

Introduction to the Sandpile Model

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This article is based on a talk given by one of us (EVI) at the conference "StatPhys-Taipei-1997". It overviews the exact results in the theory of the sandpile model and discusses shortly yet unsolved problem of calculation of avalanche distribution exponents. The key ingredients include the analogy with the critical reaction-diffusion system, the spanning tree representation of height configurations and the decomposition of the avalanche process into waves of topplings.

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I. INTRODUCTION

The concept of Self-Organized Criticality [1]; one of the most fundamental concepts of modern physics of non-equilibrium phenomena; actually has a long history, dating back about a half of the century, when Kolmogorov put forward his seminal theory of isotropic and homogeneous turbulence [2]. The cornerstone of his very simple, but surprisingly robust dimensional analysis, was the assumption that the fluid being driven by a random force evolves to a stationary state where the velocity correlation functions are universal and obey power laws, provided that the viscosity of the fluid is small enough to ensure the existence of the so-called inertial range of scales. It was assumed that within this inertial interval the energy is being transferring from large eddies to small ones without any dissipation whatsoever. Such a non-equilibrium but stationary state, with a constant flux of conserved quantities, is usually called the Flux State.

Later on, Kardar, Parisi and Zhang [3] retraced very nearly the same path on an absolutely different ground. Namely, they considered the simple type of growth exemplified by a vapor-deposition process, where the growth rate is locally determined by the flux of particles arriving ballistically at the surface. They found that this system evolves to the non-equilibrium stationary state where spatial and temporal fluctuations of the growing surface obey power laws.

After a careful analysis of all these rather phenomenological theories, Bak, Tang and Wiesenfeld [1] have squeezed the concept of Self-Organized Criticality out in its most purified form. They introduced a cellular automaton now commonly known as "sandpile" because of the crude analogy between its dynamical rules and the way sand topples when building a real sand pile.

The formulation of this model is given in terms of integer height variables z_i at each site of the square lattice \mathcal{L} . Particles are added randomly and the addition of a particle increases the height at that site by one. If this height exceeds the critical value $z_c = 4$, then the site topples, on toppling its height decreases by 4 and the heights at each of its nearest neighbors increases by 1.

These may become unstable in their turn and the relaxation process continues. This chain reaction propagates up to the moment when all sites become stable again. One assumes the updating to be done concurrently, with all sites updated simultaneously. Open boundary conditions are usually assumed, so that the toppled boundary site gives one particle to each of its three neighbors while one grain drops out of the system.

This system also evolves stochastically into a critical state with a constant flux of particles on which it exhibits properties similar to that of a second order phase transition [4]. It lacks therein any characteristic length or time scale and obeys power-law distributions. The critical properties of the flux state are independent of the initial configuration of the system and are determined completely by the flux rate. Unlike ordinary critical phenomena, no fine tuning of any other control parameters is necessary to arrive at this state.

II. MEAN-FIELD APPROXIMATION

A. Flux State

For our derivation of the mean-field equations it will be even more convenient to generalize the dynamical rules of the model [5]. Namely, let us suppose that at each site of the lattice one of the species A, B, C or D is living. These species represent respectively four stable heights of original model. Then, due to the external driving force, some new particles (we will call them φ) are added randomly into the system initiating avalanches. Actually, φ are the only mobile species in this model and the avalanche process can be described in terms of their propagation through the correlated media of immobile species A, B, C and D. In this propagation the φ -species mutate all others in the sites they have visited and topple the critical ones according to the following rules



$$D + \varphi \rightarrow \begin{cases} p_1 : D + \bar{\varphi} \\ p_2 : C + 2\bar{\varphi} \\ p_3 : B + 3\bar{\varphi} \\ p_4 : A + 4\bar{\varphi}. \end{cases}$$

Here φ and $\bar{\varphi}$ denote the particles obtained by the site and the particles transferred to the neighboring sites after toppling, respectively. These processes can formally be reinterpreted as an irreversible chemical reaction which takes place at each site of the lattice or, in other words, as a branching process which is characterized by the so-called branching probabilities

$$\mathbf{p} = (p_1, p_2, p_3, p_4), \quad p_1 + p_2 + p_3 + p_4 = 1. \quad (2)$$

This generalized model also describes the critical flux state where the matter is conserved. It (obviously) corresponds to the total conservation of particles in the original formulation of the sandpile model. The latter can be reproduced exactly if we choose $\mathbf{p} = (0, 0, 0, 1)$.

The simplest possible description of the critical flux state can be achieved within the so-called mean-field approximation. There are a lot of different methods to calculate mean-field critical exponents [6–9]. All of them are simple enough and have many features in common that, when applied to our model of chemical reactions, can be summarized as follows.

At first, we introduce the concentrations of species A, B, C and D

$$\mathbf{n} = (n_A, n_B, n_C, n_D), \quad n_A + n_B + n_C + n_D = 1. \quad (3)$$

The normalization condition comes from the constraint that we always have only one species in each given site.

Then, following the standard prescriptions of chemical physics, we can write kinetic equations corresponding to this scheme of chemical reactions

$$\dot{n}_A = n_\varphi (p_4 n_D - n_A), \quad (4a)$$

$$\dot{n}_B = n_\varphi (p_3 n_D + n_A - n_B), \quad (4b)$$

$$\dot{n}_C = n_\varphi (p_2 n_D + n_B - n_C), \quad (4c)$$

$$\dot{n}_D = n_\varphi (p_1 n_D + n_C - n_D), \quad (4d)$$

$$\dot{n}_\varphi = n_\varphi (\bar{p} n_D - 1) + \bar{p} \nu \nabla^2 (n_\varphi n_D) + \eta(\mathbf{r}, t) \quad (4e)$$

where $\bar{p} = p_1 + 2p_2 + 3p_3 + 4p_4$ is equal to the average number of particles φ leaving the cell on toppling and \mathbf{r} is the position vector of the site in the 2D space. The noise term $\eta(\mathbf{r}, t)$, being non-negative, mimics the random addition of particles to the system. The diffusion term $\nabla^2 (n_\varphi n_D)$ describes the transfer of particles into the neighboring cells, and the diffusion coefficient ν for the discrete Laplacian on the square lattice is equal to $1/4$.

