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THE FINITE-TEMPERATURE RENORMALIZATION GROUP APPLIED TO $\lambda\phi^4$ THEORY AND QCD

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ABSTRACT

In this paper we apply the finite-temperature renormalization group from the point of view of "environmentally friendly" renormalization. We study both $\lambda\phi^4$ theory and the magnetic sector of QCD. At one loop level the complete temperature range of $\lambda\phi^4$ is successfully described in terms of the parameters of the zero temperature theory. We show also how the critical temperature can be calculated in terms of the latter. For the magnetic sector of QCD, in distinction to $\lambda\phi^4$, a one-loop finite temperature renormalization group improvement is not sufficient to describe the high-temperature regime.

1. Introduction

Since its introduction the finite-temperature Renormalization Group[1] has been repeatedly studied and applied. Here we consider it as an example of "environmentally friendly" renormalization[2]. The latter is based on the notion that the effective degrees of freedom (fluctuations) of a system are sensitive to the environment. With environmentally friendly renormalization one renormalizes (reparameterizes) the theory in an environment dependent way so as to enable the renormalized parameters to track the evolving nature of the effective degrees of freedom as a function of scale in a perturbatively controllable manner.

Temperature is an interesting example of a relevant environmental parameter, as a field theory at very high temperatures exhibits qualitatively different effective degrees of freedom than those of the zero-temperature theory. In earlier work[2,3] this crossover of the effective degrees of freedom was accessed completely for $\lambda\phi^4$ theory, both in the symmetric and broken phases, using an environmentally friendly renormalization group. The running parameter used was the finite-temperature mass (inverse screening length) $m(T)$, the limit $m(T) \to 0$ corresponding to a second order phase transition. Although the latter is a natural parameter it is often the case that only the zero-temperature parameters are experimentally known. To relate these to the finite-temperature parameters one must run a parameter other than $m(T)$. The temperature itself is the obvious candidate, or rather one runs an arbitrary renormalized temperature scale τ, and after the renormalization group equations are solved, τ is set equal to the physical temperature T. This is necessary in any case in order that perturbation theory in terms of the renormalized parameters be well behaved. As we will show, by running the environment itself in this way, one may answer questions such as: what is the critical temperature?

2. Running the Environment for $\lambda\phi^4$ Theory

In this section we consider the renormalization of $\lambda\phi^4$ theory at an arbitrary fiducial temperature scale τ. We apply the normalization conditions

$$\begin{aligned}
\Gamma^{(2)}(p=0,m_\tau,\lambda_\tau,\bar{\phi}^0_\tau,T=\tau) &= m_\tau^2+\frac{\lambda_\tau}{2}(\bar{\phi}^0_\tau)^2 \\
\Gamma^{(4)}(p=0,m_\tau,\lambda_\tau,\bar{\phi}^0_\tau,T=\tau) &= \lambda_\tau \\
\frac{d}{d\vec{p}^2}\Gamma(p_0=0,\vec{p},m_\tau,\lambda_\tau,\bar{\phi}^0_\tau,T=\tau)\Big|_{\vec{p}=\vec{0}} &= 1
\end{aligned} \tag{1}$$

where $\bar{\phi}^0$ represents the minimum of the effective potential, i.e. it satisfies the equation of state

$$\Gamma^{(1)}(p=0,m_\tau,\lambda_\tau,\bar{\phi}^0_\tau,T=\tau)=0. \tag{2}$$

The beta functions, obtained by differentiating these renormalization conditions with respect to τ for fixed bare parameters, depend on derivatives $d\bar{\phi}^0_\tau/d\tau$. These can be eliminated with the equation of state (2)[4]. The equations relevant for the symmetric phase are

$$\tau\frac{dm_\tau^2}{d\tau} = \frac{\lambda_\tau(N+2)\tau^2}{24\pi^2}\sum_{n=1}^{\infty}\int_0^\infty\frac{ds}{s^2}e^{-\frac{m_\tau^2}{\tau^2}s}e^{-\frac{n^2}{4s}}(1+\frac{m_\tau^2}{\tau^2}s) \tag{3}$$

$$\tau\frac{d\lambda_\tau}{d\tau} = -\frac{\lambda_\tau^2(N+8)}{96\pi^2\tau^2}\sum_{n=-\infty}^{\infty}\int_0^\infty ds\, e^{-\frac{m_\tau^2}{\tau^2}s}e^{-\frac{n^2}{4s}}(2m_\tau^2-\tau\frac{dm_\tau^2}{d\tau}) \tag{4}$$

