

On the Baxterisation of Braid Group Representations of Rational Conformal Field Theories*

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Abstract

We discuss aspects of Coulomb gas representations of rational conformal field theories which might be useful for the Baxterisation of braid group representations appearing in this context. Trigonometric solutions of the Yang-Baxter equations emerging from one-component Coulomb gases -with underlying quantum group $SU(2)_q$ - are constructed explicitly.

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Prologue: In the lectures given by the author orally in Razlog dealing with various facets of relations between rational conformal field theories and integrable systems several problems were posed as exercises for the attending students. One of the problems mentioned was the recursive construction of solutions of the Yang-Baxter equations taking advantage of Coulomb Gas representations of conformal field theories. I take the opportunity to collect the material which might be useful in this context and discuss the solution of the problem in the simplest possible framework of a one-component Coulomb gas representation. I hope that these notes may be helpful for somebody who wishes to work out the exercise in its full complexity.

1 Introduction

Given the progress made in understanding $d=2$ conformal field theories on one side and of $d=2$ integrable spin systems on the other side much effort has been spent in recent years to find spin systems whose critical behaviour is governed by a particular conformal field theory, or, vice versa, to determine the universality class of a given spin system, that is, to identify the field theory that describes the given spin system at criticality. A short cut version of such a two-sided program has been advocated by Gepner [1] some time ago. This author conjectures that all representations of the braid group appearing in rational conformal field theories (RCFT) can in a rather specific

way be "Baxterised". The latter terminology has been coined [2] to denote the attachment of spectral parameters to representations of the braid group s.t. the Yang-Baxter equations (YBE) are satisfied. Solutions of the YBEs give rise to integrable vertex models and integrable "interaction round a face" models [3]. It is commonly believed that the so constructed spin systems lead in the scaling limit at criticality back to the RCFT one started from. Taking Gepner's conjecture and the previous prejudice for granted one has a neat connection indeed between RCFTs and integrable spin systems. The purpose of these notes is to dwell on Gepner's conjecture.

The formulas proposed in [1] can easily be verified to be true for minimal conformal field theories with central charge smaller than one. This can be seen by noting 1.) that Gepner's formulas are identical in structure with Jimbo's trigonometric solutions of the YBE [4] for the quantum group $Su(2)_q$, and that 2.) according to Gomez and Sierra (GS) [5] the Coulomb gas representation of the minimal theories mediate $SU(2)_q$ representations.

I have not found any indication that Gepner's conjecture - or rather a suitable generalisation of it - can be deduced from the general principles of $d=2$ conformal field theory. It seems most likely that one needs for this purpose more specific information. Our basic assumption will be that there exists a Coulomb gas representation of the RCFT under consideration. But also this seems in general not to supply enough structural insight to proceed with the Baxterisation. One needs, as far as I can see, the underlying structure of a quantum group. We are therefore finally lead to advertise the Coulomb gas representa-

tion as a calculational tool to construct Yang-Baxter matrices intertwining arbitrary representations of any quantum group. The plan of these notes is as follows : Some material about RCFT is collected in the following section. The third section is devoted to a discussion of different bases of contour integrals within Coulomb gas representations. We will deal there also with the GS representation of the braid group. The Baxterisation is discussed in the last section.

2 RCFT

The basic data of a $d=2$ conformal field theory are a symmetry algebra -which may be the Virasoro algebra or a larger algebra in which the Virasoro algebra is imbedded- and a finite number of highest weight representations of this algebra, among them the vacuum representation with the vacuum being the unique projectively invariant state of the respective theory. Let ψ_i denote the finitely many primary operators (we deal with a RCFT !) intertwining the vacuum representation with the other highest weight representations. Vacuum correlation functions of the operators ψ_i decompose into sums of products of holomorphic and antiholomorphic functions, the so-called conformal blocks.

$$\langle \psi(z_1, \bar{z}_1) \dots \psi(z_n, \bar{z}_n) \rangle = \sum F \cdot \bar{F} \quad (1)$$

The operators can accordingly be factored into holomorphic and anti-holomorphic parts. We will deal here only with the

holomorphic parts of the operators which will be denoted by ϕ_i .

Conformal blocks are conveniently described in terms of so-called chiral vertex operators(CVO) [6]. Let P_i be the projector onto the representation space of the symmetry algebra which is reached by acting with ϕ_i on the vacuum state. A CVO is defined by $V_{j_i}^k = P_k \phi_j P_i$. The holomorphic blocks F figuring in Eq.(1) emerge from the (iterated) partition of unity through projectors P_i ,

$$F_{i_n \dots i_1}^{k_{n-1} \dots k_2}(z_n, \dots, z_1) =$$

$$\langle P_0 \phi_{i_n}(z_n) P_{k_{n-2}} \dots P_{k_2} \phi_{i_2}(z_2) P_{i_1} \phi_{i_1}(z_1) \rangle = \langle V_{i_n k_{n-1}}^0(z_n), \dots, V_{i_1 0}^{i_1}(z_1) \rangle \quad (2)$$

with P_0 denoting the projector onto the vacuum sector. A function F is exactly then non-vanishing if the operators ϕ_{i_x} and $\phi_{k_{x-1}}$ couple in the short distance algebra with a non-zero expansion coefficient to ϕ_{k_x} , $x=1, \dots, n$. The specification of the operator algebra completes the data characterising a particular RCFT. It may happen that a representation occurs more than once at one place in the block decomposition, Eq.(2). We do not bother to indicate this potential degeneracy.

The conformal blocks in Eq.(2) are analytic functions with respect to each argument in the complex plane as long as no pair of arguments coincides. The analytic continuation of the block functions generates representations of the braid group [6]. As generating move for the braid group may be taken the interchange of neighbouring CVOs in the operator product scheme

scheme of Eq.(2). This gives the linear transformation,

$$V_{xk}^f(z)V_{x'i}^k(z') = \sum_{k'} R^\pm(fxx'i)_k^{k'} V_{x'k'}^f(z') V_{xi}^{k'}(z) \quad (3)$$

where \pm on the representation matrix R refer to the two homotopically inequivalent ways to move the neighbouring CVOs around each other, cf. Fig.1. Note that two of the arguments in brackets on the r.h.s. of Eq.(3) indicate the primary operators which are moved, (x,x') , and two (f,i) , the surrounding in which they are moved.

