Correlation Functions of Dense Polymers and $c = -2$ Conformal Field Theory

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The model of dense lattice polymers is studied as an example of non-unitary Conformal Field Theory (CFT) with $c = -2$. "Antisymmetric" correlation functions of the model are proved to be given by the generalized Kirchhoff theorem. Continuous limit of the model is described by the free complex Grassmann field with null vacuum vector. The fundamental property of the Grassmann field and its twist field (both having non-positive conformal weights) is that they themselves suppress zero mode so that their correlation functions become non-trivial. The correlation functions of the fields with positive conformal weights are non-zero only in the presence of the Dirichlet operator that suppresses zero mode and imposes proper boundary conditions.

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Introduction.—In spite of the remarkable success of unitary CFT's in predicting the critical properties of different lattice spin models [1], the non-unitary theories, although of no less importance for statistical physics, so far were not fully understood. It already becomes obvious that some of the axioms of unitary CFT have to be sacrificed in this case [2]. Still, it remains unclear where one has to modify the foundations and not to destroy the whole building of CFT.

The general idea of the Letter is not to study the non-unitary CFT's on their own but, instead, to analyze one particular model of dense polymers on the lattice whose continuous limit corresponds to the non-unitary $c = -2$ CFT. We believe that at least some of the results obtained on this way should be universal and applicable to other non-unitary CFT's.

The model of dense polymers actually has a long history, dating back more then a century, when Kirchhoff proved a beautiful theorem that the number of one-components spanning trees (polymers) on the lattice of N sites is given by the principal minors of the $N \times N$ matrix of discrete Laplacian [3,4]. Another fundamental result was due to Fortuin and Kasteleyn [5,6] who observed that the partition function Z_N of the q-component Potts model can be represented as a dichromatic polynomial that continuously depend on q. Although the partition function of the model vanishes in the formal limit $q \to 0$ owing to zero mode of the discrete Laplacian, its derivative with respect to q does not and gives the partition function of one-component spanning trees.

The purpose of the Letter is to show that:

(i) The $q \to 0$ limit of the Potts model can be carried on in two steps. The first, $\lambda \to 0$, leads to the model of lattice polymers with arbitrary number of components γ; the second, $\kappa \to 0$, to their dense phase. Although the partition function of the model again vanishes in the limit, some "antisymmetric" 2γ-point correlation functions survive.

(ii) These correlation functions are given exactly by the minors of rank $(N - \gamma)$ of the Laplacian matrix. These can be rewritten in terms of integrals over anti-commuting variables and in continuous limit coincide with the correlation functions of the free complex Grassmann field.

(iii) The vacuum vector of the field theory have to be defined as having zero norm. The fundamental property of the Grassmann field and its twist field (both are primary with non-positive conformal weights) is that their operator products define the Dirichlet operator that suppresses zero mode and imposes proper boundary conditions for the primary fields with positive conformal weights. This does not change other basic principles of CFT and leads to the logical and self-consistent theory.

Dense Phase of Lattice Polymers.—Let lattice $\mathcal{L}$ has N sites labeled $1, 2, ..., N$. With each site i we associate a spin variable σ_i which can take q values, say $1, 2, ..., q$.

Then the average of any operator $\mathcal{A}(\sigma)$ in the q-component Potts model we define as (without normalization factor!)

$$\langle \mathcal{A}(\sigma) \rangle = \sum_{\sigma} \mathcal{A}(\sigma) \exp \left\{ \beta J \sum_{(ij)} \delta(\sigma_i, \sigma_j) \right\} . \qquad (1)$$

Here the σ-summation is over all the spins $\sigma_1, ..., \sigma_N$; the second summation is over all edges of the lattice. It has been shown that Z_N can be expressed as a dichromatic polynomial [5,6]. To fix notations we briefly repeat the derivation of the result. Set $v = \exp(\beta J) - 1$, then the partition function can be rewritten as

$$Z_N = \langle 1 \rangle = \sum_{\sigma} \prod_{(ij)} [1 + v\delta(\sigma_i, \sigma_j)] . \qquad (2)$$

Let E be the number of edges of the lattice $\mathcal{L}$. Then the summand in Eq.(2) is a product of E factors. Each factor is the sum of two terms: 1 and $v\delta(\sigma_i, \sigma_j)$, so the product can be expanded as the sum of 2^E terms.