In fig. 1 we present a plot of the solutions of these equations for the case where the system is in a state of broken symmetry at $T=0$. We have chosen the arbitrary temperature $\tau=T$. This is not simply because T is the relevant physical scale but also because without this choice perturbation theory would break down in the vicinity of a phase transition. The place where the mass curve is zero determines the critical temperature T_c in terms of the initial conditions for the flow equations (3) and (4). For temperatures $T\gg T_c$ the mass increases linearly with temperature and a "mean field" regime is reached where infrared (IR) fluctuations are suppressed due to the large thermally induced mass. The solution of these equations can be perturbatively controlled at higher order (though a resummation technique should be used in the vicinity of the phase transition) thereby providing a technique by which the critical temperature can be calculated to all orders. Notice that it is in no way assumed that the behaviour near the critical point is three dimensional. One starts off at zero temperature, heats the theory up, then examines what occurs without any prejudice as to the expected behaviour. We believe this to be an important advantage of environmentally friendly methods. In fig. 1 the coupling constant goes to zero as the critical point is approached. As has previously been emphasized[2] this doesn't imply that the interactions there vanish. The appropriate coupling constant in the vicinity of the phase transition is $\lambda T/m(T)$ and this approaches a non-zero fixed point.

The results here are completely consistent with those found by environmentally friendly methods wherein the finite-temperature mass is the running parameter. It is important to realize that in the above equations we have evaluated the appropriate diagrams with propagators of mass $m(\tau)$. Hence if we write $d/d\tau = \partial/\partial\tau + (\tau dm_\tau^2/d\tau)\partial/\partial m_\tau^2$, the latter acts non-trivially. Diagramatically one can think of this as being equivalent to summing up all "daisy" insertions into the internal lines of the β function diagrams. Such a maneuver is crucial in that without it the tadpole insertions, which constitute the largest temperature effects in the daisy diagrams, would ensure a breakdown in perturbation theory.

3. The QCD case

We now turn our attention to QCD. We have used as a renormalization condition that the static (i.e. zero energy), spatial three-gluon vertex equals the tree-level vertex in the symmetric momentum configuration

$$\Gamma^{abc}_{ijk}(p_i^0 = 0, \vec{p}_i, g_{\kappa,\tau}, T = \tau)\Big|_{\substack{\text{symm.}\\ \kappa}} = g_{\kappa,\tau} f^{abc}\left[g_{ij}(p_1 - p_2)_k + \text{cycl.}\right]. \qquad (5)$$

In contradistiction to the previously discussed case this chosen renormalization condition depends on two parameters, the momentum scale κ, and the temperature scale τ. Therefore we can perform a renormalization group analysis with respect to both parameters, i.e. we can run more than one environmental parameter at the same time.

For the calculation we have used the Landau gauge Background Field Feynman rules[5] resulting from the Vilkovisky-de Witt effective action in order to get rid of ambiguities arising from gauge dependence[6]. Due to the corresponding Ward Identities the calculation is simplified in that we only have to calculate the transverse gluon self energy function Π^{Tr} in the static limit. In terms of the coupling $\alpha_{\kappa,\tau} := g^2_{\kappa,\tau}/4\pi^2$ the β functions are then

$$\kappa\frac{d\alpha_{\kappa,\tau}}{d\kappa} = \alpha_{\kappa,\tau}\ |\vec{p}|\,\frac{d\Pi^{\text{Tr}}}{d\,|\vec{p}\,|}\bigg|_{\substack{|\vec{p}|=\kappa\\ T=\tau}}, \qquad \tau\frac{d\alpha_{\kappa,\tau}}{d\tau} = \alpha_{\kappa,\tau}\ T\frac{d\Pi^{\text{Tr}}}{dT}\bigg|_{\substack{|\vec{p}|=\kappa\\ T=\tau}}. \qquad (6)$$

The τ renormalization group is needed to draw conclusions about the temperature dependence of the coupling. This can not be done using the κ-scheme alone without assuming something about the temperature dependence of the initial value of the coupling used in solving the differential equation.

