N)$. We will refer to those classes with $h = 1$ as *Normal* and those with $h \neq 1$ as *Anomalous* i.e. the fixing group of $T_g(\tau)$ is of type $n + e_1, e_2, \dots$ for normal classes and of type $n|h + e_1, e_2, \dots$ for anomalous classes in the notation of [CN]. This terminology is motivated by whether the corresponding twisted sector $\mathcal{V}_g$ described below has a global phase anomaly or not.

For a normal element $g \in \mathbb{M}$ of prime order p (there is only one anomalous prime class of order 3 with $h = 3$) we find either $\Gamma_g = \Gamma_0(p)$ or $\Gamma_0(p)+ = \langle\Gamma_0(p), W_p\rangle$. $\Gamma_0(p)$ is of genus zero only when $(p-1)|24$. There is a corresponding class of $\mathbb{M}$, denoted by $p-$, for each such prime with this Thompson series e.g. the involution r^* above belongs to the class $2-$. $\Gamma_0(p)+$ is of genus zero for all the prime divisors of the order of $\mathbb{M}$. There is a class of $\mathbb{M}$, denoted by $p+$, for each such prime with Thompson series fixed by $\Gamma_0(p)+$. In general all the classes of $\mathbb{M}$ can be divided into Fricke and non-Fricke classes according to whether or not $T_g(\tau)$ is invariant under the Fricke involution W_N. It is also observed that the Thompson series for Fricke classes have non-negative integer coefficients whereas the coefficients of non-Fricke Thompson series are integers of mixed sign. There are a total of 51 non-Fricke classes of which 38 are normal and there are a total of 120 Fricke classes of which 82 are normal.

For each of the 38 constructions above based on classes $\{\overline{a}\}$ satisfying the conditions (i)-(iii) we can compute the dual automorphism Thompson series T_{a^*} and this agrees precisely with the genus zero series for the 38 non-Fricke normal classes of the Monster which also obey the centraliser relationship (4.1). Likewise, we can identify the other 13 anomalous non-Fricke classes and find the corresponding correct genus zero Thompson series [T2]. This is further evidence for the assertion that $\mathcal{V}_{orb}^{\langle a\rangle} = \mathcal{V}^{\natural}$ implied by the uniqueness conjecture for $\mathcal{V}^{\natural}$ which we will now assume to be true from now on.

We now turn to the interpretation of a Thompson series as an orbifold trace with $T_g(\tau) = Z(1,g)$ as in (3.2) where now $\mathcal{V} = \mathcal{V}^{\natural}$. For g in a non-Fricke class, we can construct the twisted sector $\mathcal{V}_g$ by choosing $g = a^*$ as above with characteristic function obeying (3.3). In particular, all the coefficients of $Z(g,1)$ are non-negative integers and hence $Z(1,g) - T_g(0)$, which is inverted up to a multiplicative constant under the Fricke involution to give $Z(g,1)(N\tau) - T_g(0)$, has mixed sign coefficients as observed. For the 38 normal non-Fricke classes we may orbifold $\mathcal{V}^{\natural}$ with respect to $\langle a^*\rangle$ to obtain $\mathcal{V}^{\Lambda}$. Then the vacuum energy $E_0^g = 0$ for the twisted sector $\mathcal{V}_g$ so that conformal weight 1 operators are reintroduced. On the other hand, for an anomalous non-Fricke class, a global phase anomaly parameterised by the parameter $h \neq 1$ occurs and we cannot obtain a MCFT from the resulting orbifolding [T1].

