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Supersymmetry in quantum mechanics with point interactions

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Abstract

We investigate supersymmetry in one-dimensional quantum mechanics with point interactions. We clarify a class of point interactions compatible with supersymmetry and present $N = 2$ supersymmetric models on a circle with two point interactions as well as a superpotential. A hidden $su(2)$ structure inherent in the system plays a crucial role to construct the $N = 2$ supercharges. Spontaneous supersymmetry breaking due to point interactions and an extension to higher N-extended supersymmetry are also discussed.

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1 Introduction

Quantum mechanics in one dimension admits point singularities as interactions of zero range. A point interaction is parameterized by the group $U(2)$ [1, 2, 3], and the varieties of connection conditions between a wavefunction and its derivative due to $U(2)$ lead to various interesting physical phenomena, such as duality [4, 5], the Berry phase [6, 7] and scale anomaly [8]. Furthermore, for some specific choices of point interactions, there occurs a double degeneracy in the energy level, which suggests the existence of supersymmetry [8].

The purpose of this paper is to examine supersymmetry in quantum mechanics on a circle with not only point interactions but also potentials regular except for point singularities and to give a brief discussion on N-extended supersymmetry with $N > 2$. We first clarify a class of connection conditions compatible with supersymmetry. We then find $N = 2$ supersymmetric models, which differ from the usual supersymmetric Witten model [9] in several ways. Our system consists of only one bosonic degree of freedom, and "bosonic" and "fermionic" states are assigned according to eigenvalues of a parity-like transformation. The derivative of a superpotential $W'(x)$ is restricted to be parity-odd, while there is no such restriction in the Witten model. Further, $W'(x)$ is allowed to have discontinuity at singularities in our models. Our system naturally possesses some discrete transformations that form an $su(2)$ algebra and they are used in constructing $N = 2$ supercharges in place of the Pauli matrices in the Witten model. Spontaneous supersymmetry breaking occurs in the Witten model if zero energy solutions are not normalizable. On the other hand, in our models, it occurs if zero energy solutions are incompatible with connection conditions at point interactions.

2 Quantum mechanics with point interactions

In this section, we give a brief review of one-dimensional quantum mechanics with point interactions and present a setup of our model.

In this paper, we consider quantum mechanics on a circle $S^1(-l < x \leq l)$ on which two point interactions are put at $x = 0$ and $x = l$. Although our analyses can equally apply for a noncompact space, characteristic features of point interactions will become more apparent in our model, namely, in discussing spontaneous breaking of supersymmetry. A point interaction is specified by a characteristic matrix $U \in U(2)$, and a wavefunction $\varphi(x)$ and its derivative are required to obey the connection condition at, say $x = 0$ [10, 8]

$$(U - \mathbf{1})\Phi + iL_0(U + \mathbf{1})\Phi' = 0, \tag{2.1}$$

where L_0 is an arbitrary nonzero constant and

$$\Phi \equiv \begin{pmatrix} \varphi(0_+) \\ \varphi(0_-) \end{pmatrix}, \quad \Phi' \equiv \begin{pmatrix} \varphi'(0_+) \\ -\varphi'(0_-) \end{pmatrix}. \tag{2.2}$$

Here, $0_\pm$ denote $0\pm\varepsilon$ with an infinitesimal positive constant ε, and $\varphi'(x) = \frac{d\varphi(x)}{dx}$. The connection condition at $x = l$ is similarly specified with a (generally different) characteristic matrix $\bar{U} \in U(2)$.

For later convenience, let us rewrite the connection condition (2.1) to make a supersymmetric structure clearer. To this end, we first note that since the four generators of $U(2)$ are taken to be $\{\mathbf{1}, \vec{\sigma}\}$, any $U(2)$ matrix can be parameterized as

$$U_g(\theta_+, \theta_-) = \exp\left\{ i\theta_+ P_g^+ + i\theta_- P_g^- \right\}, \tag{2.3}$$

where

$$\begin{aligned} P_g^\pm &= \frac{1}{2}(\mathbf{1} \pm g), && (2.4) \\ g &= \vec{\alpha} \cdot \vec{\sigma} \quad \text{with } (\vec{\alpha})^2 = 1. && (2.5) \end{aligned}$$