Each of these 2^E terms can be associated with a bond-graph on the lattice $\mathcal{L}$. To do this, note that the term is the product of E factors, one for each edge. The factor for edge (ij) is either 1 or $v\delta(\sigma_i, \sigma_j)$: if it is the former, leave the edge empty, if the later, place a bond on the edge. Do this for all edges (ij). We then have a one-to-one correspondence between bond-graphs on $\mathcal{L}$ and terms in the expansion of the product in Eq.(2).

Consider a typical bond-graph $\mathcal{G}$, containing N sites, L bonds, γ connected components and ω internal cycles. These are not independent, but must satisfy Euler's relation

$$L + \gamma = N + \omega. \tag{3}$$

Then the corresponding term in the expansion contains a factor v^L, and the effect of delta functions is that all sites within a component must have the same spin σ. Summing over all independent spins and over all bond-graphs $\mathcal{G}$ that can be drawn on $\mathcal{L}$ we obtain [5,6]

$$Z_N = \sum_{\mathcal{G}} q^\gamma v^L. \tag{4}$$

Note that here q need not be an integer. We can allow it to be any real number and, in particular, to consider formal limit $q \to 0$. Since we are going to deal with not only one- but arbitrary γ-component spanning trees, we have to treat the limit in a way different from [5].

At first we consider the limit $\lambda, q, v \to 0$ while $\kappa = q/\lambda$ and $x = v/\lambda$ remain finite. As a result we obtain the partition function of lattice polymers

$$\tilde{Z}_N = \lim_{\lambda\to 0} \lambda^{-N} Z_N = \lim_{\lambda\to 0} \sum_{\mathcal{G}} \kappa^\gamma \lambda^\omega x^L = \sum_{\mathcal{T}} \kappa^\gamma x^L. \tag{5}$$

Here the last summation is over all bond-graphs $\mathcal{T}$ that has no internal cycles, i.e. $\omega = 0$. Such graphs are usually called spanning trees (polymers). The number of bonds L of the spanning tree is related to the number of its components γ as $L = N - \gamma$. Hence, the partition function can be rewritten as

$$\tilde{Z}_N = \sum_{\gamma} \kappa^\gamma \sum_{\mathcal{T}_\gamma} x^L = \sum_{\gamma} \mathcal{N}_\gamma \kappa^\gamma x^{N-\gamma}, \tag{6}$$

where symbol $\mathcal{T}_\gamma$ denotes the set of different γ-component spanning trees and $\mathcal{N}_\gamma$ is their total number. To simplify further notations we take $x \equiv 1$ without loss of generality.

The second limit $\kappa \to 0$ leads to the so-called dense phase of the polymer model. Since $\gamma \geq 1$ the partition function (6) obviously tends to zero in this limit. Nevertheless, the correlation functions do not necessarily vanish. Indeed, repeating all the steps leading to Eq. (6) one can calculate the following correlation functions

$$\lim_{\kappa,\lambda\to 0} \langle 1 \rangle = 0, \tag{7a}$$

$$\lim_{\kappa,\lambda\to 0} \langle \delta_{kl} \rangle = \mathcal{N}_{(kl)} = \text{const}, \tag{7b}$$

$$\lim_{\kappa,\lambda\to 0} \left\langle \begin{vmatrix} \delta_{kl} & \delta_{kq} \\ \delta_{pl} & \delta_{pq} \end{vmatrix} \right\rangle = \mathcal{N}_{(kl)(pq)} - \mathcal{N}_{(kq)(pl)}, \tag{7c}$$

$$\vdots$$

Here $\delta_{kl} = (v/q)\delta(\sigma_k, \sigma_l)$; $\mathcal{N}_{(kl)}$ is the number of one-component spanning trees with both the sites k and l belonging to the same component (this number, obviously, does not depend on the position of the sites); $\mathcal{N}_{(kl)(pq)}$ is the number of two-component spanning trees with sites k, l belonging to one component and sites p, q to the other; etc. The antisymmetric combination of δ's in each 2γ-point correlation function is designed to guard against any contribution of spanning trees with the number of components less then γ (otherwise this would be divergent). So, only γ-component spanning trees contribute to the 2γ-point correlation function in the limit $\kappa \to 0$.

The importance of these correlation functions is justified by the following result.