The representation matrices have to meet a number of consistency conditions [7],[8],[9], (which may be comprised in the requirement that homotopically equivalent moves lead to identical linear transformations). Of these we only quote the "hexagon equation", which states the equivalence of two threefold moves of three neighbouring CVOs, cf. Fig.2 ,

$$\sum_l R(fxx'm)_l^l R(l'xx''i)_m^{\bar{m}} R(fx'x''\bar{m})_l^{\bar{l}} = \sum_{m'} R(lx'x''i)_m^{m'} R(fxx''m')_l^{\bar{l}} R(\bar{l}xx'i)_m^{\bar{m}'} \quad (4)$$

with the understanding that all matrices appearing here are either R^+ or R^- representatives. One should note that Eq.(4) is not a conventional matrix equation, since it comprises the contraction of environmental labels.

The braid group transformations become simple vis-á-vis the vacuum if one supposes that there is no multiple occurrence in the operator algebra of one and the same representation of the symmetry algebra. It has been shown [8] that under these circumstances normalisations can be chosen s.t. holds

$$R^\pm(fxx0)_k^{k'} = \delta_{k,x} \delta_{k',x'} \cdot e^{\pm\pi i(d_f - d_x - d_{x'})} \quad (5)$$

The label 0 refers here to the vacuum state. With d_i are denoted the scaling dimensions of the corresponding states. Specialising Eq.(4) to $x = x' = x''$ and $i = 0$ one obtains, taking into account Eq.(5),

$$R^+(\cdot xx\cdot)DR^+(\cdot xx\cdot) = DR^+(\cdot xx\cdot)D \quad (6)$$

and analogous relations for R^- . We use here the notation

$$D_{k,k'} = \delta_{k,k'} \exp \pi i (d_k - 2d_x)$$

Noting that

$$R^+(\cdot xx\cdot)^{-1} = R^-(\cdot xx\cdot)$$

one can rewrite Eq.(6) in the form

$$R^+(\cdot xx\cdot) = DR^+DR^-D^{-1} \quad (7)$$

It shows that the hexagon equation on the vacuum supplies a spectral decomposition of the braid group matrices in the case that no multiple appearances of representations occur. One needs otherwise more specific information to handle the arising degeneracies.

3 Coulomb Gas Representations

Let $\phi_i, i=1, \dots, n$ denote the chiral parts of massless scalar free fields in two Euclidean dimensions and $\alpha^x = (\alpha_1^x, \dots, \alpha_n^x)$ n -component real-valued vectors. The basic objects of a Coulomb gas representation are normal ordered exponential vertex operators (VO) of the free fields:

$$V_\alpha(z) =: \exp i\alpha \cdot \phi :$$

$\alpha^x \cdot \phi = \sum_i \alpha_i^x \phi_i$. Free field vacuum expectation values of the exponentials read as

$$\langle V_{\alpha^1}(z_1) \dots V_{\alpha^k}(z_k) \rangle \cong \prod (z_i - z_j)^{2(\alpha^i \cdot \alpha^j)} \quad (8)$$

The other ingredient of a Coulomb gas representation is a system of spin fields. But these behave neutrally under braid group transformations and will therefore not be taken into consideration in these notes.

CVOs of RCFT are built up by attaching to the free field exponentials contour-integrated so-called screening operators [6]. The latter, which will be denoted J_x , are also built out of free field exponentials (and of spin field polynomials) s.t. their scaling dimension is equal to one. Correlation functions of RCFT have in the emerging Coulomb gas representation the appearance:

$$\int \prod_{i=1}^N dt_i \langle V_{\alpha^1}(z_1) \dots V_{\alpha^k}(z_k) J_{x_1}(t_1) \dots J_{x_N}(t_N) \rangle \quad (9)$$

One may choose different bases of independent contours. Some of the possibilities are depicted in Fig.'s 3 -5.

Lines stand in Fig.'s 3-5 for integration contours, crosses for positions of CVOs and circles for screening currents .

The basis of path ordered contours between neighbouring CVOs ,cf.Fig.3, has been introduced by Dotsenko and Fateev (DF) [10]. It is particularly suited for explicit calculations.

Let us assume for simplicity that we deal with one species of screening currents.(The results in the present section are easily extended to the case of an arbitrary variety of screening currents). A picture of the basic braiding moves,as they are realised in the DF contour base, is shown in Fig.6a and Fig.6b

. It should be evident from these figures that the constitutive rule of braiding consists in the recipe to move contours with the points they are attached to.

Let N be the total number of screening currents of the configuration pictured in Fig.6a,b. We take the number of screening currents in the upper half of the figures as state-label. Let the braiding matrices going along with the processes of Fig.s 6a, 6b be denoted by R_1 and R_2 resp..These matrices can be evaluated by relating the deformed contours back to the undeformed contours. One reads from the figures that R_1 and R_2 are triangular (with respect to the above mentioned labeling) and that they are similiar to each other. With T denoting the anti-diagonal matrix $T_{ij} = \delta_{N-i,j}$, $0 \leq i, j \leq N$

we have

$$R_2 = TR_1T \quad (10)$$

The hexagon equations (4) ,as realised in the DF basis, have the nice comic strip representation of Fig.7 . It is easy to verify that the phase factors picked up along the lower and upper pathes of of Fig.7 are the same and that they are in fact independent of the particular configuration chosen.(An anlogous result holds also for the case of an arbitrary number of different species of screening currents.) Normalisations can be chosen s.t. one has $R_1R_2R_1 = T$ (in a given sector with a fixed number of screening currents). One brings the hexagon equations,using Eq.(10), into the form

$$R_1R_2R_1 = T = T^3 = TR_1R_2R_1T = R_2R_1R_2 \quad (11)$$

, where one has also used the relation $T^2 = 1$.

The basis of contours depicted in Fig.4 has been introduced

by Felder because of its immediate interpretation in terms of CVOs. We will call it the F-basis. Our interest in this basis is related to the fact that one achieves with Felder's contours a diagonalisation of the braid matrices. We exploited this already -without mentioning it- in drawing Fig.3 for representing the DF-basis. It is there tacitly assumed that the DF -contours are imbedded into F-contours as shown in Fig.8 .

