DIAS-04-06

A gauge invariant UV-IR mixing and the corresponding phase transition for $U(1)$ fields on the Fuzzy Sphere

P.Castro-Villarreal[*+] , R.Delgadillo-Blando[*+] , Badis Ydri[*].

[*] *School of Theoretical Physics, Dublin Institute for Advanced Studies, Dublin, Ireland.*
[+] *Dept. de Fisica, Centro de Investigaciones de Estudios Avanzados del IPN, Apdo. Postal 14-740, 07000, Mexico D.F., Mexico.*

Abstract

From a string theory point of view the most natural gauge action on the fuzzy sphere $\mathbf{S}^2_L$ is the Alekseev-Recknagel-Schomerus action which is a particular combination of the Yang-Mills action and the Chern-Simons term . The differential calculus on the fuzzy sphere is 3−dimensional and thus the field content of this model consists of a 2-dimensional gauge field together with a scalar fluctuation normal to the sphere . For $U(1)$ gauge theory we compute the quadratic effective action and shows explicitly that the tadpole diagrams and the vacuum polarization tensor contain a gauge-invariant UV-IR mixing in the continuum limit $L\longrightarrow\infty$ where L is the matrix size of the fuzzy sphere. In other words the quantum $U(1)$ effective action does not vanish in the commutative limit and a noncommutative anomaly survives . We compute the scalar effective potential and prove the gauge-fixing-independence of the limiting model $L = \infty$ and then show explicitly that the one-loop result predicts a first order phase transition which was observed recently in simulation . The one-loop result for the $U(1)$ theory is exact in this limit . It is also argued that if we add a large mass term for the scalar mode the UV-IR mixing will be completely removed from the gauge sector . It is found in this case to be confined to the scalar sector only. This is in accordance with the large L analysis of the model . Finally we show that the phase transition becomes harder to reach starting from small couplings when we increase M .

1 Introduction and results

The fuzzy sphere $\mathbf{S}^2_L$ as an approximation of the ordinary sphere is due originally to Madore [13]. See also Hoppe [14]. Unlike naive lattice prescription this approximation preserves all symmetries of the continuum theory such as rotational symmetry of the ordinary sphere, local gauge symmetry of the standard model [15–17] and most notably supersymmetry [18]. In particular it is shown that chiral symmetry is maintained without fermion doubling [17, 19] and that this approximation captures most of the topology of the original commutative sphere such as monopole configurations, integer winding numbers and the index theorem [20, 21].

It is believed that field theories on continuum manifolds can always be regularized in this fashion, i.e by replacing the underlying space with a finite dimensional (fuzzy) matrix model.

Extension to 4 dimensions for example entails the use of either $a)$ the Cartesian product of two fuzzy spheres $\mathbf{S}^2 \times \mathbf{S}^2$ or $b)$ a fuzzy $\mathbf{CP}^2$ [22, 23]. Fuzzy $\mathbf{S}^4$ as obtained from squashed $\mathbf{CP}^3$ is also a candidate for $4-$dimensional fuzzy physics [24]. From a practical point of view the spaces $\mathbf{S}^2_L$, $\mathbf{S}^2_L \times \mathbf{R}$, $\mathbf{S}^2_L \times \mathbf{R}^2$ and $\mathbf{S}^2_L \times \mathbf{S}^2_L$ are the most useful for analytical manipulation since clearly they will only involve the well known $SU(2)$ Clebsch-Gordan coefficients [22]. We note in passing that other higher dimensional fuzzy spaces can also be formulated [25].

The motivation for studying quantum field theories on fuzzy models is therefore two-fold. Firstly this is clearly a novel way of (possibly) simulating ordinary gauge theories, QCD in particular, based on random matrix models which is potentially superior to current methods because of the symmetry-topology arguments outlined above. Secondly fuzzy spaces because of their close connection to Moyal-Weyl noncommutative spaces could provide a systematic way of regularizing and then renormalizing Moyal-Weyl quantum field theories. Indeed and as it turns out field theories on the noncommutative Moyal-Weyl spaces can also be regularized by replacing them with finite dimensional (fuzzy) matrix models [16, 22]. In either cases the limit of interest is a continuum large L limit where L is the size of the matrices approximating (say) in two dimensions $\mathbf{S}^2$ or $\mathbf{R}^2_\theta$.

It is well known that perturbation theory of fuzzy scalar models are plagued by the so-called UV-IR mixing. On fuzzy spaces the mixing is defined by the requirement that the fuzzy quantum actions do not approach the corresponding quantum effective actions on the commutative spaces [22, 26]. We remark that this criterion for the existence of the UV-IR mixing on fuzzy spaces is different from the criterion on the noncommutative Moyal-Weyl planes found in [4]. However the fuzzy UV-IR mixing can be viewed as a regularized version of the UV-IR mixing on the noncommutative plane which will reduce to it in some appropriate flattening limit. This was shown explicitly for $\lambda\phi^4$ theory in 2 and 4 dimensions in [22, 27].

The presence of this mixing on the fuzzy sphere is however a major problem from a theoretical point of view since it means that the scalar model on the fuzzy sphere does not really approximate (as it should) the corresponding scalar model on the ordinary sphere. A priori any simulation of such fuzzy models will therefore give wrong results. Several ways of dealing with this problem were devised [22, 26] and a numerical study to probe the nonperturbative properties of the model was undertaken in [28].

In this article we will study $U(1)$ gauge theory on the fuzzy sphere S^2_L. The main results are summarized as follows. The differential calculus on the fuzzy sphere is three dimensional and as a consequence a spin 1 vector field $\vec{A}$ is intrinsically $3-$dimensional. Each component A_a, $a = 1, 2, 3$, is an element of some matrix algebra Mat_{L+1}. Thus $U(1)$ symmetry will be implemented by $U(L+1)$ transformations U as follows $A_a \longrightarrow U A_a U^+ + U[L_a, U^+]$ where L_a are the generators of $SU(2)$ in the irreducible representation $\frac{L}{2}$ of the group.

On the fuzzy sphere S^2_L it is not possible to split the vector field $\vec{A}$ in a gauge-covariant fashion into a tangent two-dimensional gauge field and a normal scalar fluctuation. This splitting is done trivially on the commutative sphere by simply writing $A_a = n_a \Phi + a_a$ where $\vec{n}$ is the unit vector on S^2, $\Phi = \vec{n} \cdot \vec{A}$ is the normal gauge-invariant component of $\vec{A}$ and $\vec{a}$ is the tangent gauge field. However although we do not have the analogue of a_a on the fuzzy sphere we can still write a gauge-covariant expression for the normal scalar component in terms of A_a which reads

$$\Phi = \frac{1}{2}\left(x_a A_a + A_a x_a + \frac{A_a^2}{\sqrt{L_a^2}}\right), \tag{1.1}$$

where $x_a = \frac{L_a}{\sqrt{L_a^2}}$ are the matrix coordinates on fuzzy S_L^2. In the limit $L \to \infty$ it is not difficult to see that the matrix coordinates x_a tend to the commutative coordinates n_a and the scalar field Φ tends to $\vec{n}\cdot\vec{A}$.

The most general action (up to quartic power in A_a) which is invariant under $U(1)$ transformations on the fuzzy sphere S_L^2 is given by

$$\begin{aligned} S_L[A_a] &= -\frac{1}{4g^2}Tr_L\left[F_{ab}^{(0)} + [A_a, A_b]\right]^2 - \frac{i}{2g^2}\epsilon_{abc}Tr_L\left[\frac{1}{2}F_{ab}^{(0)}A_c + \frac{1}{3}[A_a, A_b]A_c\right] \\ &+ \frac{2M^2}{g^2}Tr_L\Phi^2 + \frac{\alpha|L|}{g^2}Tr_L\Phi. \end{aligned} \tag{1.2}$$

$F_{ab} = F_{ab}^{(0)} + [A_a, A_b]$ is the $U(1)$ covariant curvature where $F_{ab}^{(0)} = [L_a, A_b] - [L_b, A_a] - i\epsilon_{abc}A_c$. In the continuum limit $L \longrightarrow \infty$ all commutators vanish and we get the action

$$\begin{aligned} S_\infty[A_a] &= -\frac{1}{4g^2}\int_{S^2}\frac{d\Omega}{4\pi}(F_{ab}^{(0)})^2 - \frac{i}{2g^2}\epsilon_{abc}\int_{S^2}\frac{d\Omega}{4\pi}\frac{1}{2}F_{ab}^{(0)}A_c + \frac{2M^2}{g^2}\int_{S^2}\frac{d\Omega}{4\pi}\Phi^2 \\ &+ \frac{\alpha|L|}{g^2}\int_{S^2}\frac{d\Omega}{4\pi}\Phi. \end{aligned} \tag{1.3}$$

$F_{ab}^{(0)}$ becomes the $U(1)$ curvature which is now given by $F_{ab}^{(0)} = \mathcal{L}_a A_b - \mathcal{L}_b A_a - i\epsilon_{abc}A_c$ where $\mathcal{L}_a = -i\epsilon_{abc}n_b\frac{\partial}{\partial n_c}$. For $U(1)$ theory this curvature is exactly gauge invariant.

The continuum action S_∞ is at most quadratic in the field A_a (which can therefore be integrated out easily in the path integral) and as a consequence the corresponding effective action will be essentially given by S_∞ itself. On the other hand the quantization of the fuzzy action S_L is much more involved and yields a non-trivial effective action. As it turns out the continuum limit of this fuzzy effective action does not tend to S_∞ for generic values of the parameters M and α. This is the signature of the UV-IR mixing in this model. In this article we computed explicitly *the quadratic* effective action for the values $M = \alpha = 0$ and found it to be given in the continuum limit $L \longrightarrow \infty$ by the expression

$$\begin{aligned} \Gamma_2 &= -\frac{1}{4g^2}\int\frac{d\Omega}{4\pi}F_{ab}^{(0)}(1 + 2g^2\Delta_3)F_{ab}^{(0)} - \frac{i}{4g^2}\epsilon_{abc}\int\frac{d\Omega}{4\pi}F_{ab}^{(0)}(1 + 2g^2\Delta_3)A_c + 4|L|\int\frac{d\Omega}{4\pi}\Phi \\ &+ \text{non local quadratic terms.} \end{aligned} \tag{1.4}$$

The operator Δ_3 is a complicated function of the Laplacian $\mathcal{L}^2$ which is defined in equation (5.8). The 1 in $1+2g^2\Delta_3$ corresponds to the classical action whereas Δ_3 is the quantum correction. By comparing (1.3) and (1.4) it is clear that $1+2g^2\Delta_3$ provides a non-local renormalization of the inverse coupling constant $1/g^2$ whereas the third term in (1.4) provides a local renormalization of the coupling constant α which acquires the value $4g^2$. The last terms in (1.4) are new non-local quadratic terms which have no counterpart in the classical action. Their explicit expression is given in (4.34). Remark that the quadratic action (1.4) is already gauge-invariant which will not be the case for $U(n)$ theories. We have thus established the existence of a gauge-invariant UV-IR mixing problem in $U(1)$ gauge theory on fuzzy S^2_L for the values $M=\alpha=0$. It is only natural to expect that the same result will also hold for generic values of the parameters M and α.

In this paper we will also show that this UV-IR mixing problem is only confined to the scalar sector of the model in the following sense. We consider the model (1.2) for $\alpha=0$ and finite M. We remark that at the level of the classical continuum action (1.3) the limit $M\longrightarrow\infty$ projects out the scalar fluctuation Φ. Indeed in this limit this field becomes infinitely heavy and thus decouples from the rest of the dynamics. If we decide to quantize the model (1.2) and then take the limit $M\longrightarrow\infty$ first and then $L\longrightarrow\infty$ then one finds that the quantum corrections depend only on the scalar field Φ [see equation (7.17)]. Hence in this limit the effective action of the two-dimensional gauge field seems to be given essentially by the classical action whereas the normal scalar field still gets non-trivial quantum contributions in the path integral due to the underlying noncommutativity. This is consistent with the case of pure scalar models studied in [22, 26, 27] but the detail structure of the remaining UV-IR mixing in here is different.

A more elegant test for the UV-IR mixing in this theory can be given in terms of the normal scalar field Φ. Let us consider the following simple normal field configuration defined by

$$A_a = (\phi-1)\,L_a. \tag{1.5}$$

The normalization is chosen for latter convenience. The real number ϕ is related to the normal scalar field Φ by $\phi=\sqrt{1+2\Phi/\sqrt{L_a^2}}$.

It is a trivial exercise to compute the classical action (1.2) for this configuration and one obtains the classical potential given by

$$S=\frac{\sqrt{L_a^2}}{2g^2}\left[\phi^4-\frac{4}{3}\phi^3+M^2\left(\phi^2-1\right)^2+\alpha\phi^2\right]. \tag{1.6}$$

The *full* effective action in the continuum large L limit is given by

$$\Gamma = S+4\sqrt{L_a^2}\log\phi. \tag{1.7}$$

For $M=0$ and $\alpha=0$ the classical potential has a minimum at $\phi=1$ for which the above normal gauge field A_a vanishes. This is the vacuum of the classical theory. To find the effect of the quantum corrections on this vacuum we take the first and second derivatives of the effective potential Γ with respect to ϕ. The condition $\Gamma^{'}=0$ will give us extrema of the model whereas the condition $\Gamma^{''}=0$ tells us when we go from bounded potential (a minimum) to unbounded potential. Solving the above two equations yield immediately the minimum $\phi_* = \frac{3}{4}$ with the corresponding critical value

$$g_*^2 = \frac{1}{8}\left(\frac{3}{4}\right)^3 . \tag{1.8}$$

We can conclude from this result that at the critical value (1.8) a first order phase transition occurs which separates the fuzzy sphere phase where ϕ has a well defined minimum from the pure matrix phase where the minimum disappears. In the fuzzy sphere phase the interpretation of a $U(1)$ gauge theory on a sphere is valid and it holds for $g < g_*$. The matrix phase is where this interpretation brakes down and it holds for $g > g_*$. This agrees nicely with the result of [11] which was however obtained by simulating the full theory. In other words the one-loop result obtained here is exact.

For $\alpha = 0$ and in the limit $M \rightarrow \infty$ we can also compute the values of ϕ and g^2 at the critical point and find them to be given by $\phi_* \sim \pm 1/\sqrt{2}$ and $g_*^2 \sim M^2/8$. In other words the phase transition happens each time at a larger value of the coupling constant when M is increased and hence it is harder for the system to reach the pure matrix phase for large enough masses if one starts from the fuzzy sphere phase.

This article is organized as follows. Section 2 introduces the fuzzy sphere. In section 3 the fuzzy gauge field is defined and its different actions are written down. In section 4 we quantize the model in the Feynman-'t Hooft Background field gauge and compute the quadratic effective action by computing tadpole graphs and the vacuum polarization tensor. In section 5 we study the continuum limit of the theory in great detail and show the existence of a gauge-invariant UV-IR mixing in the limit. In section 6 we compute the effective potential of the scalar mode and show the presence of a first order phase transition in the model. The critical point in the strict limit $L = \infty$ is written down and we show its gauge-fixing independence. In section 7 we study the large mass limit of the model and show that in this limit the scalar mode is decoupled from the gauge modes and correspondingly the gauge-sector of the theory is UV-IR free in the continuum limit. We also show that in the presence of a large mass term the first order phase transition of the model becomes harder to reach from small couplings as we increase the mass. In the appendices we give the detail of our calculation.

2 The Fuzzy Sphere

In here we define the non-commutative fuzzy sphere by Connes spectral triple $(Mat_{L+1}, H_L, \Delta_L)$ [10]. Mat_{L+1} is the algebra of $(L+1)\times(L+1)$ matrices which acts on an $(L+1)-$dimensional Hilbert space H_L with inner product $(M,N) = \frac{1}{L+1}Tr(M^\dagger N)$ where $M, N \in Mat_{L+1}$. Δ_L is the Laplacian on the fuzzy sphere which we will define shortly. Matrix coordinates on $\mathbf{S}_L^2$ are defined by

$$x_1^2 + x_2^2 + x_3^2 = 1 \ , \ [x_a, x_a] = \frac{i}{|L|}\epsilon_{abc}x_c, \tag{2.1}$$

with

$$x_a = \frac{L_a}{|L|}. \tag{2.2}$$

L_a are the generators of the irreducible representation $\frac{L}{2}$ of $SU(2)$, i.e $[L_a, L_b] = i\epsilon_{abc}L_c$, $\sum_a L_a^2 = |L|^2 \equiv \frac{L}{2}(\frac{L}{2}+1)$. The Hilbert space H_L is naturally associated with this representation. These definitions are motivated by the fact that $\mathbf{S}^2$ is nothing but the co-adjoint orbit $SU(2)/U(1)$ which is thus a symplectic manifold and hence it can be quantized in a canonical fashion by simply quantizing the volume form $\omega = dcos\theta \wedge d\phi$ [16].

"Fuzzy" functions on $\mathbf{S}^2_L$ are linear operators in the matrix algebra while derivations are inner defined by the generators of the adjoint action of $SU(2)$, in other words the derivative of the fuzzy function $\phi \in Mat_{L+1}$ in the space-time direction a is the commutator $[L_a, \phi]$. This can also be put in the form

$$AdL_a(\phi) \equiv [L_a, \phi] = (L_a^L - L_a^R)(\phi) \equiv \mathcal{L}_a(\phi), \tag{2.3}$$

where L_a^L's and $-L_a^R$'s are the generators of the IRR $\frac{L}{2}$ of $SU(2)$ which act respectively on the left and on the right of the algebra Mat_{L+1}, i.e $L_a^L\phi \equiv L_a\phi$, $L_a^R\phi \equiv \phi L_a$ for any $\phi \in Mat_{L+1}$.

A natural choice of the Laplacian operator Δ_L on the fuzzy sphere is therefore given by the following Casimir operator

$$\Delta_L = (L_a^L - L_a^R)^2 \equiv \mathcal{L}^2. \tag{2.4}$$

Thus the algebra of matrices Mat_{L+1} decomposes under the action of the group $SU(2)$ as $\frac{L}{2} \otimes \frac{L}{2} = 0 \oplus 1 \oplus 2 \oplus .. \oplus L$ (The first $\frac{L}{2}$ stands for the left action of the group while the other $\frac{L}{2}$ stands for the right action). It is not difficult to convince ourselves that this Laplacian has a cut-off spectrum of the form $k(k+1)$ where $k = 0, 1, ..., L$. As a consequence a general function on $\mathbf{S}^2_L$, i.e scalar fields, can be expanded in terms of polarization tensors as follows

$$\phi = \sum_{k=0}^{L} \sum_{m=-k}^{k} \phi_{km} \hat{Y}_{km}. \tag{2.5}$$

For an extensive list of the properties of $\hat{Y}_{km}(l)$ see [2].