More details of the ingredients of the calculation can be found in reference [7]. The result is

$$\kappa \frac{d\alpha_{\kappa,\tau}}{d\kappa} = \beta_{vac} + \beta_{th}, \qquad \tau \frac{d\alpha_{\kappa,\tau}}{d\tau} = -\beta_{th}, \tag{7}$$

where the vacuum contribution is, as usual,

$$\beta_{vac} = \alpha_{\kappa,\tau}^2 \left(-\tfrac{11}{6}N_c + \tfrac{1}{3}N_f\right), \tag{8}$$

and where, in terms of the IR and UV convergent integrals

$$F_n^\eta = \int_0^\infty dx \frac{x^n}{e^{\kappa x/2\tau} - \eta} \left[\log\left|\frac{x+1}{x-1}\right| - 2\sum_{k=0}^{\frac{n}{2}-1} \frac{x^{2k+1}}{2k+1}\right] \tag{9}$$

and

$$G_n^\eta = \int_0^\infty dx \frac{1}{e^{\kappa x/2\tau} - \eta} \, \mathrm{P} \frac{x}{(x^2-1)^n}, \tag{10}$$

the thermal contribution is given by

$$\begin{aligned} \beta_{th} = \alpha_{\kappa,\tau}^2 \Big[& \left(\tfrac{21}{16}F_0^1 + \tfrac{3}{4}F_2^1 - \tfrac{3}{2}G_0^1 - \tfrac{25}{8}G_1^1 - G_2^1\right) N_c + \\ & \left(\tfrac{1}{4}F_0^{-1} + \tfrac{3}{4}F_2^{-1} - \tfrac{3}{2}G_0^{-1} - G_1^{-1}\right) N_f \Big]. \end{aligned} \tag{11}$$

Small differences with the thermal gluon contribution obtained by Antikainen et.al.[8] (which contains $\frac{28}{16}F_0^1 + \frac{3}{4}F_2^1 - \frac{3}{2}G_0^1 - \frac{32}{8}G_1^1$) are probably caused by the use of the Background Field Feynman gauge, as may possibly be checked with Gauge Dependence Identities[9].

Because the two beta functions (7) are not exactly each other's opposite the renormalization group improved coupling is not just a function of the ratio κ/τ. There is another dimensionful scale (such as Λ_{QCD}) that comes from an initial condition for these differential equations. The solution of the set of coupled differential equations can be written in the form

$$\alpha_{\kappa,\tau} = \frac{1}{\left(\frac{11}{6}N_c - \frac{1}{3}N_f\right)\ln\frac{\kappa}{\Lambda_{QCD}} - f\left(\frac{\kappa}{\tau}\right)} \tag{12}$$

where the function f satisfies $\beta_{th} = \alpha_{\kappa,\tau}^2 \kappa df/d\kappa$ with the initial condition $\lim_{\tau\downarrow 0} f = 0$ so that we can identify Λ_{QCD} with the usual zero-temperature QCD scale. Actually this function f can be found in terms of the functions F and G:

$$f = \left(\tfrac{21}{16}F_0^1 + \tfrac{1}{4}F_2^1 + \tfrac{7}{8}G_1^1\right) N_c + \left(\tfrac{1}{4}F_0^{-1} + \tfrac{1}{4}F_2^{-1}\right) N_f. \tag{13}$$

Fig. 2 is a contour plot of the effective coupling as a function of both momentum- and temperature-scale. The results are best trusted in places where the coupling is small. For physical reasons we have to restrict ourselves in any case to the region where the coupling is positive, which is below the uppermost line in the graph.

The high-temperature behaviour (i.e. for $\tau \gg \kappa$) is determined by

$$f \longrightarrow N_c \tfrac{21\pi^2}{16}\tfrac{\tau}{\kappa} + \left(\tfrac{11}{6}N_c - \tfrac{1}{3}N_f\right)\ln\tfrac{\kappa}{\tau} + O(1). \tag{14}$$