Consider next $f \in \mathbb{M}$, a Fricke element of order n. The corresponding twisted sector can be constructed when f is lifted from a lattice automorphism. We will assume that $\mathcal{V}_f$ exists in each case obeying (3.3)-(3.5). For normal elements, no global phase anomaly occurs and we may orbifold $\mathcal{V}^{\natural}$ with respect to $\langle f\rangle$ to obtain a self-dual MCFT $\mathcal{V}_{orb}^{\langle f\rangle}$. Assuming $T_f(\tau)$ is a hauptmodul we then find that $\mathcal{V}_{orb}^{\langle f\rangle} = \mathcal{V}^{\natural}$ for each normal Fricke element. The converse is also true, where given that $\mathcal{V}_{orb}^{\langle f\rangle} = \mathcal{V}^{\natural}$ for some $f \in \mathbb{M}$ then T_f is the hauptmodul for a genus zero modular group containing the Fricke involution [T2]. In general, we find (assuming the uniqueness conjecture for $\mathcal{V}^{\natural}$) that for all normal elements of $\mathbb{M}$

$$\mathcal{V}^{\Lambda} \underset{\overleftarrow{\langle a^*\rangle}}{\overset{\langle a\rangle}{\longrightarrow}} \mathcal{V}^{\natural} \overset{\langle f\rangle}{\longleftrightarrow} \mathcal{V}^{\natural} \quad \Leftrightarrow \quad T_{a^*},\, T_f \text{ are hauptmoduls} \tag{4.3}$$

(4.3) can be understood briefly for the prime ordered normal Fricke classes as follows. Suppose that f is a $p+$ element with Fricke invariant hautpmodul $T_f(\tau)$. Then $Z(f,1)(\tau) = Z(1,f)(\tau/p) = q^{-1/p} + 0 + \dots$ so that $\mathcal{V}_f$ has vacuum energy $E_0^f = -1/p$, degeneracy $D_f = 1$, contains no conformal weight 1 operators and has non-negative integer coefficients. Thus $\mathcal{P}_{\langle f\rangle}\mathcal{V}_f$ does not reintroduce conformal weight 1 operators. Similarly $\mathcal{P}_{\langle f\rangle}\mathcal{V}_{f^k}$, $k \neq 0 \bmod p$, contains no such operators (f and f^k are conjugate) so that $\mathcal{V}_{orb}^{\langle f\rangle} = \mathcal{V}^{\natural}$ since the characteristic function is $J(\tau)$.

Conversely, if $\mathcal{V}_{orb}^{\langle f\rangle} = \mathcal{V}^{\natural}$ for a prime p ordered element f then since f and f^k are conjugate, $T_f(\tau)$ is automatically $\Gamma_0(p)$ invariant. The fundamental region $H/\Gamma_0(p)$ for $\Gamma_0(p)$ has only two cusp points [Gu] at $\tau = \infty$ where $T_f(\tau)$ has a simple pole of order 1 from (2.2) and at $\tau = 0$ which is singular iff $E_0^f < 0$ with residue D_f from (3.3). We can then argue that since $\mathcal{V}_{orb}^{\langle f\rangle} = \mathcal{V}^{\natural}$, $E_0^f = -1/p$ with $D_f = 1$. This follows by considering the dual automorphism $f^* \in \mathbb{M}$ to f as in (3.9) and showing that $T_{f*} = T_f$. Then the corresponding centralisers must be equal which implies that $D_f = 1$, since no extension occurs. Furthermore, $\mathcal{V}_f$ contains no conformal weight 1 operators which implies that either $E_0^f = -1/p$ or $E_0^f > 0$. The latter possibility is ruled out because then $T_f(\tau) = q^{-1} + 0 + O(q)$ would be a hauptmodul for $\Gamma_0(p)$ which implies $E_0^f = 0$ when the constant term of T_f is zero. Thus we must have $E_0^f = -1/p$ with $D_f = 1$. Finally, consider $\phi(\tau) = T_f(\tau) - T_f(W_p(\tau))$ which is $\Gamma_0(p)$ invariant and is holomorphic on the compactification of $H/\Gamma_0(p)$, which is a compact Riemann surface. Hence $\phi(\tau)$ is a constant which is zero since it is odd under W_p. Hence $T_f(\tau)$ is $\Gamma_0(p)+$ invariant and has a unique simple pole on $H/\Gamma_0(p)+$ and is therefore a hauptmodul for $\Gamma_0(p)+$.

