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Quantization of Interacting WZW Systems

by

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1. Introduction

An interesting property of non-abelian Toda systems [1][2] is that they are also conformal-invariant interacting WZW (Wess-Zumino-Witten) systems. This suggests considering conformal-invariant interacting WZW systems in their own right, and a large class of such systems, which we call Toda-like systems, is obtained by relaxing the constraints on the coupling constants in the Toda systems. The resultant systems may not be integrable, but they are conformally-invariant (at least classically).

We quantize the Toda-like systems by canonical methods and thus extend previous work [3][4][5] on abelian and Toda systems and provide an alternative to BRST quantization. We show that the the Toda-like systems are conformal-invariant at the quantum level and exhibit the Virasoro algebra. The results for the Virasoro centre illuminate the results obtained previously for Toda systems. The formalism can be extended to curved spaces in the functional integral version and this extension provides a new insight into the role of the centre.

For simplicity, we consider the $sl(n)$ Toda-like systems associated with integral $sl(2)$ embeddings, with for which the relevant subalgebras (little algebras in $sl(n)$ of the grading generator M_o of the $sl(2)$ embedding) are $G_o = G_c \oplus \sum_a \oplus G_\alpha$ where G_c is the centre, and the G_α are $sl(n_\alpha)$ subalgebras. In terms of G_o the Toda-like systems may be described as systems with Lagrangians of the form

$$L = \sum_\alpha k_\alpha L^w(g_\alpha) + \frac{1}{2}\sum k_{ab}(\partial_\mu\phi_a\partial^\mu\phi_b) + \sum_a \mathrm{tr}(g_\alpha M_+^{(a)} g_{\alpha+1}^{-1} M_-^{(a)})e^{\phi^a} \qquad (1.1)$$

where $L^w(g_\alpha)$ are the WZW Lagrangians for the simple subalgebras, the k_α and k_{ab} are arbitrary coupling constants, and the $M_\pm^{(a)}(\neq 0)$ are constant matrices that intertwine G_α and $G_{\alpha+1}$. Throughout, the index a will be associated with $(\alpha, \alpha+1)$. The central fields $\phi^a(x) = \mathrm{tr}(\sigma^a, \phi(x))$ are given by the duals σ^a of the generators σ_a of G_c defined by $[\sigma_a, M_\pm] = \pm M_\pm^{(a)}$. From (1.1) one sees that the generalization from Toda to Toda-like systems consists of keeping the Toda subalgebra G_o fixed but relaxing the Toda condition that the (classical) coupling constants k_α and k_{ab} be determined by one overall coupling constant k, and that the intertwining matrices $M_\pm = \sum M_\pm^{(a)}$ be the generators of the embedded $sl(2)$.

2. The Virasoro Algebra

The problem with conformal invariance in the presence of a conformal scalar interaction V is that the canonical energy-momentum tensor $T_{\mu\nu} = T_{\mu\nu}^f + g_{\mu\nu}V$, where $T_{\mu\nu}^f$ is the free WZW energy-momentum tensor, is not traceless. Equivalently, the equal-time commutators of $T_{\pm\pm} = T_{\pm\pm}^f + V$ do not close to form a Virasoro algebra. The solution to this problem in all cases treated so far to add an 'improvement' term $t_{\mu\nu}$ to $T_{\mu\nu}$ such that $T_{\mu\nu} + t_{\mu\nu}$ is traceless but still conserved. Then $L_\pm = T_{\pm\pm} + t_{\pm\pm} + V$

generate the Virasoro algebra. The role of the improvement term is to convert the potential from a conformal scalar (with respect to $T_{\pm\pm}$) to a conformal tensor of weight (1,1) (with respect to $T_{\pm\pm}+t_{\pm\pm}$) and the important point to note is that *it is the (1,1) character of V that allows the Virasoro algebra to close.*
At the classical level the improvement term for the Toda-like systems takes is the same as for the Toda systems namely, $t_{\pm\pm}(x)=\mathrm{tr}(M_o,J'(x))$ where prime denotes differentiation with respect to the space coordinate. This is not true at the quantum level, however, because at that level V is not a scalar, or even a tensor of definite conformal weight. However, V takes the form $V=\sum_a V_a$ where each V_a has a definite (anomalous) conformal weight $\hbar(\lambda_a,\lambda_a)$. Thus in the quantum case an improvement term is needed to convert the anomalous conformal weights $\hbar(\lambda_a,\lambda_a)$ to $(1,1)$. Our main result is that such an improvement term exists and takes the form

$$t_{\pm\pm}=\mathrm{tr}(I,J'_{\pm}(x)) \quad \text{where} \quad I=M_o-\hbar M_q=\sum_a(1-\hbar d_a)\sigma_a \tag{2.1}$$