It is evident from the picture that the outer F-contours do not interfere with the action of the braid group inside these contours. The action of the braid group becomes, in other words, through this choice of contours partially diagonalized. It is in the same vein obvious that in going from the DF-basis to the F-basis one diagonalizes R_1 (for the case of one species of screening currents) and with the help of Eq.(10) also R_2 . Let B and b resp. denote the phase factors generated by braiding the free field VOs in the inner circle of Fig.4 and by braiding a screening current with a VO (supposing for simplicity that these VOs are of the same type) and let q denote the phase factor emerging from the braiding of two screening currents. A configuration with a fixed number, say k , of screening currents integrated along the inner contours of Fig.4 is an eigenvector of R_1 with eigenvalue

$$\lambda_k = B b^{2k} q^{k(k-1)/2} \quad (12)$$

I want to digress, even if I will not pursue it further, in order to comment on the diagonalisation of R-matrices in the case that several species of screening currents exist. It is then convenient to use the basis of contours displayed in Fig.9 . The monodromy move of the inner, connected piece of Fig.9 (that is, the 2π rotation of this piece) is represented by a diagonal matrix whereas

the braid action relates two configurations respectively to each other. This twofold connection is easily diagonalized by hand.

The basis of contours of Fig.5 has been used by Gomez and Sierra (GS) [5] to realise quantum group representations on Coulomb gas contours. GS have worked out the case of one species of screening currents which leads to representations of $SU(2)_q$. The case of higher rank groups has been considered in [12]. The contours in Fig.5 may either extend in the complex plane to infinity or may end at a finite point. To comply with $SU(2)_q$ conventions we stipulate (aiming at the spin j representation of $SU(2)_q$) that the constants in Eq.(12) are given by

$$B = q^{j^2} \quad (13)$$

$$b = q^{-j} \quad (14)$$

$j \in Z_+$. We use, following [5] and [12], the notation $e_j^n(z)$ for a VO at position z with n attached contours of the type displayed in Fig.5. A contour creation operator F is defined by setting

$$FV_j^n = 1/(1 - q^{-1}) \int_{GS} dt J(t) e_j^n(z) = e_j^{n+1}(z) \quad (15)$$

where the index GS is meant to indicate that the added contour is again of the GS type. One finds inductively

$$e_j^n = F^n e_j^0 =$$

$$\prod_{i=1}^n \int_{GS} dt_i J(t_i) e_j^0(z) = [2j]_q [2j-1]_q \dots [2j-n+1]_q P^n e_j^0(z) \quad (16)$$

with $[x]_q = (1 - q^{-x})/(1 - q^{-1})$. $P^n e_j^0(z)$ denotes here the VO with n attached straight contours, as they are shown in

Fig.10. We deduce from Eq.(16) that

$$F^n e_j^0 = 0, \text{ for } n \geq 2j \quad (17)$$

We define furthermore-following again [5] and [12]- a phase operator k by

$$k^2 e_j^n = q^{-j+n} e_j^n \quad (18)$$

The eigenvalue q^{-j+n} is the phase factor one picks up in transporting a screening current around a VO e_j^n . Comparison of Eq's(15) and (17) yields the relation

$$kF = q^{1/2} Fk \quad (19)$$

The conclusion to be drawn from Eq's (15),(17) and (19) is that k and F generate a Borel subalgebra of $SU(2)_q$ and that a spin j representation is built up with e_j^0 as highest weight vector. It is possible to complete the $SU(2)_q$ algebra by defining a suitable contour destruction operator, cf. [5],[12]. We will here only make use of the above defined Borel subalgebra.

The main achievement of the GS approach is a natural representation of the comultiplication. The comultiplication law for the generator F is depicted in Fig.11. This reads in formulas as

$$\begin{aligned} \Delta F(e_j^{n_1}(z_1) \otimes e_j^{n_2}(z_2)) = & \quad (20) \\ F e_j^{n_1}(z_1) \otimes e_j^{n_2}(z_2) + k^2 e_j^{n_1}(z_1) \otimes F e_j^{n_2}(z_2) \end{aligned}$$

The two terms on the r.h.s. of (20) arise through decomposition of the large contour of Fig.11 into separate contours around each of the VOs. The obvious comultiplication law for k is

$$\Delta k(e_j^{n_1} \otimes e_j^{n_2}) = k e_j^{n_1} \otimes k e_j^{n_2} \quad (21)$$

(The phase factor picked up along a path encircling two VOs is equal to the product of phase factors picked up on separate tours around each of the VOs).Eqs (20) and (21) are equivalent to the operator relations

$$\Delta F = F \otimes 1 + k^2 \otimes F \quad (22)$$

$$\Delta k = k \otimes k \quad (23)$$

It is straightforward to verify that (22) and (23) map the Borel algebra we started from homomorphically.

Practitioners of q-analysis often quote Eq.(22) in a slightly different form.Defining

$$f \equiv F \cdot k^{-1}$$

one obtains (exploiting the fact that Δ is an algebra homomorphism)

$$\Delta f = f \otimes k^{-1} + k \otimes f \quad (24)$$

The action of the comultiplied quantum algebra interchanges with the braiding of the two involved VOs, as one directly infers from Fig.11 and the very definition of $\Delta(k)$, Eq.(21).

We want to derive still another intertwining property of the braid group generators, which will be used in the subsequent section. Let us inspect the braiding of two VOs in the GS basis,as it is illustrated in Fig.12 . The deformed contours may be handled,one after another,cf.[5], [12],in a way which is demonstrated for the outermost contour of one of the VOs in Fig.13 . Iterating the operation displayed in Fig.13 one arrives at the formula

$$R(e_j^{n_1}(z_1) \otimes e_j^{n_2}(z_2)) = \sum_{k=0}^{n_1} c_k(n_1)(\Delta F)^{n_1-k} e_j^{n_2+k}(z_2) \otimes e_j^0(z_1) \quad (25)$$