The physical meaning of these equations is transparent. When the concentration n_φ of species φ is equal to zero, all toppling processes die. Then, due to the noise term $\eta(\mathbf{r}, t)$, the particles are added randomly into the

system initiating a branching process directed towards the open boundary of the system. This process mutates species in the cells it has visited and topples the critical ones. Finally, the system will reach the steady state where the probability that the activity will die is on average balanced by the probability that the activity will branch. Thus, the chain reaction maintains this stationary state and all further avalanches cannot change the concentrations of species A, B, C, and D. Therefore, the steady state is characterized by the conditions that

$$\dot{n}_A = \dot{n}_B = \dot{n}_C = \dot{n}_D = 0 \quad (5)$$

and Eqs. (4a-4d) lead to the following relations between concentrations of species \mathbf{n} at the stationary state and branching probabilities \mathbf{p}

$$n_A^* = p_4 / \bar{p}, \quad (6a)$$

$$n_B^* = (p_3 + p_4) / \bar{p}, \quad (6b)$$

$$n_C^* = (p_2 + p_3 + p_4) / \bar{p}, \quad (6c)$$

$$n_D^* = (p_1 + p_2 + p_3 + p_4) / \bar{p} = 1 / \bar{p}. \quad (6d)$$

Here, within the mean-field approximation, we obviously neglected all spatial and temporal fluctuations of concentrations \mathbf{n} .

B. Branching Process

The next question is how the avalanche process can be described within the same approximation. It can naturally be represented as the critical branching process. To describe it we first subdivide the total number of sites toppled in the avalanche into the four terms: N_A , N_B , N_C and N_D . These correspond respectively to the numbers of sites that contain species A, B, C and D after toppling in the avalanche. Their sum $N = N_A + N_B + N_C + N_D$ corresponds to the total number of topplings in the avalanche at the moment t .

Then, we introduce the probability

$$P_t(N_A, N_B, N_C, N_D), \quad (7)$$

of having such numbers after t time steps of avalanche process and the corresponding generating function

$$G_t(\lambda_A, \lambda_B, \lambda_C, \lambda_D) = \sum_N P_t(N_A, N_B, N_C, N_D) \lambda_A^{N_A} \lambda_B^{N_B} \lambda_C^{N_C} \lambda_D^{N_D}. \quad (8)$$

Initially, at the moment $t = 0$, the only nonzero element of the probability distribution is $P_0(0, 0, 0, 0) = 1$, which correspond to the value $G_0 = 1$ of the generating function.

On the next time step, after the toppling of the first site, all nonzero probabilities are follows:

$$P_1(1, 0, 0, 0) = p_4 n_D, \quad (9)$$

$$P_1(0, 1, 0, 0) = p_3 n_D + n_A, \quad (10)$$

$$P_1(0, 0, 1, 0) = p_2 n_D + n_B, \quad (11)$$

$$P_1(0, 0, 0, 1) = p_1 n_D + n_C, \quad (12)$$

with corresponding generating function

$$G_1(\lambda_A, \lambda_B, \lambda_C, \lambda_D) = \quad (13)$$

$$(n_D p_4) \lambda_A + (n_D p_3 + n_A) \lambda_B + \quad (14)$$

$$(n_D p_2 + n_B) \lambda_C + (n_D p_1 + n_C) \lambda_D, \quad (15)$$

Now, it is easy to check directly that the generating function obeys the following recursion relation

$$G_{t+1} = (n_A \lambda_B + n_B \lambda_C + n_C \lambda_D) \quad (16)$$

$$+ n_D (p_1 \lambda_D G_t + p_2 \lambda_C G_t^2 + p_3 \lambda_B G_t^3 + p_4 \lambda_A G_t^4).$$

In the limit of large avalanches we believe that this generating function tends to some universal function $G_\infty(\lambda_A, \lambda_B, \lambda_C, \lambda_D)$. Thus, (16) becomes a closed equation for this function. This equation depends on two set of parameters \mathbf{n} and \mathbf{p} . In the critical state, however, these are not independent. The relation between them can easily be found if we note that “below” the critical point, when all avalanches are finite, the average number of species A, B, C and D in the toppled sites can be simply related to the generating function

$$\langle N_A \rangle = \left[\frac{\partial}{\partial \lambda_A} G_\infty(\lambda_A, \dots) \right]_{\lambda=1} = \frac{p_4 n_D}{1 - n_D \bar{p}} \quad (17a)$$

$$\langle N_B \rangle = \left[\frac{\partial}{\partial \lambda_B} G_\infty(\lambda_B, \dots) \right]_{\lambda=1} = \frac{n_A + p_3 n_D}{1 - n_D \bar{p}} \quad (17b)$$

$$\langle N_C \rangle = \left[\frac{\partial}{\partial \lambda_C} G_\infty(\lambda_C, \dots) \right]_{\lambda=1} = \frac{n_B + p_2 n_D}{1 - n_D \bar{p}} \quad (17c)$$

$$\langle N_D \rangle = \left[\frac{\partial}{\partial \lambda_D} G_\infty(\lambda_D, \dots) \right]_{\lambda=1} = \frac{n_C + p_1 n_D}{1 - n_D \bar{p}} \quad (17d)$$

As expected, these numbers become divergent at the critical state. Calculating the concentrations of species A, B, C and D after the avalanche and equating them to the concentrations before the avalanche

$$\frac{\langle N_A \rangle}{\langle N_A \rangle + \langle N_B \rangle + \langle N_C \rangle + \langle N_D \rangle} = n_A, \dots \quad (18)$$

we will reproduce again the relations (6), thus proving the self-consistency of this mean-field description.