The $P_g^\pm$ can be regarded as projection matrices,

$$(P_g^\pm)^2 = P_g^\pm, \quad P_g^\pm P_g^\mp = 0, \quad P_g^+ + P_g^- = \mathbf{1}. \tag{2.6}$$

Let us next introduce three discrete transformations. The first is the parity transformation $\mathcal{P}$ defined by

$$\mathcal{P}: \quad \varphi(x) \to \mathcal{P}\varphi(x) = \varphi(-x). \tag{2.7}$$

The half-reflection transformation $\mathcal{R}$ is inherent in quantum mechanics with point singularities and is defined by

$$\mathcal{R}: \quad \varphi(x) \to \mathcal{R}\varphi(x) = (\Theta(x) - \Theta(-x))\varphi(x), \tag{2.8}$$

where $\Theta(x)$ is the Heaviside step function. The third transformation $\mathcal{Q}$ is defined by $\mathcal{Q} \equiv -i\mathcal{R}\mathcal{P}$. An important observation is that the set $\{\mathcal{P}_1 = \mathcal{P}, \mathcal{P}_2 = \mathcal{Q}, \mathcal{P}_3 = \mathcal{R}\}$ forms the $su(2)$ algebra of spin $\frac{1}{2}$, i.e.

$$\begin{aligned} [\mathcal{P}_i, \mathcal{P}_j] &= 2i \sum_{k=1}^{3} \epsilon_{ijk} \mathcal{P}_k, && (2.9) \\ \{\mathcal{P}_i, \mathcal{P}_j\} &= 2\delta_{ij}. && (2.10) \end{aligned}$$

It turns out that they are essential ingredients to construct $N = 2$ supercharges in our formulation, as we will see later. By use of $\mathcal{P}_j$, we can introduce an $su(2)$ element $\mathcal{G}$ associated with $g = \vec{\alpha} \cdot \vec{\sigma}$ as

$$\mathcal{G} \equiv \vec{\alpha} \cdot \vec{\mathcal{P}}. \tag{2.11}$$

Since $\mathcal{G}^2 = 1$, we can decompose any wavefunction $\varphi(x)$ into two eigenfunctions $\varphi_\pm(x) \equiv \frac{1}{2}(1 \pm \mathcal{G})\varphi(x)$ satisfying

$$\mathcal{G}\varphi_\pm(x) = \pm\varphi_\pm(x). \tag{2.12}$$

It is not difficult to show that the connection condition (2.1) at $x = 0$ splits into

$$\begin{aligned} \sin\frac{\theta_+}{2}\,\varphi_+(0_+) + L_0\cos\frac{\theta_+}{2}\,\varphi'_+(0_+) &= 0, \\ \sin\frac{\theta_-}{2}\,\varphi_-(0_-) - L_0\cos\frac{\theta_-}{2}\,\varphi'_-(0_-) &= 0. \end{aligned} \tag{2.13}$$

The connection condition at $x = l$ is assumed to be specified by the characteristic matrix $U_g(\bar{\theta}_+, \bar{\theta}_-)$, i.e.

$$\begin{aligned} \sin\frac{\bar{\theta}_+}{2}\,\varphi_+(l) + L_0\cos\frac{\bar{\theta}_+}{2}\,\varphi'_+(l) &= 0, \\ \sin\frac{\bar{\theta}_-}{2}\,\varphi_-(-l) - L_0\cos\frac{\bar{\theta}_-}{2}\,\varphi'_-(-l) &= 0, \end{aligned} \tag{2.14}$$

where $(\bar{\theta}_+, \bar{\theta}_-)$ are, in general, different from (θ_+, θ_-).

3 Compatibility with supersymmetry

In this section, we show that the connection conditions (2.13) and (2.14) are, in general, inconsistent with supersymmetry transformations, and clarify how compatibility with supersymmetry restricts the values of (θ_+, θ_-) and $(\bar{\theta}_+, \bar{\theta}_-)$ in eqs.(2.13) and (2.14).