,with $c_k(n_1)$ denoting constants which are independent of n_2 . The representation of braiding through the last equation and the commutation relation,Eq.(19), allow to deduce the intertwining relation

$$R(k^{-1} \otimes f) = (f \otimes k)R \quad (26)$$

4 Baxterisation

To "Baxterize" a given representation of the braid group means that one finds matrices $R_{i,i+1}(x)$ acting on a tensor product of vector spaces $V_i \otimes V_{i+1}, 1 \leq i \leq n - 1$ depending on a spectral parameter x s.t. they coincide for $x = 0$ with the matrices of the given braid group representation,

$$R_{ij}(x = 0) = R_{ij} \quad (27)$$

and that they satisfy for general values x,y of the spectral parameters the YB equations

$$R_{i,i+1}(x)R_{i+1,i+2}(xy)R_{i,i+1}(y) = R_{i+1,i+2}(y)R_{i,i+1}(xy)R_{i+1,i+2}(x) \quad (28)$$

These equations appear in particular in the context of integrable spin models as sufficient condition for the consistency of so-called Yang-Baxter-Faddeev-Zamolodchikov algebras [13]. The latter are defined through relations of the form

$$R(x/y)L(x) \otimes L(y) = L(y) \otimes L(x)R(x/y) \quad (29)$$

for some linear operators $L(\cdot)$.Eq.(29) suggests(it may even though not always be necessary)that R obeys the boundary condition

$$R_{ij}(x = 1) \cong 1_{ij} \quad (30)$$

with 1_{ij} denoting the unit operator in $V_i \otimes V_j$. Solutions of Eq.(28) with the property (30) are called "regular",[13]. We demand regularity in the following.

A consequence of Eqs.(28) and (30) is the "unitarity relation"

$$R_{ij}(x)R_{ij}(1/x) \cong 1_{ij} \quad (31)$$

Taking into account the boundary condition (27) and the unitarity relation one has also

$$R_{ij}(x) \rightarrow R_{ij}^{-1}, \text{ for } x \rightarrow \infty \quad (32)$$

The conditions (27),(30),(31)and (32) are easily adapted in an ansatz involving the spectral decomposition of the matrices R_{ij} ,

$$R_{ij} = \sum \lambda_\nu (P_\nu)_{ij}$$

(One should remember from the previous section that an explicit spectral decomposition of the R matrices is available if one has to do with a Coulomb gas representation of a RCFT. It is then just a matter of relating the basis of contours under consideration to Felder's basis.) The ansatz for $R(x)$ reads as

$$R_{ij}(x) = \sum_{nu} \rho_\nu(x) (P_\nu)_{ij} \quad (33)$$

The conditions (27),(30)and(32) are satisfied if one chooses the the functions $\rho_\nu(x)$ s.t.

$$\rho_\nu(x) \rightarrow \lambda_\nu, \text{ for } x \rightarrow 0 \quad (34)$$

$$\rho_\nu(x) \rightarrow 1/\lambda_\nu, \text{ for } x \rightarrow \infty \quad (35)$$

$$\rho_\nu(x) \rightarrow \text{constant}, \text{ for } x \rightarrow 1 \quad (36)$$

holds. Condition (31) is more restrictive than the others. We are aiming at solutions of the YB equations which depend polynomially on the spectral parameter. (Those are called trigonometric solutions within the general classification scheme for solutions of YB equations[14].) Eq.(31) can only be satisfied if all functions $\rho_\nu(x)$ have as polynomials in the spectral parameter x the same degree, say k . Moreover, inserting representations of the polynomials in the form

$$\rho_\nu(x) \cong \prod_{j=1}^k (x - a_{\nu,j})$$

, where $a_{\nu,j}$ denote some constants, into Eq.(31) one finds that the $a_{\nu,j}$ are restricted to

$$a_{\nu,j} = b_j^{\epsilon(j,\nu)}, \epsilon(j,\nu) = \pm 1 \quad (37)$$

with $b_j, j=1, \dots, k$ denoting some ν -independent constants. Let R_i^F and R_i^{DF} , $i=1, 2$, denote the braiding matrices $R_{i,i+1}$, evaluated in the F- and DF basis respectively. Restricting to the case of a one component Coulomb gas, of which the parameters are adapted to a spin j representation of $SU(2)_q$ - Eqs.(13),(14)- we have

$$\begin{aligned} (R_1^F)_{k,k'} &= \delta_{k,k'} q^{k(k-1)/2 + j^2 - kj} \\ &\equiv D_{k,k'} \end{aligned} \quad (38)$$

Let A denote the matrix connecting the DF and F bases, $R_1^{DF} = ADA^{-1}$.

The ansatz for the Baxterisation, Eq.(33), becomes

$$R_1^{DF}(x) = AD(x)A^{-1} \quad (39)$$

with $D(x)_{k,k'} = \delta_{k,k'}\rho_k(x)$ and
 $R_2(x) = TR_1(x)T, R \equiv R^{DF}$,
cf.Eq.(10).Using the notation
 $Q(x, y) = R_1(x)TR_1(xy)TR_1(y)$
we rewrite the YB equations (28) in the form

$$Q(x, y) = TQ(y, x)T \quad (40)$$

The task of finding of finding trigonometric solutions of the YB equations being connected with $SU(2)_q$ has been reduced by Jimbo[4] to a linear problem (which has its origin in an integrable Toda model).This author suggests to construct matrices $R(x)$ with the intertwining property

$$R(x)(x \cdot f \otimes k + k^{-1} \otimes f) = (f \otimes k + x \cdot k^{-1} \otimes f)R(x) \quad (41)$$

To solve this equation we insert the ansatz (39) and multiply from left and right with projectors P_ν and $P_{\nu'}$ resp. to obtain

$$\rho_\nu(x)P_\nu(x \cdot f \otimes k + k^{-1} \otimes f)P_{\nu'} = \rho_{\nu'}(x)P_{\nu'}(f \otimes k + x \cdot k^{-1} \otimes f)P_\nu \quad (42)$$

This can be simplified with the help of Eq.(26)¹, which is the the $x=0$ version of Eq.(34) to :

$$P_\nu f \otimes k P_{\nu'} \rho_\nu(x + \lambda_{\nu'}/\lambda_\nu) = P_{\nu'} f \otimes k P_\nu \rho_{\nu'}(x)(1 + \lambda_{\nu'}/\lambda_\nu x) \quad (43)$$

