A new theory of superfluidity

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Abstract

The understanding of superfluidity represents one of the most challenging problems in modern physics. From the observations of [1–3], in various respects the Bogoliubov theory [4–8] is not appropriate as the model of superfluidity for Helium 4. His outstanding achievement, i.e., the derivation of the Landau-type excitation spectrum [9, 10] from the full interacting Hamiltonian, is based on a series of recipes or approximations, which were shown to be wrong, even from their starting point [11–14]. We therefore present some very promising new results performed in [15]. In particular, we explain a new theory of superfluidity at all temperatures. At this point we then touch one of the most fascinating problems of contemporary mathematical physics - the proof of the existence of superfluidity in interacting (non-dilute) systems.

Keywords : Bogoliubov, helium, superfluidity, Landau spectrum, Cooper, Bose condensation.

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1. Superfluidity of Helium 4 versus the Bogoliubov theory

1.1. Historical remarks

The past years has seen a increasing interest in the analysis of quantum phase transitions, driven by experiments on the cuprate superconductor, the heavy fermion materials, helium liquid and dilute trapped Bose gases.

One of the first important quantum phase transitions is Bose-Einstein condensation, a phenomenon, first predicted by Einstein in 1925 and experimentally realized in 1995 [16–18]. A very interesting liquid, long associated with this phenomenon in correlation with another quantum phase transition, the superfluidity, is liquid helium. Actually, its first isotope, Helium 4, is unique in having two liquid forms. The normal liquid form is called Helium I and exists at temperatures T between 2.17 K and its boiling point 4.21 K. Below the "λ-transition", i.e., for $T \leq T_\lambda = 2.17$ K, Kapitza [19] and Allen, Misener [20] in 1938 discovered that Helium-4 becomes superfluid (i.e., its viscosity, or resistance to flow, nearly vanishes). Its thermal conductivity becomes more than 1,000 times greater than that of copper. By contrast, the second isotope, Helium 3, forms three distinguishable liquid (quantum) phases, of which two are superfluids. This last phenomenon in Helium-3 was first observed only in 1972 [21,22].
The system in question in this paper is related to ^{4}He which are *bosons,* but at the end (Section 4.3), we explain how this theory may also be a starting point for a microscopic theory of superfluidity for ^{3}He within the framework of Fermi systems.

In theoretical physics, Landau understood for the first time that the properties of ^{4}He, which rest liquid (under normal pressure) even for $T \to 0$ K, can be explained only by a new kind of quantum arguments. In particular, the Landau phenomenology is based on the following assumptions [4,6,7,23–26]:

- quantum liquid is still fluid even for zero-temperature;
- at low temperatures, apart translations (flow), the state of this liquid is entirely described by the spectrum of collective (elementary) excitations;
- through thermodynamic data [26,27] (e.g. specific heat capacity) this spectrum for ^{4}He should be a phonon-like for the long-wave length collective excitations and should be above a straight line with positive slope with ("roton") minimum in the vicinity of $\|k_{\rm rot}\| \simeq 2\ \mathring{\rm A}^{-1}$ (figure 1.1).

This structure of spectrum agrees with the low-temperature thermodynamic properties of liquid ^{4}He, such as specific heat and others [9,10,26]. Moreover, the two last assumptions ensure that this liquid is *superfluid* via the famous Landau's criterion of superfluidity [9,10].

Within the framework of mathematical physics, the key-problem to explain the Landau's assumptions is to show that quantum mechanical behaviour of the *macroscopic* system of about $n = 10^{23}$ atoms of ^{4}He is such that the *spectrum* of the corresponding *microscopic Hamiltonian* $H_\Lambda^{(n)}$ for n particles is close to the one proposed by Landau. More precisely, this homogeneous system of bosons interacting

Figure 1.1: *The Landau-type excitation spectrum E_L as a function of the momentum p.*

via a (real) *two-body* interaction potential $\varphi(x) = \varphi(\|x\|)$ is interpreted via the following self-adjoint (s.-a.) extension

$$H_{\Lambda}^{(n)} = \left\{ \sum_{j=1}^{n} \left(-\frac{\hbar^2 \Delta_j}{2m} \right) + \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n} \varphi(x_i - x_j) \right\}_{\text{s-a}}$$

on the *symmetrized* n-particle Hilbert spaces

$$\mathcal{H}_B^{(n)} \equiv \left(L^2(\Lambda^n) \right)_{\text{symm}}, \quad \mathcal{H}_B^{(0)} = \mathbb{C},$$

appropriate for bosons [28,29]. We denote by $V \equiv |\Lambda| = L^{d=3}$ the volume of the box Λ. The one-particle energy spectrum is $\varepsilon_k \equiv \hbar^2 k^2 / 2m$ and, using periodic boundary conditions,

$$\Lambda^* \equiv \left\{ k \in \mathbb{R}^3 : \; k_\alpha = \frac{2\pi n_\alpha}{L}, \; n_\alpha = 0, \pm 1, \pm 2, ..., \; \alpha = 1, 2, 3 \right\}$$

is the set of wave vectors.

Within the framework of the second quantization, the standard device often used in quantum many-body problems, the corresponding Hamiltonian acting on the boson Fock space

$$\mathcal{F}_\Lambda^B \equiv \bigoplus_{n=0}^{+\infty} \mathcal{H}_B^{(n)}$$

is equal to

$$H_{\Lambda, \lambda_0 > 0} = T_\Lambda + \widetilde{U}_\Lambda + U_\Lambda^{MF}, \tag{1.1}$$

with

$$T_\Lambda \equiv \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k,$$

$$\widetilde{U}_\Lambda \equiv \frac{1}{2V} \sum_{k_1,k_2,q\neq 0\in\Lambda^*} \lambda_q a^*_{k_1+q} a^*_{k_2-q} a_{k_1} a_{k_2},$$

$$U_\Lambda^{MF} \equiv \frac{\lambda_0}{2V} \sum_{k_1,k_2\in\Lambda^*} a^*_{k_1} a^*_{k_2} a_{k_2} a_{k_1} = \frac{\lambda_0}{2V}\left(N_\Lambda^2 - N_\Lambda\right). \tag{1.2}$$

Here

$$\lambda_k = \int_{\mathbb{R}^3} d^3x \varphi(x)\, e^{-ikx},\ k \in \mathbb{R}^3,$$

is the (real) Fourier transformation of $\varphi(x)$, whereas $N_\Lambda \equiv \sum_{k\in\Lambda^*} a_k^* a_k$ is the particle number operator, with $a_k^\# = \{a_k^* \text{ or } a_k\}$ defined as the usual boson creation/annihilation operators in the one-particle state $\psi_k(x) = V^{-\frac{1}{2}} e^{ikx}, k \in \Lambda^*,\ x \in \Lambda$, acting on $\mathcal{F}_\Lambda^B$.

To ensure the thermodynamics properties of $H_{\Lambda,\lambda_0>0}$ via its superstability [29], as usual, it is assumed that $\varphi(x) \in L^1(\mathbb{R}^3)$ and $\lambda_0 > 0,\ 0 \leq \lambda_k = \lambda_{-k} \leq \lim_{\|k\|\rightarrow 0^+} \lambda_k$ for $k \in \mathbb{R}^3$. However, notice that the helium liquid is a Bose system with strong interactions, i.e., it does not corresponds to a dilute Bose gas. The interaction potential $U_{\text{th}}(r)$ is of Lennard-Jones type [29] and was found by Slater et Kirkwood [30] using the electronic structure of ^{4}He (see figure 1.2 with $U_{\text{th}}(r)$ in Kelvin and also [31]).

Figure 1.2: *The theoretical interaction potential of ^{4}He*

The exact formula for the interaction potential $U_{\text{th}}(r)$ given in [31] is valid only for strictly positive r, whereas close to zero it is given by a polynomial interaction like in figure 1.2. A first caricature of this interaction is the hard sphere interaction potential (who gives an estimation of the condensate fraction surprisingly close to the experiments [32, 33]). However, to ensure the thermodynamics properties of our models, we should mimic an interaction potential $\varphi(x)$ close to $U_{\text{th}}(r)$. In particular, in contrast with the hard sphere potential the value of $\varphi(x)$ for $x = 0$ has to be given and has not to be infinite. A standard way to do it is to cut $U_{\text{th}}(r)$ when

$r \longrightarrow 0^+$ as follows:

$$\varphi(x) = \varphi(r = \|x\|) = \begin{cases} U_{\text{th}}(r) & \text{for } r > r_{\text{min}}. \\ U_{\text{th}}(r_{\text{min}}) & \text{for } 0 \leq r \leq r_{\text{min}}. \end{cases}$$

With this cut interaction potential, the Fourier transformation $\lambda_0(r_{\text{min}})$ of $\varphi(x)$ for the mode $k = 0$ drastically depends on r_{min} (specially when $r_{\text{min}} \longrightarrow 0^+$), i.e.,

$$\lim_{r_{\text{min}} \longrightarrow 0^+} \lambda_0(r_{\text{min}}) = +\infty. \tag{1.3}$$

Moreover, the influence of r_{min} should only correspond to a small (specially when $r_{\text{min}} \longrightarrow 0^+$) perturbation of the Fourier transformation for $k \neq 0$ of $U_{\text{th}}(r)$. In fact one should choose $r_{\text{min}} << r_{mean}$ where $r_{mean} \sim \rho^{-1/3}$ is the average length of the inter-particle distance at density $\rho > 0$.