It is hardly possible to solve the fourth order equation (16) explicitly. We can simplify it further (still within the mean-field approximation) if we take into account the fact that the average number of particles leaving the site on toppling is equal to \bar{p} and each generates on average equal sub-avalanches. Then, our equation for the universal generating function becomes

$$G_\infty(\lambda) = \lambda((1 - n_D) + n_D [G_\infty(\lambda)]^{\bar{p}}), \quad (19)$$

where we put $\lambda_A = \lambda_B = \lambda_C = \lambda_D \equiv \lambda$. A similar recursion relation naturally appears in any mean-field description of the avalanche process [6–9].

In the simplest case $\bar{p} = 2$ the solution of equation (19) can be written in the closed form

$$G_\infty(\lambda) = \frac{1 - \sqrt{1 - 4\lambda^2 n_D (1 - n_D)}}{2\lambda n_D}. \quad (20)$$

which, after expanding it as a series in λ , leads to the well known mean-field probability distribution for the sizes of avalanches at the critical state

$$P(N) \sim N^{-3/2}. \quad (21)$$

Actually, this asymptotic behavior does not depend on the value of \bar{p} and the mean-field exponent $3/2$ is universal for any critical branching process [11].

III. SPANNING TREES IN THE SANDPILE MODEL

A. Basic Properties

Now let us go back to the original formulation of the sandpile model. It has been shown by Dhar [12] that this model is actually exactly solvable.

To derive these results it is convenient to reformulate the dynamical rules of the sandpile model as follows. We consider the model on a square lattice \mathcal{L} of N sites labeled by $1, 2, \dots, N$. Each boundary site is connected by a bond to the additional site \star (the *root*) which plays the role of the sink for the particles. This auxiliary site, although unphysical, will be very convenient for all our further constructions. The $N \times N$ matrix of the discrete Laplacian Δ_{ij} has non-zero diagonal elements Δ_{ii} equal to the number of neighboring sites of i and non-zero off-diagonal elements $\Delta_{ij} = -1$ for all pairs of adjacent sites i and j . The addition of sand corresponds to increasing the height of the pile by unity at a site of the lattice chosen at random (except \star). If the height at any site i exceeds its critical value Δ_{ii} , that site topples and all the variables z_j , ($j = 1, \dots, N$) are updated according to the rule

$$z_j \rightarrow z_j - \Delta_{ij} \quad (22)$$

The process stops when all the heights of the resulting configuration $\psi = \{z_i\}$ do not exceed their critical values. Such a configuration is called a stable configuration.

Yet, it is not clear whether the dynamics of the model is well defined because during the toppling process a configuration may occur with two or more unstable sites. We have to make sure that the resulting stable configuration does not depend on the order of their topplings. This can easily be verified if we note that after the toppling of two

unstable sites i and j in an arbitrary order one gets a configuration in which height z_k at any site k of the lattice \mathcal{L} decreases by $\Delta_{ik} + \Delta_{jk}$. This expression is invariant under the exchange of i and j . By a repeated use of this argument, one obtains that after any avalanche the same final stable configuration is produced irrespectively of the order of topplings of unstable sites [12].

Another possible ambiguity is due to the fact that the updating procedure may in principle enter a nontrivial infinite cycle. Nevertheless, one can easily prove that this is impossible by noting that topplings in the interior of the lattice does not change the total amount of sand in the system and every toppling on the boundary decreases this value. Here, no cycle can have topplings at the boundary. Next, the sand on the boundary will monotonically increase if there is any toppling one site away. This cannot happen in the infinite cycle, thus there can be no topplings one site away from the edges. By induction, there can be no topplings at an arbitrary distance from the boundary, thus, there can be no infinite cycle [13].

The Markovian evolution of the model implies that the set of all stable configurations S can be divided into two subsets: recurrent R and transient $S \setminus R$. By definition, the first subset includes those stable configurations that can be reached from all others by sequential addition of particles. This subset is not empty because there is at least one such a configuration $\psi_0 = \{z_i = \Delta_{ii}\}$ with all the heights equal to their critical values. Then, the very general result of the theory of Markovian chains states that once the system gets into the set of recurrent configurations, it will never get out under the further evolution. All non-recurrent stable configurations are usually called transient.

To calculate the number of recurrent configurations [12], let us consider the space of all possible (including unstable) configurations obtainable from ψ_0 by addition of particles. Then, we define two such configurations $\{z_i\}$ and $\{z'_i\}$ as equivalent if and only if under topplings they evolve to the same stable configuration. This means that there exists some integers r_j such that

$$z'_i = z_i - \sum_{j=1}^N r_j \Delta_{ji}, \quad \text{for all } i. \quad (23)$$

Thus, if we represent configurations $\{z_i\}$ by points in a N -dimensional hypercubical lattice with basis vectors \bar{e}_i , the set of equivalent points forms a super-lattice with basis vectors

$$\bar{E}_i = \sum_{j=1}^N \Delta_{ij} \bar{e}_j, \quad i = 1 \text{ to } N. \quad (24)$$

Since every class of equivalent configurations corresponds to some unique recurrent configuration, the volume of the unit cell of the super-lattice must be equal to the number of distinct recurrent configurations. Thus one gets

$$\mathcal{N}_R = \det \Delta. \quad (25)$$

It is important to note that, although the geometrical shape of the set R in this N -dimensional space is quite nontrivial, the copies of R can be arranged to give a simple periodic tilling of the space. In other words, the set R of all recurrent configurations has the topology of a N -dimensional torus. Therefore, the addition of a particle to an arbitrary site i of the lattice can be represented by a jump of the point on this torus along the corresponding unit vector \bar{e}_i (in the positive direction). Then, if one drops particles onto different sites of the lattice with equal probabilities, one forces the representing point on the torus to move randomly (but always in the positive directions of the axes) from one site of the torus to another. Once this random motion covers the torus uniformly, all the recurrent configurations will appear in the Markovian evolution of the system with equal probability.