We reach so the conclusion that

$$\rho_\nu(x)/\rho_{\nu'}(x) = \frac{(1 + \lambda_{\nu'}/\lambda_\nu x)}{(x + \lambda_{\nu'}/\lambda_\nu)} \quad (44)$$

in the case that $P_\nu f \otimes k P_{\nu'}$ is non-vanishing.To figure out,when this happens, we observe that the quantum group interpretation of the P_ν is that they provide the projection onto the different irreducible representations of the comultiplied quantum

¹In solving Eq.(41) we adopt the strategy developed in[15].

group. The operator $f \otimes k$ on the other hand carries (quantum) spin one. It follows, [4], that

$$\begin{aligned} P_\nu k \otimes f P_{\nu'} &\neq 0 \quad \text{for } \nu' = \nu, \nu + 1 \\ &= 0, \text{ otherwise} \end{aligned} \quad (45)$$

The last relations show that Eq.(41) has a solution which is unique up to an overall normalisation. One may also straightforwardly verify that the conditions (27),(30),(31) and (32) are satisfied by the solution of Eq.(41). It has finally to be shown that the YB equations are a consequence of Eq.(40). We introduce for this purpose the notations

$$Q_1(x, y) = R_1(x)R_2(xy)R_1(y)$$

$$Q_2(y, x) = R_2(y)R_1(xy)R_2(x)$$

(Remind that one has $TQ_1(x, y)T = Q_2(y, x)$ in the DF basis.)

$$X(x, y) = (xy \cdot f \otimes k \otimes k + x \cdot k^{-1} \otimes f \otimes k + k^{-1} \otimes k^{-1} \otimes f)$$

$$Y(x, y) = (f \otimes k \otimes k + x \cdot k^{-1} \otimes f \otimes k + xy \cdot k^{-1} \otimes k^{-1} \otimes f)$$

Using the fact that braiding commutes with the operators $k^\pm \otimes k^\pm$ and applying repeatedly Eq.(41) we arrive at the relations

$$Q_i(x, y)X(x, y) = Y(x, y)Q_i(x, y), \quad i = 1, 2 \quad (46)$$

This lends itself to a proof of the YB equations through an induction in the number of integrated screening currents (since the operations represented by X and Y add one integrated screening current to a given configuration.)

Suppose that Eq.(46) is satisfied for all configurations with the number of screening currents less or equal to n. Eq.(46) shows that the same is true for configurations with (n+1) screening currents, which can be reached from configurations with n currents through application of X(x,y). The same holds, because

of Eq.(40),for configurations which can be reached through TX(y,x). But those two operations can be shown to generate all configurations with (n+1) currents.The induction is there-with complete.

Remarks:1.) The quantum group has effectively only be used to guarantee the validity of Eq.(45).

2.)Our original hope was to find a recipe of Baxterisation for a braid group representation generated by an arbitrary Coulomb gas.The restricting power of relations (31) and (45) (or a condition substituting the latter relation) leads us to the conjecture that a trigonometric Baxterisation associated with a simple linear problem is only possible within the realm of quantum group representations.

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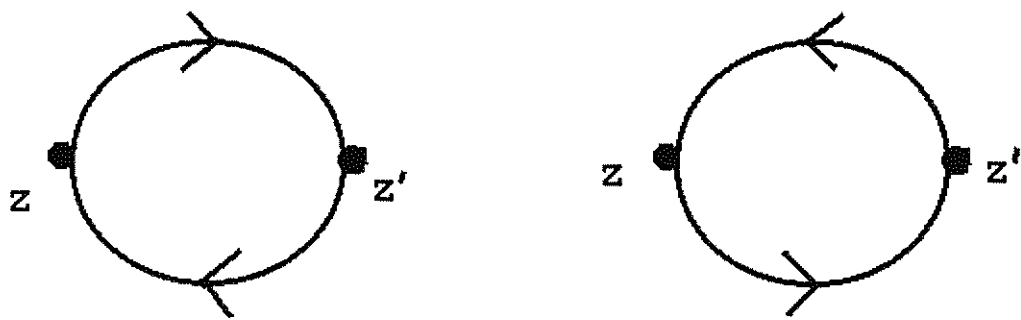