Note that M_q lies in G_c but, in contrast to M_o, it is not uniform in a. With this improvement term the quantum Virasoro centre takes the form

$$C=\hbar\Big[n_c+\sum\frac{k_\alpha n_\alpha}{k_\alpha+\hbar g_\alpha}\Big]-12\sum_{ab}k_{ab}\mathrm{tr}\Big(\sigma^a I\sigma^b I\Big) \tag{2.2}$$

where n_c is the dimension of G_c and g_α are the Coxeter numbers of $sl(n_\alpha)$. The contribution in the square bracket is the standard anomalous contribution of the free WZW system and the term with the double-summation is due to the potential.

3. Anomalous Conformal Weights

We compute the anomalous conformal weights λ_a using canonical formalism. The main point is that, by using a suitable formalism for the normal-ordering, it possible to define 'group elements' g and g^{-1} such that

$$\frac{1}{i\hbar}[J_r,g(y)]=\sigma_r g(y)\delta(x-y) \qquad \frac{1}{i\hbar}[J_r,g^{-1}(y)]=-g^{-1}(y)\sigma_r\delta(x-y), \tag{3.1}$$

just as in the classical case. More precisely, it is possible to show that (3.1) is compatible with the Kac-Moody algebra for the currents J_a and the (normal-ordered version of) the differential relationship between g and J. Once (3.1) is established it is easy to show that with the usual definition of the free-WZW Virasoro operator $L^f(x)=T_{\pm\pm}+t_{\pm\pm}$ we have

$$\frac{1}{i\hbar}[L^f(x),g(y)]=g'(y)\delta(x-y)-\frac{\hbar c}{2(k+\hbar g)}g(y)\delta'(x-y) \tag{3.2}$$

where c is the value of the Casimir operator $\Sigma\sigma^r\sigma_r$. Equation (3.2) shows that in the quantum theory g acquires an anomalous conformal weight $\hbar c/(2k+\hbar g)$. It follows that the potentials V_a of (1.4) acquire anomalous conformal weights of the form $\hbar d_a$ where $2\lambda_a=\hbar(c_\alpha/\kappa_\alpha+c_{\alpha+1}/\kappa_{\alpha+1}+c_a/k_a)$, and 2κ is shorthand for $2k+\hbar g$, the c's for the simple subgroups are just the Casimirs $c_\alpha=(n_\alpha^2-1)/n_\alpha$ in the fundamental representation and the constants $c_a = \left(n_\alpha^{-1} + n_{\alpha+1}^{-1}\right)$ are the values of the Casimir for the *adjoint* action of G_c on $M_+^{(\alpha)}$. Thus finally

$$\lambda_a = \frac{(n_\alpha^2 - 1)}{n_\alpha(2k_\alpha + \hbar g_\alpha)} + \frac{(n_{\alpha+1}^2 - 1)}{n_{\alpha+1}(2k_{\alpha+1} + \hbar g_{\alpha+1})} + \frac{(n_\alpha + n_{\alpha+1})}{2n_\alpha n_{\alpha+1} k_a} \tag{3.3}$$

4. Comparison with Reduction Formula

To compare the general formula (2.2) for the Virasoro centre with the standard formula [2][4] for the Toda case we first note that in the Toda case the coupling constants satisfy [2][5] the universality conditions $k_\alpha + \hbar g_\alpha = k + \hbar g$ and $k_{ab} = (k + \hbar g)\mathrm{tr}(\sigma_a \sigma_b)$ (not $k_\alpha = k$ and $k_{ab} = k\mathrm{tr}(\sigma_a\sigma_b)$). In that case the general formula (2.3) reduces to

$$C = \hbar\Big[n_c + \sum_\alpha \frac{k_\alpha n_\alpha}{2k_\alpha + \hbar g_\alpha}\Big] - 12(2k+\hbar g)\mathrm{tr}\Big(M_o - \hbar M_q\Big)^2 \qquad M_q = \frac{n_\alpha + n_{\alpha+1}}{(2k + \hbar g)} \tag{4.1}$$