Generalized Kirchhoff Theorem.—Given a lattice $\mathcal{L}$ with N sites labeled $1, 2, ..., N$, the $N \times N$ matrix of discrete Laplacian Δ_{ij} has the elements: Δ_{ii} =number of edges incident to i, Δ_{ij} = −number of edges with end points i and j. The minor $\Delta^{(k)(l)}$ of rank $(N-1)$ is obtained from the matrix Δ by deleting k-th column and l-th row; similarly, the minor $\Delta^{(kp)(lq)}$ of rank $(N-2)$ is obtained by deleting columns k, p and rows l, q; etc. Then

$$\det \Delta = 0, \tag{8a}$$

$$\det \Delta^{(k)(l)} = \mathcal{N}_{(kl)} = \text{const}, \tag{8b}$$

$$\det \Delta^{(kp)(lq)} = \mathcal{N}_{(kl)(pq)} - \mathcal{N}_{(kq)(pl)}, \tag{8c}$$

$$\vdots$$

Here one immediately recognizes the "antisymmetric" correlation functions (7). The standard proof of the original version of the theorem (first two lines of the sequence) can be found in Ref. [3]. Priezzhev [4] proposed an alternative proof of the original version in the spirit of the combinatorial solution of Ising model. His method is simpler and can also be generalized to prove all other lines of the sequence (8).

Free Complex Grassmann Field.—Using the matrix representation we can reinterpret the partition function of lattice polymers as being the partition function of some artificial statistical system. To this end we define at each site i of the lattice $\mathcal{L}$ the pair of anti-commuting variables θ_i and θ_i^* (its complex conjugate). Then, using Berezin's definition of the integral over anti-commuting variables [7] we can rewrite the determinant of the matrix Δ as

$$\det \Delta = \int d\theta_1^* ... d\theta_N \exp \sum_{ij} \theta_i^* \Delta^{ij} \theta_j \tag{9}$$

$$= \int d\theta_1^* ... d\theta_N \exp \sum_{ij} (\theta_i^* - \theta_j^*)(\theta_i - \theta_j).$$

In continuous limit this partition function defines field theory with the action

$$\mathcal{S}[\theta] = \frac{1}{4\pi} \int \partial_\mu \theta^* \partial^\mu \theta \; \mathrm{d}^2\mathbf{r}. \tag{10}$$

The average of any operator $\mathcal{A}[\theta]$ we define as

$$\langle\mathcal{A}[\theta]\rangle = \int [d\theta^* d\theta]\ \mathcal{A}[\theta]\ \exp -\mathcal{S}[\theta]. \qquad (11)$$

Then, although the average of the identity operator is equal to zero due to the presence of zero mode, all other correlation functions of the field θ are non-trivial and can be normalized so that

$$\langle 1\rangle = 0, \qquad (12a)$$
$$\langle\theta_1^*\theta_2\rangle = 1, \qquad (12b)$$
$$\langle\theta_1^*\theta_2^*\theta_3\theta_4\rangle = \ln\left(\eta_{34}^{12}\right), \qquad (12c)$$
$$\langle\theta_1^*\theta_2^*\theta_3^*\theta_4\theta_5\theta_6\rangle = \begin{vmatrix} \ln\left(\eta_{45}^{12}\right) & \ln\left(\eta_{56}^{12}\right) \\ \ln\left(\eta_{45}^{23}\right) & \ln\left(\eta_{56}^{23}\right)\end{vmatrix}, \qquad (12d)$$
$$\vdots$$

The field θ is scalar ant its correlation functions depend only on the projectively invariant cross-ratios

$$\eta_{34}^{12} = \left(\frac{r_{13}r_{24}}{r_{14}r_{23}}\right)^2, \cdots \qquad (13)$$

Here $\theta_1 \equiv \theta(\mathbf{r}_1)$; $r_{12} \equiv |\mathbf{r}_1 - \mathbf{r}_2|$. These correlation functions are nothing but asymptotics of the "antisymmetric" correlation functions (7) in the continuous limit.

The surprising thing is that the Grassmann field itself suppresses zero mode of the Laplacian operator. In spite of this unusual property it still can be considered as a primary conformal field with the weight $h_\theta = 0$.

The stress-energy tensor,

$$T(z) = :\partial\theta^*\partial\theta: = \lim_{w\to z}\left\{\partial\theta^*(z)\partial\theta(w) + \frac{1}{(z-w)^2}\right\}, \qquad (14)$$

satisfies standard operator product expansion,

$$T(z)T(w) = \frac{-1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \qquad (15)$$

One can assure himself by direct calculation that the stress-energy tensor is indeed the generator of conformal transformations in the sense that for any correlation function $\langle X\rangle = \langle\theta_1^*\ldots\theta_{2N}\rangle$ from the sequence (12) its transformation law is given by

$$\delta_\epsilon\langle X\rangle = \oint_C dz\ \epsilon(z)\ \langle T(z)X\rangle + \oint_C d\bar{z}\ \bar{\epsilon}(\bar{z})\ \langle\bar{T}(\bar{z})X\rangle. \quad (16)$$

The correlation functions of the field θ satisfy the third-order differential equation coming from the condition of degeneration of the operator $(1,3)$ with the weight $h_{1,3} = 0$ on the third level. This equation actually becomes of the second order for the field $\partial\theta$ and, in its turn, coincides with the condition of degeneration of the operator $(2,1)$ with the weight $h_{2,1} = 1$ on the second level.