3 Classical gauge fields on $\mathbf{S}^2_L$

3.1 The Alekseev-Recknagel-Schomerus action.

It was shown in [5] that the dynamics of open strings moving in a curved space with $\mathbf{S}^3$ metric in the presence of a non-vanishing Neveu-Schwarz B-field and with Dp-branes is equivalent to leading order in the string tension to a gauge theory on a noncommutative fuzzy sphere with a Chern-Simons term. The full $U(n)$ action on the fuzzy sphere they found is given by the combination

$$S_{ARS}[D_a] \;=\; S_{YM}[D_a]+S_{CS}[D_a], \tag{3.1}$$

where D_a is the covariant derivative with curvature $F_{ab}=[D_a,D_b]-i\epsilon_{abc}D_c$ and the Yang-Mills and Chern-Simons actions are given respectively by

$$\begin{aligned} S_{YM}[D_a] &= -\frac{1}{4g^2}Tr_Ltr\;\;F_{ab}^2 \\ S_{CS}[D_a] &= -\frac{1}{6g^2}Tr_Ltr\left[i\epsilon_{abc}F_{ab}D_c+(D_a^2-L_a^2)\right]. \end{aligned} \tag{3.2}$$

This result is simply an extension of the original result of [3] in which strings moving in a flat space in the presence of a constant N-S B-field are described in the limit $\alpha^{'}\longrightarrow 0$ by a Moyal-Weyl noncommutative gauge theory. From string theory point of view the above combination of Yang-Mills and Chern-Simons actions is therefore the most natural candidate for a gauge action on the fuzzy sphere. In most of this paper we will thus work with the action

$$S_{ARS}[D_a]=\frac{1}{g^2}Tr_Ltr\left[-\frac{1}{4}[D_a,D_b]^2+\frac{i}{3}\epsilon_{abc}[D_a,D_b]D_c\right]+\frac{1}{6g^2}Tr_Ltr|L|^2. \tag{3.3}$$

Gauge transformations are implemented here by the unitary transformations $U\in U_{n(L+1)}$ as follows $D_a{\rightarrow}D_a^{'}=UC_aU^{-1}$. In above we have adopted the convention $Tr_L=\frac{Tr}{L+1}$ and set $R=1$ where R is the radius of the sphere . tr is the trace over the gauge group. S_{ARS} is the Alekseev-Recknagel-Schomerus action.

The first remark about this action is the fact that there is no quadratic term, i.e the term D_a^2 from the YM part cancels exactly the term D_a^2 from the CS. Furthermore we remark (as we will show shortly) that in the Feynman gauge $\xi=1$, the kinetic term reduces to $\mathcal{L}^2$: This is simply the inverse propagator in the plane which can already be seen at the level of equations of motion. Indeed varying the action yields

$$\delta S_{ARS}=-\frac{1}{g^2}Tr_Ltr\delta D_a\left([D_b,[D_a,D_b]]-i\epsilon_{abc}[D_b,D_c]\right), \tag{3.4}$$

and thus we obtain the equations of motion

$$[D_b,F_{ab}]=0\;,\;\;F_{ab}=[D_a,D_b]-i\epsilon_{abc}D_c. \tag{3.5}$$

As it was shown in [5] classical solutions in the presence of the Chern-Simons term which are also absolute minima of the action are characterized by $SU(2)$ IRR. (3.5) can also be solved with general $SU(2)$ representations as well as with diagonal matrices.

A final remark about the action (3.3) is to note that it has the extra symmetry $D_a{\longrightarrow}D_a+\alpha_a\mathbf{1}_{n(L+1)}$ where α_a are constant real numbers . This symmetry needs to be fixed by restricting the covariant derivative D_a to be traceless, i.e by removing the zero mode [6,7]. This symmetry manifests itself also in the form $A_a{\longrightarrow}A_a+\alpha_a\mathbf{1}_{n(L+1)}$ where A_a is the gauge field defined by

$D_a = L_a + A_a$. Remark however that for $D_a = B_a \mathbf{1}_{n(L+1)}$ the action takes the value $\frac{1}{6g^2} n|L|^2$ whereas for $A_a = B_a \mathbf{1}_{n(L+1)}$ the action is identically zero.

As it turns out the action (3.3) on its own does not describe in the continuum limit pure gauge fields. Indeed we can show that in the continuum limit the gauge field A_a decomposes as $A_a = a_a + n_a \phi$ where a_a is the field tangent to the sphere while ϕn_a is the normal component and as a consequence the gauge action becomes

$$S_{ARS} = -\frac{1}{4g^2} \int_{\mathbf{S}^2} \frac{d\Omega}{4\pi} tr\left[f_{ab}^2 + 4i\epsilon_{abc} f_{ab} n_c \phi + 2[\mathcal{L}_a + a_a, \phi]^2 - 4\phi^2 \right]. \tag{3.6}$$

f_{ab} is clearly the curvature of the field a_a, i.e $f_{ab} = \mathcal{L}_a a_b - \mathcal{L}_b a_a - i\epsilon_{abc} a_c + [a_a, a_b]$. As one can immediately see this theory consists of a 2−dimensional gauge field with a Higgs particle.

3.2 Scalar action.

Next we show how to suppress the scalar fluctuation ϕ in order to reduce the model to a purely two-dimensional Yang-Mills theory. It is obvious that the 3−component gauge field is an element of the full projective module $Mat_{n(L+1)} \otimes C^3$. This is due to the fact that our description of the limiting commutative sphere uses global coordinates n_a, $a = 1, 2, 3$ instead of local patches. Nevertheless strictly two-dimensional gauge fields can still be defined in the fuzzy as elements of the tangent projective module $P^T(Mat_{n(L+1)} \otimes C^3)$ where P^T is the projector given by

$$P^T_{ab} = \delta_{ab} - x_a x_b. \tag{3.7}$$

The meaning of this projector can be explained as follows. The algebra of matrices $Mat_{n(L+1)}$ represents both the space, i.e the sphere, and the gauge group $U(n)$. It is clear therefore that in the presence of a spin 1 field the algebra of matrices Mat_{L+1} decomposes under the action of the rotation group $SU(2)$ as follows

$$\Gamma_{\frac{L}{2}} \otimes \Gamma_{\frac{L}{2}} \otimes \Gamma_1 = \Gamma_{\frac{L}{2}} \otimes \left(\Gamma_{\frac{L}{2}+1} \oplus \Gamma_{\frac{L}{2}} \oplus \Gamma_{\frac{L}{2}-1} \right). \tag{3.8}$$

The two IRR $\Gamma_{\frac{L}{2}}$ stand for the left and right actions of the group on the algebra Mat_{L+1} whereas Γ_1 stands for the spin 1 structure we want to add. It is rather a trivial exercise to compute the projectors on the spaces $\Gamma_{\frac{L}{2}+1}$, $\Gamma_{\frac{L}{2}-1}$ and verify that $P^T = P_+ + P_-$. In other words $P_+ A$ and $P_- A$ are the components of the gauge field tangent to the sphere whereas the normal component can be simply defined by $P_0 A$ where $P_0 \equiv P^N = 1 - P^T$ which clearly projects on the space $\Gamma_{\frac{L}{2}}$ and reads in terms of components

$$P^N_{ab} = x_a x_b. \tag{3.9}$$

By analogy with the language of continuum manifolds one can think of P^T as the projector onto the fuzzy tangent bundle (In fact in the continuum limit P^T is precisely the projector onto the tangent bundle $\mathbf{TS}^2$). This projector clearly satisfies $(P^T)^2 = P^T = P^{T+}$, $P^T_{ab}x_b = 0$, $x_a P^T_{ab} = 0$ and $P^T_{ab}P^T_{ba} = 2$ which translates the fact that $P^T_{ab}A_b$ is indeed a $2-$dimensional gauge field.

A more practical way of implementing the projection P^T is to constrain in a gauge-covariant way the gauge field A_a to satisfy an extra condition and as a consequence reduce the number of its independent components from 3 to 2 . We adopt here the prescription of [9], i.e we impose on the gauge field A_a the gauge-covariant condition

$$D_a D_a = |L|^2. \tag{3.10}$$

This means that on the fuzzy sphere we allow only gauge configurations D_a which themselves live on a sphere of radius $|L|$ in order for the model to describe a two-dimensional Yang-Mills theory. This constraint reads explicitly

$$\Phi = \frac{1}{2}\left(x_a A_a + A_a x_a + \frac{A_a^2}{|L|}\right) = 0, \tag{3.11}$$

and thus it is not difficult to check that in the continuum limit $L{\longrightarrow}\infty$ the normal component of the gauge field is zero, i.e $\Phi{\equiv}\vec{n}.\vec{A} = 0$. In fact (3.11) is the correct definition of the normal scalar field on the fuzzy sphere which is only motivated by gauge invariance. Following [8] we incorporate the constraint (3.11) into the theory by adding the following scalar action

$$S_M[D_a] = \frac{1}{2g^2}\frac{M^2}{|L|^2}Tr_L tr(D_a^2 - |L|^2)^2 = \frac{2M^2}{g^2}Tr_L tr\Phi^2, \tag{3.12}$$

to (3.3) where M is a large mass. This term in the continuum theory changes the mass term of the Higgs particle appearing in (3.6) from $\sqrt{2}$ to $\sqrt{2(1+2M^2)}$ and hence in the large M limit the normal scalar field simply decouples. The limit of interest in the remainder of this paper is therefore $M{\longrightarrow}\infty$ first then $L{\longrightarrow}\infty$.

The most general $U(n)$ gauge action on the fuzzy sphere which is at most quartic in the fields is therefore given by

$$S[D_a] \;=\; S_{ARS}[D_a] + S_M[D_a] + \frac{\alpha}{2g^2}Tr_L tr(D_a^2 - |L|^2) - \frac{1}{6g^2}Tr_L tr|L|^2 \tag{3.13}$$

or equivalently

$$\begin{aligned} S \;=\; & \frac{1}{g^2}Tr_L tr\left[-\frac{1}{4}[D_a, D_b]^2 + \frac{i}{3}\epsilon_{abc}[D_a, D_b]D_c\right] + \frac{1}{2g^2}\frac{M^2}{|L|^2}Tr_L tr(D_a^2 - |L|^2)^2 \\ + \; & \frac{\alpha}{2g^2}Tr_L tr(D_a^2 - |L|^2), \end{aligned} \tag{3.14}$$

where we have also added a linear term in the scalar field with parameter α while the constant is added so that the action vanishes for pure gauges $D_a = UL_aU^+$.

4 The Feynman-'t Hooft background field gauge

4.1 The effective action.

The partition function of the theory depends therefore on 4 parameters, the Yang-Mills coupling constant g, the mass M of the normal scalar field, the linear coupling constant α as well as the size of the fuzzy sphere L, viz

$$Z_L\left[J\right] \equiv Z_L\left[J; g, M, \alpha\right] = \int \prod_{a=1}^{3} \left[dC_a\right] e^{-S[C]-\frac{1}{g^2}Tr_L J_a C_a}. \tag{4.1}$$

The equations of motion derived from the action (3.14) read in terms of the gauge-covariant current J_a as follows

$$[C_b, F_{ab}] = \frac{M^2}{|L|^2}[C_a, C_b^2 - |L|^2]_+ + \alpha C_a + J_a\ . \tag{4.2}$$

Remark that the above action (3.14) for $M\neq 0$ and/or $\alpha\neq 0$ does not enjoy the symmetry $C_a \longrightarrow C_a + \alpha_a \mathbf{1}_{n(L+1)}$ and thus we can not simply remove the zero mode in this model.

Now we adopt the background field method to the problem of quantization of the $U(1)$ theory and then extend the results to the $U(n)$ theory in a later communication. This method consists in making a perturbation of the field around the classical solution and then quantizing the fluctuation. Towards this end we first separate the field as $C_a = D_a + Q_a$ and write the action in the form

$$\begin{aligned} S[C_a] &= S[D_a] + \frac{1}{g^2}Tr_L\big(\hat{J}_a - J_a\big)Q_a - \frac{1}{2g^2}Tr_L[D_a, Q_b]^2 + \frac{1}{2g^2}(1 + \frac{M^2}{|L|^2})Tr_L[D_a, Q_a]^2 \\ &+ \frac{1}{g^2}Tr_L Q_a[F_{ab}, Q_b] + \frac{1}{2g^2}Tr_L Q_a\left[\alpha\delta_{ab} + \frac{2M^2}{|L|^2}(D_c^2 - |L|^2)\delta_{ab} + \frac{4M^2}{|L|^2}D_a D_b\right]Q_b \\ &+ \text{(higher order terms in } \mathrm{Q_a}) \end{aligned} \tag{4.3}$$

where $\hat{J}_a = -[D_b, F_{ab}] + \frac{M^2}{|L|^2}[D_a, D_b^2 - |L|^2]_+ + \alpha D_a + J_a$, $F_{ab} = [D_a, D_b] - i\epsilon_{abc}D_c$ and where we have only written down explicitly linear and quadratic terms in Q_a. This action is invariant under the gauge transformations $D_a \longrightarrow D_a$, $Q_a \longrightarrow U Q_a U^+ + U[D_a, U^+]$. Remark that the background vector field D_a appears here to play the same role as the role played by L_a in ordinary perturbation theory. This means in particular that in order to fix the gauge in a consistent way the gauge fixing term should be covariant with respect to the background field. For example instead of the Lorentz gauge $[L_a, Q_a] = 0$ we impose here the covariant Lorentz gauge $[D_a, Q_a] = 0$. The gauge fixing term and the Faddeev-Popov term are therefore given by

$$S_{g.f} + S_{gh} = -\frac{1}{2g^2}Tr_L\frac{[D_a, Q_a]^2}{\xi} + \frac{1}{g^2}Tr_L b^+[D_a, [D_a, b]]. \tag{4.4}$$

We will choose now for simplicity the gauge $\xi^{-1} = 1 + \frac{M^2}{|L|^2}$ which will cancel the 4th term in (4.3). This gauge becomes Feynman gauge $\xi = 1$ in the limit $L \longrightarrow \infty$ and the Landau gauge

$\xi = 0$ in the limit $M \longrightarrow \infty$. Furthermore we will assume that the background field D_a satisfies the classical equations of motion (4.2) and hence the 2nd term of (4.3) also vanishes. The partition function becomes then

$$Z_L[J] \;=\; e^{-S[D_a]-\frac{1}{g^2}Tr_L D_a J_a} \det\left(\mathcal{D}^2\right) \int \prod_{a=1}^{3} [dQ_a] \; e^{-\frac{1}{2g^2}Tr_L Q_a \Omega_{ab} Q_b + \dots}, \tag{4.5}$$

where $Det\left(\mathcal{D}^2\right)$ comes obviously from the integration over the ghost field whereas the Laplacian Ω_{ab} is defined by

$$\Omega_{ab} \;=\; \alpha\delta_{ab} + \mathcal{D}_c^2\delta_{ab} + 2\mathcal{F}_{ab} + \frac{2M^2}{|L|^2}(D_c^2 - |L|^2)\delta_{ab} + \frac{4M^2}{|L|^2}D_a D_b. \tag{4.6}$$

In above the notation $\mathcal{D}_a$ and $\mathcal{F}_{ab}$ means that the covariant derivative D_a and the curvature F_{ab} act by commutators, i.e $\mathcal{D}_a(M) = [D_a, M]$, $\mathcal{F}_{ab}(M) = [F_{ab}, M]$ where $M \in Mat_{n(L+1)}$. Similarly $\mathcal{D}^2(f) = [D_a, [D_a, M]]$. Performing the Gaussian path integral we obtain the one-loop effective action

$$\Gamma[D_a] \;=\; S[D_a] + \frac{1}{2}Tr_3 TR \log\Omega - TR\log\mathcal{D}^2. \tag{4.7}$$

Note that the trace TR appearing in this action is not the trace Tr_L over the indices of matrices but it is a trace over 4 indices corresponding to the left action and right action of operators on matrices . Tr_3 means a trace associated with $3-$dimensional rotations. As we will show the above result holds also for $U(n)$ theories on the fuzzy sphere where only the meaning of the different symbols becomes of course different.

4.2 The $U(1)$ theory.

In the following we will concentrate on the $U(1)$ model on the fuzzy sphere with $M = \alpha = 0$. It is not difficult to see that the theory with $\alpha \neq 0$ involves adding the constant α to the propagator and hence it is expected to have the same limiting qualitative behaviour, whereas the theory with $M \neq 0$ will be studied in more detail in the next sections. We introduce the $U(1)$ gauge field by writing $D_a = L_a + A_a$. Although we are not going to use the Feynman rules explicitly it will be instructive to write them down here for completeness. They are extracted from the last two terms of (4.7) or more precisely from the action

$$-\frac{1}{2g^2}Tr_L Q_a \Omega_{ab} Q_b + \frac{1}{g^2}Tr_L b^+ \mathcal{D}^2 b. \tag{4.8}$$

The propagators of the fluctuation fields Q_a and the ghost fields b^+ and b are found to be given by the inverse of the Laplacian $\mathcal{L}^2$ (see Figure 1a and Figure 1b respectively). Indeed it is not difficult to see from (4.8) that the quadratic actions reads

$$-\frac{1}{2g^2}Tr_L Q_a \mathcal{L}^2 Q_a + \frac{1}{g^2}Tr_L b^+ \mathcal{L}^2 b. \tag{4.9}$$

There are also cubic vertices involving the fluctuation field Q_a , the gauge field A_a and the ghost fields b and b^+. The AQQ vertex , the $\mathcal{F}^0QQ$ and the b^+bA vertex are given respectively by the operators

$$
\begin{aligned}
&-\frac{1}{g^2}Tr_LQ_a\left(\frac{\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L}}{2}\right)Q_a\\
&-\frac{1}{g^2}Tr_LQ_a\mathcal{F}^{(0)}_{ab}Q_b\\
&+\frac{1}{g^2}Tr_Lb^+(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L})b.
\end{aligned}
\tag{4.10}
$$

The relevant Feynman graphs are given in Figures $2a$, $2b$ and $2c$. Let us remark here that there is no coupling between the ghost fields and the curvature. The quartic vertices are given on the other hand by the interactions $AAQQ$ and AAb^+b. Explicitly they are given by the following terms in the action

$$
\begin{aligned}
&-\frac{1}{g^2}Tr_LQ_a\left(\frac{[A_c,[A_c,\ \cdot]]\delta_{ab}+2[[A_a,A_b],\ \cdot]}{2}\right)Q_b\\
&+\frac{1}{g^2}Tr_Lb^+\mathcal{A}^2b.
\end{aligned}
\tag{4.11}
$$

The corresponding Feynman graphs are given in $3a$, $3b$.