Finding a Landau-type excitation spectrum from $H_\Lambda^{(n)}$ or H_{Λ,λ_0}, the two modellings of the liquid Helium 4, was a challenging program, since such questions pose enormous problems even for a *few-body* quantum system (e.g. atomic or molecular spectra beyond the hydrogen atom). Faced with this important difficulty, Bogoliubov was guided by the Landau's bench-mark that (at least) the low energy part of the spectrum of about 10^{23} atoms of ^{4}He is defined by coherent collective movements of the system instead of *individual* ones. His starting point was to find a physical (or mathematical) mechanism which as in crystals with phonons, favors the collective motions of the "helium jelly", via some kind of ordering or coherence.

Since the atoms of ^{4}He are bosons (in contrast to ^{3}He, which are fermions), a plausible conjecture relates to the *Bose-Einstein condensation* predicted for the *Perfect Bose Gas* (PBG) by Einstein [34] in 1925. In fact, it was *originally* suggested by Fritz London [35] in 1938, since the transition of the *normal* liquid ^{4}He (called He I) to *superfluid* phase He II takes place at a temperature $T_\lambda = 2.17$ K, whereas, if the liquid ^{4}He is treated as a Perfect Bose Gas, its temperature of Bose-Einstein condensation T_c would be very close to T_λ: $T_c = 3.14$ K. Experimentally, a fraction of condensate in liquid ^{4}He was only found in the sixties, almost 30 years after the London's idea of genius, via deep-inelastic neutron scattering, see [36, 37].

However, in spite of arguments of Tisza and London [38], the spectrum $\varepsilon_k \equiv \hbar^2 k^2 / 2m$ of the Perfect Bose Gas does not satisfy the Landau criterion of superfluidity. From more recent experiments, the Bose condensate represents at $T = 0$ K only 9% of the system, whereas there is 100% of Bose-Einstein condensation in the Perfect Bose Gas! In fact, Bogoliubov stressed [4–8] that, in spite of a long-range coherence of the condensate, elementary excitations in the Perfect Bose Gas correspond to movements of individual atoms, i.e. "quasi-particles" simply coincide with particles. He accepted the idea that the Bose condensation plays a crucial role in decoding a nature of superfluidity but he insisted that "an energy level scheme based on the solution of the quantum mechanical many-body problem with interactions, must be found" (in [7]: Part 3.4).

To summarize, the Bose condensation *together* with interaction between bosons will *transform* individual excitations of the Perfect Bose Gas into collective excitations of the "helium jelly" with a Landau-type spectrum.

For more details concerning the Bogoliubov approach, including its history, see [2].

1.2. The Bogoliubov theory of superfluidity (1947)

Inspired by these observations, Bogoliubov proposed his famous microscopic theory of superfluidity in [4–8] . The first main idea was the following. If one supposes that Bose-Einstein condensation, which occurs in the Perfect Bose Gas for $k = 0$, persists for a weak interaction $\varphi(x)$ then, according to Bogoliubov, the most important terms in (1.1) should be those in which at least two operators a_0^*, a_0 appear. We are thus led to consider the following *truncated* Hamiltonian, i.e., the Bogoliubov Hamiltonian or the Weakly Imperfect Bose Gas (WIBG) (see [7], Part 3.5, eq. (3.81)):

$$H^B_{\Lambda,\lambda_0>0} \equiv T_\Lambda + U^D_\Lambda + U^{ND}_\Lambda + U^{BMF}_\Lambda, \tag{1.4}$$

with

$$U^D_\Lambda \equiv \frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k a_0^* a_0 \left(a_k^* a_k + a_{-k}^* a_{-k}\right), \tag{1.5}$$

$$U^{ND}_\Lambda \equiv \frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k \left(a_k^* a_{-k}^* a_0^2 + a_0^{*2} a_k a_{-k}\right), \tag{1.6}$$

$$U^{BMF}_\Lambda \equiv \frac{\lambda_0}{2V} a_0^{*2} a_0^2 + \frac{\lambda_0}{V} a_0^* a_0 \sum_{k\in\Lambda^*\setminus\{0\}} a_k^* a_k. \tag{1.7}$$

The Bogoliubov model (1.4) is "simpler" than the full Hamiltonian (1.1) but it is still *nondiagonal with an unknown spectrum.*

Let us consider the grandcanonical ensemble (β, μ) where $\beta \equiv (\mathrm{k_B T})^{-1}$ ($\mathrm{k_B}$ is the Boltzmann constant) and μ are the inverse temperature and the chemical potential respectively. For a density of Bose condensate $|c|^2 > 0$, a very ingenious Bogoliubov treatment to solve this problem was to consider the two operators $a_0/\sqrt{V}$, $a_0^*/\sqrt{V}$ as complex numbers:

$$a_0/\sqrt{V} \to c, \ a_0^*/\sqrt{V} \to \bar{c}, \tag{1.8}$$

since for large volume V, $a_0/\sqrt{V}$ and $a_0^*/\sqrt{V}$ almost commute. This second assumption is called the Bogoliubov approximation. Then, by *imposing*

$$\mu = \lambda_0 |c|^2 > 0 \tag{1.9}$$

at zero-temperature, Bogoliubov gets the Landau-type excitation spectrum (figure 1.1).

1.3. A critical discussion of the Bogoliubov theory

His Weakly Imperfect Bose Gas (WIBG) arising from the truncation of a full interacting gas, was a starting point for this theory, see (1.4). Different questions on this approach should be analyzed.

First, the assumption that we *do* have a Bose condensation in the Bogoliubov WIBG must be proven, whereas, in the *full* Hamiltonian, this problem is still open. A challenge to mathematical physics is to *prove* this fact at least for the *truncated* WIBG-model related to the Bogoliubov theory. The second hypothesis (1.8) is also too straightforward [2]: it sounds as "approximation" of *unbounded* creation/annihilation operators by some *bounded c-numbers.* Moreover, it seems to exclude all quantum fluctuations related to the condensed mode, which may be a cause of indirect effective interactions between bosons in this mode as well as outside of it. Attempts of mathematical justification of this procedure and its intimate connection with representations of the Canonical Commutations Relations (CCR) was the subject of several papers, see e.g. [28,39,40]. A very interesting analysis was done by Ginibre [41], where he thermodynamically treated this problem for the full Hamiltonian (1.1). In particular, Ginibre proves that the Bogoliubov approximation has to be done with a fixed c as a function of the inverse temperature β and the chemical potential μ, which satisfies a variational principle (i.e., maximization of the pressure) rather different from (1.9)!

Actually, neither the first, nor the two other hypotheses have been ever justified rigorously for the WIBG by Bogoliubov. He stressed many times in [6, 7, 42] that, since the perturbation theory around the Perfect Bose Gas is highly singular, a Bose-gas with any interaction, however weak, may be qualitatively rather *different* from the non-interacting system.

Only very few rigorous results concerning his WIBG and ansätze were known until 1998-2000. One of the first important rigorous result concerning the Bogoliubov WIBG was performed in 1992 in a very interesting analysis [1]. From thermodynamic estimations in the grandcanonical ensemble, the author have shown that the assumption (1.9), which is crucial to get a gapless spectrum, is *wrong*, in the sense that the theory is not rigorously consistent. Indeed, the condition (1.9) for $|c|^2 > 0$ involves a positive chemical potential where the pressure of the original Bogoliubov Hamiltonian $H^B_{\Lambda,\lambda_0>0}$ (1.4) does not exist!

Then, the recent papers [11–14, 43] expressed in 1998-2000 a rigorous analysis of this Bogoliubov model (WIBG) in the sense that the grand-canonical thermodynamic behavior is finally given at all temperatures and densities. In particular, it is shown that the Bogoliubov approximation (1.8) on the model $H^B_{\Lambda,\lambda_0>0}$ is in fact true in terms of the thermodynamic behavior. However the thermodynamically relevant spectrum of the original Hamiltonian $H^B_{\Lambda,\lambda_0>0}$ always has a gap for any chemical potential μ in the existence domain of the pressure, i.e., the equality (1.9) is inexact.

In the canonical ensemble, Bogoliubov [6] suggests a different but similar way corresponding to a canonical Bogoliubov theory of superfluidity, where the assumption (1.9) disappears. Indeed, in this analysis, the (grandcanonical) Bogoliubov approximation (1.8) is replaced by the following transformations (the canonical Bogoliubov approximation):

$$\frac{N_0}{V} \to |c|^2, \quad \frac{N_0^{1/2}(N_0 - \mathrm{I})^{1/2}}{V} \to |c|^2, \quad N_0 \equiv a_0^* a_0. \tag{1.10}$$

Meantime, at zero-temperature he used the approximation

$$\frac{N_0}{V} \simeq \frac{N_\Lambda}{V} \to \rho = |c|^2, \tag{1.11}$$

where ρ is the fixed full particle density in the canonical ensemble (β, ρ). We should be very doubtful concerning (1.11). The approximation (1.11), taken in terms of operators, change the original Bogoliubov Hamiltonian $H^B_{\Lambda,\lambda_0>0}$ drastically, whereas $\rho = |c|^2$ in (1.11) imposes a completely condensed particle density. This last assumption is not true for liquid helium 4, where we recall that, experimentally (cf. [36,37]), an estimate of the fraction of condensate in liquid ^{4}He at zero-temperature is *only* 9% and *not* 100%! Bogoliubov (and Zubarev) himself realized the difficulty with his ansätze of 100% of condensate: his u-v transformation (see (2.3) below) implies a diminution of condensate because of repulsion between particles. Some discussions corresponding to the problem of the condensate depletion can be found in [7,8,44–48].