The twist field $\sigma(z,\bar{z})$ can be defined with the use of the standard operator product expansion

$$\partial\theta(z)\sigma(w,\bar{w}) \sim \frac{\tau(w,\bar{w})}{\sqrt{z-w}}. \qquad (17)$$

Alternatively, on the lattice it can be defined by means of the construction similar to that for the disorder operator in Ising model [8].

Conformal properties of the twist field σ are similar to those of the Grassmann field θ. Namely, its correlation functions are non-trivial even in the presence of zero mode. Its correlation functions can be found from the condition of degeneration of the operator $(1,2)$ with the weight $h_{1,2} = -1/8$ on the second level [8]

$$\langle\sigma_1\sigma_2\rangle = \sqrt{r_{12}}, \qquad (18a)$$
$$\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle = \pi\sqrt{r_{12}r_{34}}\sqrt{|\eta(1-\eta)|} \times \left\{F(\eta)\bar{F}(1-\bar{\eta}) + \bar{F}(\bar{\eta})F(1-\eta)\right\}, \quad (18b)$$

where $F(\eta) = {}_2F_1(\frac{1}{2},\frac{1}{2};1;\eta)$; and $\eta = (z_{13}z_{24})/(z_{12}z_{34})$. Mixed four-point correlation function of the fields θ and σ can also be found using standard techniques of CFT

$$\langle\theta_1^*\theta_2\sigma_3\sigma_4\rangle = 2\sqrt{r_{34}}\left\{H(\eta) + \bar{H}(\bar{\eta})\right\}. \qquad (19)$$

Here $H(\eta) = \ln\left(\sqrt{\eta} + \sqrt{\eta - 1}\right)$.

This means that both the Grassmann field θ and its twist field σ can be considered as primary conformal fields with the weights $h_\theta = 0$ and $h_\sigma = -1/8$ provided that the vacuum state has been defined as having zero norm. These fields are unique in having both the property and non-positive conformal weights.

Dirichlet Operator and Green Function.—There is a simple relation between the four-point correlation function (12c) and the Green function of the Laplacian operator. The most straightforward way to understand this is follows. Let us consider a conducting plane with a current $I = 1$ entering the plane at a point $\mathbf{r}_1$ and leaving it at a point $\mathbf{r}_2$. Then the voltage difference between sites $\mathbf{r}_3$ and $\mathbf{r}_4$ on the plane is given by the four-point function $\langle\theta_1^*\theta_2^*\theta_3\theta_4\rangle$.

The Green function of the Laplacian operator with the Dirichlet boundary conditions at the point $\mathbf{r}_0$ can be defined quite similarly. Consider the same conducting plane earthed at the point $\mathbf{r}_0$. This means that the voltage at this point is always maintained to be equal to zero. If a current $I = 1$ enters the plane at a point $\mathbf{r}_1$ (and leaves it at the earthed point $\mathbf{r}_0$) then the voltage at a site $\mathbf{r}_2$ is given by the Green function $G_0(\mathbf{r}_1,\mathbf{r}_2)$.

The operator $\mathcal{D}_0$ that corresponds to the earthed point $\mathbf{r}_0$ can, obviously, be considered as the product of the field θ_0 with its complex conjugate θ_0^* at the same point. We will call it the *Dirichlet operator* since it imposes the Dirichlet boundary conditions on the Grassmann field. With the help of this operator the Green function can be represented as

$$G_0(\mathbf{r}_1, \mathbf{r}_2) = \langle \mathcal{D}_0 \theta_1^* \theta_2 \rangle. \tag{20}$$

The Dirichlet operator can formally be defined through the following operator products

$$\mathcal{D}_0 = \lim_{1\to 0} \{\theta_0^* \theta_1\} = \lim_{1\to 0} \left\{ \frac{\sigma_0 \sigma_1}{\sqrt{r_{01}}} \right\}. \tag{21}$$