This argument can be generalised to any normal Fricke element $f \in \mathbb{M}$ of order n. Then (4.3) is equivalant to the fact that (i) $\mathcal{V}_f$ has vacuum energy $E_0^f = -1/n$ and degeneracy $D_f = 1$ and (ii) if f^r is Fricke then so is f^s with $s = n/(r, n)$ where r and s must be co-prime. These conditions are then sufficient to supply all the poles and residues of $T_f(\tau)$ so that $T_f(\tau)$ is a hauptmodul for some genus zero fixing group [T1,T2]. Finally, the genus zero property for an anomolous class of $\mathbb{M}$, which follows from the Harmonic formula of [CN], is described in [T2].

5. Generalised Moonshine from Abelian Orbifolds. Let us now consider the more general set of conjectures suggested by Norton [N] concerning Moonshine for centralisers (or extensions thereof) of elements of the Monster $\mathbb{M}$. Specifically, in the notation of (3.7a) we consider:

$$Z(g, h) = \mathrm{Tr}_{\mathcal{H}_g}(hq^{L_0 - 1}) \tag{5.1}$$

for $h \in C_g \equiv Aut(\mathcal{V}_g)$. For all Fricke elements, the twisted Hilbert space vacuum, which we now denote by $\mathcal{H}_g|_0$, is unique and hence $C_g = \langle g \rangle \times G_n$ where $G_n = C(g|\mathbb{M})/\langle g \rangle$ whereas for g non-Fricke $C_g = \hat{K}.G_n$ (for some extension $\hat{K}$). Norton has conjectured :

Generalised Moonshine Conjecture. *$Z(g, h)$ is either constant or is a hauptmodul for some genus zero fixing group for every pair of commuting elements $g, h \in \mathbb{M}$.*

This conjecture has been explicitly verified for an orbifold construction based on the Mathieu group M_{24} [DM1]. In terms of the orbifold picture reviewed in the earlier sections we can note the following properties for $Z(g, h)$:

(i) $Z(g, h) = Z(g^{-1}, h^{-1})$.

(ii) $Z(g, h) = Z(g^x, h^x)$ for conjugation with respect to any $x \in \mathbb{M}$.

(iii) $S : T_g(\tau) \to Z(g, 1) = D_g q^{E_0^g} + \ldots$ is a series with non-negative integer coefficients decomposible into positive sums of the dimensions of the irreducible representations of C_g where E_0^g is twisted vacuum energy and D_g is the vacuum degeneracy, the dimension of $\mathcal{H}_g|_0$, the twisted Hilbert space vacuum.

(iv) From (3.7) we find that $\gamma : Z(g, h) \to Z(h^{-c}g^a, h^d g^{-b})$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Note that $g = \exp(-2\pi i E_0^g)$ on $\mathcal{H}_g|_0$. In particular, for a normal Fricke element of order m, $g = \omega = e^{2\pi i/m}$ and each $h \in C_g$ acts as some element of $\langle \omega \rangle$ on $\mathcal{H}_g|_0$.

(v) As a consequence of (iv), $Z(g, h)$ is invariant up to roots of unity under $\Gamma(m, n) = \{\gamma \in \Gamma | a = 1 \ mod \ m, b = 0 \ mod \ m, c = 0 \ mod \ n, d = 1 \ mod \ n\}$ where

$m = O(g)$, $n = O(h)$. These extra factors appear if $h^{-c}g^a$ is anomalous for some co-prime a and c.

(vi) The value of $Z(g,h)$ at any parabolic cusp a/c (a and c co-prime) is determined by the vacuum energy of the $k = g^a h^{-c}$ twisted sector from (iv). In particular, only the Fricke classes are responsible for singular cusp points [N]. The residue of these cusps is determined by the action of $h^d g^{-b}$ on $\mathcal{H}_k|_0$. We will assume, as discussed in section 3, that $Z(g,h)$ is holomorphic at all other points on H.

Thus given any commuting pair of elements g, h as above, the location of any singularities for $Z(g,h)$ is known by finding which of the classes $k = g^a h^{-c}$ is Fricke for $(a,c) = 1$. The strength of the pole is then determined by the corresponding vacuum energy E_0^k. However, the residue for each singular cusp still needs to be found. We will argue below that this extra information is also supplied by the constraints of (4.3), at least in the simplest non-trivial prime cases. Once these singularities are known, then $Z(g,h)$ can be shown in each case to be either constant or to be the hauptmodul for a genus zero modular group.