On the other hand the standard formula for the central charge in the Toda theory is

$$C = \hbar \mathrm{dim} G_o - 12(2k + \hbar g)\mathrm{tr}\Big(M_o + \frac{\hbar m_o}{2k + \hbar g}\Big)^2 \tag{4.2}$$

where M_o and m_o are the grading operators corresponding to the actual and the principle $sl(2)$ embeddings respectively. The expression (4.2) cannot be compared with (4.1) immediately because it is not in the canonical form in which the free contribution is seperated. However, it can be converted to the canonical form by decomposing m_o into the principle m_o^α's for the seperate $sl(n_\alpha)$. i.e. by writing $m_o = \sum_\alpha m_o^\alpha + \bar{M}$ where $\bar{M}$ is a remainder consisting of block averages of m_o. Since the m_o^α are trace-orthogonal to M_o and to $\bar{M}$ we can write (4.2) in the canonical form

$$\begin{aligned} C &= \hbar\Big[n_c + \sum_\alpha \Big(n_\alpha - 12\frac{\mathrm{tr}(m_o^\alpha)^2}{2k_\alpha + \hbar g_\alpha}\Big)\Big] - 12(2k + \hbar g)\mathrm{tr}\Big(M_o + \frac{\hbar \bar{M}}{2k + \hbar g}\Big)^2 \\ &= \hbar\Big[n_c + \sum_\alpha \frac{k_\alpha n_\alpha}{(k_\alpha + \hbar g_\alpha)}\Big] - 12(2k + \hbar g)\mathrm{tr}\Big(M_o + \frac{\hbar \bar{M}}{2k + \hbar g}\Big)^2 \end{aligned} \tag{4.3}$$

where we have used the 'strange' formula $12\mathrm{tr}(m_o^\alpha)^2 = g_\alpha n_\alpha$. Comparing (4.3) with (4.1) we see that the two expressions agree if $\bar{M} = -(2k + g)M_q$, and from (4.1) and the definition of $\bar{M}$ as a block-average it can be verified that this is indeed the case. The result also shows that (4.4) and not (4.3) is the natural form of the Virasoro centre for the Toda case.

5. Curved Space Interpretation

An interesting insight into the role of the conformal group may be obtained by embedding the theory in a curved space [6] with metric $g_{\mu\nu}$ and gauging it with respect to the Weyl group. This leads to the partition-function

$$Z(g_{\mu\nu}) = \int d(g^{\frac{1}{4}}\phi) e^{\sqrt{g}\left(L + R\sum \phi_\alpha\right)} \tag{5.1}$$

where L is the Lagrangian (1.1), R is the curvature and a Polyakov factor of the form $\exp(-\frac{1}{2}\gamma R\Delta^{-1}R)$, where $\gamma = \mathrm{tr}(k^{ab})$ and k^{ab} is the inverse of k_{ab}, has been factored out for normalization purposes. As usual, $Z(g)$ has a Weyl anomaly and it takes the form

$$\frac{\delta Z(g)}{\delta\sqrt{g}} = -cR + m \quad \text{where} \quad c = \gamma + c_q, \tag{5.2}$$

c_q is a finite quantum correction to γ, and m is a renormalization constant. Integrating (5.2) with respect to $\sqrt{g}$, and using the diffeomorphic invariance of $Z(g)$ one obtains

$$Z(g) = e^{\int \sqrt{g}\left(-\frac{c}{2}R\Delta^{-1}R+m\right)} \tag{5.3}$$

The relevance of this for the Virasoro algebra is that the *improved* energy-momentum tensor $< T_{\mu\nu} >$ is obtained from $Z(g)$ by differentiation with respect to the metric,

$$\frac{\delta Z(g)}{\delta g^{\mu\nu}(x)} =< T_{\mu\nu}(x) > \tag{5.3}$$

Thus the Virasoro algebra may also be thought of as a differential algebra operating on $Z(g)$. In particular, to leading order in $(x-y)^{-1}$ and up to a universal constant,

$$\frac{\delta Z(g)}{\delta g^{++}(x)\delta g^{++}(y)} =< T_{++}(x)T_{++}(y) >= \frac{}{(x-y)^4} \tag{5.5}$$

and thus c may be identified as the Virasoro centre. Thus the Virasoro centre may also be computed as the coefficient of R in the Weyl anomaly.

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