The first term in (4.7) gives the full tree-level action of the gauge field A_a. This reads

$$S[D_a]=-\frac{1}{4g^2}Tr_LF_{ab}^2-\frac{i}{2g^2}\epsilon_{abc}Tr_L\big[\frac{1}{2}F^{(0)}_{ab}A_c+\frac{1}{3}[A_a,A_b]A_c\big]=S_2+S_3+S_4,\tag{4.12}$$

where $F_{ab}=F^{(0)}_{ab}+[A_a,A_b]$, $F^{(0)}_{ab}=\mathcal{L}_aA_b-\mathcal{L}_bA_a-i\epsilon_{abc}A_c$ and S_2, S_3 and S_4 are the quadratic, cubic and quartic actions respectively of the gauge field A_a. In particular the quadratic action S_2 reads

$$S_2=-\frac{1}{2g^2}Tr_L[L_a,A_b]^2+\frac{1}{2g^2}Tr_L[L_a,A_a]^2.\tag{4.13}$$

We will apply directly the result (4.7) to find quantum corrections to this quadratic action. This will of course capture all quantum corrections to the vacuum polarization tensor as well as tadpole corrections. The quadratic effective action is given by

$$\Gamma_2\ =\ S_2+\frac{1}{2}TR\log\mathcal{L}^2+\frac{1}{2}TR\left(\Delta^{(1)}+\Delta^{(2)}-\frac{1}{2}(\Delta^{(1)})^2\right)-\frac{1}{4}TR(\Delta^{(j)})^2_{aa}\tag{4.14}$$

where we have only kept constant, linear and quadratic terms in the gauge field A_a and where $\Delta^{(1)}$, $\Delta^{(2)}$ and $\Delta^{(j)}$ are defined by

$$\Delta^{(1)}=\frac{1}{\mathcal{L}^2}\big(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L}\big)\ ,\ \Delta^{(2)}=\frac{1}{\mathcal{L}^2}\mathcal{A}^2\ ,\ \Delta^{(j)}_{ab}=\frac{2}{\mathcal{L}^2}\mathcal{F}^{(0)}_{ab}.\tag{4.15}$$

In the appendices we will give the detail of the computation and we summarize the corresponding results below. In this computation we use extensively the following Green's function

$$\left(\frac{1}{\mathcal{L}^2}\right)^{AB,CD} = \frac{1}{L+1}\sum_{lm}\frac{1}{l(l+1)}\hat{Y}_{lm}^{AB}(\hat{Y}_{lm}^{+})^{DC}. \tag{4.16}$$

Remark that since the propagator $(\mathcal{L}^2)^{-1}$ acts on the algebra of matrices it carries 4 indices. We will also need to use the following identities

$$TR\big(M\frac{1}{\mathcal{L}^2}\mathcal{O}\big) = \sum_{lm}\frac{1}{l(l+1)}Tr_L\hat{Y}_{lm}^{+}\mathcal{O}\big(M\hat{Y}_{lm}\big) \tag{4.17}$$

and

$$TR\big(M\frac{1}{\mathcal{L}^2}\mathcal{O}\frac{1}{\mathcal{L}^2}N\big) = \sum_{lm}\sum_{kn}\frac{1}{l(l+1)}\frac{1}{k(k+1)}Tr_L\left[\hat{Y}_{lm}\hat{Y}_{kn}^{+}NM\right]Tr_L\left[\hat{Y}_{lm}^{+}\mathcal{O}(\hat{Y}_{kn})\right]. \tag{4.18}$$

In above M and N are two arbitrary matrices and $\mathcal{O}$ is some operator acting on the space of these matrices.

Tadpole contribution. Quantum correction to the tree-level linear term which is identically zero, i.e $S_1 = 0$, is given by the combination of the two tadpole diagrams of Figure 4. These diagrams are also equal to the third term in expansion (4.14), viz $\Gamma_1 = \frac{1}{2}TR\Delta^{(1)}$. By writing $A_a = \sum_{pn}A_a(pn)\hat{Y}_{pn}$ and Γ_1 as

$$\Gamma_1 = \frac{1}{2}\sum_{pn}\frac{Tr_L\left[L_a,\hat{Y}_{pn}^{\dagger}\right]\left[A_a,\hat{Y}_{pn}\right]}{p\,(p+1)} \tag{4.19}$$

we can compute, using the different identities of [2], the action Γ_1. One finds the result

$$\Gamma_1 = \frac{4}{\sqrt{3}}|L|A_{-\mu}(1-\mu) = 4|L|Tr_Lx_aA_a. \tag{4.20}$$

Now we can use the definition of the normal scalar field on the fuzzy sphere given by $\phi = \frac{1}{2}(x_aA_a + A_ax_a + \frac{A_a^2}{|L|})$ to rewrite this expression in the form

$$\Gamma_1 = 4|L|Tr_L\phi - 2Tr_LA_a^2. \tag{4.21}$$

This identity is exact and as it turns out this separation is crucial in establishing covariance of the $U(1)$ theory in the fuzzy setting in the sense of equation (4.43) below.

Vacuum polarization tensor. *The* $4-$*vertex contribution* to the vacuum polarization tensor is given by the diagrams of Figure 5 which are also equal to the 4th term in the expansion (4.14), i.e $\Gamma_2^{(4)} = \frac{1}{2}TR\Delta^{(2)}$. This can be put in the form

$$\Gamma_2^{(4)} = -\frac{1}{2}\sum_{l_1m_1}\frac{Tr_L[A_a,\hat{Y}_{l_1m_1}^{\dagger}][A_a,\hat{Y}_{l_1m_1}]}{l_1(l_1+1)} \tag{4.22}$$

After a long calculation we get the explicit answer

$$\Gamma_2^{(4)} = Tr_L A_a \mathcal{L}^2 \Delta_4 A_a, \tag{4.23}$$

with the conservation law that $p_1 + l_1 + l_2$ must be an odd number and where the eigenvalues of the operator $\Delta_4 \equiv \Delta_4(\mathcal{L}^2)$ on the eigenvectors $\hat{Y}_{p_1 n_1}$ are given by

$$\Delta_4(p_1) = \sum_{l_1,l_2} \frac{2l_1+1}{l_1(l_1+1)} \frac{2l_2+1}{l_2(l_2+1)} (1-(-1)^{l_1+l_2+p_1})(L+1) \left\{ \begin{array}{ccc} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{array} \right\}^2 \frac{l_2(l_2+1)}{p_1(p_1+1)}. \tag{4.24}$$

The $3-$*vertex contribution* comes from three different diagrams. The contribution of the $\mathcal{F}$ term is given by the diagram of Figure 6b and it corresponds to the last term in expansion (4.14), namely $\Gamma_2^{(3F)} = -\frac{1}{4} TR(\Delta^{(j)})^2_{aa}$ whereas the 5th term in expansion (4.14), i.e $\Gamma_2^{(3A)} = -\frac{1}{4} TR(\Delta^{(1)})^2$, corresponds to the combination of the diagrams displayed in Figure 6a. The diagram of Figure 6b can be represented by

$$\Gamma_2^{(3F)} = \sum_{k_1 m_1} \sum_{k_2 m_2} \frac{Tr_L \left[F_{ab}^{(0)} [\hat{Y}_{k_2 m_2}, \hat{Y}^{\dagger}_{k_1 m_1}] \right] Tr_L \left[F_{ab}^{(0)} [\hat{Y}_{k_1 m_1}, \hat{Y}^{\dagger}_{k_2 m_2}] \right]}{k_1\,(k_1+1)\,k_2\,(k_2+1)} \tag{4.25}$$

For this diagram a long calculation yields the explicit result

$$\Gamma_2^{(3F)} = Tr_L F_{ab}^{(0)} \Delta_F F_{ab}^{(0)}, \tag{4.26}$$

where the operator $\Delta_F \equiv \Delta_F(\mathcal{L}^2)$ is defined by its spectrum

$$\Delta_F(p_1) \;=\; 2 \sum_{l_1,l_2} \frac{2l_1+1}{l_1(l_1+1)} \frac{2l_2+1}{l_2(l_2+1)} (1-(-1)^{l_1+l_2+p_1})(L+1) \left\{ \begin{array}{ccc} l_1 & l_2 & p_1 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{array} \right\}^2. \tag{4.27}$$

The quantum action $\Gamma_2^{(4)}$ given by (4.23) is clearly a correction to the first term of (4.13) whereas the action $\Gamma_2^{(3F)}$ given by (4.26) contains a correction to both terms in (4.13) plus a mass term.

Similarly the diagrams of Figure 6a admit the representation

$$\begin{aligned} \Gamma_2^{(3(A))} = & -\frac{1}{2} \sum_{l_1 m_1} \sum_{l_2 m_2} \frac{Tr_L [L_a, \hat{Y}_{l_1 m_1}][A_a, \hat{Y}^{+}_{l_2 m_2}] Tr_L [L_b, \hat{Y}_{l_2 m_2}][A_b, \hat{Y}^{+}_{l_1 m_1}]}{l_1(l_1+1) l_2(l_2+1)} \\ & -\frac{1}{2} \sum_{l_1 m_1} \sum_{l_2 m_2} \frac{Tr_L [L_a, \hat{Y}_{l_1 m_1}][A_a, \hat{Y}^{+}_{l_2 m_2}] Tr_L [L_b, \hat{Y}^{+}_{l_1 m_1}][A_b, \hat{Y}_{l_2 m_2}]}{l_1(l_1+1) l_2(l_2+1)} \end{aligned} \tag{4.28}$$

Explicitly we find for these diagrams the result

$$\begin{aligned} \Gamma_2^{(3A)} = & -2 \sum_{p_1 n_1} \sum_{p_2 n_2} A_{-\mu}(p_1 n_1) A_{-\nu}(p_2 n_2) (-1)^{n_1+\nu} \sum_{l_1,l_2} \frac{2l_1+1}{l_1(l_1+1)} \frac{2l_2+1}{l_2(l_2+1)} (L+1) \\ & \times \left\{ \begin{array}{ccc} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{array} \right\} \left\{ \begin{array}{ccc} p_2 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{array} \right\} f^{(A)}(lpn; \mu, \nu). \end{aligned} \tag{4.29}$$

In this case we must also have the conservation laws $l_1+l_2+p_1=$ odd number, $l_1+l_2+p_2=$ odd number which means in particular that p_1+p_2 can only be an even number. The function $f^{(A)}$ is of the form $f^{A}=f^{A_1}+f^{A_2}$ where in particular

$$f^{A_1}=-\frac{C^{p_1m}_{p_1n_11\mu}C^{p_1-m}_{p_2n_21\nu}\delta_{p_1p_2}}{2p_1(p_1+1)}\left[l_2(l_2+1)-l_1(l_1+1)\right]\left[l_2(l_2+1)-l_1(l_1+1)-p_1(p_1+1)\right]. \quad (4.30)$$

We can see therefore that the contribution $\Gamma_2^{(3A)}$ splits into two parts, a canonical gauge contribution $\Gamma_2^{(3A_1)}$ plus a non-trivial part $\Gamma_2^{(3A_2)}$ corresponding to whether f is equal to f^{A_1} or f^{A_2} respectively. The canonical gauge part is explicitly given by the expression

$$\Gamma_2^{(3A_1)}=-Tr_LA_a\mathcal{L}_a\Delta_3\mathcal{L}_bA_b, \quad (4.31)$$

where again the operator $\Delta_3\equiv\Delta_3(\mathcal{L}^2)$ is defined by its spectrum

$$\begin{aligned}\Delta_3(p_1) &= \sum_{l_1,l_2}\frac{2l_1+1}{l_1(l_1+1)}\frac{2l_2+1}{l_2(l_2+1)}(1-(-1)^{l_1+l_2+p_1})(L+1)\left\{\begin{array}{ccc} p_1 & l_1 & l_2\\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}^2\\ &\times \frac{l_2(l_2+1)}{p_1^2(p_1+1)^2}\big(l_2(l_2+1)-l_1(l_1+1)\big). \quad (4.32)\end{aligned}$$

This is clearly a correction to the kinetic term in the tree-level action (4.13). We remark that so far all quantum corrections to the vacuum polarization tensor given by equations (4.23), (4.26) and (4.31) are written in terms of the operator

$$\Delta(p_1,p_2) = \sum_{l_1l_2}\frac{2l_1+1}{l_1(l_1+1)}\frac{2l_2+1}{l_2(l_2+1)}(L+1)\left\{\begin{array}{ccc} p_1 & l_1 & l_2\\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}\left\{\begin{array}{ccc} p_2 & l_1 & l_2\\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}X(l_1,l_2,p_1,p_2), \quad (4.33)$$

where X , for all these actions (4.23), (4.26) and (4.31, is of the form $X(l_1,l_2,p_1,p_2)=\delta_{p_1p_2}\bar{X}(l_1,l_2,p_1)$ and where the sums are always over l_1 and l_2 such that $l_1+l_2+p_1$ is an odd number.

The last correction to the vacuum polarization tensor is given by equation (4.29) where the function $f^{(A)}$ is replaced by $f^{(A_2)}$. This leads to a more complicated contribution which we can write in the form

$$\begin{aligned}\Gamma_2^{(3A_2)} &= \sum_{p_1n_1}\sum_{p_2n_2}A_{-\mu}(p_1n_1)A_{-\nu}(p_2n_2)(-1)^{n_1+\nu}\Big[C^{p_1-1m}_{p_1n_11\mu}C^{p_1-1-m}_{p_2n_21\nu}\big(\Lambda^{(-)}(p_1,p_2)+\Sigma^{(-)}(p_1,p_2)\big)\\ &+ C^{p_1+1m}_{p_1n_11\mu}C^{p_1+1-m}_{p_2n_21\nu}\big(\Lambda^{(+)}(p_1,p_2)+\Sigma^{(+)}(p_1,p_2)\big)\Big]. \quad (4.34)\end{aligned}$$

The functions $\Lambda^{(\pm)}(p_1,p_2)$ are of the form (4.33) with some X such that $\Lambda^{(\pm)}(p_1,p_2)=\delta_{p_1,p_2}\bar{\Lambda}^{(\pm)}(p_1)$ whereas the functions $\Sigma^{(\pm)}(p_1,p_2)$ are the form (4.33) but with X's such that $\Sigma^{(\pm)}(p_1,p_2)=$

$\delta_{p_1\pm 2,p_2}\bar{\Sigma}^{(\pm)}(p_1)$. The explicit expressions of the corresponding X's is not important for the purpose of this section and thus we simply skip writing them down here . They are given in appendix C . Finally we can see by inspection that the Clebsch-Gordan coefficients appearing in the action $\Gamma_2^{(3A_2)}$ are exactly those which appear in the scalar mass term in the action . Indeed we can compute

$$\frac{1}{4}Tr_L[x_a,A_a]_+^2=\sum_{p_1n_1}\sum_{p_2n_2}A_{-\mu}(p_1n_1)A_{-\nu}(p_2n_2)(-1)^{n_1+\nu}\Big[C^{p_1-1m}_{p_1n_11\mu}C^{p_1-1-m}_{p_2n_21\nu}$$
$$\times\quad\big(\lambda^{(-)}(p_1,p_2)+\sigma^{(-)}(p_1,p_2)\big)+C^{p_1+1m}_{p_1n_11\mu}C^{p_1+1-m}_{p_2n_21\nu}\big(\lambda^{(+)}(p_1,p_2)+\sigma^{(+)}(p_1,p_2)\big)\Big],\tag{4.35}$$

where $\lambda^{(\pm)}(p_1,p_2)$ and $\sigma^{(\pm)}(p_1,p_2)$ are some other functions which are such that $\lambda^{(\pm)}(p_1,p_2)=\delta_{p_1,p_2}\bar{\lambda}^{(\pm)}(p_1)$, $\sigma^{(\pm)}(p_1,p_2)=\delta_{p_1\pm 2,p_2}\bar{\sigma}^{(\pm)}(p_1)$. These functions are classical and hence they are not loops of the form (4.33) as it must be obvious (see appendix C). By comparing (4.34) and (4.35) we can immediately deduce that the action $\Gamma_2^{(3A_2)}$ in position space must involve anticommutators of x_a and A_a instead of commutators and hence it is of a scalar-like type. As we will show this action will still contain (in the limit) terms which describe non-local interactions between the scalar and the gauge fields. In configuration it reads

$$\Gamma_2^{(3A_2)}\;=\;\sum_{ij}Tr_L[\nabla_i(A_a),x_a]_+\Delta_{ij}\big([\nabla_j(A_b),x_b]_+\big).\tag{4.36}$$

The operators Δ_{ij} are some combinations of the operators $\Lambda^{(\pm)}$ and $\Sigma^{(\pm)}$ whereas the operators ∇_i are sign operators of the form $\nabla_i=(-1)^{\alpha_i\hat{N}}$, $\hat{N}\hat{Y}_{lm}=l\hat{Y}_{lm}$. Again the corresponding explicit expressions are found in appendix C.