The basic idea is to consider the orbifolding of $\mathcal{V}^\natural$ with respect to $\langle g,h\rangle$ and to re-express this as the composition of two $\mathbb{Z}_n$ orbifoldings. If $\langle g,h\rangle = \mathbb{Z}_k$, $k = mn/(m,n)$, then $Z(g,h)$ can always be related to a regular Thompson series via an appropriate modular transformation e.g. for m,n co-prime with $am+bn=1$ then $\langle g,h\rangle = \mathbb{Z}_{mn}$ and $\begin{pmatrix} 1 & 1 \\ -bn & am \end{pmatrix} : Z(1,gh) \to Z(g,h)$ from (iv). For $\langle g,h\rangle \neq \mathbb{Z}_k$ we will consider here the simplest non-trivial case where $\langle g,h\rangle$ contains only normal prime order p elements. Then $Z(g,h)$ is $\Gamma(p) \equiv \Gamma(p,p)$ invariant from (v). We will further assume that $h \overset{C_g}{\sim} h^k$ for $k \neq 0 \bmod p$ i.e. conjugate in C_g. This is sufficient to ensure that the coefficients in the q expansion of $Z(g,h)$ are rational since all the irreducible characters are rational. Furthermore, this condition restricts the possible conjugacy classes in $\mathbb{M}$ generated by g and h to just three i.e. $g \overset{\mathbb{M}}{\sim} g^a$, $h \overset{\mathbb{M}}{\sim} h^b$ and $gh \overset{\mathbb{M}}{\sim} g^a h^b$ for $a,b \neq 0 \bmod p$. From (iv), we have that $Z(g,h)$ is fixed by $\Gamma_0^0(p) = \{\gamma \in \Gamma | b = c = 0 \bmod p\} \sim \Gamma_0(p^2)$ (under conjugation by $diag(1,p)$ so that $Z(g,h)(p\tau)$ is $\Gamma_0(p^2)$ invariant). $Z(g,h)$ therefore has parabolic cusps on $H/\Gamma_0^0(p)$ at $\tau = i\infty, 0, 1, \ldots, p-1$ [Gu] with behaviour determined, from (vi), by the vacuum energy of the sectors twisted by $g, h, g^{p-1}h, \ldots g^2h, gh$ respectively where the last $p-1$ classes are conjugate. Within these assumptions we then find that there are 5 possible cases (up to relabelling) that may occur for any p as follows.

Case 1 : $g, h, gh = p-$. None of the cusps are singular and therefore $Z(g,h)$ is holomorphic on $H/\Gamma_0^0(p)$ and hence is constant.

We may now assume for the remaining 4 cases (without loss of generality by relabelling) that $g = p+$ so that

$$Z(g,h) = q^{-1/p} + 0 + O(q^{1/p}) \tag{5.2}$$

We also note from (iv) that g acts as $\omega = e^{2\pi i/p}$ on $\mathcal{H}_g|_0$.

Case 2 : $g = p+$, $h, gh = p-$. In this case $Z(g,h)$ has a unique simple pole at $q = 0$ as in (5.2) on $H/\Gamma_0^0(p)$ and therefore $Z(g,h)$ is a hauptmodul. This is only possible for $p = 2,\ 3,\ 5$ (where $Z(g,h)(p\tau)$ is a hauptmodul for $\Gamma_0(p^2)$). For $p = 5$, no such Generalised Moonshine function is actually observed which, interestingly, is also the case for regular Monstrous Moonshine where 25− is one of the so-called ghost elements [CN].