Let us first discuss the quantum system only with point interactions, so that the Hamiltonian is given by

$$H = -\frac{1}{2}\frac{d^2}{dx^2} \tag{3.1}$$

except for the singular points. Here we have set $\hbar = 1$ and the mass $m = 1$ for simplicity. An extension to models with potential terms will be given in Section 5. If the system is supersymmetric, the Hamiltonian will be written, in terms of a supercharge Q, as $H = 2Q^2$. It follows that the supercharge is expected to be proportional to the derivative, $Q \propto \frac{d}{dx}$. This, however, causes a trouble to construct supercharges because $\varphi'(x)$ does not, in general, obey the same connection condition as $\varphi(x)$, and hence the state $Q\varphi(x)$ would not belong to the Hilbert space of the model.

To find a class of connection conditions compatible with supersymmetry, let us examine a supersymmetric partner $\chi(x) \equiv Q\varphi(x)$ of any state $\varphi(x)$ that satisfies the connection condition (2.1) and the Schrödinger equation

$$H\varphi(x) = E\varphi(x). \tag{3.2}$$

Since the supercharge is proportional to $\frac{d}{dx}$, $\chi(0_\pm)$ will be given, in general, by a linear combination of $\varphi'(0_+)$ and $\varphi'(0_-)$ such as

$$\Phi_\chi \equiv \begin{pmatrix} \chi(0_+) \\ \chi(0_-) \end{pmatrix} = M \begin{pmatrix} \varphi'(0_+) \\ -\varphi'(0_-) \end{pmatrix} \tag{3.3}$$

for some invertible constant matrix M. Since $\varphi(x)$ satisfies the Schrödinger equation (3.2), $\varphi''(x)$ is proportional to $\varphi(x)$. This fact implies that $\chi'(0_\pm)$ should be related to $\varphi(0_\pm)$ as

$$\Phi'_\chi \equiv \begin{pmatrix} \chi'(0_+) \\ -\chi'(0_-) \end{pmatrix} = E\tilde{M} \begin{pmatrix} \varphi(0_+) \\ \varphi(0_-) \end{pmatrix} \tag{3.4}$$

for some invertible constant matrix $\tilde{M}$. Here, we have explicitly shown the energy dependence in the above relation. Substituting eqs.(3.3) and (3.4) into eq.(2.1) leads to the connection condition for $\chi(x)$, i.e.

$$(U - \mathbf{1})\tilde{M}^{-1}\Phi'_\chi + iEL_0(U + \mathbf{1})M^{-1}\Phi_\chi = 0. \tag{3.5}$$

Since the connection condition must be independent of the energy E and since Φ_χ and Φ'_χ cannot vanish simultaneously, we conclude that the eigenvalues of U must be ± 1.[4] The case of $U = \pm\mathbf{1}$ turns out to lead to no nontrivial models because Φ_χ and Φ'_χ would vanish if we require $\chi(x)$ to satisfy the same connection condition as $\varphi(x)$. The remaining possibility is that the two eigenvalues of U are $+1$ and -1. Then, the general form of U is given by

$$U = \exp\left\{ i\frac{\pi}{2}(\mathbf{1} + \vec{\alpha}\cdot\vec{\sigma}) \right\} \quad \text{with } (\vec{\alpha})^2 = 1. \tag{3.6}$$

This corresponds to the choice $(\theta_+, \theta_-) = (\pi, 0)$ in eq.(2.3). Thus, in terms of the eigenfunctions $\varphi_\pm$ of $\mathcal{G} = \vec{\alpha}\cdot\vec{\mathcal{P}}$, the connection condition is reduced to

$$\text{type A}: \quad \varphi_+(0_+) = 0 = \varphi'_-(0_-). \tag{3.7}$$

If we replace $\vec{\alpha}$ by $-\vec{\alpha}$, the role of φ_+ and φ_- is exchanged, so that we have another type of allowed connection conditions.

$$\text{type B}: \quad \varphi'_+(0_+) = 0 = \varphi_-(0_-). \tag{3.8}$$

Repeating the same argument given above, we obtain two allowed connection conditions at $x = l$, i.e.

$$\text{type A}: \quad \varphi_+(l) = 0 = \varphi'_-(-l), \tag{3.9}$$

$$\text{type B}: \quad \varphi'_+(l) = 0 = \varphi_-(-l). \tag{3.10}$$

In the next section, we examine the models with the connection conditions obtained above, and explicitly construct $N = 2$ supercharges.