B. Forbidden Sub-Configurations and Spanning Trees

To describe in detail the set of recurrent configurations, we introduce first the important concept of *forbidden sub-configuration* [14]. It is defined as an arbitrary subset of sites of the lattice $\mathcal{F} \subseteq \mathcal{L}$ such that all its corresponding height variables $\{z_j\}$, $j \in \mathcal{F}$, satisfy the inequalities

$$z_i \leq \deg_{\mathcal{F}}(i) \equiv \sum_{j \in \mathcal{F}, j \neq i} (-\Delta_{ij}), \quad \text{for all } i \in \mathcal{F}, \quad (26)$$

where we denote by $\deg_{\mathcal{F}}(i)$ the number of bonds connecting the site i with the other sites of the subset \mathcal{F} .

Any stable height configuration that contains no forbidden sub-configurations is called an allowed configuration. We will prove soon that the set of all allowed configurations A is the same as the set of all recurrent configurations R .

First, let us show that the set of allowed configurations is closed under the dynamics of the sandpile model [12]. In other words, we want to show that once the system gets into the set of allowed configurations it will never get out. Assume the contrary. We then note that the addition of particles only increases heights and, hence, cannot create a forbidden sub-configuration. This means that there exists an allowed configuration ψ such that after a single toppling it becomes the configuration ψ' which contains a forbidden sub-configuration \mathcal{F} . Suppose the toppling occurs at site $i \in \mathcal{F}$. From the toppling rules and the definition of forbidden sub-configuration one gets that if \mathcal{F} is a forbidden sub-configuration in ψ' , then the set $\mathcal{F} \setminus i$ (obtained from \mathcal{F} by deleting the site i) is forbidden in the initial configuration ψ . This contradicts our assumption that ψ is allowed. Finally, since the recurrent configuration ψ_0 is allowed and all other recurrent

configurations are reachable from this particular one, it follows that all recurrent configurations are allowed

$$R \subseteq A \text{ and } \mathcal{N}_R \leq \mathcal{N}_A. \quad (27)$$

Another important notion is that of *toppling from the root*. It gives a simple recursive procedure determining whether a given stable configuration ψ is allowed.

To organize such a toppling, we topple the auxiliary site \star as if it were an ordinary site of the lattice. We could equivalently define it in terms of the simultaneous dropping of particles onto each boundary site of the lattice (i.e. one particle at each site on the edges and two particles onto each corner site). If no boundary sites become unstable then the original configuration ψ is not allowed. Using the definition one can easily show that in this case the lattice itself plays the role of forbidden sub-configuration: $\mathcal{F} = \mathcal{L}$. More generally, if the avalanche initiated by the toppling from the boundary has stopped after a few topplings and some set \mathcal{F} of lattice sites remains untoppled, then this set \mathcal{F} is nothing but the forbidden sub-configuration associated to the initial configuration ψ .

In such a toppling no lattice sites can topple more than once [15]. To prove this, let us assume that some site $i \in \mathcal{L}$ has toppled for the first time after its nearest neighbor j . Then, i would topple for the second time only after the topplings at all its neighbors, including j . Therefore, to topple i twice, we have first to topple the neighboring site j twice. Using the same arguments for j and for sites that toppled earlier than j , we conclude that to topple i more than once, we have to topple some boundary sites a second time. This is obviously impossible due to the loss of particles after the first toppling of boundary sites.

Thus, we have proved that the initial configuration ψ is allowed if and only if the toppling from the root generates an avalanche process under which each site of the lattice topples only once and, hence, the height configuration remains unchanged. If the initial configuration is not allowed, then some untoppled sites will survive and will form a forbidden sub-configuration.

It is possible to visualize the avalanche process initiated at the root. To this end let us consider the dynamical process step by step. Initially only some of the sites at the boundary can become unstable. They will topple at the moment $t = 1$ and transfer particles to neighbors that can become unstable in their turn. Similarly, all sites unstable at the moment t topple simultaneously and produce new unstable sites that will topple at the next time step $t + 1$. Consider an arbitrary site i . Let t be the time step at which this site becomes unstable and $t + 1$ the moment at which it topples. This means that at least one of its nearest neighbors toppled at the time step t . Let ξ be the number of such neighbors that toppled at the moment t . The height z_i should obey the following inequalities

$$\Delta_{ii} - \xi < z_i \leq \Delta_{ii}, \quad (28)$$

because otherwise it could not topple at the moment $t + 1$ as was assumed. Now, if $\xi = 1$, we simply mark in red the only bond connecting the site i with the neighboring site that toppled at the earlier moment t . In the other case, when $\xi > 1$, we select from these ξ possibilities only one bond to mark in red, dependent on the height z_i at site i . To this end we create an order of preferences by enumerating all bonds incident to the site i in an arbitrary but fixed manner. We then choose from the ξ candidates the bond that occupies the $(z_i + \xi - \Delta_{ii})$ -th position in this list of preferences. For example; if the northward bond of the site i has been allocated the number 1; the eastward 2; the southward 3; and the westward 4 and if the two neighboring sites of i at the north and at the south topple at the time step t , then if $z_i = 4$, we mark the southward bond (numbered by 3) red as it is the greater of the two in the list of preferences. This algorithm makes it possible to avoid any ambiguity in the choice of red bonds and the construction of a unique graphical representation of a given allowed configuration.