Actually, from the beginning, the Bogoliubov theory with his truncation is far from being exact. The Bogoliubov model $H^B_{\Lambda,\lambda_0>0}$ manifests, for high densities, a coexistence of two Bose condensations in the grand-canonical ensemble [11–14]. The first Bose condensation appears on the single mode $k = 0$ due to the nondiagonal interaction U^{ND}_Λ cf. [11–13,43]. But it saturates for high densities and then coexists with a conventional Bose-Einstein condensation on modes next to the zero-mode ($\|k\| = 2\pi/L$), see [14]. Then, for high densities, to be at least self-consistent in this procedure, the terms in (1.1) involving the 6 modes $\|k\| = 2\pi/L$ should not have been neglected in the truncation of the full interaction!

The paper [43] is very useful to point out the origin of this second (conventional) Bose condensation for the Bogoliubov Hamiltonian $H^B_{\Lambda,\lambda_0>0}$ (1.4). Indeed, the apparition of the second (conventional) Bose-Einstein condensation for the Bogoliubov WIBG comes from the term of repulsion

$$\frac{\lambda_0}{2V} a_0^{*2} a_0^2 = \frac{\lambda_0}{2V} \left(N_0^2 - N_0\right), \text{ with } N_0 \equiv a_0^* a_0, \tag{1.12}$$

which implies the saturation of the first (non-conventional) Bose condensation by excluding particles in the zero mode, since for any $k \neq 0$ the similar terms of repulsion

$$\left\{\frac{\lambda_0}{2V} a_k^{*2} a_k^2 = \frac{\lambda_0}{2V} \left(N_k^2 - N_k\right), \text{ with } N_k \equiv a_k^* a_k\right\}_{k\in\Lambda^*\backslash\{0\}} \tag{1.13}$$

in the full Hamiltonian (1.1) are neglected in the Bogoliubov truncation. All terms (1.12)-(1.13) come from the Mean-Field (also called the "forward scattering") interaction U^{MF}_Λ (1.2). Consequently, keeping the interaction (1.2) or avoiding all these terms for any $k \in \Lambda^*$ seem to be necessary to avoid the appearance of a second Bose condensation, which would be inconsistent with this truncation.

A deeper analysis of the Bogoliubov theory and its attempts of generalization, including all recent studies [11–14, 43, 49] has been done from the point of view of rigorous results in [2]. A constructive criticism of the Bogoliubov theory is performed with more details in [3], which is the origin of our new approach explained in the next section.

2. Our model for superfluidity

2.1. Setup of the appropriate model

Before we embark on a strong revision of the Bogoliubov theory, we want to make precise the definition of the excitation spectrum of a system of particles. In particular, which is the relevant ensemble of the two Boltzmann ensembles - the canonical and grandcanonical one, in terms of physical excitation spectrum?

It is clear that the spectrum of excitations should be understood as the spectrum of the corresponding Hamiltonian. Considering for example the Perfect Bose Gas, this spectrum is given by $\{\varepsilon_k\}_{k\in\mathbb{R}}$ in the canonical ensemble, whereas in the grand-canonical ensemble it equals $\{\varepsilon_k - \mu\}_{k\in\mathbb{R}}$, i.e. the spectrum has a gap for $\mu < 0$. Of course, the presence of this gap comes only from the *Lagrange multiplier* μ associated with the operator N_Λ/V [50]. The excitation spectrum of the Perfect Bose Gas is then $\{\varepsilon_k\}_{k\in\mathbb{R}}$. The chemical potential μ has no physical relevance in terms of spectrum of excitations, i.e. the *physical* spectrum of excitations should be seen *only* in the canonical ensemble.

An absence of gaps in the grand-canonical ensemble is *only a specific case.* For example, it is only in the presence of the conventional Bose-Einstein condensation that this property holds for the Perfect Bose Gas and then for the Mean-Field Bose Gas or the Imperfect Bose Gas, see [31,51–57]. This fact can also not be generalized to any Bose system having a Bose condensation, i.e. a gap on the spectrum in the grand-canonical ensemble may appear even if no gap exists in the canonical ensemble. For the Bogoliubov microscopic theory of superfluidity, the spectrum in the two ensembles gives the same result . However, it is only because of the *drastic* Bogoliubov assumption (1.9), that all effects of the chemical potential on the spectrum are removed in the grand-canonical ensemble (β, μ).

Consequently, in terms of the spectrum of excitations, a Bose system should be thermodynamically analyzed *only in the canonical ensemble.* Within this framework, considering the existence of a Bose condensation in the zero-kinetic energy state, one should partially truncate the full interaction, i.e., without taking into account the Mean-Field interaction *since it is a constant in the canonical ensemble.* This procedure implies the non-diagonal Hamiltonian:

$$H_{\Lambda,0}^{B} \equiv T_\Lambda + U_\Lambda^{D} + U_\Lambda^{ND}, \tag{2.1}$$

with U_Λ^D and U_Λ^{ND} defined by (1.5) and (1.6) respectively. Actually, without any Bose condensation, the model should be equal to the Mean-Field model, i.e., the Perfect Bose gas in the canonical ensemble. Whereas, in presence of Bose condensation, the interaction $\widetilde{U}_\Lambda$ should play a crucial role on the thermodynamics. Formally, the Mean-Field interaction U_Λ^{MF} does not change the "physical properties" of a Bose system (cf. [3, 57, 58]) and the "physical" effect of the interaction potential should express itself by the other terms of interaction, i.e., by the interaction $\widetilde{U}_\Lambda$.

The Bogoliubov procedures [4–8] always involved the truncation of the Mean-Field interaction U_Λ^{MF}. The interaction U_Λ^{BMF} (1.7) in the WIBG comes directly from the Bogoliubov truncation of the Mean-Field interaction. Avoiding this interaction term allows us to solve the problem of the second condensation explained in the previous section.

By doing on $H_{\Lambda,0}^{B}$ the usual Bogoliubov approximation (1.8) note that the new Hamiltonian does not commute with the particle number operator N_Λ. Following suggestions of Bogoliubov [6], we first use the new set of operators

$$\zeta_k = a_0^* \left(N_0 + \mathrm{I}\right)^{-1/2} a_k,\ \zeta_k^* = a_k^* \left(N_0 + \mathrm{I}\right)^{-1/2} a_0,\ k \in \Lambda^*.$$

This set $\{\zeta_k\}_{k\in\Lambda^*\setminus\{0\}}$ satisfies the Canonical Commutation Relations. Then, for $H_{\Lambda,0}^{B}$ the next step corresponds to do the canonical Bogoliubov approximation (1.10). It implies a bilinear form in Bose-operators $\{\zeta_k\}_{k\in\Lambda^*\backslash\{0\}}$:

$$\begin{aligned} H^B_{\Lambda,0}(c) &= \sum_{k\in\Lambda^*\backslash\{0\}} \varepsilon_k \zeta_k^* \zeta_k + \frac{1}{2}\sum_{k\in\Lambda^*\backslash\{0\}} \lambda_k |c|^2 \left[\zeta_k^*\zeta_k + \zeta_{-k}^*\zeta_{-k}\right] \\ &+\frac{1}{2}\sum_{k\in\Lambda^*\backslash\{0\}} \lambda_k \left[c^2 \zeta_k^*\zeta_{-k}^* + \bar{c}^2 \zeta_k \zeta_{-k}\right]. \end{aligned} \tag{2.2}$$

This Hamiltonian commutes with the particle number operator N_Λ. After the canonical gauge transformation to boson operators

$$\zeta_k e^{-i\arg c},\ k\in\Lambda^*\backslash\{0\},$$

the model $H^B_{\Lambda,0}(c)$ only depends on $x\equiv|c|^2$. Then, the Bogoliubov canonical u-v transformation diagonalizes it by using a new set of boson operators $\{b_k, b_k^*\}_{k\in\Lambda^*\backslash\{0\}}$ defined by

$$\zeta_k = \mathbf{u}_k b_k - \mathbf{v}_k b^*_{-k},\ \zeta_k^* = \mathbf{u}_k b_k^* - \mathbf{v}_k b_{-k}. \tag{2.3}$$

The real coefficients $\{\mathbf{u}_k = \mathbf{u}_{-k}\}_{k\in\Lambda^*\backslash\{0\}}$ and $\{\mathbf{v}_k = \mathbf{v}_{-k}\}_{k\in\Lambda^*\backslash\{0\}}$ satisfy:

$$\mathbf{u}_k^2 - \mathbf{v}_k^2 = 1,\ 2\mathbf{u}_k\mathbf{v}_k = \frac{x\lambda_k}{\sqrt{\varepsilon_k(\varepsilon_k+2x\lambda_k)}},\ \mathbf{u}_k^2+\mathbf{v}_k^2 = \frac{\varepsilon_k}{\sqrt{\varepsilon_k(\varepsilon_k+2x\lambda_k)}}.$$

It follows that the Hamiltonian $H^B_{\Lambda,0}(c\neq 0)$ corresponds to the perfect Bose gas of quasi-particles defined by

$$\begin{aligned} \mathrm{H}^{PBG}_{\Lambda,0}\left(x\equiv|c|^2\right) &= \sum_{k\in\Lambda^*\backslash\{0\}} \sqrt{\varepsilon_k(\varepsilon_k+2x\lambda_k)}\, b_k^* b_k \\ &+\frac{1}{2}\sum_{k\in\Lambda^*\backslash\{0\}} \left(\sqrt{\varepsilon_k(\varepsilon_k+2x\lambda_k)} - (\varepsilon_k + x\lambda_k)\right). \end{aligned} \tag{2.4}$$

In other words, *if we consider that this "canonical Bogoliubov approximation" is true*, we directly get the well-known Bogoliubov gapless spectrum, i.e., the Landau-type excitation spectrum of figure 1.1, for a Bose condensate density $x=|c|^2>0$.