The coefficient of the dominant contribution is the same as in Landsman's result[10], but others found different coefficients[8,11] and even different signs[12] with a strong dependence on the gauge parameter and the details of the renormalization condition[6]. The sign of this coefficient is of crucial significance for the behaviour of the coupling in this limit. For increasing temperatures at fixed momentum scale, our sign makes the coupling grow to a pole, an indication that we are entering a strong-coupling regime, whereas the opposite sign would lead to asymptotic freedom in this limit. Stimulated by the original belief[13] that high-temperature QCD would be asymptotically free as in the high-momentum situation, Landsman suggested that this sign would be an artifact of the one-loop calculation and that a higher-loop calculation or a resummation could change it. We however believe that this will not happen, as the sign appears quite naturally if one realises that this limit $\tau/\kappa \to \infty$ is an IR limit where confinement takes place. Unless at higher loop order the magnetic mass increases quickly enough with temperature in order to act as an sufficient IR cutoff, we cannot get around this problem without actually solving confinement. We believe this to be an important consideration when considering phase transitions which involve non-abelian gauge fields.

In the regime $\tau \gg \kappa$ the beta functions behave as in a three-dimensional theory so that we designate this as the region where dimensional reduction occurs. Here it is natural, as for $\lambda\phi^4$, to use a different dimensionless coupling $u = \alpha_{\kappa,\tau}\frac{\tau}{\kappa}$ since then fixed points may turn up more clearly. However in this case such a reparametrization cannot remove the pole and will not give a different behaviour.

If we allow the momentum-scale to change with temperature simultaneously, the high-temperature limit can be taken in many ways. In the region $\tau \gg \kappa$ the shape of the contours is given by $\tau \sim \kappa \ln\frac{\kappa}{\Lambda_{QCD}}$. This characterizes exactly along which paths in the (τ,κ)-plane the coupling increases or decreases. For example at a fixed ratio τ/κ (no matter what this ratio is) we eventually find a coupling that decreases like $1/\ln\kappa$, much in the same way as at zero temperature. This is a natural contour to consider for a weak-coupling regime[14] where one could treat the quark-gluon plasma as a perfect gas, as then the thermal average of the momentum of massless quanta at temperature T is proportional to the temperature. However at low momenta the assumption of weak coupling breaks down. Furthermore, instead of considering quantities at the average momentum it is more appropriate to use thermal averages of the quantities themselves as a weighted integral over all momenta[12]. But once again one runs into problems at the low-momentum end as long as we cannot treat the strong-coupling regime.

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5. References

1. H. Matsumoto, Y. Nakano and H. Umezawa, *Phys. Rev.* **D29** (1984) 1116.
2. Denjoe O'Connor and C.R. Stephens,*Nuc. Phys.* **B360** (1991) 237; "Environmentally Friendly Renormalization"; Preprin THU-93/14, DIAS-STP-93-19; (to be published in *Int. Jou. Mod. Phys.* **A**).
3. Denjoe O'Connor, C.R. Stephens and F. Freire, *Mod. Phys. Lett* **A25** (1993) 1779; *Class. Quan. Grav.* **23** (1993) S243; F. Freire and C.R. Stephens, *Z. Phys.* **C60** (1993) 127.
4. P. Elmfors, NORDITA preprint 92/38P (1992).
5. A. Rebhan, *Z. Phys.* **C30** (1986) 309.
6. N. Ashida, A. Niegawa, H. Nakkagawa and H. Yokota, *Phys. Rev.* **D44** (1991) 473.
7. M.A. van Eijck, *Can. J. Phys.* **71** (1993) 237; M.A. van Eijck, C.R. Stephens and C.W. van Weert, *Mod. Phys. Lett.* **A9** (1994) 309.
8. J. Antikainen, M. Chaichian, N.R. Pantoja and J.J. Salazar, *Phys. Lett.* **B242** (1990) 412.
9. R. Kobes, G. Kunstatter and A. Rebhan, *Nucl. Phys.* **B355** (1991) 1.
10. N.P. Landsman, *Phys. Lett.* **B232** (1989) 240.
11. R. Baier, B. Pire and D. Schiff, *Z. Phys.* **C51** (1991) 581.
12. K. Enquist and K. Kainulainen, *Z. Phys.* **C53** (1992) 87.
13. J.C. Collins and M.J. Perry, *Phys. Rev. Lett.* **34** (1975) 1353.
14. K. Enquist and K. Kajantie, *Mod. Phys. Lett.* **A2** (1987) 479.

Figure Captions

Fig. 1: Solutions of the coupled differential equations (3,4) for the running mass m and coupling λ of the one-component $\lambda\phi^4$ model as a function of temperature T. At the critical temperature the coupling goes to zero simultaneously with the mass.