Case 3 : $g, h = p+$, $gh = p-$. $Z(g,h)$ has two singularities at $\tau = i\infty$ and 0. Under $S : \tau \to -1/\tau$ we have

$$Z(g,h) \to Z(h, g^{-1}) = \omega^{-k_g} q^{-1/p} + 0 + O(q^{1/p}) \tag{5.3}$$

where $g = \omega^{k_g}$ on $\mathcal{H}_g|_0$, $k_g \in \mathbb{Z}_p$. We may conjugate h to h^{-1} in C_g so that $Z(h, g^{-1}) = Z(h^{-1}, g^{-1}) = Z(h, g)$, from (i), which implies that $2k_g = 0 \bmod 2$. Hence for $p > 2$, $k_g = 0 \bmod p$. For $p = 2$ we will show below that $k_g = 0 \bmod 2$ also. Consider $f(\tau) = Z(g,h)(\tau) - Z(g,h)(S(\tau)) = 0 + O(q^{1/p})$. $f(\tau)$ is $\Gamma_0^0(p)$ invariant without any poles on $H/\Gamma_0^0(p)$ and hence is constant and equal to zero. Therefore, $Z(g,h)$ is $\langle \Gamma_0^0(p), S\rangle \sim \Gamma_0(p^2)+$ invariant with a unique simple pole and is a hauptmodul. This is only possible for $p = 2,\ 3,\ 5,\ 7$. For $p = 7$, no such Generalised Moonshine function is observed which corresponds to the ghost element 49+ of Monstrous Moonshine !

To understand the $p = 2$ case it is necessary to consider the interpretation of $Z(g,h)$ in terms of a $\langle g,h\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolding of $\mathcal{V}^\natural$. The orbifold so obtained is meromorphic self-dual (since no anomalous Monster elements occur) and is explicitly

$$\mathcal{V}_{orb}^{\langle g,h\rangle} = \mathcal{P}_{\langle g,h\rangle}(\mathcal{V}^\natural \oplus \mathcal{V}_g \oplus \mathcal{V}_h \oplus \mathcal{V}_{gh}) \tag{5.4}$$

where $\mathcal{P}_{\langle g,h\rangle} = (1+g+h+gh)/4 = \mathcal{P}_{\langle g\rangle}\mathcal{P}_{\langle h\rangle}$. We can consider this as two successive $\mathbb{Z}_2$ orbifoldings

$$\mathcal{V}_{orb}^{\langle g,h\rangle} = \mathcal{P}_{\langle g\rangle}(\mathcal{P}_{\langle h\rangle}(\mathcal{V}^\natural \oplus \mathcal{V}_h) \oplus \mathcal{P}_{\langle h\rangle}(\mathcal{V}_g \oplus \mathcal{V}_{gh})) \tag{5.5}$$

i.e. $\mathcal{V}_{orb}^{\langle g,h\rangle}$ is a Z_2 orbifolding with respect to $\langle g\rangle$ of $\mathcal{V}_{orb}^{\langle h\rangle} \equiv \mathcal{P}_{\langle h\rangle}(\mathcal{V}^\natural \oplus \mathcal{V}_h)$. Since $h = 2+$, we know that $\mathcal{V}_{orb}^{\langle h\rangle} = \mathcal{V}^\natural$ and hence $\mathcal{P}_{\langle h\rangle}(\mathcal{V}_g \oplus \mathcal{V}_{gh})$ is a g twisted sector for $\mathcal{V}^\natural$ for g of order two by the assumed uniqueness of the twisted sectors. Thus $g = 2+$ or $2-$ when acting on $\mathcal{V}_{orb}^{\langle h\rangle}$. However, we can determine from (5.5) that the character for this g twisted sector is $[Z(g,1) + Z(g,h) + Z(gh,1) + Z(gh,h)]/2 = q^{-1/2} + \ldots$ using (5.2). This implies that g is Fricke when acting on $\mathcal{V}_{orb}^{\langle h\rangle}$ and hence $\mathcal{V}_{orb}^{\langle g,h\rangle} = \mathcal{V}^\natural$ with $g = 2+$. We can represent this sequence of orbifoldings diagramatically as follows:

$$\begin{array}{ccccc} & & \mathcal{V}^\natural & & \\ & \overset{\langle h\rangle}{\nearrow} & & \overset{\langle g\rangle}{\searrow} & \\ \mathcal{V}^\natural & & \xrightarrow{\langle g,h\rangle} & & \mathcal{V}^\natural \end{array} \tag{5.6}$$

where each copy of $\mathcal{V}^\natural$ is orbifolded with respect to the denoted group.