2.2. A polemical discussion of this new approach

The main problem of the previous attempts (Bogoliubov *et al*, see for example [1, 4–8, 59, 60]) is to assume, à priori, the Bose condensation by directly doing the Bogoliubov approximation with an *arbitrary* choice of $|c|^2$, without exactly solving it in terms of the thermodynamic behavior. In particular, the "canonical Bogoliubov approximation" (1.10) applied on $H^B_{\Lambda,0}$ *has to be proven.* For example, Bogoliubov made the wrong assumption of 100% of Bose condensate at zero-temperature in the canonical ensemble, but what is our value of $x=|c|^2$ after the approximation (1.10)?

Actually, these questions are solved in [15] since the canonical thermodynamic behavior of the nondiagonal Hamiltonian $H^B_{\Lambda,0}$ is *rigorously* performed. In particular, it is shown that the "canonical Bogoliubov approximation" (1.10) is true in the following sense: the thermodynamics of $H^B_{\Lambda,0}$ corresponds, at the thermodynamic level, to the perfect Bose gas (2.4) of quasi-particles for $k\in\Lambda^*\backslash\{0\}$ with a Bose condensate density $\widehat{x}(\beta,\rho)$ on $k=0$ (cf. theorems 3.1 and 3.2). In fact we prove [15] that the model $H_{\Lambda,0}^{B}$ solves, in the canonical ensemble, the problems of the previous Bogoliubov theory and implies a new microscopic theory of superfluidity at all temperatures explained in the next Section 3.

2.3. Additional remarks

As it is explained in Section 1.3, the first important problem of the Bogoliubov theory was highlighted by Angelescu, Verbeure and Zagrebnov in 1992 [1]. It concerns the instability for positive chemical potential $\mu > 0$ of the Bogoliubov Hamiltonian $H_{\Lambda,\lambda_0>0}^{B}$. Therefore, a "minimal" stabilization of the Bogoliubov Hamiltonian is to add the "forward scattering" interactions between particles above zero-mode, i.e., to save during the truncation the Mean-Field interaction U_{Λ}^{MF} (1.2). This quite interesting approach was first developed in the papers [1, 59, 60] and leads to the model

$$H_{\Lambda,\lambda_0}^{SB} = H_{\Lambda,0}^{B} + U_{\Lambda}^{MF}, \tag{2.5}$$

called the AVZ-Hamiltonian or the superstable Bogoliubov model. In the canonical ensemble, the two models $H_{\Lambda,0}^{B}$ and $H_{\Lambda,\lambda_0}^{SB}$ are equivalent, i.e. their Gibbs states are equal for all (β,ρ).

Their main object was of course to correct the instability for positive chemical potentials of the Bogoliubov Hamiltonian $H_{\Lambda,\lambda_0}^{B}$ but also to find a gapless Bogoliubov spectrum in the *grandcanonical* ensemble. In [1], they use a Bogoliubov approximation partially applied on $H_{\Lambda,\lambda_0}^{SB}$, i.e., they save the Mean-Field interaction U_{Λ}^{MF} (1.2), whereas in [60], the authors use a "generalized" Bogoliubov approximation. This "generalized" Bogoliubov approximation corresponds to partially change the operators $\left\{a_0/\sqrt{V}, a_0^*/\sqrt{V}\right\}$ by a suitable function $\left\{b(c), \overline{b(c)}\right\}$ in (2.5) *except* in the Mean-Field interaction U_{Λ}^{MF}. Then, they prove a Bose condensation in zero-mode via second-order phase transition and a linear asymptotic of the elementary excitation spectrum in condensed phase for $\|k\| \to 0$, see also discussions in Section 3.4 of [2].

In [61] it is shown that the two procedures [1, 60] are inexact, in the sense that they are equivalent to some drastic modifications of the original Hamiltonian $H_{\Lambda,\lambda_0}^{SB}$. For example, as Bogoliubov did, they were forced in [1] to add some *additional assumptions* to find a gapless spectrum. As it is explained in Section 2.1, it was unlikely that the *exact solution* of $H_{\Lambda,\lambda_0}^{SB}$, in the *grandcanonical* ensemble, had a gapless spectrum even in the presence of Bose condensation. In fact, we prove [61] that, on the thermodynamic level, the spectrum always has a gap in the grandcanonical ensemble.

Actually, as the review [2] explains in the "outline" section, we should be discouraged *"from performing sloppy manipulations with Bose condensations, quantum fluctuations and different kinds of ansätze"*. The analysis performed in [61] provides another strong warning in doing it.

3. A new theory of superfluidity at all temperatures [15]

To fix the notations, we recall that $\beta = (\mathrm{k_B T})^{-1} > 0$ is here the inverse temperature and $\rho > 0$ the fixed full particle density, whereas $n = [\rho V]$ defined as the integer of ρV, is the number of particles in the canonical ensemble.

3.1. Rigorous thermodynamics in the canonical ensemble

The aim of this section is to examine the Hamiltonian $H^B_{\Lambda,0}$ (2.1) in the canonical ensemble specified by (β, ρ). It is essential here to note that in the canonical ensemble the conditions relating to the interaction potential $\varphi(x)$ may be relaxed as follows. The model is independent of the Fourier transformation of $\varphi(x)$ for $k = 0$, which may be infinite for some specific interaction potentials. However, the (effective coupling) constant

$$g_{00} \equiv -\frac{1}{4(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{\lambda_k^2}{\varepsilon_k} < 0, \tag{3.1}$$

and $\varphi(0)$ have to exist. In particular, the Fourier transformation $\lambda_0(r_{\min})$ of $\varphi(x)$ for the mode $k = 0$ which drastically depends on $r_{\min}$ (specially when $r_{\min} \to 0^+$), see (1.3), has *no* influence on the canonical thermodynamic behavior of $H^B_{\Lambda,0}$.
Now we give all promised properties of the Hamiltonian $H^B_{\Lambda,0}$ in the canonical ensemble.

1. Let $f^B_{\Lambda,0}(\beta, \rho)$ be the corresponding free-energy density defined for a fixed particle density $\rho > 0$ by

$$f^B_{\Lambda,0}(\beta, \rho) \equiv -\frac{1}{\beta V} \ln Tr_{\mathcal{H}_B^{(n)}} \left(\left\{ e^{-\beta H^B_{\Lambda,0}} \right\}^{(n=[\rho V])} \right). \tag{3.2}$$

Recall that the "canonical Bogoliubov approximation" (1.10) implies the model $H^B_{\Lambda,0}(c)$ (2.2). Here, for technical considerations, we use the operator $\widetilde{H}^B_{\Lambda,0}(c)$ corresponding in $H^B_{\Lambda,0}(c)$ to replace again the operators $\{\zeta_k\}_{k\in\Lambda^*\setminus\{0\}}$ by $\{a_k\}_{k\in\Lambda^*\setminus\{0\}}$. The Hamiltonian $\widetilde{H}^B_{\Lambda,0}(c)$ is well-defined on the boson Fock space

$$\mathcal{F}'_B \equiv \bigoplus_{n_1=0}^{+\infty} \mathcal{H}^{(n_1)}_{B,k\neq 0}$$

of the symmetrized n-particle Hilbert spaces $\mathcal{H}^{(n)}_{B,k\neq 0}$ for non-zero momentum bosons. The Bogoliubov canonical u-v transformation gives also for $k \in \Lambda^* \setminus \{0\}$ the perfect Bose gas (2.4) of quasi-particles for a Bose condensate ($k = 0$) density x. Even if $\widetilde{H}^B_{\Lambda,0}(c)$ does not commute anymore with $\sum_{k\neq 0} a_k^* a_k$, we consider its (infinite volume) free-energy density defined by

$$f^B_0(\beta, \rho_1, x) \equiv \lim_{\Lambda} -\frac{1}{\beta V} \ln Tr_{\mathcal{H}^{(n_1)}_{B,k\neq 0}} \left(\left\{ e^{-\beta \widetilde{H}^B_{\Lambda,0}(c)} \right\}^{(n_1=[\rho_1 V])} \right), \tag{3.3}$$

for any $\beta > 0$, $\rho_1 > 0$ and $x = |c|^2 \geq 0$. The (infinite volume) pressure of this gas of quasi-particles is

$$\begin{aligned} p_0^B(\beta,\alpha,x) &\equiv \lim_{\Lambda} \frac{1}{\beta V} \ln Tr_{\mathcal{F}'_B} e^{-\beta\left(\widetilde{H}^B_{\Lambda,0}(c)-\alpha\left(\sum_{k\in\Lambda^*\setminus\{0\}} a_k^* a_k + x\right)\right)} \\ &= \sup_{\rho_1>0} \left\{\alpha\left[\rho_1 + x\right] - f_0^B(\beta,\rho_1,x)\right\} \\ &= \alpha x + \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \left\{\frac{1}{\beta}\ln\left(1-e^{-\beta E^B_{k,0}}\right)^{-1} + \frac{1}{2}\left(f_{k,0} - E^B_{k,0}\right)\right\} d^3k \end{aligned} \tag{3.4}$$

for $\alpha \leq 0$ with

$$f_{k,0} = \varepsilon_k - \alpha + x\lambda_k,\ E^B_{k,0} = \sqrt{(\varepsilon_k - \alpha)(\varepsilon_k - \alpha + 2x\lambda_k)}. \tag{3.5}$$