As all sites of the allowed configuration must topple, the graph \mathcal{T}_\star formed by the red bonds will cover all sites of the lattice. Such a graph is usually called the *spanning graph*. The fact that each site of the lattice has toppled only once implies that the spanning graph \mathcal{T}_\star contains no loops. Such a graph is called a *spanning tree*. A spanning tree having one site (the *root* \star) distinguished from all others is called a *rooted spanning tree*. Since there is only one path from any site of the rooted spanning tree to the root, we may uniquely orient this path so that all of its red bonds will be supplied with arrows in the direction of the root [17].

It is not difficult to check that, given a rooted spanning tree on the lattice, one can easily reconstruct the height configuration using the same list of preferences as above.

Thus, starting with the definition of forbidden sub-configurations, we have proved the one-to-one correspondence between allowed sandpile configurations on the lattice and rooted spanning trees on the same lattice [14].

C. Correlation Functions and the Continuous Limit

The spanning tree is actually a strongly correlated object. This explains why the absolutely uncorrelated heights of an arbitrary initial configuration become correlated in the recurrent state during the course of relaxation. The tree analogy provides a useful representation for the determination of the statistical properties of the sandpile model in the critical state. The tool used to enumerate rooted spanning trees is given by the famous

Kirchhoff's theorem [17]. If to any bond of the lattice \mathcal{L} , whose adjacent sites i and j are different from the

root, the weight x_{ij} is ascribed, then the determinant of the matrix

$$\Delta_{ij}(x) = \begin{cases} \sum_k x_{ik} & , \text{ if } i = j \\ -x_{ij} & , \text{ if } |i - j| = 1 \\ 0 & , \text{ otherwise.} \end{cases} \quad (29)$$

is a partition function of the rooted spanning trees on the lattice.

The fact that we treat here the weights x_{ij} and x_{ji} as different variables implies that we consider the arrow outgoing from site i and directed towards j as different from its opposite. When $x_{ij} = 1$ for all adjacent sites i and j this matrix coincides with the discrete Laplacian and its determinant gives the total number of rooted spanning trees. Hence the number of distinct allowed configurations is given by

$$\mathcal{N}_A = \det \Delta. \quad (30)$$

The equivalence between the set A of all allowed configurations and the set R of all recurrent configurations now follows immediately from Eqs.(25,27,30).

Using the Kirchhoff's theorem one can determine, in principle, all of the correlations between different branches of spanning trees. Although the method of such a calculation is far from being novel, it is still worth recalling the principal ideas.

Any modification of the weights of a finite number of lattice bonds is called a local defect of the lattice. For example deleting the bonds or inserting additional ones can be considered as a proper local defect. The difference between a discrete Laplacian of the new lattice Δ' and that of the initial one Δ is referred to as the defect matrix δ . The locality condition simply implies that only a finite number of the rows and columns of the defect matrix δ have non-zero elements. It follows that the ratio of the number of spanning trees on the new lattice \mathcal{L}' to the number of all trees on the original one \mathcal{L} is given by an easily calculable determinant

$$\text{Prob}(\delta) = \frac{\det \Delta'}{\det \Delta} = \det(\mathbf{1} + G\delta) \quad , \quad (31)$$

where $\mathbf{1}$ is the unit matrix and $G = \Delta^{-1}$ is the lattice Green's function. On the planar square lattice the Green's function only depends on the distance between the sites i and j and has the following integral representation [18]

$$G(n, m) - G(0, 0) = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos n\alpha \cos m\beta - 1}{2 - \cos \alpha - \cos \beta} d\alpha d\beta. \quad (32)$$

Here n and m are the numbers of columns and rows between the sites i and j respectively. This integral can be calculated explicitly with the results

$$G(0, 1) - G(0, 0) = G(1, 0) - G(0, 0) = -\frac{1}{4}, \quad (33)$$

$$G(n, n) - G(0, 0) = -\frac{1}{\pi} \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} \right). \quad (34)$$

Using these values and the discrete Laplace equation as a recursion relation, one can find all other elements of the matrix G_{ij} .

For example, let us find the probability of having the height $z_i = 1$ at some site i deep within the lattice [19]. One can notice that decreasing the height at the site i by 1 — so that its height becomes equal to 0 — one ends up the forbidden sub-configuration that consists of only the site i itself. In the language of spanning trees this corresponds to those trees that cover site i by a leaf bond (by deleting the bond site i becomes disconnected from the rest of the tree). Due to the equivalence of the 4 positions of the leaf bond, the unit height probability we are interested in can be expressed as

$$\text{Prob}(z_i = 1) = \frac{1}{4} \frac{\mathcal{N}'}{\mathcal{N}} = \frac{1}{4} \frac{\det \Delta'}{\det \Delta}. \quad (35)$$

Here \mathcal{N}' is the number of spanning trees on the new lattice \mathcal{L}' where all the bonds connecting the site i with its four nearest neighbors are deleted and, instead, the site i is directly connected to the root \star . Labeling the nearest neighbors of the site i in a clockwise fashion as N , E , S and W , we can write the corresponding defect matrix

$$\delta = \begin{matrix} & i & N & E & S & W \\ \begin{matrix} i \\ N \\ E \\ S \\ W \end{matrix} & \begin{pmatrix} -3 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix} \end{matrix} \quad (36)$$

Direct evaluation of the determinant Eq.(31) then leads to the result

$$\text{Prob}(z_i = 1) = \frac{2}{\pi^2} \left(1 - \frac{2}{\pi} \right) \approx 0.073 \ 636. \quad (37)$$

The calculation of all other height probabilities requires more sophisticated ideas and can be found in [20].

Similarly, we could calculate the asymptotic probability that sites i and j (both deep within the lattice) separated by the distance r will both have height 1 in the recurrent state. It is more instructive however to employ another approach to find the asymptotics of this two-point correlation function.

