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Exact velocity of dispersive flow in the asymmetric avalanche process

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Using the Bethe ansatz we obtain the exact solution for the one-dimensional asymmetric avalanche process. We evaluate the velocity of dispersive flow as a function of driving force and the density of particles. The obtained solution shows a dynamical transition from dispersive to continuous flow.

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The avalanche dynamics is a basic scenario of relaxation of unstable states in extremal systems where each movable element is near a border of stability. A typical long-tailed distribution of avalanche sizes leads to the dispersive transport of particles [1]. As an illustrative example, granular systems exhibit intermittent avalanches what enables one to use granular piles (sand piles, rice piles) for explanation of self-organized criticality in generic dissipative systems [2]. In the past decade, it has become clear that the dispersive transport can be recast in terms of interface depinning [3], [4] and various growth models [5]. Recently, a dynamical transition from intermittent to continuous flow in a random sandpile model has been revealed [6]. Nevertheless, an explicit theoretical description of stochastic avalanche processes and an exact evaluation of characteristics of dispersive flow remains an open problem.

Despite drastic simplifications which were introduced to mimic real avalanches, exact results are scarce even for the determimistic dynamics. As to stochastic dynamics, it is especially difficult as it is beyond the class of abelian models [7], where asymmetric processes appear to be solvable [8]. The situation is to be compared with the theory of exclusion processes where many properties, such as steady states, average current, diffusion constant etc. have been calculated for an asymmetric one-dimensional case [9], [10], [11], [12]. The usual presentation of the asymmetrical exclusion process (ASEP) is given by a master equation for the probability $P_t(x_1, ..., x_P)$ of finding P particles at time t on sites $x_1, ..., x_P$ of a ring consisting of N sites. During any time interval dt, each particle jumps with probability dt to its right if the target site is empty. This elementary restriction leads to a non-trivial problem of evaluation of the steady state properties, which can be solved by the Bethe ansatz.

In a similar way, the simplest asymmetric avalanche process (ASAP) can be formulated as follows. In a stable state, each of N sites on a ring is either occupied by one particle or empty. The total number of particles P is fixed. During time interval dt, each particle jumps with probability dt to its right. In the course of time, some site x may get unstable with occupation number $n > 1$. Then it must relax immediately to the stable state by transferring to its right either n particles with probability $q(n)$ or $n-1$ particles with the probability $1-q(n)$. The quantity $q(n)$ can be associated with a driving force acting on the unstable group of n particles.

The main difference between the ASEP and ASAP lies in the depth of reconstruction of a configuration $C = \{x_1, ..., x_P\}$ during the time interval dt. In the ASEP, the total distance Y_t covered by all particles between time 0 and t increases by 1 during dt if the configuration C differs from a new one C' or remains unchanged if the motion is forbidden. In the ASAP, the motion of a particle is always possible and increase of Y_t is not bounded. Thus, the configuration C may be completely different from C' depending on numbers of particles spilled to right from each unstable site.

The present formulation of the ASAP is inspired by works [13] where a model of activated random walks is introduced and [15] where the directed avalanche dynamics is formulated in terms of continuous variables. Under an assumption about independence of variations of the avalanche size at each time step, the probability distribution of avalanche sizes is found exactly [15]. However, the configurational space of the continuous model is too complicated to determine steady state features.

Here, extending the Bethe anzatz approach to exclusion processes, we obtain the expression for the generating function of Y_t for the discrete ASAP in the thermodynamic limit of large N for a fixed density of particles $\rho = P/N$. We find two phases corresponding to a dispersive flow and a continuous flow, and evaluate the exact average velocity in the whole range of parameters of the first phase. We determine the separation line between two phases where avalanches are critical.

Consider the ASAP consisting of P particles on a ring of N sites and denote by $P_t(C)$ the probability of finding at time t the system in a configuration C. The probability $P_t(C)$ satisfies

$$\frac{d}{dt}P_t(C) = \sum_{C'}[M_0(C, C') + M_1(C, C']P_t(C') \qquad (1)$$

where $M_1(C, C')dt$ is the probability of going from C' to C during the time interval dt, and M_0 is a diagonal matrix

$$M_0(C, C) = -\sum_{C' \neq C} M_1(C', C) \qquad (2)$$

Before using the Bethe ansatz, it is instructive to note that in the region where the distances between each two neighbouring particles exceed 1, the master equation (1) becomes "free":