4 Construction of $N = 2$ supercharges

In the previous section, we have found that compatibility with supersymmetry restricts a class of connection conditions. For every $su(2)$ element $g = \vec{\alpha}\cdot\vec{\sigma}$ (or $\mathcal{G} = \vec{\alpha}\cdot\vec{\mathcal{P}}$),

[4] This situation resembles the discussion of the Weyl scaling invariance [8].

there are four types of the allowed connection conditions; type (A,A), (B,B), (A,B) and (B,A). (The first (second) entry denotes the type of the connection condition at $x = 0$ ($x = l$).) In this section, we explicitly construct $N = 2$ supercharges for the models, and show that the models of type (A,B) and (B,A) possess no supersymmetric vacua, so that supersymmetry is spontaneously broken for those models.[5]

Let us first examine the type (A,A) model whose connection conditions are given by

$$\varphi_+(0_+) = \varphi'_-(0_-) = \varphi_+(l) = \varphi'_-(-l) = 0. \tag{4.1}$$

Since $\mathcal{G}$ commutes with H, any energy eigenfunction $\varphi_E(x)$ can be a simultaneous eigenfunction of $\mathcal{G}$. The eigenfunctions are easily found as[6]

$$\begin{aligned}
\varphi_{+,E_n}(x) &= \Theta(x)A_n \sin\left(\frac{n\pi}{l}x\right) - \Theta(-x)A_n\frac{\alpha_1 + i\alpha_2}{1+\alpha_3}\sin\left(\frac{n\pi}{l}x\right), \\
&\qquad\qquad \text{for } n = 1, 2, 3, \cdots, \qquad (4.2) \\
\varphi_{-,E_n}(x) &= -\Theta(x)A_n\frac{\alpha_1 - i\alpha_2}{1+\alpha_3}\cos\left(\frac{n\pi}{l}x\right) + \Theta(-x)A_n \cos\left(\frac{n\pi}{l}x\right), \\
&\qquad\qquad \text{for } n = 0, 1, 2, \cdots, \qquad (4.3)
\end{aligned}$$

where A_n are normalization constants and the energy eigenvalues E_n are given by

$$E_n = \frac{1}{2}\left(\frac{n\pi}{l}\right)^2 \quad \text{for} \quad n = 0, 1, 2, \cdots. \tag{4.4}$$

Thus, we found that the vacuum has vanishing energy and all the excited states are doubly degenerate between even and odd $\mathcal{G}$-parity states. To show the existence of supersymmetry, let us construct $N = 2$ hermitian supercharges Q_a $(a = 1, 2)$ satisfying

$$\{Q_a, Q_b\} = H\delta_{ab}, \tag{4.5}$$
$$(Q_a)^\dagger = Q_a. \tag{4.6}$$

It is important to note that the above relations are not enough to guarantee $N = 2$ supersymmetry. We have to further require that for any state $\varphi(x)$ satisfying the connection conditions (4.1), the states $Q_a\varphi(x)$ $(a = 1, 2)$ obey the same connection conditions (4.1), otherwise Q_a would not regarded as physical operators of the system.

The connection conditions (4.1) and the fact that Q_a are proportional to $\frac{d}{dx}$ strongly suggest that the supercharges Q_a connect φ_+ and φ_-, i.e. $Q_a\varphi_\pm \propto \varphi_\mp$. This implies that Q_a should exchange the eigenvalues of $\mathcal{G}$ and hence anticommute with $\mathcal{G}$,

$$Q_a\mathcal{G} = -\mathcal{G}Q_a \quad \text{for } a = 1, 2. \tag{4.7}$$

[5] Our results for $g = \sigma_3$ reproduce those obtained in ref.[11].