Namely, we can reinterpret the partition function of the spanning trees (given by Kirchhoff's theorem) as being the partition function of some artificial statistical system. To this end we define at each site i of the lattice \mathcal{L}

the pair of anticommuting variables θ_i and θ_i^* (its conjugate). Then, using Berezin's definition of the integral over anticommuting variables [21] we can rewrite the determinant of the matrix Eq.(29) as

$$Z = \det \Delta^{ij} = \int d\theta_1 \dots d\theta_N^* \exp \sum_{ij} \theta_i^* \Delta^{ij} \theta_j. \quad (38)$$

In the continuous limit this partition function defines a conformal field theory with the central charge $c = -2$ and the Hamiltonian (after integration by parts)

$$\mathcal{H} = \int \partial_\mu \theta^* \partial^\mu \theta \, d^2 \mathbf{r}. \quad (39)$$

The Green function of the field θ and its conjugate θ^* coincides with the asymptotic behavior of the lattice Green functions G_{ij}

$$\langle \theta^*(\mathbf{r}_1) \theta(\mathbf{r}_2) \rangle = G(\mathbf{r}_1 - \mathbf{r}_2) = G(\mathbf{0}) - \frac{1}{2\pi} \ln |\mathbf{r}_1 - \mathbf{r}_2|. \quad (40)$$

Changing the weight of a given directed bond in Eq.(38), say x_{ij} , and then again taking the continuous limit we will get the current operator corresponding to the fixed arrow on the spanning tree

$$j_\mu = \theta^* \partial_\mu \theta. \quad (41)$$

Hence we can immediately determine the asymptotic behavior of the arrow-arrow correlation function on the spanning tree

$$\langle j_\mu(\mathbf{r}) j_\nu(\mathbf{0}) \rangle = \langle j_\mu \rangle \langle j_\nu \rangle + \frac{1}{4\pi^2} \frac{x_\mu x_\nu}{r^4}. \quad (42)$$

Similarly, using the defect matrix δ (36), we can define the local energy operator

$$\varepsilon = \partial_\mu \theta^* \partial^\mu \theta \quad (43)$$

and calculate its two-point correlation function

$$\langle \varepsilon(\mathbf{r}) \varepsilon(\mathbf{0}) \rangle = \langle \varepsilon \rangle^2 - \frac{1}{4\pi^2} \frac{2}{r^4}. \quad (44)$$

This formula gives the asymptotic behavior of the two-point correlation function of unit heights in the sandpile model [19] (up to proper normalization of the local energy operator in the continuous limit). Moreover, on the boundary of the lattice only this operator determines the correlations of all other heights [22,23].

D. Waves of topplings

The study of avalanches requires a further extension of the spanning tree representation. In order to make it

possible we have to reorganize the topplings inside the avalanche into successive *waves of topplings* [15]. As has already been mentioned, the dynamics of the sandpile model admits an arbitrary order of topplings of unstable sites during an avalanche. We choose a particular — but quite natural — order amongst all others. Namely, let us drop a particle onto the critical site i in an allowed configuration ψ . We topple it once and then topple all sites that become unstable keeping the initial site i out of the second toppling. We call the set of sites toppled in this way *the first wave of topplings*, denote it \mathcal{F}_1 . After the first wave has gone out we topple the site i a second time and continue the avalanche, never permitting this site to topple a third time. The set of relaxed sites in the period after the first wave is called *the second wave of topplings*, denote it \mathcal{F}_2 . The process continues producing intermediate configurations ψ_1, ψ_2, \dots , until the site i undergoes the maximum number of topplings and the avalanche stops.

In complete analogy with the case of *toppling from the root*, the sites of any set \mathcal{F}_k topple only once during the k -th wave. The set \mathcal{F}_k has no holes: otherwise the subset of sites corresponding to the hole would form a forbidden sub-configuration of the initial recurrent configuration ψ . The compact cluster \mathcal{F}_k of sites toppled in the k -th wave forms the forbidden sub-configuration \mathcal{F}_k of the configuration ψ_k . Indeed, the inequality (26) holds for the height of every site in the cluster provided that all other sites of the lattice remain untoppled during that wave.

Thus, the avalanche starting at i is represented as a collection of T waves, $T \geq 1$, which we also denote by \mathcal{F}_k , ($1 \leq k \leq T$). All waves with numbers $1, \dots, T-1$ have the site i strictly inside \mathcal{F}_k . The cluster \mathcal{F}_T of the last wave, T , has the site i on its boundary. Indeed, the avalanche can stop only if the site i has at least one neighboring site not belonging to the last wave and therefore giving no contribution to the growth of the height z_i in ψ_T .

To find the tree representation of waves [15], we consider the sandpile model on a new lattice that consists of the original one \mathcal{L} and the additional bond ($\star i$) connecting a given site i with the root \star . Accordingly, we have to change one element of the matrix of the discrete Laplacian which determine the toppling rules

$$\Delta_{ii} \rightarrow \Delta_{ii} + 1, \quad (45)$$

Now, to construct the spanning tree corresponding to the wave of topplings, let us consider again the toppling from the root \star . In this toppling, particles drop onto the boundary sites of the lattice and one particle goes along the bond connecting the root \star with the origin of the avalanche i . Let us consider these topplings separately (again choosing a particular order of the topplings). We start by sending one particle to the site i . The avalanche starting at the site i spreads over some part \mathcal{F}_1 of the

lattice. The site i can topple only once (as it is always connected to the root in this new lattice) hence \mathcal{F}_1 is the first wave of topplings. We drop then all other particles onto the boundary sites causing another avalanche that should cover the rest of the lattice $\mathcal{L} \setminus \mathcal{F}_1$. This avalanche returns the configuration updated after the first wave to the initial state and can be termed the *inverse wave*.