Fig. 1

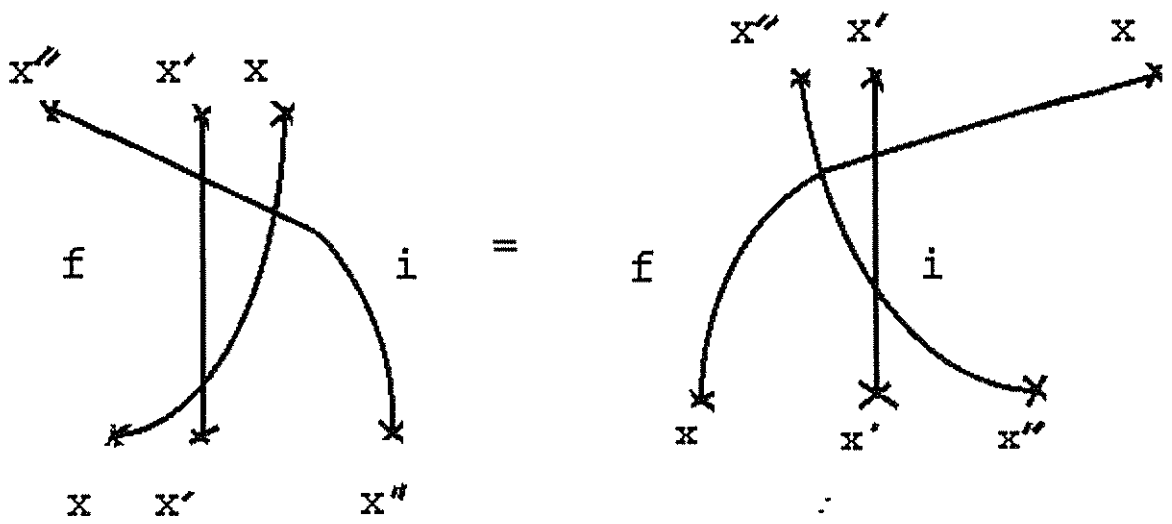


Fig.2



Fig. 3

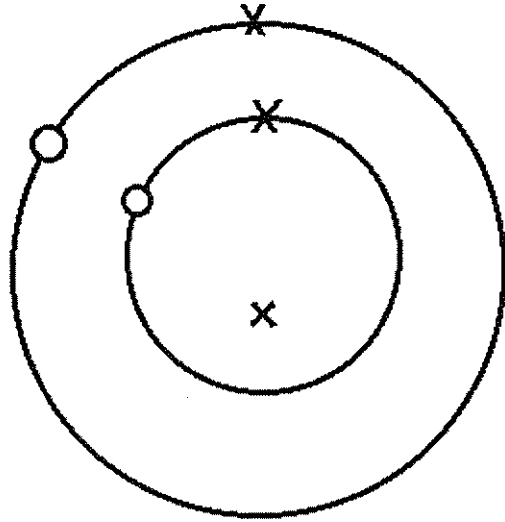


Fig. 4

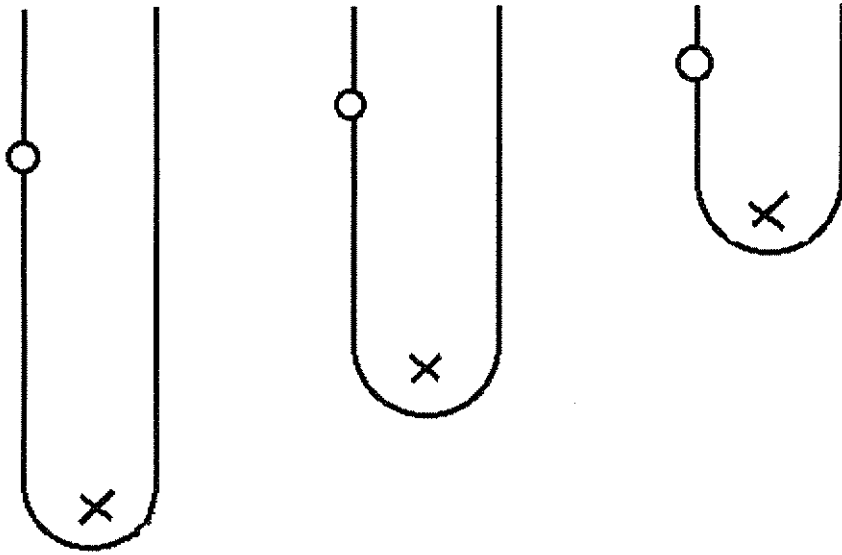


Fig. 5

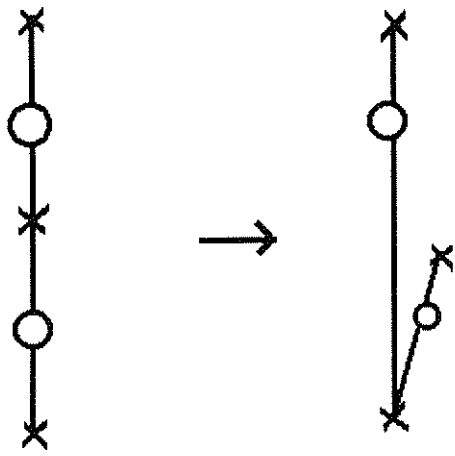