4.3 Gauge covariance on S^2_L.

By putting together equations (4.21), (4.23), (4.26), (4.31) and (4.36) we obtain the full quadratic $U(1)$ effective action on $\mathbf{S}^2_L$, namely

$$\Gamma_2\;=\;S_2+\frac{1}{2}TRlog\mathcal{L}^2+4|L|Tr_L\phi+Tr_LA_a\big(\mathcal{L}^2\Delta_4-2\big)A_a-Tr_LA_a\mathcal{L}_a\Delta_3\mathcal{L}_bA_b$$
$$+\;\;Tr_LF^{(0)}_{ab}\Delta_FF^{(0)}_{ab}+\Gamma_2^{(3A_2)}.\tag{4.37}$$

In is rather clear that the first 3 terms are gauge invariant. Naturally we also expect that the 4th and 5th terms in (4.37) to become gauge invariant in the continuum limit. To check this property explicitly we rewrite these two terms as follows

$$Tr_LA_a\big(\mathcal{L}^2\Delta_4-2\big)A_a-Tr_LA_a\mathcal{L}_a\Delta_3\mathcal{L}_bA_b\;=\;-\frac{1}{2}Tr_LF^{(0)}_{ab}\Delta_3F^{(0)}_{ab}$$
$$-\;\;\frac{i}{2}\epsilon_{abc}Tr_LF^{(0)}_{ab}\big(\Delta_3+\mathcal{L}^2(\Delta_3-\Delta_4)+2\big)A_c$$
$$+\;\;i\epsilon_{abc}Tr_L\mathcal{L}_aA_b\big(\mathcal{L}^2(\Delta_3-\Delta_4)+2\big)A_c.\tag{4.38}$$

The first two terms in this expression are now exactly gauge invariant in the continuum limit whereas the third term it can not be gauge invariant unless it vanishes identically. We expect therefore by the requirement of gauge invariance alone that we have the asymptotic behaviour $\mathcal{L}^2(\Delta_3-\Delta_4)+2\longrightarrow 0$, $L\longrightarrow\infty$. As it turns out this statement is true for all finite values of L, in other words we have in fact the identity

$$\mathcal{L}^2(\Delta_3-\Delta_4)+2=0. \tag{4.39}$$

Indeed we have from (4.24) and (4.32) the difference

$$\begin{aligned}
\Delta_3(p_1)-\Delta_4(p_1) &= \frac{(L+1)}{p_1^2(p_1+1)^2}\sum_{l_1,l_2}\frac{2l_1+1}{l_1(l_1+1)}\frac{2l_2+1}{l_2(l_2+1)}(1-(-1)^{l_1+l_2+p_1})\left\{\begin{array}{ccc} p_1 & l_1 & l_2\\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}^2\\
&\times\ l_2(l_2+1)\left[l_2(l_2+1)-l_1(l_1+1)-p_1(p_1+1)\right]\\
&= -\frac{2(L+1)}{p_1^2(p_1+1)^2}\sqrt{p_1(p_1+1)(2p_1+1)}\sum_{l_1}\frac{2l_1+1}{l_1(l_1+1)}\sqrt{l_1(l_1+1)(2l_1+1)}\\
&\times\ \sum_{l_2}(2l_2+1)(1-(-1)^{l_1+l_2+p_1})\left\{\begin{array}{ccc} p_1 & l_1 & l_2\\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}^2\left\{\begin{array}{ccc} l_2 & l_1 & p_1\\ 1 & p_1 & l_1\end{array}\right\}\\
&= -\frac{2\,(L+1)}{p_1^2(p_1+1)^2}\sqrt{p_1(p_1+1)(2p_1+1)}\sum_{l_1}\frac{2l_1+1}{l_1(l_1+1)}\sqrt{l_1(l_1+1)(2l_1+1)}\\
&\times\ \left[\left\{\begin{array}{ccc} p_1 & p_1 & 1\\ \frac{L}{2} & \frac{L}{2} & l_1\\ \frac{L}{2} & \frac{L}{2} & l_1\end{array}\right\}-(-1)^{l_1+p_1+1}\left\{\begin{array}{ccc} \frac{L}{2} & \frac{L}{2} & 1\\ p_1 & p_1 & \frac{L}{2}\end{array}\right\}\left\{\begin{array}{ccc} \frac{L}{2} & \frac{L}{2} & 1\\ l_1 & l_1 & \frac{L}{2}\end{array}\right\}\right],
\end{aligned} \tag{4.40}$$

where we have used in the second line the identity

$$\left\{\begin{array}{ccc} l_2 & l_1 & p_1\\ 1 & p_1 & l_1\end{array}\right\}=-\frac{1}{2}\frac{l_2(l_2+1)-l_1(l_1+1)-p_1(p_1+1)}{\sqrt{l_1(l_1+1)(2l_1+1)p_1(2p_1+1)(p_1+1)}}, \tag{4.41}$$

then performed the sum over l_2 by using equations (5) and (6) on page 305 of [2]. Similarly we can use the explicit expressions of the resulting $9j$ and $6j$ symbols and then do the sum over l_1 to obtain the final result

$$\Delta_3(p_1)-\Delta_4(p_1)=-\frac{2}{p_1(p_1+1)}. \tag{4.42}$$

This is exactly equation (4.39). Thus the quadratic effective action on the fuzzy sphere reads

$$\begin{aligned}
\Gamma_2 &= S+\frac{1}{2}\sum_{l=1}^{L}(2l+1)Log l(l+1)+4|L|Tr_L\phi+Tr_LF_{ab}^{(0)}\big(\Delta_F-\frac{1}{2}\Delta_3\big)F_{ab}^{(0)}\\
&- \frac{i}{2}\epsilon_{abc}Tr_LF_{ab}^{(0)}\Delta_3A_c+\Gamma_2^{(3A_2)}.
\end{aligned} \tag{4.43}$$

From this last expression it is now obvious that the $4th$ and $5th$ terms in this effective action become gauge invariant in the continuum limit. It remains now to establish gauge invariance of the last term $\Gamma_2^{(3A_2)}$. To see this let us recall that $U(1)$ gauge transformations on the continuum sphere act as follows $A_a \longrightarrow A_a^{\Lambda} = A_a - i\mathcal{L}_a(\Lambda)$. Let us also observe that in equation (4.36) the operators ∇_i depend only on $\mathcal{L}^2$ and since $[\mathcal{L}^2, \mathcal{L}_a] = 0$ we can show that $\nabla_i(A_a^{\Lambda}) = \nabla_i(A_a) - i\mathcal{L}_a(\nabla_i(\Lambda))$. Hence in the limit we will have the identity $x_a\nabla_i(A_a^{\Lambda}) = x_a\nabla_i(A_a)$ and as a consequence the action $\Gamma_2^{(3A_2)}$ is gauge invariant as expected.

This in fact establishes gauge invariance of the above quadratic effective action in the continuum large L limit. However in order to show gauge invariance of the whole model for finite L we need also to compute the quantum corrections to the cubic and quartic vertices.

5 The continuum limit and the UV-IR mixing

5.1 The UV-IR mixing.

The criterion for the existence of a UV-IR mixing phenomena on the fuzzy sphere is defined by the requirement that the fuzzy quantum effective action does not approach the quantum effective action on the commutative sphere. For a $U(1)$ theory on ordinary S^2 the action (3.6) is quadratic in the fields a_a and ϕ and thus the quantum corrections are trivial, i.e the effective action on S^2 is essentially equal to the classical action (3.6). On the other hand the quantum corrections on the fuzzy sphere yield the action Γ_2. So to show the existence of a UV-IR mixing phenomena in this model we need only to check that some (or all) of the operators in the above effective action (4.43) do not vanish in the continuum limit. In other words we need to show that Γ_2 does not tend to S in the limit. Since each term of the action (4.43) will be gauge invariant in the continuum limit it is safe to study the continuum limit of the individual operators Δ_3, Δ_4, Δ_F and those appearing in $\Gamma_2^{(3A_2)}$.

In this section we concentrate only on the operators Δ_3, Δ_4 and Δ_F. We go back to equation (4.24) and rewrite the loop $\Delta_4(p_1)$ in the form

$$\Delta_4(p_1) = \frac{(L+1)}{p_1(p_1+1)}\sum_{l_1}\frac{2l_1+1}{l_1(l_1+1)}\sum_{l_2}(2l_2+1)(1-(-1)^{l_1+l_2+p_1})\begin{Bmatrix} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{Bmatrix}^2 \frac{l_2(l_2+1)}{p_1(p_1+1)}. \tag{5.1}$$

We can immediately notice that it is possible in fact to do the sum over l_2 using the results

$$\sum_{l_2}(2l_2+1)\begin{Bmatrix} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{Bmatrix}^2 = \frac{1}{L+1} \tag{5.2}$$

and

$$\sum_{l_2}(2l_2+1)(-1)^{l_2}\begin{Bmatrix} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{Bmatrix}^2 = (-1)^L\begin{Bmatrix} p_1 & \frac{L}{2} & \frac{L}{2} \\ l_1 & \frac{L}{2} & \frac{L}{2} \end{Bmatrix}. \tag{5.3}$$

Δ_4 takes therefore the simple form

$$\Delta_4(p_1) = \frac{1}{p_1(p_1+1)}\sum_l \frac{2l+1}{l(l+1)}\left[1-(L+1)(-1)^{l_1+p_1+L}\left\{\begin{array}{ccc} p_1 & \frac{L}{2} & \frac{L}{2} \\ l_1 & \frac{L}{2} & \frac{L}{2}\end{array}\right\}\right]. \qquad (5.4)$$

As it turns out we can use in the large L limit the same approximation used in [27], namely

$$\left\{\begin{array}{ccc} p_1 & \frac{L}{2} & \frac{L}{2} \\ l_1 & \frac{L}{2} & \frac{L}{2}\end{array}\right\} \simeq \frac{(-1)^{L+p_1+l_1}}{L}P_{p_1}(1-\frac{2l_1^2}{L^2}), \qquad (5.5)$$

where P_p are the Legendre polynomials which are defined by the generating function

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{p=0}^{\infty} P_p(x)t^p. \qquad (5.6)$$

It is quite straightforward to conclude that in the limit we must have

$$\Delta_4(p_1) = -\frac{1}{p_1(p_1+1)}\int_{-1}^{1}\frac{dx}{1-x}\left[P_{p_1}(x)-1\right] \equiv -\frac{h(p_1)}{p_1(p_1+1)}, \qquad (5.7)$$

while by equation (4.42) we must also have

$$\Delta_3(p_1) = -\frac{h(p_1)+2}{p_1(p_1+1)}. \qquad (5.8)$$

The number $h(p_1)$ can be evaluated using the following trick. We regularize the integral by replacing the upper bound by $1-\delta$. By using (5.6) one have the following result

$$\sum_{p=1}^{\infty} h(p)t^p = \frac{2}{1-t}\int_{1+\bar{\delta}}^{\frac{1+t}{1-t}}\frac{d\alpha}{\alpha^2-1} + \frac{1}{1-t}ln\frac{\delta}{2} = \frac{1}{1-t}ln\frac{t\delta}{\bar{\delta}} = \frac{2}{1-t}ln(1-t)\ ,\ \bar{\delta} = \frac{t}{(1-t)^2}\delta. \qquad (5.9)$$

We can further write the above equation as follows

$$\sum_{p=1}^{\infty} h(p)t^p = -2\sum_{p=0}^{\infty}\sum_{k=1}^{\infty}\frac{t^{p+k}}{k} = -2\sum_{p=1}^{\infty}\sum_{k=1}^{p}\frac{t^p}{k},$$

and thus $h(p_1) = -2\sum_{l=1}^{p_1}\frac{1}{l}$. Hence one obtains

$$\Delta_4(p_1) = \frac{2\sum_{l=1}^{p_1}\frac{1}{l}}{p_1(p_1+1)}\ ,\ \Delta_3(p_1) = \frac{2\sum_{l=2}^{p_1}\frac{1}{l}}{p_1(p_1+1)}. \qquad (5.10)$$

Putting the above results together we obtain in the continuum the effective action

$$\begin{aligned}\Gamma_2 &= S_2 + \frac{1}{2}\sum_{l=1}^{L}(2l+1)\log l(l+1) - \frac{1}{2}\int\frac{d\Omega}{4\pi}F_{ab}^{(0)}\Delta_3 F_{ab}^{(0)} - \frac{i}{2}\epsilon_{abc}\int\frac{d\Omega}{4\pi}F_{ab}^{(0)}\Delta_3 A_c \\ &+ 4|L|\int\frac{d\Omega}{4\pi}\phi + \Gamma_2^{(3A_2)}. \qquad (5.11)\end{aligned}$$

In above we have also used the result that the operator Δ_F approaches zero in the continuum limit (see below for the proof). The first correction to the classical action S_2 is just an infinite constant. The 2nd and 3th of the effective action (5.11) corrections clearly give rise to a non-trivial quantum contribution to the $U(1)$ action on $\mathbf{S}^2$ due to the fuzzy sphere, i.e the $U(1)$ theory on $\mathbf{S}^2$ obtained as a limit of a $U(1)$ theory on $\mathbf{S}^2_L$ is not a simple Gaussian theory. These two terms reflect the existence of a gauge invariant UV-IR mixing on the fuzzy sphere which survives the continuum limit. The 4th correction is only relevant for the scalar sector of the theory and thus it does not lead to any noncommutative anomaly or UV-IR mixing in the 2$-$dimensional gauge sector. However we still have to study the continuum limit of the last term $\Gamma_2^{(3A_2)}$ in some more detail.

5.2 The continuum limit.

In this subsection we will study in detail the continuum limit of the effective action $\Gamma_2^{(3A_2)}$. As we have said before the operators Δ_{ij} appearing in (4.36) are of the form (4.33) with some complicated X's. It is expected that the loop (4.33) will be dominated in the large L limit by the UV region of the internal momenta l_1 and l_2, i.e we should be able to assume for all practical purposes that l_1 and l_2 are much larger compared to 1. To be precise let us split the sum $\sum_{l_1l_2}$ as follows $\sum_{l_1=0}^{L}\sum_{l_2=|l_1-p_1|}^{l_1+p_1}=\sum_{l_1=0}^{\Lambda}\sum_{l_2=|l_1-p_1|}^{l_1+p_1}+\sum_{l_1=\Lambda}^{L}\sum_{l_2=|l_1-p_1|}^{l_1+p_1}$ where Λ is an intermediary cut-off which is such that $1<<\Lambda<<L$. Under the first sum, and provided we concentrate on the regime where the external momenta p_1 and p_2 are much less than the cut-off Λ, we can treat the internal momenta l_1, l_2 as small compared to the cut-off L . As a consequence it is easy to check that the contribution of the first sum will indeed go to zero in the limit since $\Lambda<<L$. In the large L limit one can therefore conclude that the full sum will be dominated by the second term corresponding to high internal momenta l_1 and l_2, viz

$$\begin{aligned}\Delta(p_1,p_2) &= \sum_{\epsilon=-\frac{p_1-1}{2}}^{\frac{p_1-1}{2}}\sum_{l=\Lambda-\epsilon}^{L-\epsilon}\frac{(2l+1)^2-4\epsilon^2}{(l^2-\epsilon^2)((l+1)^2-\epsilon^2)}(L+1)\\ &\times\left\{\begin{array}{ccc} l & \frac{L}{2}+\epsilon & \frac{L}{2}\\ \frac{L}{2}-\epsilon & l & p_1\end{array}\right\}\left\{\begin{array}{ccc} l & \frac{L}{2}-\epsilon & \frac{L}{2}\\ \frac{L}{2}+\epsilon & l & p_2\end{array}\right\}X(l_1,l_2,p_1,p_2),\end{aligned}\tag{5.12}$$

where for convenience we have also made the following change of variables $l=\frac{1}{2}(l_1+l_2)$, $\epsilon=\frac{1}{2}(l_1-l_2)\equiv\frac{n_1^{'}}{2}$ and then used Regge symmetries of the $6j$ symbols . Since $\frac{L}{2}$, $\frac{L}{2}\pm\epsilon$ and $l>>p_1,-2\epsilon$ we can use for the asymptotic behaviour of the $6j$ symbols the Edmonds' Formula [2], namely

$$\left\{\begin{array}{ccc} l & \frac{L}{2}+\epsilon & \frac{L}{2}\\ \frac{L}{2}-\epsilon & l & p_1\end{array}\right\}=\frac{1}{2}\frac{(-1)^{L+p_1+l}}{\sqrt{2l+1}}\left[\frac{(-1)^{-\epsilon}}{\sqrt{(L+1+2\epsilon)}}d^{p_1}_{-n_1^{'},0}(\theta)+\frac{(-1)^{\epsilon}}{\sqrt{(L+1-2\epsilon)}}d^{p_1}_{n_1^{'},0}(\theta)\right].\tag{5.13}$$

where $d^{p_1}_{-n_1^{'},0}(\theta)$, $d^{p_1}_{n_1^{'},0}(\theta)$ are rotation matrices and θ is the angle defined by

$$cos\theta=\frac{l(l+1)+L\epsilon+\epsilon(\epsilon+1)}{\sqrt{l(l+1)(L+2\epsilon)(L+2\epsilon+1)}}\simeq\frac{l}{L}\ ,\ \text{for } L\longrightarrow\infty. \tag{5.14}$$

In the continuum large L limit it is not difficult therefore to conclude that the leading contribution to the above loop takes the form

$$\begin{aligned}\Delta(p_1,p_2)\ &=\ \sum_{l=\Lambda}^{L}\frac{1}{2l^3}\sum_{n_1^{'}=-p_1}^{p_1}\big((-1)^{n_1^{'}}-(-1)^{p_1}\big)\Big[(-1)^{n_1^{'}}d^{p_1}_{n_1^{'},0}(\theta)d^{p_2}_{n_1^{'},0}(\theta)+d^{p_1}_{-n_1^{'},0}(\theta)d^{p_2}_{n_1^{'},0}(\theta)\Big]\\ &\times\ X(l,n_1^{'},p_1,p_2).\end{aligned} \tag{5.15}$$

Without any loss of generality we have also assumed for simplicity that $X(l,-n_1^{'},p_1,p_2)=X(l,n_1^{'},p_1,p_2)$. Now we use the result

$$d^{p_2}_{n_1^{'},0}(\theta)=\sqrt{\frac{4\pi}{2p_2+1}}(-1)^{n_1^{'}}Y_{p_2-n_1^{'}}(\theta,0)\ , \tag{5.16}$$

and the fact that for $\phi=0$ we have $Y_{p-n}=(-1)^nY_{pn}$, $Y^*_{pn}=Y_{pn}$ to rewrite the above result in the equivalent form

$$\begin{aligned}\Delta(p_1,p_2)\ &=\ \frac{4\pi}{\sqrt{(2p_1+1)(2p_2+1)}}\sum_{l=\Lambda}^{L}\frac{1}{l^3}\sum_{n_1^{'}=-p_1}^{p_1}\big(1-(-1)^{n_1^{'}+p_1}\big)Y_{p_1n_1^{'}}(\theta,0)Y^*_{p_2n_1^{'}}(\theta,0)\\ &\times\ X(l,n_1^{'},p_1,p_2)\\ &=\ \frac{1}{L^2}\frac{4\pi}{\sqrt{(2p_1+1)(2p_2+1)}}\int_0^{\frac{\pi}{2}-\delta}\frac{\sin\theta d\theta}{\cos^3\theta}\sum_{n_1^{'}=-p_1}^{p_1}\big(1-(-1)^{n_1^{'}+p_1}\big)Y_{p_1n_1^{'}}(\theta,0)Y^*_{p_2n_1^{'}}(\theta,0)\\ &\times\ X(l,n_1^{'},p_1,p_2).\end{aligned} \tag{5.17}$$

In above we have replaced the sum $\sum_l$ by an integral $\int dl$ and then made the change of variable $l=L\cos\theta$. In particular the value $l=\Lambda-\epsilon$ corresponds to the angle θ such that $\cos\theta\simeq\frac{\Lambda}{L}$, i.e $\theta=\frac{\pi}{2}-\delta$ where $\delta=\frac{\Lambda}{L}$ while the value $l=L-\epsilon$ corresponds to the angle $\theta=0$. At this stage we must also use the explicit expressions of the functions X. By inspection all these functions are found to be dominated in the large L limit by quadratic terms in the variable $l=L\cos\theta$ for which the above loop is finite and non-zero as one might easily check. Contribution of the linear and constant terms in X are found to be vanishingly small in the limit. We skip the corresponding trivial algebra for simplicity. This result means in particular that the operator Δ_F approaches zero in the continuum limit whereas the action $\Gamma_2^{(3A_2)}$ survives the continuum limit and thus it provides non-trivial highly non-local interactions between the scalar field and the $2-$dimensional gauge field.