We can similarly consider the orbifolding of $\mathcal{V}^\natural$ with respect to $g = 2+$ followed by h. The resulting orbifold must be $\mathcal{V}_{orb}^{\langle g,h\rangle} = \mathcal{V}^\natural$ and hence $\mathcal{P}_{\langle g\rangle}(\mathcal{V}_h \oplus \mathcal{V}_{gh})$ must be a 2+ twisted sector. This forces $g = 1$ on $\mathcal{H}_h|_0$ as was claimed earlier. In general, for any p, it is straightforward to see that $\mathcal{V}^\natural \xrightarrow{\langle g\rangle} \mathcal{V}_{orb}^{\langle g\rangle} = \mathcal{V}^\natural \xrightarrow{\langle h\rangle} \mathcal{V}_{orb}^{\langle g,h\rangle} = \mathcal{V}^\natural$ in this case.

Case 4 : $g, gh = p+$, $h = p-$. In this case $Z(g,h)$ has singular cusps at $i\infty, 1, ..., p-1$ on $H/\Gamma_0^0(p)$. We can find the residues of these poles by decomposing the orbifolding with respect to $\langle g,h\rangle$ to obtain $\mathcal{V}^\natural \xrightarrow{\langle g\rangle} \mathcal{V}^\natural \xrightarrow{\langle h\rangle} \mathcal{V}_{orb}^{\langle g,h\rangle}$ since $g = p+$ where we necessarily find that either $\mathcal{V}_{orb}^{\langle g,h\rangle} = \mathcal{V}^\natural$ or $\mathcal{V}^\Lambda$ from (4.3). If we alternatively orbifold with respect to $h = p-$ first we then obtain $\mathcal{V}^\natural \xrightarrow{\langle h\rangle} \mathcal{V}^\Lambda \xrightarrow{\langle g\rangle} \mathcal{V}_{orb}^{\langle g,h\rangle}$. In order that $\mathcal{V}_{orb}^{\langle g,h\rangle} = \mathcal{V}^\natural$, it is necessary that the g twisted sector of $\mathcal{V}^\Lambda$ so obtained, $\mathcal{P}_{\langle h\rangle}(\oplus_{k=0}^{p-1}\mathcal{V}_{gh^k})$, has positive vacuum energy from condition (ii) of section 4. However, from (5.2), this is impossible since $\mathcal{P}_{\langle h\rangle}\mathcal{V}_g$ has character $q^{-1/p} + 0 + O(q^{1/p})$. Hence $\mathcal{V}_{orb}^{\langle g,h\rangle} = \mathcal{V}^\Lambda$ in this case.

We can similarly decompose $\mathcal{P}_{\langle g,h\rangle} = \mathcal{P}_{\langle gh\rangle}\mathcal{P}_{\langle f\rangle}$ for $f = g^a h$ a $p+$ element with $a \neq 0, 1 \ mod\, p$. Then $\mathcal{V}^\natural \xrightarrow{\langle f\rangle} \mathcal{V}^\natural \xrightarrow{\langle gh\rangle} \mathcal{V}_{orb}^{\langle g,h\rangle} = \mathcal{V}^\Lambda$. This implies that gh must act as a $p-$ element on $\mathcal{V}_{orb}^{\langle f\rangle} = \mathcal{V}^\natural$ and hence the corresponding gh twisted sector $\mathcal{P}_f(\oplus_{k=0}^{p-1}\mathcal{V}_{ghf^k})$ has zero vacuum energy from (4.3). In particular, this implies that $\mathcal{P}_f\mathcal{H}_{gh}|_0 = 0$ so that $f = g^a h \neq 1$ on $\mathcal{H}_{gh}|_0$ for any $a \neq 0 \ mod\, p$ (noting that $gh = \omega$ on $\mathcal{H}_{gh}|_0$). Let $h = \omega^r$ be the action on $\mathcal{H}_{gh}|_0$ (from (iv)) so that $g = \omega^{1-r}$. But we can always choose $a \neq 0 \ mod\, p$ such that $g^a h$ acts as unity on $\mathcal{H}_{gh}|_0$ unless $r = 0 \ mod\, p$. Hence the orbifolding is only consistent when $h = 1$ on $\mathcal{H}_{gh}|_0$. In general, by conjugation, we then find that $h = 1$ on $\mathcal{H}_{g^a h^b}|_0$ for all $b \neq 0 \ mod\, p$. Hence, the residue of any of the singular cusps is known. This allows us to find the full fixing modular group.