Fig. 6a

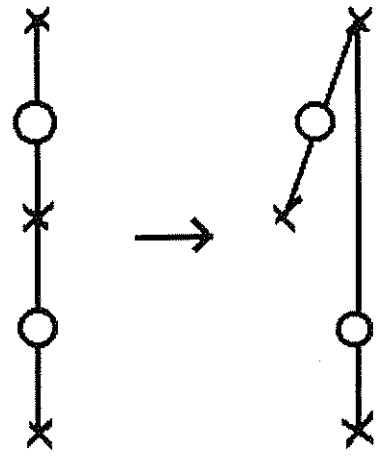


Fig. 6b

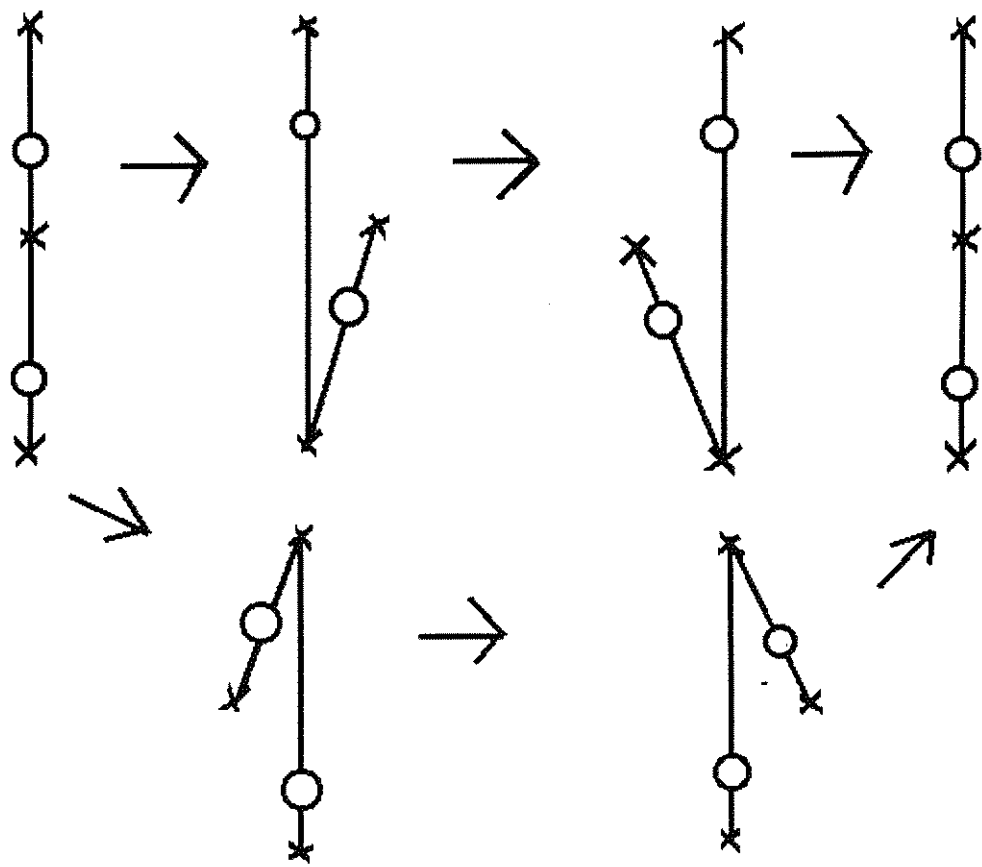


Fig. 7

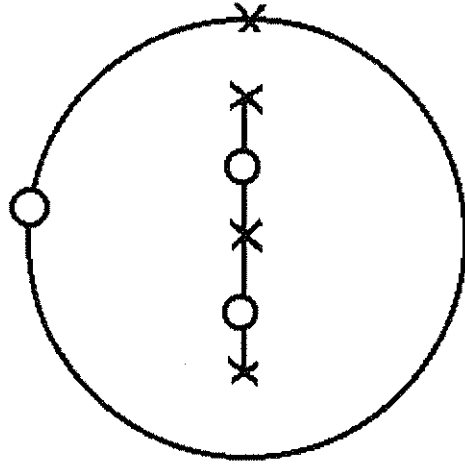


Fig. 8

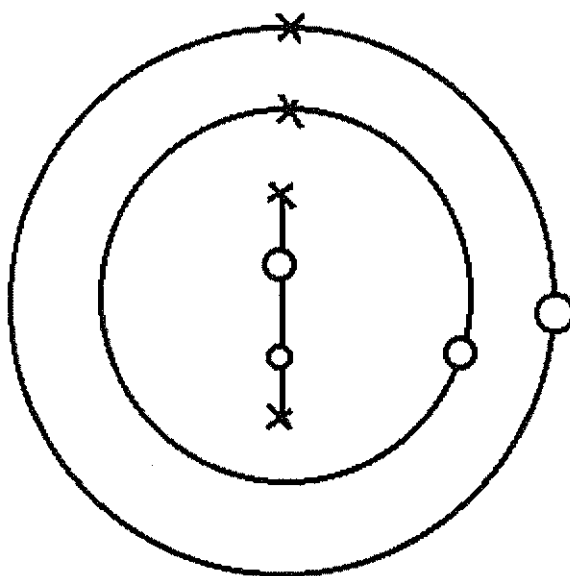