We can generalize the procedure described above. Let us perform k topplings from the root on the new lattice with the additional bond $(\star i)$ and allow k particles to pass through the bond $(\star i)$. They will initiate exactly k waves $\mathcal{F}_1, \dots, \mathcal{F}_k$ having i as origin. If we then drop particles once onto the boundary sites, thus causing an inverse wave, we will return the configuration of heights to the previous one (ψ_{k-1}) , corresponding to the $(k-1)$ -th wave. This means that there is a complete commutativity between waves and inverse waves and they can be performed in an arbitrary order. Moreover, starting from a recurrent configuration on the new lattice, by deleting one particle at site i and initiating only inverse waves, one can reverse completely the evolution of the sandpile model.

We have seen in the previous sections that the toppling process naturally draws spanning trees on the lattice. Applying the algorithm described above to the recurrent configurations on our new lattice with the additional bond $(\star i)$, we get a new set of spanning trees. All spanning trees now can be divided into two sets. The first one consists of only those spanning trees that do not contain the bond $(\star i)$ and, therefore, on deleting the bond coincides with the set of *one-rooted* spanning trees \mathcal{T}_\star on the old lattice \mathcal{L} . The second set consists of those trees that contain the bond $(\star i)$. On deleting this bond (thus returning to the original lattice \mathcal{L}) a subtree \mathcal{T}_i gets disconnected. Considering the site i as the second root of component \mathcal{T}_i we obtain a *two-rooted* situation, where a spanning tree on the lattice consists of two disconnected clusters \mathcal{T}_i and \mathcal{T}_\star .

Thus, in addition to the one-to-one correspondence between the recurrent states and one-rooted spanning trees, we get the one-to-one correspondence between all waves of topplings and all two-rooted spanning trees. The avalanche is displayed now as a collection of successive two-rooted trees.

E. Waves and Green functions

The graph representation of waves enable us to link the toppling process and the lattice Green function G_{ij} .

Theorem [15]. For an arbitrary lattice \mathcal{L} with the root \star , the Green's function is given by the ratio

$$G_{ij} = \frac{\mathcal{N}^{(ij)}}{\mathcal{N}}, \quad (46)$$

where $\mathcal{N}^{(ij)}$ is the number of two-rooted spanning trees having the roots \star and i , such that both vertices i and j belong to the same one-rooted cluster and \mathcal{N} is the total number of one-rooted spanning trees on \mathcal{L} .

As a simple consequence of the theorem we conclude that G_{ij} is equal to the expected number of topplings at the site j during the avalanche caused by adding a particle at site i [12]. Indeed, as each wave corresponds to only one toppling of its sites, then the expected number of topplings should coincide with the expected number of waves involving the site j and, hence, should be equal to G_{ij} .

Yet another result is necessary to derive the probability distributions of waves. Namely, as is known from conformal field theory arguments [24–26], the area s and the perimeter l of a finite cluster of two-rooted spanning tree (or any wave of topplings) are related to its radius r as

$$s \sim r^2 \quad \text{and} \quad l \sim r^{5/4}. \quad (47)$$

To find the size distribution of general waves [15], we consider all possible waves that belong to the avalanches starting at a fixed point i deep within the lattice. All these waves are in one-to-one correspondence with the recurrent configurations of the sandpile model on the lattice with an additional bond connecting the sites \star and i . As all recurrent configurations have the same probability it follows that all general waves are also equally likely.

Let us now consider waves on the lattice initiated at site i without any reference to the particular avalanches they belong to. In other words, we consider a sequence of avalanches initiated at the same site i . These avalanches consist of waves. We ignore the pauses between the avalanches and study the properties of the collection of waves in general. Using the above theorem we can estimate the probability that the radius r of the wave is not less than the distance between the sites i and j as

$$\text{Prob}(r \geq |i - j|) \sim G_{ij} \sim \ln |i - j|. \quad (48)$$

The corresponding probability density

$$P_w(r) dr \sim \frac{dr}{r}, \quad (49)$$

is only normalizable provided that the size of the lattice L (upper cutoff) and its spacing a (lower cutoff) are fixed. Using scaling relations Eq.(47) and

$$P_w(s) ds \sim P_w(l) dl \sim P_w(r) dr, \quad (50)$$

we can rewrite this probability distribution either in terms of the area s or the perimeter l of the waves of topplings.

We have seen above that the last wave corresponds to the rooted subtree having its root at the boundary. The alternative way to get its tree representation is to cut a

bond and disconnect a branch from the one-rooted tree representing the recurrent configuration that appears after the last wave.

A general wave differs from the last one only in the location of the root. The position of the root in the general wave is distributed uniformly over the whole area, in contrast to its location at the boundary for the last wave. Therefore, given a general wave of area s and perimeter l , an additional factor $l/s \sim s^{-3/8}$ appears in the probability distribution of the last waves. Hence, we get the size probability distribution of the last waves [27]

$$P_l(s) ds \sim P_w(s) s^{-3/8} ds \sim s^{-11/8} ds. \quad (51)$$

The critical exponents of the general waves are not related directly to the exponents of the avalanche distributions because by considering the general wave we lose information as to the concrete avalanche to which it belongs. Fortunately, there exists a situation where all avalanches consist of one and only one wave and the avalanche distribution coincides with the distribution of general waves. Namely, avalanches starting at the boundary consist of only one wave because the second toppling is impossible due to loss of particles.