Let $\gamma_p = \begin{pmatrix} 1 & -p \\ 1 & 1-p \end{pmatrix}$ which is of order p in $\Gamma_0^0(p)$. γ_p permutes the p cusps of $Z(g,h)$ where $\gamma_p : Z(g,h) \to Z(gh, h^{-1}) = q^{-1/p} + 0 + O(q^{1/p})$ and similarly for the other singular twisted sectors. Then $f(\tau) = Z(g,h)(\tau) - Z(g,h)(\gamma_p(\tau))$ is holomorphic on $H/\Gamma(p)$ and is therefore zero. Hence $Z(g,h)$ is $\langle \Gamma_0^0(p), \gamma_p\rangle \sim \Gamma_0(p)$ invariant with a unique simple pole and is a hauptmodul. This is only possible for $p = 2,\ 3,\ 5,\ 7,\ 13$. Once again, the largest possible case is not observed, $p = 13$, although this does not correspond to a ghost element for regular Moonshine.

Case 5 : $g, h, gh = p+$. In this last case all sectors are Fricke and $Z(g,h)$ has singular cusps at $i\infty, 0, 1, ..., p-1$. With the assumption that all $h \overset{C_g}{\sim} h^k$ for $k \neq 0 \ mod\, p$ we need only in practice consider $p = 2, 3$ and 5 where $C_g = \langle g\rangle \times G_p$ for $G_p = B, Fi_{24}'$ or HN respectively [CCNPW]. Following a general argument as in Case 3, it is easy to see again that $\mathcal{V}_{orb}^{\langle g,h\rangle} = \mathcal{V}^\natural$ since both g and h are Fricke and (5.2) is obeyed.

For $p = 2$ we again decompose the orbifolding of $\mathcal{V}^\natural$ with respect to $\langle g,h\rangle$. Referring to (5.5), we note that $\mathcal{P}_{\langle h\rangle}(\mathcal{V}_g \oplus \mathcal{V}_{gh})$ has a $2+$ character and that gh is Fricke. Hence $h = -1$ and $g = 1$ on $\mathcal{H}_{gh}|_0$. Similarly, we can orbifold with respect to g first and find that $\mathcal{P}_{\langle g\rangle}(\mathcal{V}_h \oplus \mathcal{V}_{gh})$ also has $2+$ character. Hence $g = -1$ on $\mathcal{H}_h|_0$. Thus all the residues of $Z(g,h)$ are known. In particular, ST of order three permutes the cusps $\{i\infty, 0, 1\}$ with $ST : Z(g,h) \to Z(gh, g) = q^{-1/2} + 0 + O(q^{-1/2})$. Then, by the usual argument, $Z(g,h)$ is a hauptmodul for $\langle \Gamma(2), ST\rangle$ of genus zero, which is of level 2 and index 2 in Γ. In fact, $Z(g,h)$ is invariant under the full modular group Γ up to ± 1 with $Z(g,h) = -Z(g, gh)$ so that $Z(g,h)(2\tau) = E_2(\tau)/\eta^{12}(\tau) - 252$ is the hauptmodul for $2|2$ in the notation of [CN].

For $p = 3, 5$ we may repeat the argument of Case 3 to show that $g = 1$ on $\mathcal{H}_h|_0$ so that $Z(h,g) = q^{-1/p} + 0 + ...O(q^{1/p})$. But $Z(g,h)$ and $Z(h,g)$ have the same singular structure and hence we may interchange g and h. Since $\mathcal{V}_{gh}$ is preserved by this interchange, $g = h$ on $\mathcal{H}_{gh}|_0$ with $gh = \omega$ so that $g = h = \omega^{(p+1)/2}$. Hence by conjugation as in (ii), all the residues of the singular cusps of $Z(g,h)$ are known.