Then we get our first main result:

Theorem 3.1. *The thermodynamic limit $f_0^B(\beta,\rho)$ exists for any $\beta > 0$ and $\rho > 0$. (i) Moreover, the Hamiltonian $H^B_{\Lambda,0}$ (2.1) is equivalent, at the thermodynamic level, to the perfect Bose gas (2.4) of quasi-particles for $k \in \Lambda^*\setminus\{0\}$ with a density $x = \widehat{x}(\beta,\rho)$ solution of the variational problem:*

$$f_0^B(\beta,\rho) = \inf_{x\in[0,\rho]} \left\{f_0^B(\beta,\rho - x, x)\right\} = \left\{f_0^B(\beta,\rho - x, x)\right\}\Big|_{x=\widehat{x}(\beta,\rho)<\rho}.$$

(ii) More explicitly the free-energy density $f_0^B(\beta,\rho)$ equals:

$$f_0^B(\beta,\rho) = \sup_{\alpha\leq 0} \left\{\alpha\rho - p_0^B(\beta,\alpha,\widehat{x})\right\} = \alpha(\widehat{x})\rho - p_0^B(\beta,\alpha(\widehat{x}),\widehat{x}).$$

with $\widehat{x} = \widehat{x}(\beta,\rho)$. Note that $f_0^B(\beta,\rho_1,x)$ (3.3) may have been directly defined as the Legendre transformation of $p_0^B(\beta,\alpha,x)$.

The solution $\alpha(\widehat{x})$ of the variational problem (ii) in the previous theorem is the unique solution of the *Bogoliubov density equation*:

$$\rho = \rho_0^B(\beta,\alpha,\widehat{x}) \text{ for } \rho > 0. \tag{3.6}$$

Here

$$\begin{aligned} \rho_0^B(\beta,\alpha,x) &\equiv \partial_\alpha p_0^B(\beta,\alpha,x) = x + \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{f_{k,0}}{E^B_{k,0}\left[e^{\beta E^B_{k,0}} - 1\right]} d^3k \\ &+ \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{x^2\lambda_k^2}{2E^B_{k,0}\left[f_{k,0} + E^B_{k,0}\right]} d^3k. \end{aligned} \tag{3.7}$$

Moreover, there is a particle density $\rho_c(\beta)$ such that the solution $\widehat{x}(\beta,\rho) = 0$ for $\rho \leq \rho_c(\beta)$, whereas for $\rho > \rho_c(\beta)$, $0 < \widehat{x}(\beta,\rho) < \rho$ (even for $\beta \to +\infty$). For a fixed particle density ρ, there is also a critical inverse temperature $\beta_c(\rho)$. An illustration of $\beta_c(\rho)$ is performed in figure 3.1. Note that $\partial_\rho f_0^B(\beta,\rho) = \alpha(\widehat{x})$ and

Figure 3.1: *Illustration of the critical inverse temperature $\beta_c(\rho)$ as a function of ρ. The dotted line corresponds to the phase diagram of the Perfect Bose Gas. The difference with the Perfect Bose Gas is always greater or equal to zero. It may be zero for all $\beta > 0$.*

$\partial_\rho f_0^B(\beta,\rho) < 0$ for $\rho \neq \rho_c(\beta)$ or $\beta \neq \beta_c(\rho)$.
For $\rho \leq \rho_c(\beta)$, remark also that the thermodynamic behavior of $H_{\Lambda,0}^B$ corresponds to the Perfect Bose Gas (excitation spectrum ε_k).
2. Now we give our main result for the thermodynamic behavior of $H_{\Lambda,0}^B$ in the *canonical ensemble* (β,ρ). In the following theorem, $\langle - \rangle_{H_{\Lambda,0}^B}(\beta,\rho)$ represents the (finite volume) canonical Gibbs state associated with $H_{\Lambda,0}^B$.

Theorem 3.2. *(i) A non-conventional Bose condensation induced by the non-diagonal interaction U_Λ^{ND} for high particle densities, or low temperatures:*

$$\lim_\Lambda \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda,0}^B} (\beta,\rho) = \widehat{x}(\beta,\rho) = \left\{ \begin{array}{l} = 0 \ \text{for}\ \rho \leq \rho_c(\beta)\ \text{ or } \beta \leq \beta_c(\rho). \\ > 0 \ \text{for}\ \rho > \rho_c(\beta)\ \text{ or } \beta > \beta_c(\rho). \end{array} \right.$$

(ii) No Bose condensation (of any type I, II or III [62–64]) outside the zero-mode for any particle densities or temperatures:

$$\left\{ \begin{array}{c} \forall k \in \Lambda^* \backslash \{0\}, \ \lim_\Lambda \left\langle \frac{a_k^* a_k}{V} \right\rangle_{H_{\Lambda,0}^B} (\beta,\rho) = 0 \\ \lim_{\delta \to 0^+} \lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^*, 0 < \|k\| \leq \delta\}} \langle a_k^* a_k \rangle_{H_{\Lambda,0}^B} (\beta,\rho) = 0 \end{array} \right\}$$

(iii) A particle density outside the zero-mode equal to:

$$\lim_\Lambda \frac{1}{V} \sum_{k \in \Lambda^* \backslash \{0\}} \langle a_k^* a_k \rangle_{H_{\Lambda,0}^B} (\beta,\rho) = \left\{ \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{f_{k,0}}{E_{k,0}^B \left[e^{\beta E_{k,0}^B} - 1 \right]} d^3k \right\} \Big|_{x=\widehat{x}, \alpha=\alpha(\widehat{x})}$$

$$+\left\{\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\frac{x^2\lambda_k^2}{2E_{k,0}^B\left[f_{k,0}+E_{k,0}^B\right]}d^3k\right\}\Bigg|_{x=\widehat{x},\alpha=\alpha(\widehat{x})}.$$

Here $f_{k,0}$ and $E_{k,0}^B$ are defined by (3.5) for a chemical potential given by the solution $\alpha(\widehat{x})$ of the variational problem (ii) in the previous theorem.

(iv) There is no discontinuity of the particle densities (density in the zero-mode (i) or outside the zero-mode (iii)).

(v) For $\rho \leq \rho_c(\beta)$ or $\beta \leq \beta_c(\rho)$ one has the Bose statistics for a corresponding chemical potential $\alpha(0) < 0$:

$$\forall k \in \Lambda^* : \|k\| \geq \delta > 0, \ \lim_{\Lambda}\langle a_k^* a_k\rangle_{H_{\Lambda,0}^B}(\beta,\rho) = \frac{1}{e^{\beta(\varepsilon_k-\alpha(0))}-1}.$$

But for $\rho > \rho_c(\beta)$ or $\beta > \beta_c(\rho)$, i.e. in the presence of a Bose condensation, we get another one, which we call the Bogoliubov statistics, for a corresponding chemical potential $\alpha(\widehat{x}) < 0$:

$$\lim_{\Lambda}\langle a_k^* a_k\rangle_{H_{\Lambda,0}^B}(\beta,\rho) = \left\{\frac{f_{k,0}}{E_{k,0}^B\left(e^{\beta E_{k,0}^B}-1\right)}+\frac{x^2\lambda_k^2}{2E_{k,0}^B\left(f_{k,0}+E_{k,0}^B\right)}\right\}\Bigg|_{x=\widehat{x},\alpha=\alpha(\widehat{x})}$$

for any $k \in \Lambda^*$ such that $\|k\| \geq \delta > 0$.

The illustrations of the particle densities inside and outside the zero-mode are given in figures 3.2 and 3.3 respectively.

Remark 3.3. *For $\rho > \rho_c(\beta)$, there is a non-conventional Bose condensation whereas no Bose condensation (of any type I, II, or III [62–64]) appears outside the zero-mode at all densities $\rho > 0$ (theorem 3.2). In contrast to the Bogoliubov theory (see for example [3]), this one is self-consistent with the corresponding truncation of the full Hamiltonian in the canonical ensemble.*

3.2. Conclusions

1. In order to obtain a microscopic theory of superfluidity we have to get a Landau-type excitation spectrum (cf. figure 1.1 [9,10]) as Bogoliubov did [4–8] for a *suitable choice* of $c-$numbers.