$$\frac{d}{dt}P_t(x_1,...,x_P) = \sum_k \nabla_k P_t(x_1,...,x_k,...,x_P) \quad (3)$$

where $\nabla_k P_t(...,x_k,...) = P_t(...,x_k-1,...) - P_t(...,x_k,...)$ To compensate the difference between (1) and (3) when $x_k - x_{k-1} = 1$ for some k, we introduce the boundary conditions

$$P_t(...,x_k,x_k,...) = \sum_{n=1}^{\infty}(1-\mu)\mu^{n-1} \times P_t(...,x_k-n,x_k-n+1,...) \quad (4)$$

where $\mu = q(2)$. This condition can be viewed as the recurrent relation

$$P_t(...,x,x,...) = (1-\mu)P_t(...x-1,x,...) + \times \mu P_t(...,x-1,x-1,...) \quad (5)$$

where the probability of a unstable configuration $P_t(...x,x,...)$ is given in terms of another unstable configuration $P_t(...x-1,x-1,...)$ and so on. Now, we can define the ASAP by (3) and (5) instead of (1) without even knowing the exact form of the matrix $M(C,C')$ which is very cumbersome for the ASAP model.

Specifying a configuration C by positions $1 \le x_1 < x_2... < x_P \le N$ of the P particles, we use the Bethe ansatz for an eigenvector of the matrix $M_0 + M_1$ in the form

$$\sum_Q A_Q \prod_{j=1}^{P} [e^{\gamma} z_{Q(j)}^{-1}]^{x_j} \quad (6)$$

where the sum is over all of the permutations Q of $1,2,...,P$ and the factor $exp(\gamma)$ is the activity of a single particle step. The condition (5) fixes the two particle S-matrix A_{ij}/A_{ji} as

$$\frac{A_{jk}}{A_{kj}} = -\frac{1-(1-\mu)e^{\gamma}z_j - \mu e^{2\gamma}z_j z_k}{1-(1-\mu)e^{\gamma}z_k - \mu e^{2\gamma}z_j z_k} \quad (7)$$

Imposing the periodic boundary conditions gives the Bethe equations

$$z_k^{-N} = (-1)^{N-1}\prod_{j=1}^{P}\frac{1-(1-\mu)e^{\gamma}z_j - \mu e^{2\gamma}z_j z_k}{1-(1-\mu)e^{\gamma}z_k - \mu e^{2\gamma}z_j z_k} \quad (8)$$

The eigenvalue $\lambda(\gamma)$ corresponding to (6) is

$$\lambda(\gamma) = -P + e^{\gamma}\sum_{i=1}^{P} z_i \quad (9)$$

An important property of the ASAP to be solvable by the Bethe ansatz, follows from the condition of ordering of particles. Due to (5), the decay of a unstable site can start from the rightmost argument in a block of n equal arguments in $P_t(...,x,x,...,x,...)$ and then proceed to left. This implies a recurrent relation for the probability $q(n)$ of removing all n particles from the n-fold unstable site, $q(n) = (1-q(n-1))\mu$.

The average velocity of the particle flow in the ASAP is determined by the average number of steps of all particles involved into an avalanche during the time interval dt and can be written as

$$v = \frac{\langle Y_t \rangle}{Pt} = \frac{1}{P}\frac{\partial \lambda}{\partial \gamma} \quad (10)$$

The rest of the letter is devoted to evaluation of v in the thermodynamic limit $N \to \infty$ for a fixed density of particles $\rho = P/N$.

For a finite N, the largest eigenvalue λ corresponds to the solution $\{z_j\}$ which converges to $z_j = 1, j = 1,...,P$ as $\gamma \to 0$. For small $\gamma > 0$, the distance $|z_j - 1|$ grows rapidly with N for all j and becomes of order of 1 in the limit $N \to \infty$. Introducing a variable α by

$$z_j = \frac{1-e^{i\alpha_j}}{1+\mu e^{i\alpha_j}}e^{-\gamma} \quad (11)$$

and assuming the solutions $\{\alpha_j\}$ are distributed along a smooth curve in the complex plane $\alpha = (u+ir)$ with endpoints $(-a+ib)$ and $(a+ib)$, we obtain the Bethe equation in the form

$$p(\alpha) = 2\pi F(\alpha) + \frac{1}{2\pi}\int_{-a+ib}^{a+ib}\theta(\alpha-\beta)R(\beta)d\beta - i\gamma \quad (12)$$