6 Scalar effective potential and phase transition on $\mathbf{S}^2_L$

In this section we will show explicitly and in some detail that the limiting theory $L=\infty$ is actually independent of the gauge fixing parameter ξ (see equation (4.4)) and hence the results obtained in the previous sections are gauge-invariant. As a result of this analysis we will also be able to identify a novel non-perturbative phase transition which occurs in the model. The effective action for generic values of ξ reads

$$\Gamma[D_a] = S[D_a]+\frac{1}{2}Tr_3TR\log\Omega_\xi - TR\log\mathcal{D}^2 \tag{6.1}$$

where now

$$(\Omega_\xi)_{ab} = \alpha\delta_{ab}+\mathcal{D}_c^2\delta_{ab}-(1+\frac{M^2}{|L|^2}-\frac{1}{\xi})\mathcal{D}_a\mathcal{D}_b+2\mathcal{F}_{ab}+\frac{2M^2}{|L|^2}(D_c^2-|L|^2)\delta_{ab}+\frac{4M^2}{|L|^2}D_aD_b \tag{6.2}$$

As before we will set for simplicity $\alpha=0$ and study the two limits $M\longrightarrow 0$ (which corresponds to the ARS action) and $M\longrightarrow\infty$ (which corresponds to projecting out the normal component of the gauge field from the theory). To simplify further we will only be interested in computing the effective potential of the scalar component of the gauge field which is in fact motivated by two other reasons. First the fact that tadpole diagrams of the model (3.14) are largely controlled by the fluctuation of this scalar field. The second motivation is the fact that the expectation value of this scalar field provides a measure of the radius of the sphere (as we will explain) and thus a zero expectation value means that the sphere has disappeared and a phase transition occured. We believe that this result reported first in [11] can be captured in one-loop perturbation theory since the one-loop result becomes exact in the large L limit as we will also argue in the following.

In the continuum theory the normal scalar field is defined in terms of arbitrary gauge configurations A_a by the linear equation $\Phi=A_an_a$. It is not difficult to check that for a constant normal scalar field we have $A_a=\rho n_a$, i.e $\Phi\equiv\rho$ where ρ is some constant real number, with a curvature given by $F_{ab}=i\rho\epsilon_{abc}n_c$. The YM and CS actions for this configuration are both equal to $\frac{1}{2g^2}\rho^2$ while the scalar action S_M is given by $\frac{2M^2}{g^2}\rho^2$. The classical continuum effective potential is therefore given by

$$U=\frac{1+2M^2}{g^2}\rho^2. \tag{6.3}$$

On the other hand and as we have explained earlier the normal scalar field on the fuzzy sphere is defined in terms of arbitrary gauge configurations A_a by the quadratic equation $\Phi=\frac{1}{2}(x_aA_a+A_ax_a+\frac{A_a^2}{|L|})$. However a constant scalar field configuration on the fuzzy sphere can still be defined by $A_a=\rho x_a$ where ρ is again some constant real number. This is because the corresponding curvature is given by a similar equation with a correct continuum limit, viz $F_{ab}=i\rho(1+\frac{\rho}{|L|})\epsilon_{abc}x_c$. In this case the normal scalar field Φ must also satisfy $\Phi=\rho+\frac{\rho^2}{2|L|}$. This can be solved for ρ in terms of Φ and one finds two solutions, namely $\frac{\rho_\pm}{|L|}=-1\pm\sqrt{1+\frac{2\Phi}{|L|}}$. Clearly in the large L limit one of the solutions converge to the actual scalar field Φ whereas the other one diverges, namely $\rho_+\longrightarrow\Phi$, $\rho_-\longrightarrow-2|L|$, i.e on the fuzzy sphere it seems that for every value of Φ we can have two different values of ρ. Obviously around the second solution ρ_- perturbation theory can not be trusted. The sum U of the Yang-Mills, Chern-Simons and scalar actions for this configuration is given by (with $\phi=1+\frac{\rho}{|L|}$)

$$V=\frac{U}{|L|^2}=\frac{1}{2g^2}\left[\phi^4-\frac{4}{3}\phi^3+M^2\left(\phi^2-1\right)^2+\alpha\phi^2\right]. \tag{6.4}$$

Quantum corrections can also be computed using equation (6.1) for generic values of α, M and ξ and one finds the one-loop complete result

$$\begin{aligned} U_{eff} &= U+\frac{1}{2}Tr_3TR\log\Big[\alpha\delta_{ab}+\phi^2\big(\mathcal{L}^2\delta_{ab}-\big[1+\frac{M^2}{|L|^2}-\frac{1}{\xi}\big]\mathcal{L}_a\mathcal{L}_b\big)+2\phi(1-\phi)(\vec{\theta}.\vec{\mathcal{L}})_{ab} \\ &+ 2M^2(\phi^2-1)\delta_{ab}+4M^2\phi^2P^N_{ab}\Big]-TR\log[\phi^2\mathcal{L}^2]. \end{aligned} \tag{6.5}$$

In above we have used the following identity $\mathcal{L}_a\mathcal{L}_b=\mathcal{L}^2\delta_{ab}-(\vec{\theta}.\vec{\mathcal{L}})_{ab}-(\vec{\theta}.\vec{\mathcal{L}})^2_{ab}$ where θ_a are the generators of $SU(2)$ in the spin 1 irreducible representation , i.e $[\theta_a,\theta_b]=i\epsilon_{abc}\theta_c$, $\sum_a\theta_a^2=2$, $(\theta_a)_{bc}=-i\epsilon_{abc}$. The total angular momentum is therefore $\vec{J}=\vec{\mathcal{L}}+\vec{\theta}$.

For $M=0$ and $\alpha=0$ the classical potential takes the simple form

$$V[\phi]\equiv\frac{U[\phi]}{|L|^2}=\frac{1}{2g^2}\phi^3\left(\phi-\frac{4}{3}\right). \tag{6.6}$$

The minimum of this potential is clearly $\phi=1$, i.e the fuzzy sphere with coordinates $D_a=L_a$. Quantum corrections are given by

$$V_{eff}[\phi]=V[\phi]+4\log\phi+\Delta V \tag{6.7}$$

where

$$\begin{aligned} \Delta V &= \frac{1}{2|L|^2}Tr_3TR\log\mathcal{H}-\frac{1}{|L|^2}TR\log\mathcal{L}^2 \\ \mathcal{H} &= \Big[\mathcal{L}^2+(\frac{1}{\phi}-1)(J^2-\mathcal{L}^2-2)+(1-\frac{1}{\xi})\big[\frac{1}{4}(J^2-\mathcal{L}^2)^2-\frac{1}{2}(J^2-\mathcal{L}^2)-\mathcal{L}^2\big]\Big]. \end{aligned} \tag{6.8}$$

The eigenvalues of $\mathcal{L}^2$ and $\vec{J}^2$ are given respectively by $l(l+1)$ and $j(j+1)$ respectively where $l=1,...,L$ and $j=l+1,l,l-1$. The corresponding eigenvectors are the vector spherical harmonics operators . Clearly the operator $\mathcal{H}$ is diagonal in this basis , explicitly we have

$$\begin{aligned} \Delta V &= \frac{1}{2|L|^2}\sum_l(2l+1)\log\big[\frac{1}{\xi}l(l+1)-2(\frac{1}{\phi}-1)\big]+\frac{1}{2|L|^2}\sum_l(2l+3)\log\big[l(l+1)+2l(\frac{1}{\phi}-1)\big] \\ &+ \frac{1}{2|L|^2}\sum_l(2l-1)\log\big[l(l+1)-2(l+1)(\frac{1}{\phi}-1)\big]-\sum_l(2l+1)\log l(l+1). \end{aligned}$$

Obviously all $\xi-$dependence of this potential ΔV is confined to the first term. In the large L limit the relevant terms are given by

$$\begin{aligned}\Delta V &= \frac{1}{2|L|^2}\sum_l (2l+1)\log\big[1-\frac{2\xi}{l(l+1)}(\frac{1}{\phi}-1)\big]+\frac{1}{2|L|^2}\sum_l (2l-1)\log\big[1-\frac{2}{l}(\frac{1}{\phi}-1)\big]\\ &+ \frac{1}{2|L|^2}\sum_l (2l+3)\log\big[1+\frac{2}{l+1}(\frac{1}{\phi}-1)\big].\end{aligned} \tag{6.9}$$

It is rather a straightforward exercise to show that in the strict limit this potential ΔV vanishes as $\frac{\log L}{L^2}$. In other words the $\xi-$dependence drops completely and the theory $L=\infty$ is gauge invariant. We end up therefore with the simple potential

$$V_{eff}=\frac{1}{2g^2}(\phi^4-\frac{4}{3}\phi^3)+4\log\phi. \tag{6.10}$$

Taking the first and second derivatives of this potential we obtain

$$\begin{aligned}V_{eff}^{'} &= \frac{1}{2g^2}(4\phi^3-4\phi^2)+\frac{4}{\phi}\\ V_{eff}^{''} &= \frac{1}{2g^2}(12\phi^2-8\phi)-\frac{4}{\phi^2}.\end{aligned} \tag{6.11}$$

The condition $V^{'}(\phi)=0$ which also read

$$\phi^4-\phi^3+2g^2=0 \tag{6.12}$$

will give us extrema of the model. These extrema are minima and thus stable if the condition $V^{''}(\phi)>0$ (or equivalently $3\phi^4-2\phi^3-2g^2>0$) is satisfied whereas they are maxima if $3\phi^4-2\phi^3-2g^2<0$. The equation

$$3\phi^4-2\phi^3-2g^2=0 \tag{6.13}$$

tell us therefore exactly when we go from a bounded potential to an unbounded potential (see the attached figure). Solving the above two equations yield immediately the minimum $\phi\equiv\phi_\infty=\frac{3}{4}$ with the corresponding critical value

$$g^2\equiv g_*^2=\frac{1}{8}\left(\frac{3}{4}\right)^3. \tag{6.14}$$

This agrees nicely with the result of [11] which was however obtained by simulating the full theory. In other words the one-loop calculation of this article reproduces their exact result. This is anyway expected because of the following simple reason. It is easily seen that in this $U(1)$ model all vertices are given in terms of commutators and thus they all vanish in the continuum limit . Hence since we know that in $2-$dimensional gauge models only one-loop diagrams can diverge we can conclude that higher loops must strictly vanish in the continuum limit and thus they do not contribute.

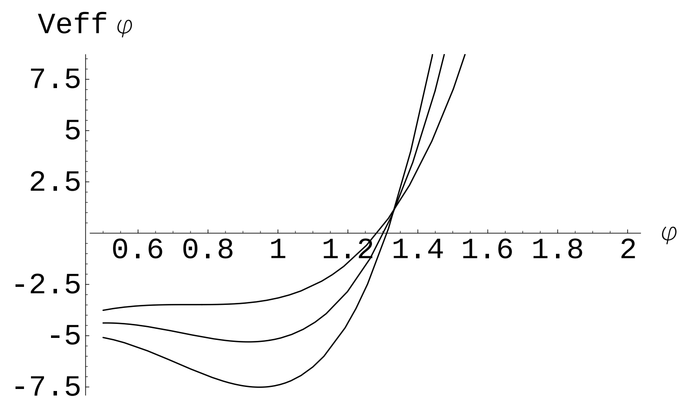


Figure 1: Effective potential for different values of g

We can clearly see from the above result that at the critical value (6.14) a first order phase transition occur which separates the fuzzy sphere phase from the pure matrix phase. The fuzzy sphere phase is where the interpretation that we have a $U(1)$ gauge theory on a (fuzzy or continuum) sphere is valid and it holds for $g < g_*$. The matrix phase is where this interpretation brakes down and it holds for $g > g_*$. Beyond this critical point the fuzzy sphere seems to disappear as the radius goes to zero (the radius here is identified with the inverse of the order parameter ϕ which diverges for $g > g_*$). We should also note here that the one-loop approximation imposes a minimum value for ϕ given by $\phi = \frac{2}{3}$. The leading contributions in the large L limit of this one-loop result can also be computed (say) from (6.9) but we postpone this as well as a thorough discussion of the phase diagram to a future communication [12] where we will also provide an alternative derivation of this phase transition.

7 The $U(1)$ theory with a large mass term.

7.1 The quadratic effective action in the limit $M \longrightarrow \infty$.

In the presence of a mass term for the scalar field the quadratic action (4.13) becomes

$$S_2 = -\frac{1}{2g^2} Tr_L [L_a, A_b]^2 + \frac{1}{2g^2} Tr_L [L_a, A_a]^2 + \frac{M^2}{2g^2} Tr_L (x_a A_a + A_a x_a)^2 \tag{7.1}$$

Whereas the quadratic effective action (4.14) becomes now given by

$$\begin{aligned} \Gamma_2 \;&=\; S_2 + TR\Big(-\frac{1}{2}\Big(\log\frac{1}{\Delta}\Big)_{aa} + \log\frac{1}{\mathcal{L}^2}\Big) + TR\Big(\frac{1}{2}\Big(\frac{1}{\Delta}\Big)_{ab}\omega^{(1)}_{ba} - \frac{1}{\mathcal{L}^2}(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L})\Big) \\ &+\; TR\Big(\frac{1}{2}\Big(\frac{1}{\Delta}\Big)_{ab}\omega^{(2)}_{ba} - \frac{1}{4}\Big(\frac{1}{\Delta}\Big)_{ac}\omega^{(1)}_{cd}\Big(\frac{1}{\Delta}\Big)_{db}\omega^{(1)}_{ba} - \frac{1}{\mathcal{L}^2}\mathcal{A}^2 + \frac{1}{2}\frac{1}{\mathcal{L}^2}(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L})\frac{1}{\mathcal{L}^2}(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L})\Big). \end{aligned} \tag{7.2}$$

The propagator of the gauge degrees of freedom is given here in terms of the Laplacian $\Delta=\mathcal{L}^2+4M^2P^N$ where P^N is the normal projector (3.9) and as a consequence the calculation of the above effective action is now more complicated and involves the propagator

$$\frac{1}{\Delta}=\frac{1}{\Delta_0}+\frac{1}{M^2}\frac{1}{\delta\Delta}\;,\;\frac{1}{\delta\Delta}=M^2\big(-\frac{1}{\Delta_0}V\frac{1}{\Delta_0}+\frac{1}{\Delta_0}V\frac{1}{\Delta_0}V\frac{1}{\Delta_0}+...\big). \tag{7.3}$$

where $\frac{1}{\Delta_0}$ is the inverse of the diagonal Laplacian

$$\Delta_0=P^T\mathcal{L}^2P^T+P^N(\mathcal{L}^2+4M^2)P^N \tag{7.4}$$

and hence $\frac{1}{\Delta_0}=P^T\frac{1}{\mathcal{L}^2}P^T+P^N\frac{1}{\mathcal{L}^2+4M^2}P^N$ whereas the vertex V is the off diagonal part of the Laplacian Δ, i.e

$$V=P^T\mathcal{L}^2P^N+P^N\mathcal{L}^2P^T. \tag{7.5}$$

In (7.2) the first quantum correction is just a number which is irrelevant in any case, the second correction gives the tadpole diagrams of Figure 1 whereas the third correction gives the vacuum polarization diagrams of Figures 2 and 3 where the operators $\omega^{(1)}$ and $\omega^{(2)}$ are given explicitly by

$$\begin{aligned}\omega^{(1)}_{ab}&=(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L})\delta_{ab}+2\mathcal{F}^{(0)}_{ab}+\frac{2M^2}{|L|^2}(LA+AL)\delta_{ab}+\frac{4M^2}{|L|^2}(L_aA_b+A_aL_b)\\ \omega^{(2)}_{ab}&=\mathcal{A}^2\delta_{ab}+2[\mathcal{A}_a,\mathcal{A}_b]+\frac{2M^2}{|L|^2}A^2\delta_{ab}+\frac{4M^2}{|L|^2}A_aA_b.\end{aligned} \tag{7.6}$$

Our interest is to compute this effective action (7.2) in the limit where we take $M\longrightarrow\infty$ first then $L\longrightarrow\infty$. To this end we combine the tadpole diagrams (Figure 1) and the diagrams corresponding to the 4−vertex correction to the vacuum polarization tensor (Figure 2) and write them in the form

$$\begin{aligned}\Gamma_1^M+\Gamma_2^{(4)M}&=\frac{1}{2}TR\Big[(\frac{1}{\Delta})_{ab}\Big((\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L}+\mathcal{A}^2)\delta_{ab}-2\mathcal{F}_{ab}+4\frac{M^2}{|L|^2}(L_bA_a+A_bL_a+A_bA_a)\\&+4\frac{M^2}{|L|}\phi\delta_{ab}\Big)\Big]-TR\frac{1}{\mathcal{L}^2}\big(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L}+\mathcal{A}^2\big).\end{aligned} \tag{7.7}$$

There are in total 8 terms to be computed in the large mass limit before we take the actual continuum limit. As it turns out it is enough to compute these terms only and then use the requirement of gauge invariance to infer the structure of the 3−vertex correction to the vacuum polarization tensor (Figure 3). These are obviously given now by

$$\Gamma_2^{(3F)M}+\Gamma_2^{(3A)M}=TR\bigg(-\frac{1}{4}\Big(\frac{1}{\Delta}\Big)_{ac}\omega^{(1)}_{cd}\Big(\frac{1}{\Delta}\Big)_{db}\omega^{(1)}_{ba}+\frac{1}{2}\frac{1}{\mathcal{L}^2}(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L})\frac{1}{\mathcal{L}^2}(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L})\bigg). \tag{7.8}$$