As Landau's predictions [9, 10], at high densities $\rho > \rho_c(\beta)$ (or sufficiently low temperatures, cf. figure 3.1) the Bose gas $H_{\Lambda,0}^B$ is equivalent to a *"gas of collective elementary excitations"* or *"quasi-particles"* (2.4) for $k \in \Lambda^*\backslash\{0\}$ with a density $\widehat{x}(\beta,\rho)$ of Bose condensate on $k = 0$, cf. theorems 3.1 and 3.2. Consequently, as stated in Section 2.1, the spectrum of excitations, which is macroscopically relevant, equals the Bogoliubov spectrum at inverse temperatures $\beta > 0$ and particle densities $\rho > 0$:

$$E_k^B(\beta,\rho) = \begin{cases} \varepsilon_k = \hbar^2k^2/2m & \text{for } \beta \leq \beta_c(\rho) \text{ or } \rho \leq \rho_c(\beta), \\ \sqrt{\varepsilon_k(\varepsilon_k+2\widehat{x}\lambda_k)} & \text{for } \beta > \beta_c(\rho) \text{ or } \rho > \rho_c(\beta), \end{cases} \tag{3.8}$$

Figure 3.2: *Illustration of the non-conventional Bose condensate density* $\widehat{x}(\beta,\rho)$ *as a function of* ρ. *The dashed dotted line corresponds to a zero-temperature, i.e., for* $\beta \to +\infty$. *The straight line is* $x = \rho$. *Note that* $\widehat{x}(\beta,\rho)$ *is originally defined as the solution of a variational problem (see theorem 3.1).*

see (2.4). The collective excitation spectrum $E_k^B(\beta,\rho)$ has *no gap for any densities or temperatures* as expected in Section 2.1. The main difficulties are to find the solution $\widehat{x}(\beta,\rho)$ of the variational problem (i) of theorem 3.1, i.e., to obtain the thermodynamic properties of the Hamiltonian $H_{\Lambda,0}^B$ in the canonical ensemble.
Note that we do not rigorously know the *exact* spectrum of $H_{\Lambda,0}^B$ even in infinite volume, since our analysis is only based on its thermodynamic properties.

To find the exact Landau-type excitation spectrum from (3.8), i.e. to get the "phonons" part and the "rotons" one, we can reason along the standard lines of Bogoliubov microscopic theory of superfluidity, see [2, 4–8]. Note that, for this approach, we have to assume some specific conditions relating to the two-body interaction potential $\varphi(x)$. In particular, λ_k is spherically-symmetric, i.e. $\lambda_k = \lambda_{\|k\|}$. Additionally, as Bogoliubov did, it is necessary to assume the absolute integrability of $x^2\varphi(x) \in L^1(\mathbb{R}^3)$ in order to have a Taylor expansion for small $\|k\|$. Then, the Bogoliubov spectrum $E_k^B(\beta,\rho)$ is a Landau-type excitation spectrum for $\rho > \rho_c(\beta)$ or $\beta > \beta_c(\rho)$ and an illustration is given by figure 3.4.

Remark 3.4. *The famous Landau's criterion of superfluidity of 1941 [9, 10] gives the following critical velocity:*

$$\begin{aligned}\inf_k \left\{ \frac{E_k^B(\beta,\rho)}{\hbar\|k\|} \right\} &= \left(\frac{1}{2m}\right)^{1/2} \left\{ \min_k \left(\varepsilon_k + 2\lambda_k^2 \widehat{x}\right) \right\}^{1/2} \\ &\equiv \mathrm{v}_0(\beta,\rho) = \left\{ \begin{array}{l} 0 \quad, \textit{ for } \beta \le \beta_c(\rho) \textit{ or } \rho \le \rho_c(\beta) \,. \\ > 0, \textit{ for } \beta > \beta_c(\rho) \textit{ or } \rho > \rho_c(\beta) \,. \end{array} \right\}\end{aligned}$$

Figure 3.3: *Illustration of the particle density outside the zero-mode* $(\rho - \widehat{x}(\beta,\rho))$ *as a function of* ρ. *Note that for* $\rho < \rho_c(\beta)$, $\widehat{x}(\beta,\rho) = 0$. *The dashed dotted line is the Bogoliubov condensate density for* $\beta \rightarrow +\infty$, *i.e., at zero-temperature.*

2. Then, the thermodynamics of the theoretical Bose gas $H^B_{\Lambda,0}$ is *qualitatively* quite similar to the one of the liquid ^{4}He:

- for small densities $\rho \leq \rho_c(\beta)$ or high temperatures $\mathrm{T} = (\mathrm{k_B}\beta)^{-1} \geq \mathrm{T}_c \equiv (\mathrm{k_B}\beta_c(\rho))^{-1}$ the thermodynamic behavior corresponds to the Perfect Bose Gas,
- for high densities $\rho > \rho_c(\beta)$ or small temperatures $\mathrm{T} < \mathrm{T}_c$ (even with $\mathrm{T} \rightarrow 0^+$, i.e., $\beta \rightarrow +\infty$), the spectrum of excitations becomes a Landau-type excitation spectrum and meantime a non-conventional Bose condensation appears via a second order transition with the (continuous) density $0 < \widehat{x}(\beta,\rho) < \rho$ (figure 3.2),
- a coexistence of particles inside and outside the Bose condensate (figure 3.3), even at zero-temperature as it is experimentally found in [36,37]:

$$\begin{cases} \lim\limits_{\beta\rightarrow+\infty}\lim\limits_{\Lambda}\dfrac{1}{V}\sum\limits_{k\in\Lambda^*\backslash\{0\}}\langle a_k^* a_k\rangle_{H^B_{\Lambda,0}}(\beta,\rho) = \left\{\dfrac{1}{(2\pi)^3}\displaystyle\int_{\mathbb{R}^3}\dfrac{x^2\lambda_k^2}{2E^B_{k,0}\left[f_{k,0}+E^B_{k,0}\right]}d^3k\right\}\Bigg|_{\substack{x=\widehat{x}\\ \alpha=\alpha(\widehat{x})}} > 0, \\ \forall k \in \Lambda^*\backslash\{0\}, \quad \lim\limits_{\beta\rightarrow+\infty}\lim\limits_{\Lambda}\langle a_k^* a_k\rangle_{H^B_{\Lambda,0}}(\beta,\rho) = \left\{\dfrac{x^2\lambda_k^2}{2E^B_{k,0}\left[f_{k,0}+E^B_{k,0}\right]}\right\}\Bigg|_{\substack{x=\widehat{x}\\ \alpha=\alpha(\widehat{x})}} > 0. \end{cases} \tag{3.9}$$

Quantitatively, the critical density $\rho_c(\beta)$ is approximately given by $\rho_c(\beta) \approx \rho_c^{PBG}(\beta)$ (cf. figure 3.1), with $\rho_c^{PBG}(\beta)$ defined as the critical density for the Perfect Bose Gas. The theoretical temperature of the phase transition T_c always verifies

Figure 3.4: *The Bogoliubov spectrum* $E_k^B(\beta,\rho)$ *for* $\beta > \beta_c(\rho)$ *or* $\rho > \rho_c(\beta)$. *Each wave-length* k *corresponds to a momentum* $\hbar k$.

$\mathrm{T}_c \geq \mathrm{T}_c^{PBG} = 3.14$ K (critical temperature evaluated for a Perfect Gas of helium particles) but is quite close to T_c^{PBG}:

$$\mathrm{T}_c \approx 3.14 \text{ K}.$$

(In fact we are not able to prove an exact equality at very *high* densities [15, 61]). The experimental transition of the *normal* liquid ^{4}He (He I) to *superfluid* phase ^{4}He II takes place at a lower temperature $\mathrm{T}_\lambda = 2.17$ K (along the vapor pressure curve), which is not so far from the one of the model $H_{\Lambda,0}^B$. However, note that the Henshaw-Woods spectrum (experimental Landau-type excitation spectrum) *does not change* drastically when the temperature crosses T_λ, whereas there is *no* superfluidity for these temperatures.

Remark 3.5. *This means that there is a temperature* $\widetilde{\mathrm{T}}_\lambda > \mathrm{T}_\lambda$ *such that the experimental "quasi-particle" system still exists for* $\mathrm{T} < \widetilde{\mathrm{T}}_\lambda$ *even if Landau's criterion of superfluidity (remark 3.4) experimentally fails at these temperatures* $\mathrm{T}_\lambda < \mathrm{T} < \widetilde{\mathrm{T}}_\lambda$.

3. To resume, this analysis is not a complete theory of "real superfluidity". In particular, the Bogoliubov phonon-maxon-roton dispersion branch is only a part of the spectrum of the full quantum-mechanical Hamiltonian of the helium system. Therefore, this theory fails in being a complete description of all thermodynamics of liquid helium. For example, at temperatures $\mathrm{T}_\lambda < \mathrm{T} < \mathrm{T}_c$, a Bose condensation still exists in $H_{\Lambda,0}^B$ but not for liquid helium even if the system of "quasi-particles" resists in liquid helium for $\mathrm{T}_\lambda < \mathrm{T} < \widetilde{\mathrm{T}}_\lambda$ (remark 3.5).