where we defined as usual a function $F(\alpha)$ such that $dF/d\alpha = -R(\alpha)/2\pi$ and $F(-a+ib) = -F(a+ib) = \rho/2$. The functions $p(\alpha)$ and $\theta(\alpha)$ are

$$p(\alpha) = -i\ln\left(\frac{1-e^{i\alpha}}{1+e^{i\alpha-2\nu}}\right) \quad (13)$$

and

$$\theta(\alpha) = -i\ln\left(\frac{\cosh(\nu+i\alpha/2)}{\cosh(\nu-i\alpha/2)}\right) \quad (14)$$

where $\nu = -\ln(\mu)/2$. Taking the derivative in (12), we get the integral equation for $R(\alpha)$

$$-R(u,b) + \frac{1}{2\pi}\int_{-a}^{a}K(u-v)R(v,b)dv = \xi(u,b) \quad (15)$$

with

$$\xi(\alpha) = \frac{\cosh\nu}{\sinh\nu - \sinh(\nu - i\alpha)} \quad (16)$$

and

$$K(\alpha) = \frac{\sinh 2\nu}{\cosh 2\nu + \cos\alpha} \quad (17)$$

All that is very similar to the equations for the asymmetric 6-vertex model [16], [17](see also [18]) with an essential exception: both terms containing z_j and $z_i z_j$

in (8) are negative, which is the reason for a dynamical transition, as we shall show below.

If $a = \pi$, equation (15) can be solved by the Fourier transformation. To evaluate $\partial_\gamma \lambda$, we have to find the solution of (15) in a vicinity of the point $a = \pi$ which corresponds to a "conical" point, considered in [17]. Following Bukman and Shore, we write the solution $R(u)$ as an expansion in $\epsilon = \pi - a$ up to order of $O(\epsilon^4)$

$$R(u) = R_0(u) + \epsilon^1 \delta R_1(u) + \epsilon^2 \delta R_2(u) + \epsilon^3 \delta R_3(u) \quad (18)$$

The necessity of such a long expansion will be seen in further calculations. The Fourier transformation is defined by

$$X(u) = \sum_{n=-\infty}^{\infty} (X)_n e^{-inu} \quad (19)$$

where X stands for $R_0, \delta R_m, \xi, K$. The non-zero Fourier coefficients of K and ξ for $b \geq -2\nu$ are

$$(K)_n = (-1)^n e^{-2\nu|n|} \quad (20)$$

$$(\xi)_n = -e^{bn}(1 - (-1)^n e^{2n\nu}), n < 0 \quad (21)$$

Then, (16) gives

$$R_0(u) = (R_0)_0 + \frac{e^{iu-b}}{1 - e^{iu-b}} \quad (22)$$

In the next order in ϵ, we have

$$(\delta R_1)_n (K_n - 1) = \frac{R_0(\pi)}{\pi} (-1)^n (K)_n \quad (23)$$

so, that $R_0(\pi) = 0, (\delta R_1)_n = 0$ and $(R_0)_0 = 1/(1+\exp b)$. The next terms in (18) are evaluated in [17]

$$\delta R_2(u) = -\frac{1}{6} R_0^{''}(\pi) \quad (24)$$

$$\delta R_3(u) = -\sum_{n \neq 0} \frac{in(K)_n R_0^{'}(\pi)}{3\pi(1-(K)_n)} e^{-inu} \quad (25)$$

$$\delta R_4 = -\frac{1}{120} R_0^{(4)}(\pi) \quad (26)$$

Thus, in the expansion of $R(u)$, $\delta R_2(u) \equiv \delta R_2$ and $\delta R_4(u) \equiv \delta R_4$ are real constants and $\delta R_3(u)$ is imaginary.