So in the following we will only compute the action (7.7) explicitly. In the large M limit we will also make use of the fact that we have

$$\frac{1}{\Delta_0}\longrightarrow P^T\frac{1}{\mathcal{L}^2}P^T+O(\frac{1}{M^2}), \tag{7.9}$$

and

$$\frac{1}{\delta\Delta}\longrightarrow -\frac{1}{4}P^T\frac{1}{\mathcal{L}^2}P^T\mathcal{L}^2P^N-\frac{1}{4}P^N\mathcal{L}^2P^T\frac{1}{\mathcal{L}^2}P^T+\frac{1}{4}P^T\frac{1}{\mathcal{L}^2}P^T\mathcal{L}^2P^N\mathcal{L}^2P^T\frac{1}{\mathcal{L}^2}P^T+O(\frac{1}{M^2}). \tag{7.10}$$

The contribution (7.7) decomposes into the sum of two parts, namely

$$\frac{1}{2}TR\Big[(\frac{1}{\Delta_0})_{ab}\Big((\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L}+\mathcal{A}^2)\delta_{ab}-2\mathcal{F}_{ab}+4\frac{M^2}{|L|^2}(L_bA_a+A_bL_a+A_bA_a)+4\frac{M^2}{|L|}\phi\delta_{ab}\Big)\Big]$$
$$-TR\frac{1}{\mathcal{L}^2}(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L}+\mathcal{A}^2) \tag{7.11}$$

and

$$\frac{2}{|L|^2}TR\Big(\frac{1}{\delta\Delta}\Big)_{ab}\big[L_bA_a+A_bL_a+A_bA_a+|L|\phi\delta_{ab}\big]\equiv\frac{2}{|L|^2}TR\Big(\frac{1}{\delta\Delta}\Big)_{ab}\big[D_bD_a+|L|\phi\delta_{ab}\big], \tag{7.12}$$

where we have also used in the above last equation (7.12) the fact that $TR(\frac{1}{\delta\Delta})_{ab}L_bL_a=0$ which can be easily seen from the expression (7.10). A long straightforward calculation shows that in the large L limit the contribution (7.11) depends only on the scalar component of the gauge field, i.e (7.11) becomes a simple scalar action given explicitly by

$$3L\int\frac{d\Omega}{4\pi}\phi+2\int\frac{d\Omega}{4\pi}\phi^2. \tag{7.13}$$

By construction the second term of (7.12) can only depend on the scalar field and thus if it does not vanish in the continuum limit it will at most provide an extra scalar quantum contribution. Hence we do not really need to compute it explicitly. However as a final exercise we compute in some detail the first term of (7.12). By using te equation (4.17) and (4.18) immediately we can compute that

$$\begin{aligned}\frac{2}{|L|^2}TR\Big(\frac{1}{\delta\Delta}\Big)_{ab}D_bD_a &= -\frac{1}{2|L|^2}\sum_{lm}\frac{1}{l(l+1)}Tr_L\hat{Y}_{lm}\mathcal{L}^2(\hat{Y}^+_{lm}P^T_{ab})P^N_{bc}D_cA_dP^T_{da}\\ &- \frac{1}{2|L|^2}\sum_{lm}\frac{1}{l(l+1)}Tr_L\mathcal{L}^2(P^T_{ab}\hat{Y}^+_{lm})\hat{Y}_{lm}P^T_{bc}A_cD_dP^N_{da}\\ &+ \frac{1}{2|L|^2}\sum_{lm,kn}\frac{1}{l(l+1)}\frac{1}{k(k+1)}\big(Tr_L\hat{Y}_{lm}\hat{Y}^+_{kn}P^T_{fb}A_bA_aP^T_{ac}\big)\\ &\times\ \big(Tr_L\mathcal{L}^2(\hat{Y}^+_{lm}P^T_{cd})P^N_{de}\mathcal{L}^2(P^T_{ef}\hat{Y}_{kn})\big).\end{aligned} \tag{7.14}$$

It is rather trivial to show that the first two terms above vanish in the continuum large L limit and as a consequence we end up effectively with the expression

$$\frac{2}{|L|^2}TR\Big(\frac{1}{\delta\Delta}\Big)_{ab}D_bD_a=\frac{1}{2|L|^2}Tr_L(A^T_c)^+M_{cf}A^T_f\ , \tag{7.15}$$

where

$$M_{cf} = \sum_{lm,kn} \frac{1}{l(l+1)} \frac{1}{k(k+1)} \left(Tr_L \mathcal{L}^2 (\hat{Y}^+_{lm} P^T_{cd}) P^N_{de} \mathcal{L}^2 (P^T_{ef} \hat{Y}_{kn}) \right) \hat{Y}_{lm} \hat{Y}^+_{kn}, \tag{7.16}$$

and where we have also defined the tangent gauge field $A^T_a = P^T_{ab} A_b$. By construction the matrix M can only be proportional to the identity matrix, i.e $M_{cf} = \frac{1}{3} M_{aa} \delta_{cf}$, and furthermore we can show that in the continuum limit the trace M_{aa} vanishes identically as $\frac{1}{L}$. This can be easily derived by using the fact that $\mathcal{L}^2(\hat{Y}^+_{lm} P^T_{cd}) P^N_{de} \mathcal{L}^2 (P^T_{ef} \hat{Y}_{kn}) \longrightarrow 16 \hat{Y}^+_{lm} \hat{Y}_{kn} x_c x_f + 8 \hat{Y}^+_{lm} \mathcal{L}_A(\hat{Y}_{kn}) \mathcal{L}_A(x_c x_f) + 4 \mathcal{L}_A(\hat{Y}^+_{lm}) \mathcal{L}_B(\hat{Y}_{kn}) \mathcal{L}_A(x_c) \mathcal{L}_B(x_f)$ when $L \longrightarrow \infty$. Hence the contribution (7.15) vanishes in this continuum limit.

Putting all these results together we conclude that the action (7.7), which gives the sum of the tadpole diagrams (Figure 1) and the 4−vertex correction to the vacuum polarization tensor (Figure 2), becomes a purely scalar action in the limit where we take $M \longrightarrow \infty$ first and then $L \longrightarrow \infty$, i.e

$$\Gamma_1^M + \Gamma_2^{(4)M} = 3L \int \frac{d\Omega}{4\pi} \phi + 2 \int \frac{d\Omega}{4\pi} \phi^2 + \frac{2}{|L|} TR \left(\frac{1}{\delta \Delta} \right)_{aa} \phi. \tag{7.17}$$

This pure scalar action should be compared with the continuum limit of the sum of the two actions (4.21) and (4.23) which as we have shown does depend in the continuum limit on the 2−dimensional gauge field. In other words suppressing the normal component of the gauge field by giving it a large mass allowed us to suppress in the limit the contribution of the tangential field to the tadpole and to the 4−vertex correction of the vacuum polarization tensor. By the requirement of gauge-invariance this suppression will also occur in the other contributions to the vacuum polarization tensor and as consequence the large mass of the scalar field regulates effectively the UV-IR mixing which is consistent with [8].

7.2 The phase transition in the limit $M \longrightarrow \infty$

We note here that the above phase transition can also be thought of as a non-perturbative representation of the UV-IR mixing phenomena. In order to see this more clearly we must take the effect of the mass into account and then calculate how the critical coupling constant scales when we start increasing M. Since the UV-IR mixing disappears in the large mass limit we must be able to see that the phase transition becomes harder to reach from small couplings when we increase M. This is indeed the case as we will now show. In the case $M \neq 0$, the classical and the effective actions are given respectively by

$$V = \frac{U}{|L|^2} = \frac{1}{2g^2} \left[\phi^4 - \frac{4}{3} \phi^3 + M^2 \left(\phi^2 - 1 \right)^2 \right] \tag{7.18}$$

and

$$V_{Meff}[\phi] = V_M[\phi] + 4 \log \phi + \Delta V_M \tag{7.19}$$

where

$$\Delta V_M = \frac{1}{2|L|^2} Tr_3 TR \log(\mathcal{H}_M + 4M^2 P^N) - \frac{1}{|L|^2} TR \log \mathcal{L}^2 \tag{7.20}$$

and

$$\begin{aligned}\mathcal{H}_M &= \mathcal{L}^2 + (\frac{1}{\phi} - 1)(J^2 - \mathcal{L}^2 - 2) + (1 + \frac{M^2}{|L|^2} - \frac{1}{\xi})\big[\frac{1}{4}(J^2 - \mathcal{L}^2)^2 - \frac{1}{2}(J^2 - \mathcal{L}^2) - \mathcal{L}^2\big] \\ &+ 2M^2(1 - \frac{1}{\phi^2}).\end{aligned} \tag{7.21}$$

As before we can argue that in the large L limit the relevant terms in the effective potential are given by

$$V_{Meff}[\phi] = V_M[\phi] + 4\log\phi. \tag{7.22}$$

Solving for the critical values using the same method outlined in a previous subsection yields the results

$$\phi_*^{\pm} = \frac{3}{8(1+M^2)}\Bigg[1 \pm \sqrt{1 + \frac{32M^2(1+M^2)}{9}}\Bigg]. \tag{7.23}$$

And

$$g_*^2 = -\frac{1}{2}(1+M^2)\phi_*^4 + \frac{1}{2}\phi_*^3 + \frac{M^2}{2}\phi_*^2. \tag{7.24}$$

Extrapolating to large masses ($M \longrightarrow \infty$) we obtain the scaling behaviour

$$\begin{aligned}\phi_*^{\pm} &\longrightarrow \pm\frac{1}{\sqrt{2}} \\ g_*^2 &\longrightarrow \frac{M^2}{8}.\end{aligned} \tag{7.25}$$

In other words the phase transition happens each time at a larger value of the coupling constant when M is increased and hence it is harder for the system to reach the pure matrix phase for large enough masses if one starts of course from the fuzzy sphere phase. Putting it differently we have virtually the fuzzy sphere interpretation at all scales of the coupling constant which is indeed what we want if this model is going to approximate a $U(1)$ gauge theory in two dimensions.

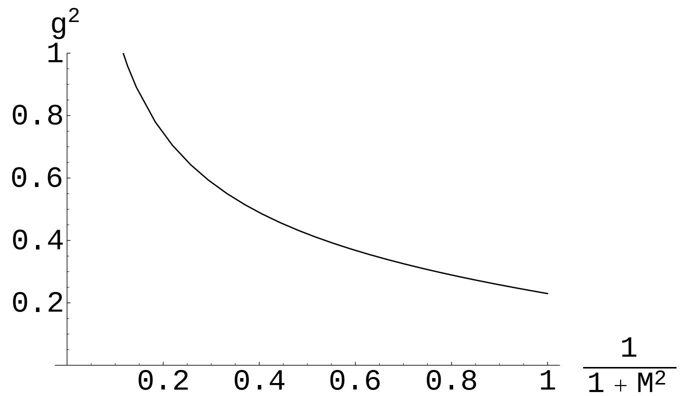


Figure 2: The phase diagram.

8 Conclusion

The fuzzy sphere $\mathbf{S}^2_L$ is a noncommutative space with finite number of degrees of freedom. As we have discussed it is an approximation of the ordinary sphere $\mathbf{S}^2$ which leaves all commutative symmetries intact. Thus it is only natural to consider this space (instead of a naive lattice) as a nonperturbative regularization of chiral gauge theories in dimension two. Generalization to $4-$dimensions is straightforward where we use $\mathbf{S}^2_{L_1}\times\mathbf{S}^2_{L_2}$ as the underlying fuzzy regularization of Euclidean compact spacetime.

It is always a good strategy to study the properties of perturbation theory on these fuzzy spaces in some detail before any serious simulation can be attempted. In particular there is this perturbative UV-IR mixing phenomena which is expected to manifest itself in some form or shape in all noncommutative (fuzzy or otherwise) field theories. The nonperturbative origin of this mixing is still unknown and a deeper interpretation of it is still lacking. Let us also recall here that the fuzzy spheres $\mathbf{S}^2_L$ and $\mathbf{S}^2_{L_1}\times\mathbf{S}^2_{L_2}$ can be thought of as regularizations of the Moyal-Weyl planes $\mathbf{R}^2_\theta$ and $\mathbf{R}^2_{\theta_1}\times\mathbf{R}^2_{\theta_2}$ respectively. Thus understanding the UV-IR mixing in the (more conceptually simple) finite setting of the fuzzy spheres may give us a better understanding of the UV-IR mixing on the noncommutative planes.

In this article we have considered for simplicity the case of a $U(1)$ gauge field on one single fuzzy sphere. Extension of this analysis to higher gauge groups $U(n)$ and to $4-$dimensions will be reported elsewhere [12].

From a string theory point of view we know that the most natural gauge action on the fuzzy sphere is the Alekseev-Recknagel-Schomerus action which is a particular combination of the Yang-Mills action and the Chern-Simons term. We computed the one-loop quadratic effective action and showed explicitly the existence of a gauge-invariant UV-IR mixing in the continuum limit $L\longrightarrow\infty$. In other words the quantum $U(1)$ effective action does not vanish in the commutative limit and a noncommutative anomaly survives. We computed also the scalar effective potential and proved the gauge-fixing-independence of the limiting model $L=\infty$ and then showed explicitly that the one-loop result predicts a first order phase transition which was observed recently in the simulation of [11]. The one-loop result for the $U(1)$ theory is therefore exact in this limit.

Since the differential calculus on the fuzzy sphere is $3-$dimensional the model contains an extra scalar fluctuation which is normal to the sphere. We have argued that if we add a large mass term for this scalar mode the UV-IR mixing will be completely removed from the gauge sector. This is in accordance with the large L analysis of the model done in [8] and shows that the UV-IR mixing of this theory is only confined to the scalar sector. This argument falls however a little short of being a rigorous proof. Correspondingly the phase transition becomes harder to reach starting from small couplings when M is increased. This suggests that the phase transition we are observing is a nonperturbative manifestation of the UV-IR mixing phenomena.

Acknowledgements The authors P. Castro-Villarreal and R. Delgadillo-Blando would like to thank Denjoe O'Connor for his supervision through the course of this study. B. Ydri would like to thank Denjoe O'Connor for his extensive discussions and critical comments while this research was in progress. B. Ydri would also like to thank A. P. Balachandran for his comments and suggestions. The work of P. C. V. and R. D. B. is supported by CONACyT México.

A Tadpole diagrams

Quantum correction to the tree-level linear term $S_1 = 0$ is given by the sum of the two tadpole diagrams of Figure 1 , viz

$$\begin{aligned}\Gamma_1 &= \frac{1}{2}TR\Delta^{(1)} = \frac{1}{2}TR\frac{1}{\mathcal{L}^2}(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L}) \\ &= \frac{1}{2}\left(\frac{1}{\mathcal{L}^2}\right)^{AB,CD}\Bigg[(\mathcal{L})^{CD,EF}(\mathcal{A})^{EF,AB}+(\mathcal{A})^{CD,EF}(\mathcal{L})^{EF,AB}\Bigg]. \qquad (A.1)\end{aligned}$$

The operators $\mathcal{L}_a(..) = [L_a, ..]$ and $\mathcal{A}_a(..) = [A_a, ..]$ carry 4 indices because they can act on matrices either from the left or from the right . We use now the identity

$$\frac{1}{L+1}\sum_{lm}\hat{Y}_{lm}^{AB}(\hat{Y}_{lm}^{+})^{CD} = \delta^{AD}\delta^{BC}, \qquad (A.2)$$

to find the $2-$point Green's function

$$\left(\frac{1}{\mathcal{L}^2}\right)^{AB,CD} = \frac{1}{L+1}\sum_{lm}\frac{\hat{Y}_{lm}^{AB}(\hat{Y}_{lm}^{+})^{DC}}{l(l+1)}. \qquad (A.3)$$

Internal momenta will always be denoted by (lm) while external momenta will be denoted by (pn) . Thus

$$\begin{aligned} \Gamma_1 &= -\sum_{lm}\frac{Tr_L[L_a,\hat{Y}^{\dagger}_{lm}][A_a,\hat{Y}_{lm}]}{l(l+1)} \\ &= -\sum_{pn}A_{-\mu}(pn)(-1)^{\mu}\sum_{lm}\frac{Tr_L[L_{\mu},\hat{Y}_{l-m}][\hat{Y}_{pn},\hat{Y}_{lm}]}{l(l+1)}. \end{aligned} \tag{A.4}$$

In above we have used the fact that $A_a=\eta^{\mu}_aA_{\mu}$ where the coefficients η^{μ}_a satisfy $\eta^{\mu}_a\eta^{\nu}_a=(-1)^{\mu}\delta_{\mu+\nu,0}$, $a=1,2,3$, $\mu=0,+1,-1$. We use now the identities

$$[L_{\mu},\hat{Y}_{lm}]=\sqrt{l(l+1)}C^{lm+\mu}_{lm1\mu}\hat{Y}_{lm+\mu} \tag{A.5}$$

$$\hat{Y}_{l_1m_1}\hat{Y}_{l_2m_2}=\sqrt{L+1}\sqrt{(2l_1+1)(2l_2+1)}\sum_{l_3m_3}(-1)^{L+l_3}\left\{\begin{array}{ccc} l_1 & l_2 & l_3\\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}C^{l_3m_3}_{l_1m_1l_2m_2}\hat{Y}_{l_3m_3}, \tag{A.6}$$

and

$$\sum_m C^{l-m+\mu}_{l-m1\mu}C^{lm-\mu}_{pnlm}=\frac{2l+1}{3}\delta_{p1}\delta_{n,-\mu} \tag{A.7}$$

$$\left\{\begin{array}{ccc} 1 & l & l\\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}=(-1)^{L+l+1}\frac{\sqrt{l(l+1)}}{\sqrt{2l+1}}\frac{1}{\sqrt{L+1}}\frac{1}{2|L|}, \tag{A.8}$$

where C^{ef}_{abcd} are the standard Clebsch-Gordan coefficients and $\{\cdots\}$ are the standard $6j$ symbols [2] to obtain the final result

$$\sum_m(-1)^mTr_L[L_{\mu},\hat{Y}_{l-m}][\hat{Y}_{pn},\hat{Y}_{lm}]=-\frac{1}{\sqrt{3}}(-1)^{\mu}\delta_{p1}\delta_{n,-\mu}\frac{(2l+1)l(l+1)}{|L|}. \tag{A.9}$$

Tadpole diagrams are given therefore by

$$\Gamma_1=\frac{4}{\sqrt{3}}|L|A_{-\mu}(1-\mu)\equiv 4|L|Tr_LA_ax_a. \tag{A.10}$$

By using the definition of the normal scalar field $\phi=\frac{1}{2}(x_aA_a+A_ax_a+\frac{A_a^2}{|L|})$ we rewrite the above action in the form (4.21) .