Let us consider the case of an infinite wedge, with internal angle γ , related to the half plane by the conformal mapping $w = z^{\gamma/\pi}$. It is known from the theory of complex variables that the Green function of the Laplacian in the region inside the wedge has the form

$$G(z, \bar{z}) \sim \text{Im}(z^{-\pi/\gamma}) \quad (52)$$

where z and \bar{z} are the complex coordinates on the plane. Thus, the function $G(r)$ decays as

$$G(r) \sim r^{-\pi/\gamma} \quad (53)$$

for all directions save the arms of the angle. This leads to the distribution

$$P_b(r) dr \sim r^{-1-\pi/\gamma} dr \quad (54)$$

Again, using the relations $s \sim r^2$ and $P_b(s) ds \sim P_b(r) dr$, we can rewrite this distribution in the form

$$P_b(s) ds \sim s^{-1-\pi/2\gamma} ds \quad (55)$$

which corresponds to the critical exponent $\tau = 1 + \pi/2\gamma$ for the area distribution of boundary avalanches [28].

The angle 2π is of special interest. In this case avalanches start at the top of a cut on the plane. The geometry of the avalanches closely resembles the one occurring deep within the lattice. So one could expect that the corresponding critical exponent $\tau = 5/4$ should be close to that occurring in the bulk of the lattice.

IV. AVALANCHE DISTRIBUTION EXPONENTS

Let R denote the linear extent (diameter of the avalanche); S , the number of distinct sites where at least one toppling occurs; V , the total number of topplings in the avalanche. We denote the number of waves of topplings it consists of as T . All these quantities are random variables. It is usually assumed that their probability distributions have the following power-law asymptotics in the thermodynamic limit

$$P_a(R) dR \sim R^{-\tau_R} dR \quad (56)$$

$$P_a(S) dS \sim S^{-\tau_S} dS \quad (57)$$

$$P_a(V) dV \sim V^{-\tau_V} dV \quad (58)$$

$$P_a(T) dT \sim T^{-\tau_T} dT \quad (59)$$

The exponents τ_R, τ_S, τ_V and τ_T can be related to each other if we make the simple scaling assumptions that R, S, V and T scale as some powers of each other, i.e. if we assume that for all avalanches with a fixed value of one variable (say R) the other variables (S, V and T) have strongly peaked conditional probability distributions [14].

Since every avalanche cluster is nothing but a superposition of compact clusters of waves of topplings, we may speculate that the avalanche clusters are also compact

$$S \sim R^2. \quad (60)$$

The total number of topplings in the avalanche can then be related to its diameter via some new exponent y

$$V \sim R^{2+y}, \quad \text{where } y \geq 0. \quad (61)$$

Finally, from the fact that in each wave of topplings the sites of the lattice topple only once, we conclude that the same exponent should appear in the relation between the number of waves in the avalanche and its linear extent

$$T \sim R^y. \quad (62)$$

The probability distribution $P_a(T)$ of the number of waves in the avalanche should be consistent with these scaling relations. It should also lead to the logarithmic dependence of the average number of waves on the lattice size, $\langle T \rangle \sim \ln L$. The only distribution that meets all these assumptions is

$$P_a(T) dT \sim \frac{dT}{T^2}. \quad (63)$$

Now, using scaling relations Eq.(60,61,62) we can express the critical exponents of all other probability distributions in terms of the only unknown scaling exponent y

$$\tau_R = 1 + y, \quad (64)$$

$$\tau_S = 1 + y/2, \quad (65)$$

$$\tau_V = (2 + 2y)/(2 + y), \quad (66)$$

$$\tau_T = 2. \quad (67)$$

In the particular case $y = 0$ these exponents coincide with those of waves of topplings. In the opposite case, $y = 1$, they reproduce the mean-field critical exponents. Thus, we may expect that the probability distribution of avalanches should correspond to some intermediate value $0 \leq y \leq 1$.

The more detailed description of the avalanche process needs new assumptions about the average space-time structure of the avalanche process and its decomposition into the waves of topplings. On the basis of numerical simulations it can be assumed that, typically, the avalanche process consists of two regimes. In the first regime one witnesses a relatively fast growth of the avalanche with each subsequent wave escaping the boundary of its predecessor, while in the second regime one observes a relatively slow squeezing of waves towards the last wave of topplings that touches the initial site. Assuming that in the thermodynamic limit the second stage completely dominates the first one can expect that the area decrement Δs_t between two successive waves s_t and s_{t+1} scales along with their average size s_k as

$$\Delta s_t \sim s_t^\alpha. \quad (68)$$

This scaling law is consistent with all our earlier assumptions. The exponent α can be related to the scaling exponent y that determine the avalanche probability distributions. Indeed, we have

$$\Delta s_t \sim s_t^\alpha \Delta t, \quad \text{and} \quad \Delta r_t \sim r_t^{2\alpha-1} \Delta t, \quad (69)$$

where t is the number of wave ($\Delta t = 1$ for two successive waves). If we integrate these relations up to the size of the avalanche S and the total number of waves T , we reproduce Eq.(62) with the scaling exponent $y = 2 - 2\alpha$. The attempt to find α from more complicated scaling arguments [30] gave $\alpha = 3/4$ and, hence, $y = 1/2$.

Although these critical exponents are in agreement with the results of the real space renormalization group calculations [29,5], it is worth mentioning again that the above construction is based only on some self-consistent hypothesis. It is not clear at the moment how they are supported by the numerical simulations.

An alternative hypothesis about inner structure of the avalanche process assume the existence of the universal asymptotics of the probability distribution of subsequent waves for a given size of the preceding wave, $P(s_{t+1}|s_t)$ [31]. Namely, on the basis of numerical simulations it was suggested that

$$P(s_{t+1}|s_t) \sim \begin{cases} s_{t+1}^\beta, & \text{if } s_{t+1} \ll s_t \\ s_{t+1}^\gamma, & \text{if } s_{t+1} \gg s_t \end{cases} \quad (70)$$

with $\beta = 3/4$ and $\gamma = 5/4$. The derivation of this scaling laws, if they exist, requires further analysis of the avalanche process.

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