[6] Without loss of generality, we can assume that $\alpha_3 \neq -1$. The choice $\alpha_3 = -1$ corresponds to $\mathcal{G} = -\mathcal{R}$. This is physically equivalent to the choice $\alpha_3 = 1$ under the exchange of $\varphi_+ \leftrightarrow \varphi_-$. The reason why the expressions (4.2) and (4.3) become ill defined for $\alpha_3 = -1$ is that in this case $\varphi_\pm(x) = \Theta(\pm x)\varphi(x)$, so that the domain of the function $\varphi_+(x)$ ($\varphi_-(x)$) is given by $0 < x < l$ ($-l < x < 0$).

It follows that $\mathcal{G}$ can be regarded as the "fermion" number operator

$$(-1)^F = \mathcal{G}. \tag{4.8}$$

Noting that $\mathcal{P}_3 \frac{d}{dx}$ commutes with $\mathcal{P}_j$ $(j = 1, 2, 3)$, we can show that the following supercharges satisfy all the desired relations:

$$Q_a = \frac{1}{2}\mathcal{G}_a \mathcal{P}_3 i \frac{d}{dx}, \quad a = 1, 2, \tag{4.9}$$

where

$$\mathcal{G}_a = \vec{\beta}_a \cdot \vec{\mathcal{P}} \quad \text{with } (\vec{\beta}_1)^2 = (\vec{\beta}_2)^2 = 1 \ \text{ and } \ \vec{\beta}_1 \cdot \vec{\alpha} = \vec{\beta}_2 \cdot \vec{\alpha} = \vec{\beta}_1 \cdot \vec{\beta}_2 = 0. \tag{4.10}$$

The vacuum state $\varphi_{-,E_0}(x)$ satisfies $Q_a \varphi_{-,E_0}(x) = 0$ for $a = 1, 2$, and hence supersymmetry is unbroken.

Let us next consider the type (B,B) model whose connection conditions are given by $\varphi'_+(0_+) = \varphi_-(0_-) = \varphi'_+(l) = \varphi_-(-l) = 0$. The type (B,B) model turns out to be a dual theory of the type (A,A) model because the transformation $\mathcal{G}_1$ (or $\mathcal{G}_2$) gives a map from the Hilbert space of the type (A,A) model onto that of the type (B,B) one due to the property $\mathcal{G}\mathcal{G}_a = -\mathcal{G}_a \mathcal{G}$ [8]. We will not discuss the type (B,B) model further.

Let us finally examine the type (A,B) model. (The type (B,A) model is physically equivalent to the type (A,B) model.) The connection conditions of the model is given by

$$\varphi_+(0_+) = \varphi'_-(0_-) = \varphi'_+(l) = \varphi_-(-l) = 0. \tag{4.11}$$

The energy eigenfunctions $\varphi_{\pm,E_n}(x)$ are found to be

$$\begin{aligned}
\varphi_{+,E_n}(x) &= \Theta(x) A_n \sin\left(\frac{(n-\frac{1}{2})\pi}{l} x\right) - \Theta(-x) A_n \frac{\alpha_1 + i\alpha_2}{1+\alpha_3} \sin\left(\frac{(n-\frac{1}{2})\pi}{l} x\right), \\
&\qquad \text{for } n = 1, 2, 3, \cdots, \qquad (4.12) \\
\varphi_{-,E_n}(x) &= -\Theta(-x) A_n \frac{\alpha_1 - i\alpha_2}{1+\alpha_3} \cos\left(\frac{(n-\frac{1}{2})\pi}{l} x\right) + \Theta(x) A_n \cos\left(\frac{(n-\frac{1}{2})\pi}{l} x\right), \\
&\qquad \text{for } n = 1, 2, 3, \cdots \qquad (4.13)
\end{aligned}$$

with

$$E_n = \frac{1}{2}\left(\frac{(n-\frac{1}{2})\pi}{l}\right)^2 \quad \text{for } n = 1, 2, 3, \cdots. \tag{4.14}$$

The supercharges $Q_a (a = 1, 2)$ are given by the same form as eq.(4.9) and lead to the relations $Q_a \varphi_{\pm,E_n} \propto \varphi_{\mp,E_n}$, as they should. All the energy eigenstates are doubly degenerate and there is no vacuum state annihilated by Q_a, so that supersymmetry is spontaneously broken in this model.