Its two-point correlation function can be found from the four-point functions Eqs. (12c,18b,19). However, one has to be careful merging different points of the four-point functions because they diverge logarithmically in the limit. These divergences have absolutely the same nature as those present in the Green function in the thermodynamic or continuous limit [7]. To treat them carefully let us first consider the correlation functions on the lattice with spacing a. This scale dictates minimal possible distance between different merging points. After arbitrary conformal transformation the lattice is no longer uniform and the area of any given fundamental square of the lattice acquires an additional factor proportional to the metric on the plane: $ds^2 = g(\mathbf{r})d\mathbf{r}^2$. Finally, we have (the factor $\sim a^2$ is absorbed into the metric!)

$$\langle \mathcal{D}_1 \rangle = 1, \quad \langle \mathcal{D}_1 \mathcal{D}_2 \rangle = \ln \frac{(r_{12})^4}{g_1 g_2}, \quad \ldots \tag{22}$$

Note, that the Dirichlet operator is scalar and its correlation functions are projectively invariant. This is natural since it has been defined as the product of scalar fields θ and θ^*. This also suits its interpretation as being the operator that determines boundary conditions.

The operator product of the Dirichlet operator with itself and with the operators θ and σ cannot be completely determined within the CFT. Indeed, merging different points of the correlation functions (12) we can only say that

$$\lim_{1\to 0} \{\theta_0 \mathcal{D}_1\} = k\ \theta_0, \tag{23a}$$

$$\lim_{1\to 0} \{\sigma_0 \mathcal{D}_1\} = k\ \sigma_0, \tag{23b}$$

$$\lim_{1\to 0} \{\mathcal{D}_0 \mathcal{D}_1\} = k\ \mathcal{D}_0, \tag{23c}$$

where k is some constant. From the point of view of the "electrical" interpretation of the operator given above the most natural choice would be $k = 1$.

As an example of the field with positive conformal weight let us consider correlation functions of the local energy operator

$$\varepsilon_0 = :\partial_\mu \theta^* \partial^\mu \theta: = \lim_{1\to 0} \{\partial_\mu \theta_0^* \partial^\mu \theta_1 - 4\pi\delta(r_{01})\}. \tag{24}$$

This is primary with conformal weight $h_\varepsilon = 1$. Its correlation functions can be found from Eqs. (12) and are all trivial, $\langle \varepsilon_1 \ldots \varepsilon_N \rangle = 0$, unless we insert the Dirichlet operator

$$\langle \mathcal{D}_0 \varepsilon_1 \varepsilon_2 \rangle = -\frac{8}{(r_{12})^4}, \tag{25a}$$

$$\langle \mathcal{D}_0 \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \rangle = \frac{64}{(r_{12} r_{34})^4} + \frac{64}{(r_{13} r_{24})^4} + \frac{64}{(r_{14} r_{23})^4}. \tag{25b}$$

This property is common to all primary operators with positive conformal weights.

Let us summarize the results of the Letter. It has been shown that the model of the free complex Grassmann field properly describes the continuous limit of the lattice model of dense polymers only provided its vacuum vector has been defined as having zero norm, $\langle 0|0\rangle = 0$. Nevertheless, it is this vacuum that has to be considered when one studies the correlation functions of the primary fields with non-positive conformal weights (θ and σ). These correlation functions, (12), imply the mode expansion

$$\theta(z, \bar{z}) = \chi_0 + 2\theta_0 \ln|z| - \sum_{n\neq 0} \left(\frac{\theta_n}{n} z^{-n} + \frac{\bar{\theta}_n}{n} \bar{z}^{-n} \right), \tag{26}$$

with the commutation relations

$$\{\chi_0^*, \chi_0\} = \Im, \quad \{\theta_n^*, \theta_m\} = \{\bar{\theta}_n^*, \bar{\theta}_m\} = n\ \delta_{n+m}, \tag{27}$$

where $\langle \Im \rangle = 1$ and $\langle \Im^2 \rangle = 0$. Operator $\Im$ is nothing but the coordinate-independent part of the Dirichlet operator. It defines yet another null vector, $|\star\rangle = \Im|0\rangle$. This can be normalized so that $\langle \star|0\rangle = 1$. Together these two vectors, $|0\rangle$ and $|\star\rangle$, define physical vacuum state for those primary fields that have positive conformal weights.

We conclude that the theory is non-trivial only due to the presence of two different null vectors that are not orthogonal to each other. This could be a general feature of other non-unitary CFT's.

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