For $p = 3$, let $\gamma_2 = T^{-1}ST$ which is of order 2 and interchanges the cusps $\{\infty, 0\} \leftrightarrow \{2,1\}$ whereas S interchanges $\{\infty, 1\} \leftrightarrow \{0,2\}$. Then $\gamma_2 : Z(g,h) \to Z(gh, h^{-1}g) = q^{-1/3} + 0 + \ldots$ and $S : Z(g,h) \to Z(h, g^{-1}) = q^{-1/3} + 0 + \ldots$ and similarly for the other cusps. By the usual argument, we then find that $Z(g,h)$ is the hauptmodul for the genus zero group $\langle \Gamma(3), S, T^{-1}ST \rangle$ of level 3 and index 3 in Γ. Further analysis shows that $Z(g,h)$ is invariant under Γ up to third roots of unity with $Z(g,h) = \omega^2 Z(g,gh) = \omega Z(g, g^2 h)$, so that $Z(g,h)(3\tau) = E_3(\tau)/\eta^8(\tau) - 368$ is the hauptmodul for $3|3$ in the notation of [CN].

For $p = 5$, let $\gamma_3 = TST^3$ which is of order 3 and cyclically permutes the cusps $\{\infty, 0\} \to \{1,4\} \to \{2,3\}$ whereas S interchanges the cusps $\{\infty, 1, 2\} \leftrightarrow \{0,4,3\}$. Then $\gamma_3 : Z(g,h) \to Z(gh^{-1}, g^3h^3) = Z(gh, (gh^{-1})^3) = q^{-1/5} + 0 + \ldots$ by conjugation and similarly for the other cusps. $Z(g,h)$ is invariant under $\Gamma(5)$ whose normaliser contains Γ and so $Z(g,h)(\tau) - Z(g,h)(\gamma(\tau))$ is holomorphic on $H/\Gamma(5)$ for both $\gamma = S$ and γ_3 and hence is zero. Thus $Z(g,h)$ is the hauptmodul for the genus zero group $\langle \Gamma_0^0(5), S, \gamma_3 \rangle$ which is of level 5 and index 5 in Γ. In the notation of [FMN], the fixing group of $Z(g,h)(5\tau)$ is $5||5$. In this case, there are five independent functions $f_k(\tau) = Z(g, g^k h), k = 0, 1, \ldots, 4$ which are permuted under Γ.

We summarise Cases 2-5 in the following table where we reproduce the genus zero fixing group for $Z(g,h)(p\tau)$ with $g = p+$ and where only the actual observed values of p are indicated.

	$gh = p-$	$gh = p+$
$h = p-$	$\Gamma_0(p^2), p = 2,3$	$\Gamma_0(p)-, p = 2,3,5,7$
$h = p+$	$\Gamma_0(p^2)+, p = 2,3,5$	$2\|2, 3\|3, 5\|\|5$

6. Conclusion. We have shown that the genus zero property for the Generalised Moonshine functions (5.1) follows from the genus zero property for Thompson series in the simplest non-trivial prime cases. It remains a much greater challenge to extend this arguments to all cases. The major difficulties of this method for general commuting elements g, h are (i) the proliferation of possible Fricke classes in $\langle g, h \rangle$ giving the location of poles and (ii) the determination of the corresponding residues. Once this information is known, then any generalised moonshine functions should be reconstructible if it is a hauptmodul. Finally, it is interesting that the ghost groups $25-$ and $49+$ are absent from the above table (as indeed is $50+50$ from the list of modular groups for the centraliser moonshine of the $5+$ or $10+$ elements of $\mathbb{M}$ where it might be expected to arise). These hauptmoduls are also distinguished by having non-quadratic irrationalities at their non-singular cusps [FMN] suggesting some possibly deeper number theoretic significance.

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