5 Supersymmetric models with a superpotential

In the previous sections, we have succeeded to construct the $N = 2$ supersymmetric models only with point interactions. In this section, we extend the analyses to models containing a superpotential. To this end, let us first recall that in the supersymmetric Witten model the supercharges are given by [9]

$$Q_1^W = \frac{1}{2}\left[\sigma_1 i\frac{d}{dx} + \sigma_2 W'(x)\right], \tag{5.1}$$

$$Q_2^W = \frac{1}{2}\left[\sigma_2 i\frac{d}{dx} - \sigma_1 W'(x)\right], \tag{5.2}$$

with the Hamiltonian

$$H^W = \frac{1}{2}\left[-\frac{d^2}{dx^2} + (W'(x))^2 - \sigma_3 W''(x)\right]. \tag{5.3}$$

A crucial observation in our formulation is that the set $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 = \mathcal{G}\}$ forms the $su(2)$ algebra of spin $\frac{1}{2}$,[7] i.e.

$$[\mathcal{G}_i, \mathcal{G}_j] = 2i\sum_{k=1}^{3}\epsilon_{ijk}\mathcal{G}_k, \tag{5.4}$$

$$\{\mathcal{G}_i, \mathcal{G}_j\} = 2\delta_{ij}. \tag{5.5}$$

Further, note that $\mathcal{P}_3\frac{d}{dx}$ and $\mathcal{P}_3 W'(x)$ commute with $\mathcal{G}_j$ $(j = 1, 2, 3)$ if $W'(x)$ is an odd function, i.e.

$$W'(-x) = -W'(x). \tag{5.6}$$

Although there is no such restriction on $W'(x)$ in the Witten model, we require $W'(x)$ to be parity-odd in order for the supercharges given below to satisfy the desired relations. Since there are no reasons that the superpotential is smooth at singularities, we allow $W'(x)$ to have discontinuity there, so that $W'(0_\pm)$ and $W'(\pm l)$ do not necessarily vanish.[8]

The above observations may tell us to take supercharges to be of the form

$$\begin{aligned} Q_1 &= \frac{1}{2}\left[\mathcal{G}_1\mathcal{P}_3 i\frac{d}{dx} + \mathcal{G}_2\mathcal{P}_3 W'(x)\right], \\ Q_2 &= \frac{1}{2}\left[\mathcal{G}_2\mathcal{P}_3 i\frac{d}{dx} - \mathcal{G}_1\mathcal{P}_3 W'(x)\right], \end{aligned} \tag{5.7}$$

which satisfy the relations (4.5) and (4.7) with the Hamiltonian

$$H = \frac{1}{2}\left[-\frac{d^2}{dx^2} + (W'(x))^2 - \mathcal{G}W''(x)\right]. \tag{5.8}$$

[7] The orthogonal unit vectors $\{\vec{\beta}_1, \vec{\beta}_2, \vec{\alpha}\}$ are chosen such that $\vec{\beta}_1 \times \vec{\beta}_2 = \vec{\alpha}$.

[8] We here assume that $W'(0_\pm)$ and $W'(\pm l)$ are finite to make the connection conditions (5.9)-(5.12) well defined, although some extension to allow divergent potentials at singularities may be possible (see ref.[12]).

Again, the "fermion" number operator is identified with $\mathcal{G}$. The correspondence between the Witten model and our model is evident: The Pauli matrices σ_j in the Witten model are replaced by the operators $\mathcal{G}_j$ in our model. Both of them satisfy the $su(2)$ algebra of spin $\frac{1}{2}$. Although physical meanings of $\mathcal{P}_3$ in front of $\frac{d}{dx}$ and $W'(x)$ are less obvious, it guarantees $\mathcal{P}_3\frac{d}{dx}$ and $\mathcal{P}_3 W'(x)$ to commute with $\mathcal{G}_j$.