B 4−Vertex correction

This is given by

$$\begin{aligned} \Gamma_2^{(4)} &= \frac{1}{2}TR\Delta^{(2)}=-\frac{1}{2}\sum_{l_1m_1}\frac{Tr_L[A_a,\hat{Y}^{\dagger}_{l_1m_1}][A_a,\hat{Y}_{l_1m_1}]}{l_1(l_1+1)} \\ &= -\frac{1}{2}\sum_{p_1n_1}\sum_{p_2n_2}A_{\mu}(p_1n_1)A_{-\mu}(p_2n_2)(-1)^{\mu}\sum_{l_1m_1}(-1)^{m_1}\frac{Tr_L[\hat{Y}_{p_1n_1},\hat{Y}_{l_1-m_1}][\hat{Y}_{p_2n_2},\hat{Y}_{l_1m_1}]}{l_1(l_1+1)}. \end{aligned} \tag{B.1}$$

This corresponds to the combination of the two diagrams of Figure 5 . The sum over m_1 can be done again by using now the identity

$$\sum_{m_1m_2}(-1)^{m_1+m_2}C^{l_2m_2}_{p_1n_1l_1-m_1}C^{l_2-m_2}_{p_2n_2l_1m_1}=\frac{2l_2+1}{\sqrt{(2p_1+1)(2p_2+1)}}(-1)^{m_1}(-1)^{l_1+l_2+p_1}\delta_{p_1p_2}\delta_{n_1,-n_2}. \tag{B.2}$$

Hence we obtain

$$\begin{aligned}\sum_{m_1}(-1)^{m_1}Tr_L[\hat{Y}_{p_1n_1},\hat{Y}_{l_1-m_1}][\hat{Y}_{p_2n_2},\hat{Y}_{l_1m_1}] &= -2(L+1)(2l_1+1)\delta_{p_1p_2}\delta_{n_1,-n_2}(-1)^{n_1}\sum_{l_2}(2l_2+1)\\ &\times\ (1-(-1)^{p_1+l_1+l_2})\left\{\begin{array}{ccc} p_1 & l_1 & l_2\\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}^2. \end{aligned} \tag{B.3}$$

We get therefore the answer

$$\Delta\Gamma_2^{(4)}=2\sum_{p_1n_1}|A_a(p_1n_1)|^2\sum_{l_1,l_2}\frac{2l_1+1}{l_1(l_1+1)}\frac{2l_2+1}{l_2(l_2+1)}(L+1)\left\{\begin{array}{ccc} p_1 & l_1 & l_2\\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}^2 l_2(l_2+1), \tag{B.4}$$

with the conservation law that $p_1+l_1+l_2$ is an odd number. In configuration space this contribution takes the form

$$\Delta\Gamma_2^{(4)}=Tr_LA_a\mathcal{L}^2\Delta_4(\mathcal{L}^2)A_a, \tag{B.5}$$

which is (4.23) .

C The $3-$Vertex contribution

In this appendix we calculate the Feynman diagrams $6a$ and $6b$ and. The contribution of the $\mathcal{F}$ term is given by the diagram of Figure $6b$, namely

$$\begin{aligned}\Gamma_2^{(3F)} &= -\frac{1}{4}Tr_3TR(\Delta^{(j)})^2=TR\frac{1}{\mathcal{L}^2}\mathcal{F}^{(0)}_{ab}\frac{1}{\mathcal{L}^2}\mathcal{F}^{(0)}_{ab}\\ &= \sum_{k_1m_1}\sum_{k_2m_2}\frac{Tr_L\left[F^{(0)}_{ab}[\hat{Y}_{k_2m_2},\hat{Y}^+_{k_1m_1}]\right]Tr_L\left[F^{(0)}_{ab}[\hat{Y}_{k_1m_1},\hat{Y}^+_{k_2m_2}]\right]}{k_1(k_1+1)k_2(k_2+1)}\\ &= \sum_{p_1n_1}\sum_{p_2n_2}F^{(0)}_{ab}(p_1n_1)F^{(0)}_{ab}(p_2n_2)\sum_{k_1m_1k_2m_2}\frac{Tr_L\hat{Y}_{p_1n_1}[\hat{Y}_{k_2m_2},\hat{Y}^+_{k_1m_1}]Tr_L\hat{Y}_{p_2n_2}[\hat{Y}_{k_1m_1},\hat{Y}^+_{k_2m_2}]}{k_1(k_1+1)k_2(k_2+1)}\end{aligned} \tag{C.1}$$

The sum over m_1 and m_2 can be easily done and one obtains

$$\begin{aligned}
&\sum_{m_1 m_2}(-1)^{m_1+m_2}Tr_L\left[\hat{Y}_{p_1n_1}[\hat{Y}_{k_2m_2},\hat{Y}_{k_1-m_1}]\right]Tr_L\left[\hat{Y}_{p_2n_2}[\hat{Y}_{k_1m_1},\hat{Y}_{k_2-m_2}]\right]\\
&= 2(2k_1+1)(2k_2+1)\delta_{p_1p_2}\delta_{n_1,-n_2}(-1)^{n_1}(L+1)\big(1-(-1)^{k_1+k_2+p_1}\big)\left\{\begin{array}{ccc} k_1 & k_2 & p_1\\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}^2 ,
\end{aligned} \tag{C.2}$$

and hence we get the contribution

$$\begin{aligned}
\Gamma_2^{(3F)} &= 2\sum_{p_1n_1}|F_{ab}^{(0)}(p_1n_1)|^2\sum_{k_1k_2}\frac{2k_1+1}{k_1(k_1+1)}\frac{2k_2+1}{k_2(k_2+1)}(L+1)(1-(-1)^{k_1+k_2+p_1})\left\{\begin{array}{ccc} k_1 & k_2 & p_1\\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}^2\\
&= 4\sum_{p_1n_1}|F_{ab}^{(0)}(p_1n_1)|^2\sum_{k_1k_2}\frac{2k_1+1}{k_1(k_1+1)}\frac{2k_2+1}{k_2(k_2+1)}(L+1)\left\{\begin{array}{ccc} k_1 & k_2 & p_1\\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}^2 ,
\end{aligned} \tag{C.3}$$

with the conservation law $k_1+k_2+p_1 =$ odd number .The next correction is given explicitly by

$$\begin{aligned}
\Gamma_2^{(3(A))} &= \frac{1}{2}TR\left(-\frac{1}{2}(\Delta^{(1)})^2\right) = -\frac{1}{4}TR\left[\frac{1}{\mathcal{L}^2}(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L})\frac{1}{\mathcal{L}^2}(\mathcal{L}\mathcal{A}+\mathcal{A}\mathcal{L})\right]\\
&= -\frac{1}{2}\sum_{l_1m_1}\sum_{l_2m_2}\frac{1}{l_1(l_1+1)l_2(l_2+1)}\Big[Tr_L[L_a,\hat{Y}_{l_1m_1}][A_a,\hat{Y}^{+}_{l_2m_2}]Tr_L[L_b,\hat{Y}_{l_2m_2}][A_b,\hat{Y}^{+}_{l_1m_1}]\\
&+ Tr_L[L_a,\hat{Y}_{l_1m_1}][A_a,\hat{Y}^{+}_{l_2m_2}]Tr_L[L_b,\hat{Y}^{+}_{l_1m_1}][A_b,\hat{Y}_{l_2m_2}]\Big].
\end{aligned} \tag{C.4}$$

This corresponds to the combination of the two diagrams displayed in Figure $6a$. Now, let us compute the two different terms separately. We have first

$$\begin{aligned}
\bar{\Gamma}_2^{(3(A_1))} &= -\frac{1}{2}\sum_{l_1m_1}\sum_{l_2m_2}\frac{Tr_L[L_a,\hat{Y}_{l_1m_1}][A_a,\hat{Y}^{+}_{l_2m_2}]Tr_L[L_b,\hat{Y}_{l_2m_2}][A_b,\hat{Y}^{+}_{l_1m_1}]}{l_1(l_1+1)l_2(l_2+1)}\\
&= -\frac{1}{2}(-1)^{\mu+\nu}\sum_{p_1n_1}\sum_{p_2n_2}A_{-\mu}(p_1n_1)A_{-\nu}(p_2n_2)\sum_{l_1l_2}\frac{1}{l_1(l_1+1)}\frac{1}{l_2(l_2+1)}\\
&\times \sum_{m_1m_2}(-1)^{m_1+m_2}Tr_L[L_\mu,\hat{Y}_{l_1m_1}][\hat{Y}_{p_1n_1},\hat{Y}_{l_2-m_2}]Tr_L[L_\nu,\hat{Y}_{l_2m_2}][\hat{Y}_{p_2n_2},\hat{Y}_{l_1-m_1}].
\end{aligned} \tag{C.5}$$

The sum over m_1 and m_2 can now be done with the help of the identity

$$\begin{aligned}
M_1 &= (-1)^{\mu+\nu}\sum_{m_1,m_2}C^{l_1m_1+\mu}_{l_1m_11\mu}C^{l_2m_2+\nu}_{l_2m_21\nu}C^{l_1-m_1-\mu}_{p_1n_1l_2-m_2}C^{l_2-m_2-\nu}_{p_2n_2l_1-m_1}\\
&= (2l_1+1)(2l_2+1)(-1)^{l_1+l_2+\nu+n_2}\sum_{km}C^{km}_{p_1n_11\mu}C^{k-m}_{p_2n_21\nu}\\
&\times \left\{\begin{array}{ccc} l_2 & l_1 & p_1\\ 1 & k & l_1\end{array}\right\}\left\{\begin{array}{ccc} l_1 & l_2 & p_2\\ 1 & k & l_2\end{array}\right\}.
\end{aligned} \tag{C.6}$$

We obtain immediately the result

$$\sum_{m_1 m_2}(-1)^{m_1+m_2}Tr_L[L_\mu,\hat{Y}_{l_1m_1}][\hat{Y}_{p_1n_1},\hat{Y}_{l_2-m_2}]Tr_L[L_\nu,\hat{Y}_{l_2m_2}][\hat{Y}_{p_2n_2},\hat{Y}_{l_1-m_1}] =$$
$$\sqrt{l_1(l_1+1)l_2(l_2+1)}\sqrt{\prod_{i=1}^2(2l_i+1)(2p_i+1)}(-1)^{l_1+l_2}(L+1)[1-(-1)^{l_1+l_2+p_1}]$$
$$\times\quad [1-(-1)^{l_1+l_2+p_2}]\left\{\begin{array}{ccc} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}\left\{\begin{array}{ccc} p_2 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}M_1, \tag{C.7}$$

and as a consequence

$$\begin{aligned}\bar{\Gamma}_2^{(3(A_1))} &= -\frac{1}{2}\sum_{p_1n_1}\sum_{p_2n_2}A_{-\mu}(p_1n_1)A_{-\nu}(p_2n_2)(-1)^{n_1+\nu}\sum_{l_1l_2}\frac{2l_1+1}{l_1(l_1+1)}\frac{2l_2+1}{l_2(l_2+1)}(L+1)\\ &\times\quad [1-(-1)^{l_1+l_2+p_1}][1-(-1)^{l_1+l_2+p_2}]\left\{\begin{array}{ccc} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}\left\{\begin{array}{ccc} p_2 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}f_1(lpn;\mu,\nu),\end{aligned} \tag{C.8}$$

where

$$\begin{aligned}f_1 = \sqrt{l_1(l_1+1)l_2(l_2+1)}&\sqrt{\prod_{i=1}^2(2l_i+1)(2p_i+1)}\\ &\times\sum_{km}C^{km}_{p_1n_11\mu}C^{k-m}_{p_2n_21\nu}\left\{\begin{array}{ccc} l_2 & l_1 & p_1 \\ 1 & k & l_1\end{array}\right\}\left\{\begin{array}{ccc} l_1 & l_2 & p_2 \\ 1 & k & l_2\end{array}\right\}\end{aligned} \tag{C.9}$$

Similarly we have

$$\begin{aligned}\bar{\Gamma}_2^{(3(A_2))} &= -\frac{1}{2}\sum_{l_1m_1}\sum_{l_2m_2}\frac{Tr_L[L_a,\hat{Y}_{l_1m_1}][A_a,\hat{Y}^+_{l_2m_2}]Tr_L[L_b,\hat{Y}^+_{l_1m_1}][A_b,\hat{Y}_{l_2m_2}]}{l_1(l_1+1)l_2(l_2+1)}\\ &= -\frac{1}{2}(-1)^{\mu+\nu}\sum_{p_1n_1}\sum_{p_2n_2}A_{-\mu}(p_1n_1)A_{-\nu}(p_2n_2)\sum_{l_1l_2}\frac{1}{l_1(l_1+1)}\frac{1}{l_2(l_2+1)}\\ &\times\quad\sum_{m_1m_2}(-1)^{m_1+m_2}Tr_L[L_\mu,\hat{Y}_{l_1m_1}][\hat{Y}_{p_1n_1},\hat{Y}_{l_2-m_2}]Tr_L[L_\nu,\hat{Y}_{l_1-m_1}][\hat{Y}_{p_2n_2},\hat{Y}_{l_2m_2}].\end{aligned} \tag{C.10}$$

The sum over m_1 and m_2 can now be done with the help of the identity

$$\begin{aligned}M_2 &= (-1)^{\mu+\nu}\sum_{m_1,m_2}(-1)^{m_1+m_2}C^{l_1m_1+\mu}_{l_1m_11\mu}C^{l_1-m_1+\nu}_{l_1-m_11\nu}C^{l_1-m_1-\mu}_{p_1n_1l_2-m_2}C^{l_1m_1-\nu}_{p_2n_2l_2m_2}\\ &= (2l_1+1)^2(-1)^{l_1+l_2+n_1+\mu}\sum_{km}(-1)^kC^{km}_{p_1n_11\mu}C^{k-m}_{p_2n_21\nu}\left\{\begin{array}{ccc} l_2 & l_1 & p_1 \\ 1 & k & l_1\end{array}\right\}\left\{\begin{array}{ccc} l_2 & l_1 & p_2 \\ 1 & k & l_1\end{array}\right\}.\end{aligned} \tag{C.11}$$

We obtain therefore the result

$$\sum_{m_1m_2}(-1)^{m_1+m_2}Tr_L[L_\mu,\hat{Y}_{l_1m_1}][\hat{Y}_{p_1n_1},\hat{Y}_{l_2-m_2}]Tr_L[L_\nu,\hat{Y}_{l_1-m_1}][\hat{Y}_{p_2n_2},\hat{Y}_{l_2m_2}]=$$

$$l_1(l_1+1)(2l_2+1)\sqrt{\prod_{i=1}^{2}(2p_i+1)}(L+1)[1-(-1)^{l_1+l_2+p_1}][1-(-1)^{l_1+l_2+p_2}]$$

$$\times\ \left\{\begin{matrix} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{matrix}\right\}\left\{\begin{matrix} p_2 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{matrix}\right\}M_2, \tag{C.12}$$

and hence

$$\begin{aligned}\bar{\Gamma}_2^{(3(A_2))} &= -\frac{1}{2}\sum_{p_1n_1}\sum_{p_2n_2}A_{-\mu}(p_1n_1)A_{-\nu}(p_2n_2)(-1)^{n_1+\nu}\sum_{l_1l_2}\frac{2l_1+1}{l_1(l_1+1)}\frac{2l_2+1}{l_2(l_2+1)}(L+1)\\ &\times\ [1-(-1)^{l_1+l_2+p_1}][1-(-1)^{l_1+l_2+p_2}]\left\{\begin{matrix} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{matrix}\right\}\left\{\begin{matrix} p_2 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{matrix}\right\}f_2(lpn;\mu,\nu),\end{aligned} \tag{C.13}$$

where

$$\begin{aligned}f_2 = l_1(l_1+1)(2l_1+1)\sqrt{\prod_{i=1}^{2}(2p_i+1)}\sum_{km}(-1)^{l_1+l_2+k}C^{km}_{p_1n_11\mu}C^{k-m}_{p_2n_21\nu}\\ \times\left\{\begin{matrix} l_2 & l_1 & p_1 \\ 1 & k & l_1\end{matrix}\right\}\left\{\begin{matrix} l_2 & l_1 & p_2 \\ 1 & k & l_1\end{matrix}\right\}.\end{aligned} \tag{C.14}$$

The final answer becomes

$$\begin{aligned}\Gamma_2^{(3(A))} &= -2\sum_{p_1n_1}\sum_{p_2n_2}A_{-\mu}(p_1n_1)A_{-\nu}(p_2n_2)(-1)^{n_1+\nu}\sum_{l_1l_2}\frac{2l_1+1}{l_1(l_1+1)}\frac{2l_2+1}{l_2(l_2+1)}(L+1)\\ &\times\ \left\{\begin{matrix} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{matrix}\right\}\left\{\begin{matrix} p_2 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{matrix}\right\}f^A(lpn;\mu,\nu),\end{aligned} \tag{C.15}$$

where we have the conservation laws $l_1+l_2+p_1=$ odd number , $l_1+l_2+p_2=$ odd number which means in particular that p_1+p_2 can only be an even number and where

$$\begin{aligned}f^A(lpn;\mu,\nu) = f_1+f_2 = \sqrt{l_1(l_1+1)(2l_1+1)}\sqrt{\prod_{i=1}^{2}(2p_i+1)}\\ \times\sum_{km}C^{km}_{p_1n_11\mu}C^{k-m}_{p_2n_21\nu}\left\{\begin{matrix} l_2 & l_1 & p_1 \\ 1 & k & l_1\end{matrix}\right\}\Bigg[\sqrt{l_2(l_2+1)}\sqrt{2l_2+1}\left\{\begin{matrix} l_1 & l_2 & p_2 \\ 1 & k & l_2\end{matrix}\right\}\\ +\sqrt{l_1(l_1+1)}\sqrt{2l_1+1}(-1)^{k+l_1+l_2}\left\{\begin{matrix} l_2 & l_1 & p_2 \\ 1 & k & l_1\end{matrix}\right\}\Bigg].\end{aligned} \tag{C.16}$$

From this equation and from the properties of the $3j$ and $6j$ symbols it is obvious that $m=n_1+\mu=-n_2-\nu$ while k takes only the values p_1+1 , p_1 and p_1-1 . Hence it is also clear that p_2 can only take the values p_1 , p_1+2 and p_1-2 [The values p_1-1 and p_1+1 do not contribute because of the restriction that p_1+p_2 must be an even number] . By using the different identities on page 311 of [2] we can see that the function f^A splits into two parts , a canonical gauge part plus a scalar-like part , i.e $f^A = f^{A_1} + f^{A_2}$ where

$$f^{A_1} = \frac{1}{2} C^{p_1 m}_{p_1 n_1 1\mu} C^{p_1 - m}_{p_2 n_2 1\nu} \eta_{p_1} \delta_{p_1 p_2} \tag{C.17}$$

and

$$\begin{aligned} f^{A_2} &= \frac{1}{2} C^{p_1-1 m}_{p_1 n_1 1\mu} C^{p_1-1-m}_{p_2 n_2 1\nu} \big(\eta_{p_1-1}\delta_{p_2,p_1} + \hat{\eta}_{p_1-1}\delta_{p_2,p_1-2}\big) \\ &+ \frac{1}{2} C^{p_1+1 m}_{p_1 n_1 1\mu} C^{p_1+1-m}_{p_2 n_2 1\nu} \big(\eta_{p_1+1}\delta_{p_2,p_1} + \hat{\eta}_{p_1+1}\delta_{p_2,p_1+2}\big). \end{aligned} \tag{C.18}$$

The functions η_{p_1} , η_{p_1-1,p_1+1} and $\hat{\eta}_{p_1-1,p_1+1}$ carry all the dependence of f^A on the internal momenta l_1 and l_2, namely

$$\begin{aligned} \eta_{p_1} &= -\frac{1}{p_1(p_1+1)}(l_2(l_2+1) - l_1(l_1+1))(l_2(l_2+1) - l_1(l_1+1) - p_1(p_1+1)) \\ \eta_{p_1+1} &= \frac{1}{(p_1+1)(2p_1+3)}(s+2)(s-2p_1)(s-2l_1+1)(s-2l_2+1) \\ \eta_{p_1-1} &= \frac{1}{p_1(2p_1-1)}(s+1)(s-2p_1+1)(s-2l_1)(s-2l_2) \\ \hat{\eta}_{p_1+1} &= -\sqrt{\frac{(s+2)(s-2p_1)(s-2l_1+1)(s-2l_2+1)}{(p_1+1)(2p_1+3)}}\sqrt{\frac{(s+3)(s-2p_1-1)(s-2l_1+2)(s-2l_2+2)}{(2p_1+3)(p_1+2)}} \\ \hat{\eta}_{p_1-1} &= -\sqrt{\frac{(s+1)(s-2p_1+1)(s-2l_1)(s-2l_2)}{p_1(2p_1-1)}}\sqrt{\frac{s(s-2p_1+2)(s-2l_1-1)(s-2l_2-1)}{(2p_1-1)(p_1-1)}}. \end{aligned} \tag{C.19}$$

In above s is defined by $s = p_1 + l_1 + l_2$.