Fig. 2: Contour plot of the running coupling $\alpha_{\kappa,\tau}(\kappa,\tau)$ for QCD with three colours and six fermion flavours. The fermions have been taken massless. Only below the curve $\alpha_{\kappa,\tau} = \infty$ (close to $\alpha_{\kappa,\tau} = 1000$) the coupling is positive and finite.

Fig. 1

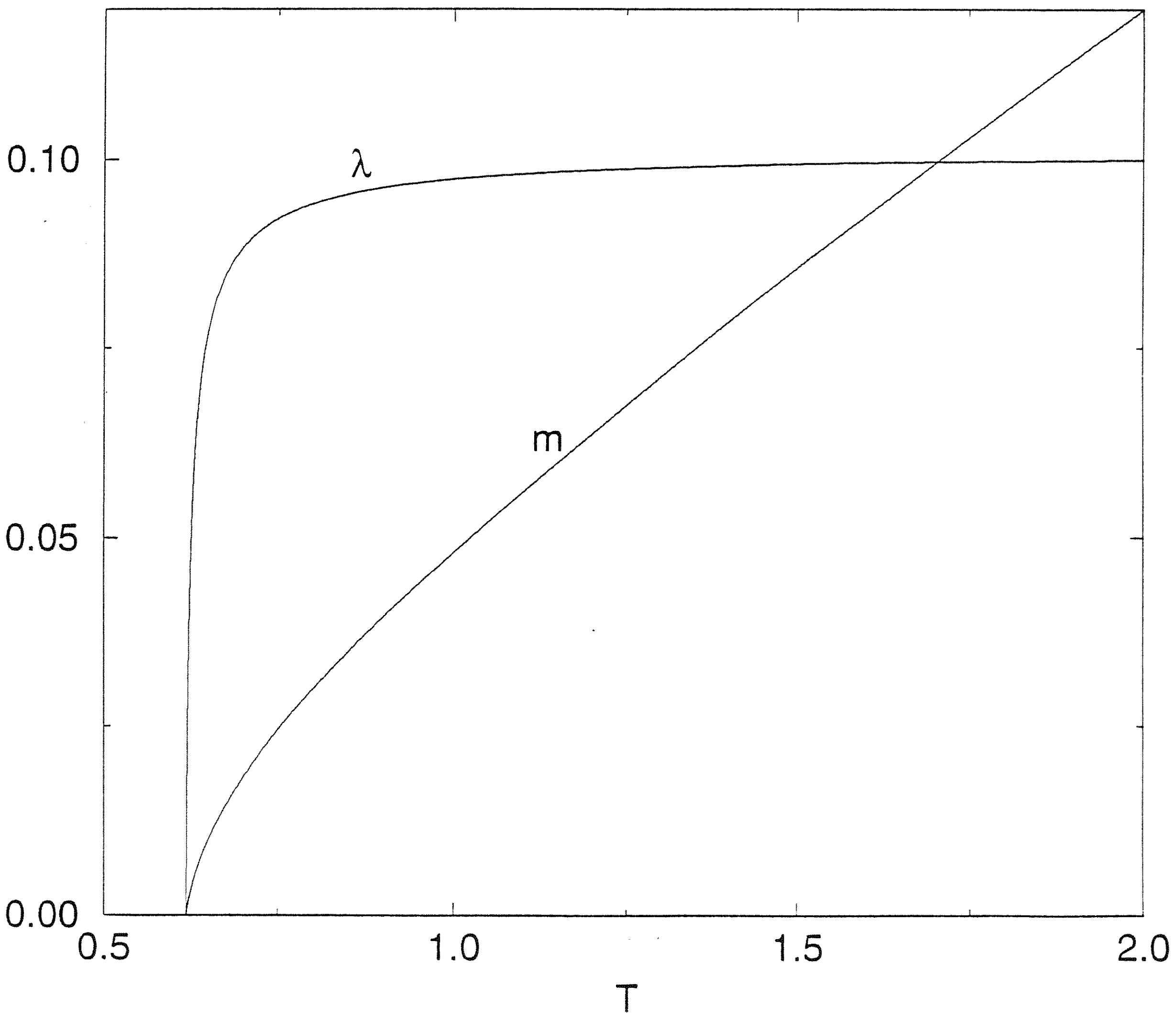

Fig. 2