We can further show that the supercharges become hermite and map any state $\varphi_+$with $\mathcal{G} = +1$ onto some state φ_- with $\mathcal{G} = -1$, and vice versa, if the connection conditions at $x = 0$ are chosen as

$$\begin{aligned} \text{type A} &: \quad \varphi_+(0+) = 0 = \varphi'_-(0_-) - W'(0_-)\varphi_-(0_-), && (5.9) \\ \text{type B} &: \quad \varphi'_+(0_+) + W'(0_+)\varphi_+(0_+) = 0 = \varphi_-(0_-), && (5.10) \end{aligned}$$

and at $x = l$

$$\begin{aligned} \text{type A} &: \quad \varphi_+(l) = 0 = \varphi'_-(-l) - W'(-l)\varphi_-(-l), && (5.11) \\ \text{type B} &: \quad \varphi'_+(l) + W'(l)\varphi_+(l) = 0 = \varphi_-(-l). && (5.12) \end{aligned}$$

Therefore, we have four types of the $N = 2$ supersymmetric models for every $su(2)$ element $\mathcal{G}$. Let us note that the above connection conditions are reduced to eqs.(3.7)-(3.10) when $W'(x) = 0$.

Let us finally discuss spontaneous breaking of supersymmetry for the models obtained above. The supersymmetric vacuum state is obtained by solving

$$Q_a\varphi_{\pm,0}(x) = 0 \quad \text{for } a = 1, 2. \tag{5.13}$$

Formal solutions to the above equations are given by $\varphi_{\pm,0}(x) \propto \exp\{\mp W(x)\}$. For a noncompact space, the normalizability of the states would remove some of them from the Hilbert space. However, since the space is compact (a circle) in our model, the solutions to eq.(5.13) are always normalizable. Nevertheless, some of them must be removed from the Hilbert space. For the type (A,A) ((B,B)) model, the state $\varphi_{+,0}(x)$ ($\varphi_{-,0}(x)$) does not satisfy the connection conditions and hence it does not belong to the Hilbert space of the model. On the other hand, $\varphi_{-,0}(x)$ ($\varphi_{+,0}(x)$) satisfies the desired connection conditions and it gives the supersymmetric vacuum. Therefore, supersymmetry is unbroken in the case of the type (A,A) and (B,B) models. For the type (A,B) and (B,A) models, both $\varphi_{\pm,0}(x)$ do not satisfy the connection conditions at $x = 0$ or $x = l$. Hence supersymmetry is spontaneously broken in these models.[9]

6 Summary and discussions

In this paper, we have investigated quantum mechanics on a circle with point interactions and clarified a class of connection conditions compatible with supersymmetry. The representation of the constructed $N = 2$ supercharges turns out to reflect the characteristics

[9]Other mechanisms of (spontaneous) supersymmetry breaking due to boundary effects have been found in ref.[13, 14].

of quantum mechanics with point singularities because the supercharges are represented in terms of the discrete transformations $\mathcal{G}_j$ ($j = 1, 2, 3$), which make wavefunctions discontinuous in general and hence are meaningless in ordinary quantum theory with no singularities.

It is interesting to note that for a special case of $\mathcal{G} = \mathcal{P}$ with any smooth odd function $W'(x)$, the connection conditions for the type (B,B) model are reduced to the conditions that wavefunctions must be smoothly connected at $x = 0$ and $x = l$. In other words, this model has no singularity at all. In this case, the form of the Hamiltonian (5.8) and the supercharges (5.7) with $\mathcal{G}_1 = \mathcal{Q}$ and $\mathcal{G}_2 = \mathcal{R}$ has been found in ref.[15], as a minimal bosonization of $N = 2$ supersymmetry. Our results may be considered as a natural extension of the minimal bosonization of $N = 2$ supersymmetry with point singularities.