In the remainder of this appendix we show explicitly that the quantum effective action $\Gamma_2^{(3A_2)}$ obtained by setting $f = f^{A_2}$ in $\Gamma_2^{(3(A))}$ can be written in the form (4.36) . We have

$$\begin{aligned} \Gamma_2^{(3A_2)} &= -2\sum_{p_1 n_1}\sum_{p_2 n_2} A_{-\mu}(p_1 n_1) A_{-\nu}(p_2 n_2)(-1)^{n_1+\nu} \sum_{l_1 l_2} \frac{2l_1+1}{l_1(l_1+1)}\frac{2l_2+1}{l_2(l_2+1)}(L+1) \\ &\times \begin{Bmatrix} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{Bmatrix} \begin{Bmatrix} p_2 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{Bmatrix} f^{A_2}(lpn;\mu,\nu). \end{aligned} \tag{C.20}$$

Using (C.18) and (C.19) we can immediately compute

$$\begin{aligned} \Gamma_2^{(3A_2)} &= \sum_{p_1 n_1}\sum_{p_2 n_2} A_{-\mu}(p_1 n_1) A_{-\nu}(p_2 n_2)(-1)^{n_1+\nu} \\ &\times \Bigg[C^{p_1-1 m}_{p_1 n_1 1\mu} C^{p_1-1-m}_{p_2 n_2 1\nu}\Big(\delta_{p_1 p_2}\Lambda^{(-)}(p_1-1) + \delta_{p_2,p_1-2}\Sigma^{(-)}(p_1-1)\Big) \\ &+ C^{p_1+1 m}_{p_1 n_1 1\mu} C^{p_1+1-m}_{p_2 n_2 1\nu}\Big(\delta_{p_1 p_2}\Lambda^{(+)}(p_1+1) + \delta_{p_2,p_1+2}\Sigma^{(+)}(p_1+1)\Big)\Bigg]. \end{aligned} \tag{C.21}$$

where

$$\Lambda^{(-)}(p_1-1) = -\sum_{l_1,l_2}\frac{2l_1+1}{l_1(l_1+1)}\frac{2l_2+1}{l_2(l_2+1)}(L+1)\left\{\begin{array}{ccc} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}^2 \eta_{p_1-1}$$

$$\Lambda^{(+)}(p_1+1) = -\sum_{l_1,l_2}\frac{2l_1+1}{l_1(l_1+1)}\frac{2l_2+1}{l_2(l_2+1)}(L+1)\left\{\begin{array}{ccc} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}^2 \eta_{p_1+1}, \quad \text{(C.22)}$$

and

$$\Sigma^{(-)}(p_1-1) = -\sum_{l_1,l_2}\frac{2l_1+1}{l_1(l_1+1)}\frac{2l_2+1}{l_2(l_2+1)}(L+1)\left\{\begin{array}{ccc} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}\left\{\begin{array}{ccc} p_1-2 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}\hat{\eta}_{p_1-1}, \quad \text{(C.23)}$$

$$\Sigma^{(+)}(p_1+1) = -\sum_{l_1,l_2}\frac{2l_1+1}{l_1(l_1+1)}\frac{2l_2+1}{l_2(l_2+1)}(L+1)\left\{\begin{array}{ccc} p_1 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}\left\{\begin{array}{ccc} p_1+2 & l_1 & l_2 \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2}\end{array}\right\}\hat{\eta}_{p_1+1}. \quad \text{(C.24)}$$

In above we must always have $l_1+l_2+p_1$ to be an odd number . In order to rewrite the above scalar action in position space let us first introduce the following operators

$$\Delta^{(1)} = \Delta^{(-)}+\Delta^{(+)}+\bar{\Delta}^{(-)}+\bar{\Delta}^{(+)}\ ,\ \Delta^{(2)} = \Delta^{(-)}+\Delta^{(+)}-\bar{\Delta}^{(-)}-\bar{\Delta}^{(+)}$$
$$\Delta^{(3)} = \Delta^{(-)}-\Delta^{(+)}+\bar{\Delta}^{(-)}-\bar{\Delta}^{(+)}\ ,\ \Delta^{(4)} = \Delta^{(-)}-\Delta^{(+)}-\bar{\Delta}^{(-)}+\bar{\Delta}^{(+)}, \quad \text{(C.25)}$$

where $\Delta^{(-)}$, $\Delta^{(+)}$, $\bar{\Delta}^{(-)}$ and $\bar{\Delta}^{(+)}$ are defined through the equations

$$\Lambda^{(-)}(p_1-1) = \frac{16p_1\big((L+1)^2-p_1^2\big)}{L(L+2)(2p_1-1)}\Delta^{(-)}(p_1-1)$$
$$\Lambda^{(+)}(p_1+1) = \frac{16(p_1+1)(L+2+p_1)(L-p_1)}{L(L+2)(2p_1+3)}\Delta^{(+)}(p_1+1)$$
$$\Sigma^{(-)}(p_1-1) = -\frac{16\sqrt{p_1(p_1-1)(L+p_1)(L+2-p_1)((L+1)^2-p_1^2)}}{L(L+2)(2p_1-1)}\bar{\Delta}^{(-)}(p_1-1) \quad \text{(C.26)}$$

and

$$\Sigma^{(+)}(p_1+1) = -\frac{16\sqrt{(p_1+1)(p_1+2)(L-p_1)(L+p_1+2)(L-p_1)(L+p_1+3)(L-p_1-1)}}{L(L+2)(2p_1+3)} \times \bar{\Delta}^{(+)}(p_1+1). \quad \text{(C.27)}$$

The above scalar action in position space is then simply given by

$$\Delta\Gamma_2^{(3A_2)} = Tr_L[A_a,x_a]_+\Delta^{(1)}(\mathcal{L}^2)[A_b,x_b]_+ + Tr_L[\nabla A_a,x_a]_+\Delta^{(2)}(\mathcal{L}^2)[\nabla^{-1}A_b,x_b]_+$$
$$+\ iTr_L[\nabla^{\frac{3}{2}}A_a,x_a]_+\Delta^{(3)}(\mathcal{L}^2)\nabla[\nabla^{\frac{3}{2}}A_b,x_b]_+ + iTr_L[\nabla^{\frac{5}{2}}A_a,x_a]_+\Delta^{(4)}(\mathcal{L}^2)\nabla[\nabla^{\frac{1}{2}}A_b,x_b]_+, \quad \text{(C.28)}$$

where ∇ is the phase operator $\nabla=(-1)^{\frac{\hat{N}}{2}}$.

References

[1] Badis Ydri, *Fuzzy Physics*, [arXiv:hep-th/0110006], Ph.D Thesis.

[2] D. A. Varshalovich, A. N. Moskalev, V. K. Khersonky, *Quantum Theory of Angular Momentum: Irreducible Tensors, Spherical Harmonics, Vector Coupling Coefficients, 3nj Symbols*, Singapore, Singapore. World Scientific (1998).

[3] N. Seiberg, E. Witten, *String Theory and Noncommutative Geometry*, *J. High Energy Phys.* **032** 9909 (1999), [arXiv:hep-th/9908142].

[4] Shiraz Minwalla, Mark Van Raamsdonk, Nathan Seiberg. *Noncommutative perturbative Dynamics.* [arXiv:hep-th/9912072].

[5] A. Y. Alekseev, A. Recknagel, V. Schomerus, [arXiv:hep-th/0003187], [arXiv:hep-th/9812193].

[6] T. Imai, Y. Kitazawa, Y. Takayama, D. Tomino, [arXiv:hep-th/0303120].

[7] N. Ishibashi, H. Kawai, Y. Kitazawa, A. Tsuchiya, [arXiv:hep-th/9612115]; S. Iso, H. Kawai, Y. Kitazawa, [arXiv:hep-th/0001027]; S. Iso, Y. Kimura, K. Tanaka, K. Wakatsuki, [arXiv:hep-th/0101102]; Y. Kitazawa, [arXiv:hep-th/0207115].

[8] H. Steinacker, *Quantized Gauge Theory on the Fuzzy Sphere as Random Matrix Model*, [arXiv:hep-th/0307075].

[9] D. Karabali , V. P. Nair and A. P. Polychronakos , *Nucl.Phys.* **B 627** (2002) 565, [arXiv:hep-th/0111249].

[10] A. Connes, *Noncommutative Geometry*, Academic Press, London , 1994 .
G. Landi, *An introduction to noncommutative spaces and their geometry*, springer (1997).
J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications* , Cambridge Press (1995).
J. M. Gracia-Bondia, J. C. Varilly, H. Figueroa, *Elements of Noncommutative Geometry*, Birkhauser (2000).

[11] T. Azuma, S. Bal, K. Nagao, J. Nishimura, [arXiv:hep-th/0401038].

[12] P. Castro-Villarreal, R. Delgadillo-Blando, Denjoe O'Connor, Badis Ydri, in progress.

[13] J. Madore, *Class. Quantum. Grav.* **9** (1992) 69.

[14] J. Hoppe, MIT Ph.D Thesis, (1982); J. Hoppe, S-T. Yau, *Commun. Math. Phys.* **195** (1998) 67-77.

[15] S. Iso, Y. Kimura, K. Tanaka, K. Wakatsuki, *Nucl. Phys. B* **604** (2001) 121-147.
D. O'Connor, *Mod.Phys.Lett.* **A18** (2003) 2423-2430.

[16] Badis Ydri, *Exact Solution of Noncommutative $U(1)$ Gauge Theory in $4-$Dimensions*, *Nucl.Phys. B* **690** (2004) 230-248; B. Ydri, *Mod. Phys. Letter A* **19** (2004) 2205.

[17] B. Ydri ,*JHEP* **0308** (2003) 046.
P. Presnajder, *J. Math. Phys.* **41** (2000) 2789.
H.Aoki, S. Iso, K.Nagao, *Phys. Rev. D* **67** (2003) 065018.

[18] S. Kurkcuoglu, [arXiv:hep-th/0311031].
H. Grosse, C. Klimcik and P. Presnajder, *Commun. Math. Phys.* **185** (1997) 155.
C. Klimcik, *Commun. Math. Phys.* **206** (1999) 587.
A. P. Balachandran, S. Kurkcuoglu, E. Rojas, *JHEP* **0207** (2002) 056.

[19] A. P. Balachandran, T. R. Govindarajan, B. Ydri, *Mod. Phys. Lett. A* **15** (2000) 1279;
A. P. Balachandran, G. Immirzi, *Phys. Rev. D* **68** (2003) 065023;
H. Aoki, S. Iso, K. Nagao, *Phys. Rev. D* **67** (2003) 085005;
A. P. Balachandran, G. Immirzi, *Int. J. Mod. Phys. A* **18** (2003) 5981;
U. Carow-Watamura, S. Watamura, *Commun. Math. Phys.* **183** (1997) 365-382; *Int. J. Mod. Phys. A* **13** (1998) 3235-3244.

[20] S. Baez, A. P. Balachandran, S. Vaidya, B. Ydri, *Commun. Math. Phys* **208** (2000) 787;
A. P. Balachandran, S. Vaidya, *Mod. Phys. A* **16** (2001) 17
H. Grosse, C. Klimcik, P. Presnajder, *Commun. Math. Phys* **178** (1996) 507.
H. Grosse, C. W. Rupp, A. Strohmaier, *J. Geom. Phys.* **42** (2002) 54-63.

[21] U. Carow-Watamura, H. Steinacker, S. Watamura, [arXiv:hep-th/0404130];
P. Valtancoli, *Mod. Phys. Lett. A* **16** (2001) 639-646; P. Valtancoli, [arXiv:hep-th/0404045], [arXiv:hep-th/0404046];
H. Aoki, S. Iso, K. Nagao *Nucl. Phys. B* **684** (2004) 162-182.

[22] Sachindeo Vaidya, Badis Ydri, *Nucl. Phys. B* **671** (2003) 401-431, Sachindeo Vaidya, Badis Ydri, [arXiv:hep-th/0209131].

[23] G. Alexanian, A. P. Balachandran, G. Immirzi, B. Ydri, *J.Geom.Phys.* **42** (2002) 28-53.
H. Grosse, A. Strohmaier, *Lett. Math. Phys.* **48** (1999) 163-179.

[24] Julieta Medina, Denjoe O'Connor, *JHEP* **0311** (2003) 051; I. Huet Hernandez, Denjoe O'Connor, in progress.

[25] A. P. Balachandran, B. P. Dolan, J. Lee, X. Martin, D. O'Connor, *J. Geom. Phys.* **43** (2002) 184-204;
S. Ramgoolam, *JHEP* **0210** (2002) 064, *Nucl. Phys. B* **610** (2001) 461;
Brian P. Dolan, Denjoe O'Connor, Peter Presnajder, *JHEP* **0402** (2004) 055;
G. Alexanian, A. Pinzul, A. Stern, *Nucl. Phys. B* **600** (2001) 531;

Y. Kitazawa, *Nucl. Phys. B* **642** (2002) 210-226;
F. Lizzi, P. Vitale, A. Zampini, [arXiv:hep-th/0306247];
B. P. Dolan, D. O'Connor, [arXiv:hep-th/0306231];
Y. Kimura, [arXiv:hep-th/0301055].

[26] S. Vaidya, *Phys. Lett. B* **512** (2001) 403; B. P. Dolan, D. O'Connor, P. Presnajder, *JHEP* **0203** (2002) 013; B. P. Dolan, D. O'Connor, P. Presnajder, [arXiv:hep-th/0204219].

[27] C. S. Chu, J. Madore, H. Steinacker, *JHEP* **0108** (2001) 038.

[28] X. Martin, *Mod. Phys. Lett. A* **18** (2003) 2389-2396; Xavier Martin, [arXiv:hep-th/0402230]; F. Garcia-Flores, D. O'Connor, in progress; W. Bietenholz, Julieta Medina, D. O'Connor, in progress.

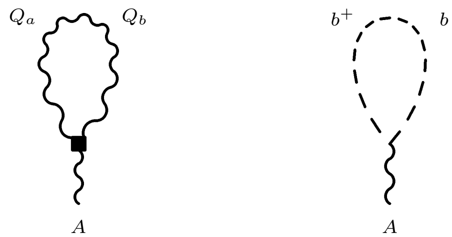


Figure 4. Tadpole diagrams

Figure 5. Vacuum polarization diagrams (the 4-vertices).

Figure 6a. Vacuum polarization diagrams (the 3-vertices with the A-field).

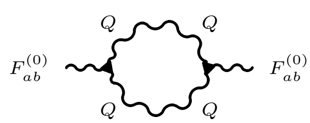
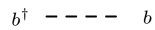


Figure 6b. Vacuum polarization diagram (the 3-vertex with the F-field).

Note: Wavy internal lines represent the fluctuation fields Q_a whereas the dashed internal lines represent the ghost fields b and $b^\dagger$. Wavy external lines are the gauge field A_a.

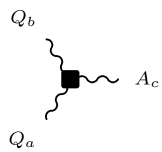
$$= \frac{1}{\mathcal{L}^2}\delta_{ab}$$

Figure $1a$



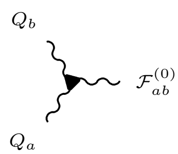
$$= \frac{1}{\mathcal{L}^2}$$

Figure $1b$



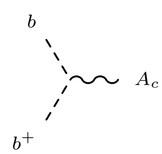
$$= -\frac{1}{g^2} Tr_L Q_a \left(\frac{\mathcal{L}_c \mathcal{A}_c + \mathcal{A}_c \mathcal{L}_c}{2} \right) Q_a$$

Figure $2a$



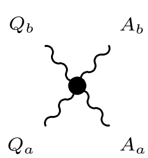
$$= -\frac{1}{g^2} Tr_L Q_a \mathcal{F}_{ab}^{(0)} Q_b$$

Figure $2b$



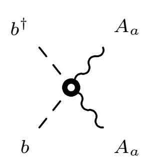
$$= \frac{1}{g^2} Tr_L b^+ (\mathcal{L}_c \mathcal{A}_c + \mathcal{A}_c \mathcal{L}_c) b$$

Figure $2c$



$$= -\frac{1}{g^2} Tr_L Q_a \left(\frac{[A_c,[A_c,\ \cdot]]\delta_{ab} + 2[[A_a, A_b],\ \cdot]}{2} \right) Q_b$$

Figure $3a$



$$= \frac{1}{g^2} Tr_L b^+ \mathcal{A}^2 b$$

Figure $3b$