Fig. 9

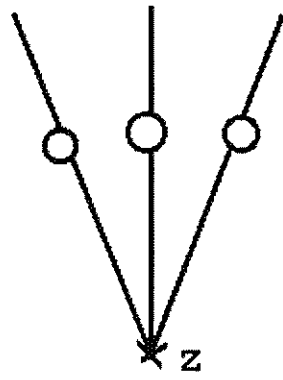


Fig.10

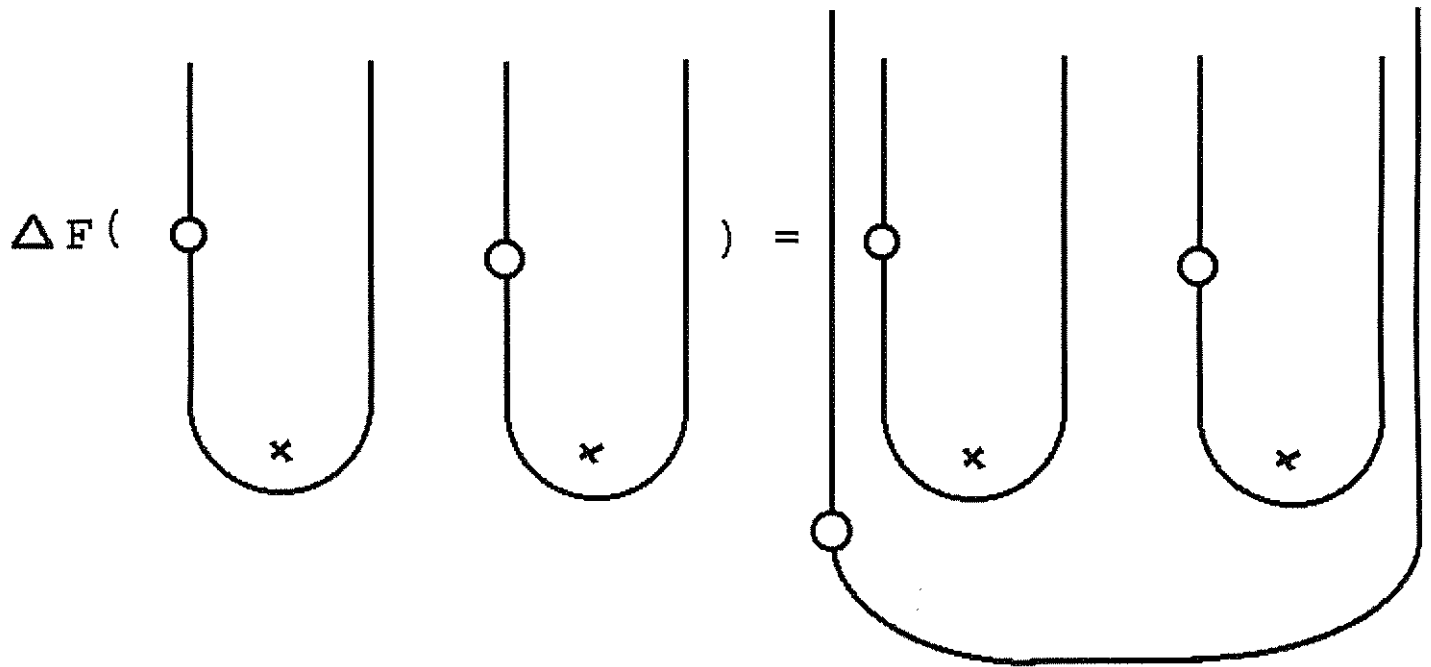


Fig. 11

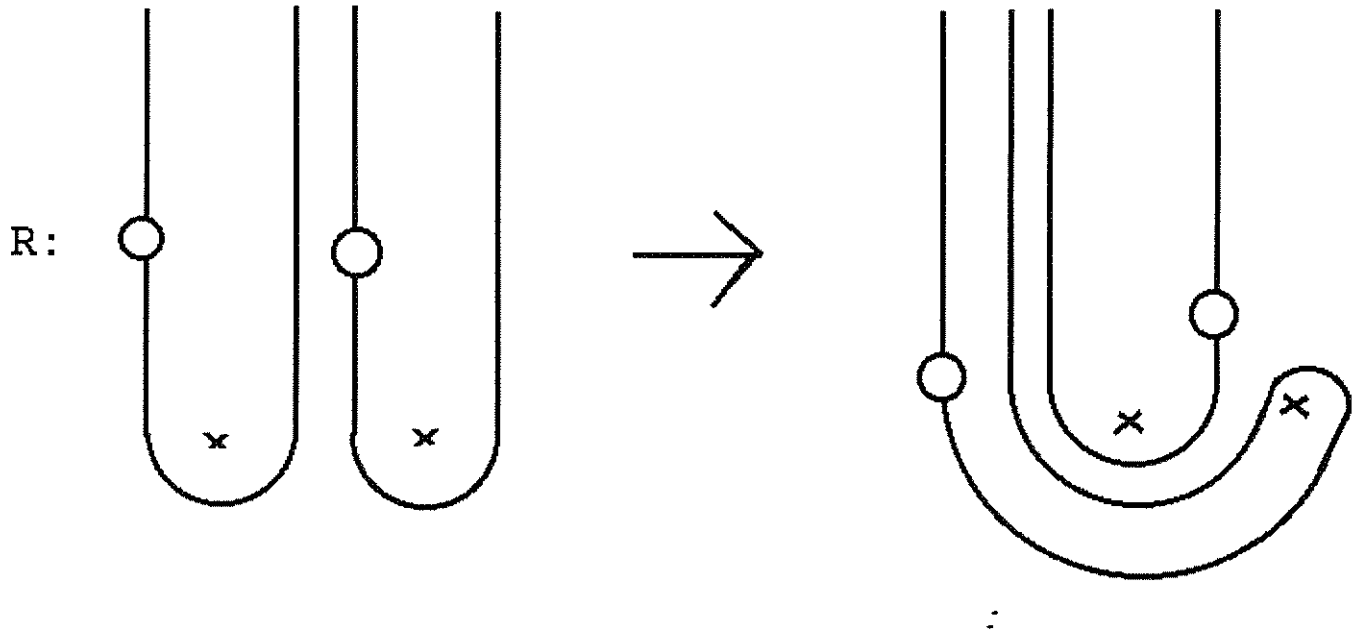


Fig.12

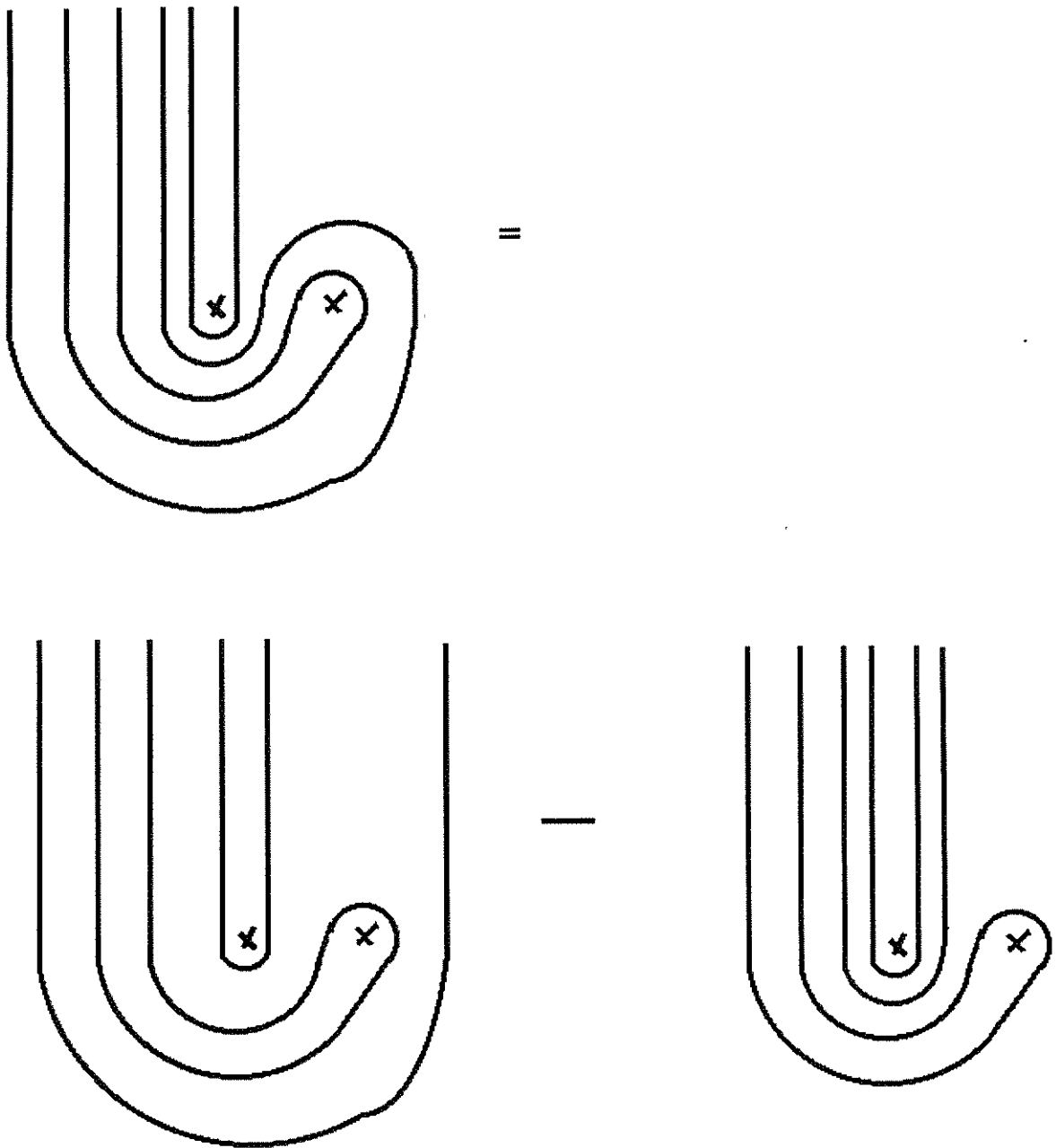


Fig. 13