In ref.[16], supersymmetry in the system of a free particle on a line $\mathbf{R}$ or an interval $[-l, l]$ with a point singularity at $x = 0$ was discussed. Although the configuration spaces are slightly different each other, some of the results overlap with ours. To see this, let us restrict the connection conditions in our model to a specific class of $g = \sigma_3$ (or $\mathcal{G} = \mathcal{R}$). Then, a point singularity associated with $g = \sigma_3$ describes a perfect wall through which no probability flow can penetrate, so that the circle S^1 can be regarded as an interval $[-l, l]$ with a point singularity at $x = 0$ in ref.[16]. We further need to restrict the superpotential to be constant ($W'(x) = -W'(-x) = b$ for $0 < x < l$), in order to have the free Hamiltonian. According to the prescription of ref.[16], we introduce a two-component wavefunction $\Psi(x) = (\psi_+(x), \psi_-(x))^T$ for $x > 0$, where $\psi_+(x) \equiv \varphi(x)$ for $x > 0$ and $\psi_-(-x) \equiv \varphi(x)$ for $x < 0$. Noting that $\psi_\pm(x)$ are the eigenfunctions of $\mathcal{R}$ with $\mathcal{R} = \pm 1$, respectively, and taking $\mathcal{G}_1 = \mathcal{P}$ and $\mathcal{G}_2 = \mathcal{Q}$, we find that the action of $Q_a (a = 1, 2)$ in eqs.(5.7) on $\Psi(x)$ can be represented by the 2×2 matrices:

$$
\begin{aligned}
Q_1 : \Psi(x) \ \rightarrow \ \tilde{Q}_1 \Psi(x) \ &= \ \left(\frac{i}{2} \frac{d}{dx} \sigma_1 + \frac{b}{2} \sigma_2 \right) \Psi(x), \\
Q_2 : \Psi(x) \ \rightarrow \ \tilde{Q}_2 \Psi(x) \ &= \ \left(\frac{i}{2} \frac{d}{dx} \sigma_2 - \frac{b}{2} \sigma_1 \right) \Psi(x).
\end{aligned} \tag{6.1}
$$

The above representation of the supercharges agrees with that of ref.[16] (see eqs.(2.21)), up to normalization. The allowed connection conditions (5.9)-(5.12) are also found in ref.[16] (see eqs.(3.17) and (3.20)).[10] Thus, our results give an extension of the work [16] for a free particle to a supersymmetric system with a superpotential. It should be stressed that although the supercharges (5.7) could be represented by 2×2 matrices, as done above for a special case, our representation of the supercharges has the advantage of clarifying the role of the discrete transformations $\mathcal{P}_j$. Our approach will be useful in analyzing more complex systems.

[10] We, however, missed $N = 1$ supersymmetry found in ref.[16]. This is due to the fact that the connection conditions at $x = 0$ and $x = l$ are taken to be the same class associated with an $su(2)$ element g in our model.

Our success of obtaining $N = 2$ supersymmetry may lead to an expectation to have higher N-extended supersymmetry by putting a number of point interactions on a circle. This turns out to be true but a class of allowed connection conditions compatible with higher N-extended supersymmetry is more restrictive [17]. A simplest example of $N = 4$ supersymmetry without a superpotential can be obtained by putting four point singularities at $x = 0, \pm l/2, l$ on a circle and by choosing all the connection conditions to be associated with $g = \sigma_3$ and the type A. Then, we find that the doubly degenerate vacua have a vanishing energy and that all the excited states are four-fold degenerate. The degeneracy results from $N = 4$ supersymmetry and we can construct four hermite supercharges. An easy way to represent the supercharges is to introduce, as a natural extension of the work [16], a four-component wavefunction $\Psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x))^T$ for $0 < x < l/2$, where $\psi_1(x) \equiv \varphi(x)$, $\psi_2(x) \equiv \varphi(-x)$, $\psi_3(x) \equiv \varphi(l-x)$, and $\psi_4(x) \equiv \varphi(-l+x)$ for $0 < x < l/2$. In this basis, the supercharges can be represented by the 4×4 matrices such as

$$\begin{aligned}
\tilde{Q}_1 &= \frac{i}{2}\frac{d}{dx}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad \tilde{Q}_2 = \frac{i}{2}\frac{d}{dx}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \\
\tilde{Q}_3 &= \frac{i}{2}\frac{d}{dx}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad \tilde{Q}_4 = \frac{i}{2}\frac{d}{dx}\begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}.
\end{aligned} \tag{6.2}$$

The extension to higher N-extended supersymmetry and the inclusion of a superpotential are possible. The subject will be reported elsewhere [17].

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