On the other hand, the understanding of superfluidity represents one of the most challenging problems in modern physics. In theoretical physics, quantum phase transitions are intrinsically complex even at equilibrium, involving the subtleties of quantum mechanics acting in concert with statistical mechanics. At equilibrium, via "thermodynamic" analysis this situation enables us to focus almost exclusively on the comparatively simple but effective model $H^{B}_{\Lambda,0}$ that shares this low-temperature physics despite providing a simplified caricature of the underlying atomic physics.

Note that, in contrast to Bogoliubov's last approach and with the caveat that the full interacting Hamiltonian is truncated, the analysis performed here is rigorous by involving for the first time [15] a complete thermodynamic analysis of a *non-trivial* continuous gas in the canonical ensemble. This *unique* truncation hypothesis is still not proven, but we show that the theory is, at least, self-consistent (remark 3.3).

4. Superfluidity theory reconsidered: new implications

4.1. Two complementary Bose systems: Cooper pairs and gas of quasi-particles

In the case of *homogeneous* systems, the previous analysis provides a new (canonical) theory of superfluidity with a gapless spectrum at all particle densities and temperatures, leading us to a deeper understanding of the Bose condensation phenomenon in liquid helium. At any temperatures $\mathrm{T} = (\mathrm{k_B}\beta)^{-1} \geq 0$ below the critical temperature T_c or above a critical density $\rho > \rho_c(\beta)$, the corresponding Bose gas is a mixture of particles inside and outside the Bose condensate (see theorem 3.2 and figures 3.2 and 3.3). Even at zero-temperature, two Bose systems coexist: the Bose condensate and a second one, which is denoted here as the *Bogoliubov condensate*, cf. (3.9), whereas at all densities $\rho > 0$ there is no Bose condensation (of any type I, II, or III [62–64]) outside the zero-mode. In the regime $\mathrm{T} < \mathrm{T}_c$ or $\rho > \rho_c(\beta)$, the system follows the Bogoliubov statistics (v) of theorem 3.2, whereas in the absence of the Bose condensation, i.e., for $\mathrm{T} \geq \mathrm{T}_c$ or $\rho \leq \rho_c(\beta)$, the (standard) Bose statistics holds.

1. The origin of the Bogoliubov statistics and also of (3.9) is a phenomenon of interaction. Actually, it has been known since [43] that the collection of particles outside the zero-mode imposes, through the non-diagonal interaction U^{ND}_{Λ}, a *glue-like attraction* between particles in the zero-mode.

Figure 4.1: *Non-diagonal-interaction vertices corresponding to* U^{ND}_{Λ}.

A natural way to see this phenomenon is to remark that the non-diagonal interaction U^{ND}_{Λ} (see figure 4.1) implies an *effective interaction term* $g_{\Lambda,00}$ for bosons with $k = 0$, see figure 4.2. Evaluated via a Fröhlich transformation in the second order [43] (see also the review [2]), $g_{\Lambda,00}$ is strictly negative. The corresponding thermodynamic limit

$$\lim_{\Lambda} g_{\Lambda,00} = g_{00} < 0,$$

see (3.1), amazingly plays a *crucial* rôle in the rigorous thermodynamic analysis of $H^{B}_{\Lambda,0}$ (see [15] or in [61]: proof of theorem 2.3). It is also essential in the rigorous study of the Weakly Imperfect Bose Gas, i.e. the Bogoliubov Hamiltonian $H^{B}_{\Lambda,\lambda_0>0}$, see [2, 11–13].

Figure 4.2: *Effective interaction for the zero-mode induced by the non-diagonal interaction* U^{ND}_{Λ}

The Bose condensate with the density $\widehat{x}\,(\beta,\rho)$ and the remaining system with the density $\{\rho-\widehat{x}>0\}$, i.e., the Bogoliubov condensate, only exist via this glue-like attraction g_{00} (figure 4.2). In fact, the particles inside the condensate pair up via the Bogoliubov condensate to form *"Cooper pairs"*. This Bose condensation constituted by Cooper pairs is then non-conventional [2,12,14,43,65,66], i.e. turned on by the Bose statistics but completely transformed by the non-diagonal interaction U^{ND}_{Λ}.

2. The coherency due to the presence of the Bose condensation is not enough to make the Perfect Bose Gas superfluid, see discussions in [4–6]. The spectrum of elementary excitations has to be collective. In this theory, the particles outside the Bose condensate (*the Bogoliubov condensate*) follow the Bogoliubov statistics (v) of theorem 3.2 and also represent a system of "quasi-particles" with the Landau-type excitation spectrum. Therefore, following Landau's criterion of superfluidity [9, 10] (remark 3.4), the Bogoliubov condensate here is *superfluid* due to phenomena of interactions which change, in the presence of the Bose condensate, the behavior of individual particles into an ideal Bose gas of *"quasi-particles"* with the given spectrum $E^{B}_{k}\,(\beta,\rho)$. Indeed, through the Bose condensate, the non-diagonal interaction U^{ND}_{Λ} *combined with* the diagonal interaction U^{D}_{Λ} creates quasi-particles from two particles respectively of modes k and $-k$ ($k \neq 0$). Formally, this can be seen via the Bogoliubov *u-v* transformation applied to $\left\{H^{B}_{\Lambda,0}(c)\right\}_{|c|^2=\widehat{x}>0}$, cf. (2.3). This gas of quasi-particles, i.e. the Bogoliubov condensate, exists *if and only if* the non-conventional Bose condensate exists too.

3. Also for high densities $\rho > 0$ we have

$$\lim_{\rho \to +\infty} \{\rho - \widehat{x}(\beta, \rho)\} = 0, \tag{4.1}$$

cf. theorem 3.2, figure 3.2. Actually, the non-diagonal interaction U^{ND}_{Λ} implies an effective *repulsion* term

$$g_{pq} \equiv \lim_{\Lambda} g_{\Lambda,pq} = \frac{\lambda_p \lambda_q}{4} \widehat{x}(\beta, \rho) \left(\frac{1}{\varepsilon_p} + \frac{1}{\varepsilon_q}\right) \geq 0, \tag{4.2}$$

inside each quasi-particle [2,43], i.e. inside each couple of particles respectively with modes q and $-q$ ($q \neq 0$) (figure 4.3). The larger the Bose condensate density $\widehat{x}(\beta, \rho)$,

Figure 4.3: *Effective interaction outside the zero-mode induced by the non-diagonal interaction* U^{ND}_{Λ}

the stronger the effective repulsion term g_{pq}. The raise of the non-conventional Bose condensate progressively destroys the Bogoliubov one, see (4.1). The Bose and Bogoliubov condensates still remain *in competition with each other.*

4. Note that the Bose condensation becomes non-conventional with the formation of Cooper pairs via the term of attraction g_{00}, i.e. because of quantum fluctuations, see figures 4.2 and 4.3. The importance of quantum fluctuations in helium systems corresponds also to the qualitative explanation for a liquid state at such extreme temperatures [31].

4.2. A polemical discussion of the Landau's criterion of superfluidity

Let us examine other interpretations of the Bose system $H^B_{\Lambda,0}$ in relation with the liquid ^{4}He. In fact, we give here two interpretations of the Bose gas $H^B_{\Lambda,0}$ obtained by following or not Landau's criterion of superfluidity [9,10] (remark 3.4). As explained above, note that the model $H^B_{\Lambda,0}$ is a caricature and may contain only a small part of the physical properties of real liquid helium. The sole purpose of these discussions is to give some new directions in light of the Bose gas $H^B_{\Lambda,0}$. At the end, note that we explain that this theory may be a starting point for a rigorous explanation of the two-fluid model of Tisza and London.

1. It is known [19,20] that below the critical temperature T_λ of the λ-transition, two fluids (^{4}He II phase) coexist: the normal fluid and the superfluid liquid. Later justified within the framework of phenomenological Landau's theory [9,10,26], the picture suggested by Tisza and London was to interpret the condensate of frozen in momentum space bosons with $p = 0$ as a "superfluid component", and the rest of particles as a "normal component" which is the carrier of the total entropy of the system. Experimentally, a Bose condensate was discovered in ^{4}He II. The apparition of this Bose condensate transition and the one of the superfluid liquid are strongly correlated to each other. Indeed, from [67–69] if γ_s is the fraction of superfluid liquid and γ_0 the one of the condensate, one has

$$\gamma_s(\mathrm{T}) \sim (\mathrm{T}_\lambda - \mathrm{T})^\eta \sim \gamma_0(\mathrm{T}), \text{ for } \mathrm{T} \to \mathrm{T}_\lambda^-, \tag{4.3}$$

see figure 4.4. However, even for zero-temperature the superfluid liquid is not in a

Figure 4.4: *The fractions, γ_s of superfluid liquid and γ_0 of the Bose condensate, as a function of the temperature* T *for ^{4}He*

full Bose condensate phase which is in contradiction with the assumption of Tisza and London.

2. *Following Landau's criterion of superfluidity* [9, 10], the theory based on $H^B_{\Lambda,0}$ might be understood as a microscopic theory of the *superfluid liquid*. Within this framework, it allows us to understand the close connection between the Bose condensate with density $\widehat{x}(\beta,\rho)$ and the Bogoliubov one with density $\{\rho-\widehat{x}>0\}$. These two systems may compose together the *superfluid liquid*, which coexists with the normal liquid for non-zero temperature at any positive velocity.
Note that Landau's criterion of superfluidity [9, 10] confronts an initial problem expressed by remark 3.5 and also a second one: the application of this criterion to the Henshaw-Woods spectrum gives for the critical velocity $\mathrm{v}_0 \approx 60\ m/s$ (remark 3.4), whereas superfluidity in capillaries disappears when velocity is of the order of *few* cm/s. Moreover, it depends sensitively on the diameter of the channel.
The attempts to explain these "misfittings" are concentrated around the idea that the Landau-type spectrum experimentally discovered by Henshaw and Woods is only a part of a plethora of other types of "elementary" excitations not covered by the Bogoliubov theory, see [37, 69].
Within the framework of the model $H^B_{\Lambda,0}$, we have seen in Section 4.1 that the Bose condensate has to exist in order to have the superfluidity property via the Bogoliubov condensate. Indeed, as soon as the non-conventional Bose condensate disappears, the collective phenomenon involved in the formation of the superfluid gas (Bogoliubov condensate) also vanishes. The introduction of a velocity in an inhomogeneous gas (in capillaries) may change the individual spectrum ε_k by increasing it. Then, the effective attraction g_{00} ((3.1), figure 4.2) becomes smaller, i.e. the *non-conventional* Bose condensate and the (*superfluid*) Bogoliubov one may be destroyed for velocities sufficiently large but smaller than v_0 (remark 3.4). Note that the non-conventional Bose condensate constituted of Cooper pairs may be changed into a conventional Bose-Einstein condensation as it exists for the Perfect Bose Gas. An experimental study of the spectrum of excitations and also of the Bose condensation phenomenon should be interesting at different velocities.

Actually, the collective behavior of this system should be quite delicate to save. A velocity may destroy the Cooper pairs and the quasi-particles. The important point is the following: the bigger the density of non-conventional Bose condensate, the stronger the robustness of Cooper pairs and quasi-particles to any perturbations.
At temperatures $\mathrm{T} < \mathrm{T}_\lambda$ even if the Bose condensate exists, its density may be not sufficiently important to keep the collective behavior for any positive velocities: some quasi-particles and Cooper pairs may be destroyed and a fraction of normal fluid appears. At temperatures $\mathrm{T}_\lambda < \mathrm{T} < \widetilde{\mathrm{T}}_\lambda$ (remark 3.5) the thermic fluctuations become sufficiently strong to destroy the non-conventional Bose condensate. Consequently, even if the quasi-particle gas resists in liquid helium for $\mathrm{T}_\lambda < \mathrm{T} < \widetilde{\mathrm{T}}_\lambda$ (remark 3.5), it is quite unstable and any perturbation of the quasi-particles (like any positive velocity) may quickly destroy the collective system and switch it to a standard liquid where no superfluidity exists.

3. Note that this last conjecture may seem a little naive since the value T_λ is very specific. Actually, the previous discussions are just phenomenological interpreta-

tions. Therefore, to conclude we examine *another* interpretation of the Bose gas $H^B_{\Lambda,0}$ *without taking into account Landau's criterion of superfluidity* [9,10], which is a *phenomenological* explanation of superfluidity.
If $\gamma^B_0(\mathrm{T}) \sim (\mathrm{T}_c - \mathrm{T})^{\eta^B}$ at temperatures $\mathrm{T} = (\mathrm{k_B}\beta)^{-1} \to \mathrm{T}_c^-$ is the fraction of Bose condensate for a fixed density $\rho > 0$, then via theorem 3.2, the fraction $\gamma^B_s(\mathrm{T}) = 1 - \rho_n/\rho$ satisfies:

$$\gamma^B_s(\mathrm{T}) \sim (\mathrm{T}_c - \mathrm{T})^{\eta^B} \sim \gamma^B_0(\mathrm{T}), \text{ for } \mathrm{T} \to \mathrm{T}_c^-, \tag{4.4}$$

where

$$\rho_n = \rho_n(\mathrm{T}) = \left\{ \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{f_{k,0}}{E^B_{k,0}\left[e^{E^B_{k,0}/\mathrm{T}} - 1\right]} d^3k \right\} \Bigg|_{x=\widehat{x},\alpha=\alpha(\widehat{x})}. \tag{4.5}$$

The relation (4.4) is strangely similar to (4.3), see figure 4.4. The fraction $\gamma^B_s(\mathrm{T})$ may be considered as the superfluid fraction of the Bose gas $H^B_{\Lambda,0}$. Therefore, at a fixed density $\rho > 0$, the superfluid density ρ_s equals

$$\rho_s(\mathrm{T}) = \left\{ x + \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{x^2 \lambda_k^2}{2E^B_{k,0}\left[f_{k,0} + E^B_{k,0}\right]} d^3k \right\} \Bigg|_{x=\widehat{x},\alpha=\alpha(\widehat{x})},$$

whereas ρ_n (4.5) is the density of normal fluid which is the carrier of the total entropy of the system. Note that $\lim_{\mathrm{T}\to 0^+} \rho_n = 0$ and within this framework there is 100% of superfluid liquid at zero-temperature with a density $\rho_s > \widehat{x} = \widehat{x}(\beta,\rho) = \widehat{x}(\mathrm{T})$. See (i) of theorem 3.2 to see the Bose condensate density at a fixed density $\rho > 0$.
In fact, this conjecture may be analyzed via the corresponding Hamiltonian with an external velocity field as it has been recently performed with dilute trapped Bose gases at zero-temperature [70]. This analysis is reserved for another paper.

4.3. Superfluidity of Fermi systems

The superfluidity of a Fermi system, i.e. the ^{3}He liquid, was discovered in 1972 for sufficiently low temperatures [21, 22]. All the previous theories concern Bose systems. However, it is remarkable to see that, via the effective coupling constant $g_{00} < 0$ (figure 4.2), the non-diagonal interaction U^{ND}_Λ of the model $H^B_{\Lambda,0}$ implies an attraction between particles in the zero-mode.
By analogy, it is well-known that the phenomenon of superconductivity comes from the effective electron-electron interaction in the BCS theory which results from the electron-phonon (non-diagonal) interaction in the second order of perturbation theory, see e.g. [71, 72]. Thus, in a superconductor, electrons can pair up in the metal crystal via phonons to form Cooper pairs which can then condense into a superconducting state. This phenomenon corresponds also to the explanation given for the existence of superfluidity in ^{3}He [73–75]. Indeed, by cooling the liquid to a low enough temperature, ^{3}He atoms can pair up, making it a boson, and therefore superfluidity can be achieved.
In the Bose gas $H^B_{\Lambda,0}$, the effective attraction characterized by $g_{00} < 0$ plays exactly the same rôle on bosons by creating Cooper pairs and may also work for Fermi

systems. Therefore, it should be interesting to study a similar Hamiltonian within the framework of Fermi systems.

Of course, the main difference comes from the Fermi statistics. In particular, the critical density $\rho_c^{PBG}(\beta)$ for the Perfect Bose Gas does *not* exist for the Perfect Fermi Gas. For the Bose system $H_{\Lambda,0}^{B}$, the corresponding *kinetic* part only *turns on* the Bose condensation phenomenon via the Bose statistics. Indeed, the corresponding "chemical potential" $\alpha(\widehat{x})$, as solution of the variational problem for a Bose condensate density $\widehat{x}(\beta,\rho)$, becomes zero when we reach the critical density as for the Perfect Bose Gas, *but* switches again to strictly negative values for $\widehat{x}>0$ (in [61]: proof of theorem 2.3). As soon as the Bose condensate appears, the non-diagonal interaction U_{Λ}^{ND} becomes sufficiently important to drastically change all thermodynamic properties of the system by instantly switching the usual Perfect Bose gas to a gas of quasi-particles: the Bose-Einstein condensation becomes non-conventional in correlation with the creation of the Bogoliubov condensate and the formation of Cooper pairs (Section 4.1).
Whereas the non-diagonal interaction U_{Λ}^{ND} is *not strong enough* to imply *alone* the Bose-condensation at the critical temperature or density of the Perfect Bose Gas, for very small temperatures it strongly dominates all thermodynamics of the system. The non-diagonal interaction U_{Λ}^{ND} obviously has a strong impact on the system (see for example the divergence of the grandcanonical pressure of $H_{\Lambda,0}^{B}$ [15]). It would have implied the non-conventional Bose condensation without the Bose statistics at sufficiently low temperatures or high densities.
In particular, if the Fermi statistics now holds, a similar non-diagonal interaction characterizing by an effective attraction g_{00} (3.1) (figure 4.2) drastically opposes the repulsion from the Pauli exclusion principle and would finally become strong enough at *sufficiently low temperatures* to imply *alone* the *non-conventional* Bose condensation (Cooper pairs) and the superfluid gas of quasi-particles explained above. This means of course that the critical temperature for the corresponding Fermi system should be quite lower than that of the Bose model $H_{\Lambda,0}^{B}$. Experimentally, the critical temperature of ^{3}He is very low in comparison with that of ^{4}He (2.17 K) : it is only two milli Kelvin for ^{3}He [21, 22].

We reserve this analysis on Fermi systems for another paper. To conclude, notice also that the ^{3}He liquid forms, at sufficiently low temperatures, several superfluid phases (A&B), which are much richer properties than those of the superfluid ^{4}He. For a complete review of properties of ^{3}He at low temperatures, see [76, 77].

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