Now, we are ready to start a direct evaluation of $\partial_\gamma \lambda$ in (10) using $\partial_\gamma \lambda = \partial_a \lambda / \partial_a \gamma$. First, we find $\partial_a \gamma$. To this end, we put $\alpha = a + ib$ in (12), and take the derivative by a at $a = \pi$. Recalling the conditions $F(a + ib) = -\rho/2$ and $\theta(0) = 0$, we obtain

$$\pi \partial_a \rho + i \partial_a \gamma = R(a,b) + (2\pi)^{-1} \theta(2a) R(-a,b) + (2\pi)^{-1} \int_{-a}^{a} \theta(a-v) \partial_a R(v,b) dv \quad (27)$$

To evaluate r.h.s. of (27), we express the values $R(a,b)$,$R(-a,b)$ and $\theta(2a)$ by their Tailor expansions at $a = \pi$ up to the order of $O(\epsilon^3)$ using (14) and (18). The integral in (27) is treated as

$$\int_{-\pi+\epsilon}^{\pi-\epsilon} f(v) dv = \int_{-\pi}^{\pi} f(v) dv + B(\epsilon) \quad (28)$$

with

$$B(\epsilon) = \sum_{m=1}^{\infty} \frac{1}{M!} \{(-\epsilon)^m f^{(m-1)}(\pi) - (\epsilon)^m f^{(m-1)}(-\pi)\}$$

and, therefore, can be evaluated by the Fourier transformation. The Fourier coefficients of $\theta(\pi - v)$ are

$$(\theta(\pi - v))_n = \frac{2\pi i}{n} (e^{-2\nu n} - (-1)^n), n \neq 0 \quad (29)$$

and $(\theta(\pi - v))_0 = 2\pi^2$. The only v-dependent part of $\partial_a R(v,b)$ is $\delta R_3(v)$. Using (24)-(26),(20) and (29), we obtain the explicit expression for r.h.s. of (27)

$$\epsilon \frac{\pi}{3} R_0^{''}(\pi) - \epsilon^2 \frac{R_0^{'}(\pi)}{\pi} + \epsilon^3 \frac{\pi}{30} R_0^{(4)}(\pi) \quad (30)$$

Due to (22), the first and third terms in r.h.s. of (30) are real, the second one is imaginary. Therefore, we have

$$\partial_a \gamma = i\epsilon^2 \frac{R_0^{'}(\pi)}{\pi} \quad (31)$$

The expression for $\partial_a \lambda$ can be found in a similar way. In this case, we take the derivative by a in

$$\frac{\lambda}{N} = \frac{1}{2\pi} \int_{-a}^{a} R(u,b)(z(u,b) - 1) du \quad (32)$$

where $z(u,b)$, according to (11), is

$$z(u,b) = \frac{1 - e^{iu-b}}{1 + e^{iu-2\nu-b}} \quad (33)$$

The obtained derivative is similar to (27) but contains $z(v,b)$ instead of $\theta(a - v)$. So, we need the Fourier coefficients of $z(v,b)$ which are

$$(z)_n = (1 + e^{2\nu})(-1)^n e^{(2\nu+b)n}, n < 0 \quad (34)$$

and $(z)_0 = 1$.Continuing as in (27), we get

$$\frac{1}{N} \frac{\partial \lambda}{\partial a} = \epsilon^2 \frac{R_0^{'}(\pi) z^{'}(\pi, b)}{\pi} - 3\epsilon^2 \sum_{n=1}^{\infty} (z)_{-n} (\delta R_3)_n \quad (35)$$

We can see that the term $\delta R_3(u)$ in (18) is relevant. As to $\delta R_4(u)$, it is sufficiently that it is a constant and does not lead to a divergency by integration.

Substituting the explicit expressions for $(z)_{-n}$ and $(\delta R_3)_n$ gives the second term in r.h.s. of (35) in the form

$$\epsilon^2(1+e^{2\nu})\frac{iR_0'(\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n e^{-(4\nu+b)n}}{1-(-1)^n e^{-2n\nu}} \tag{36}$$

Due to (31), $R_0'(\pi)$ is cancelling in $\partial_a\lambda/\partial_a\gamma$. Then, using (34) and the identity $\rho=(R_0)_0=1/(1+\exp b)$ we obtain the final result

$$v=\frac{(1-\rho)(1+\mu)}{(1-\rho(1+\mu))^2}+\frac{1+\mu}{\mu\rho}F(\mu,\rho) \tag{37}$$

with

$$F(\mu,\rho)=\sum_{n=1}^{\infty}\frac{(-1)^n n\mu^{2n}}{1-(-1)^n\mu^n}\left(\frac{\rho}{1-\rho}\right)^n \tag{38}$$

The velocity of flow v diverges at $\rho_c=1/(1+\mu)$ which implies a transition to the phase of continuous flow. The value of critical density ρ_c can be easily understood from the condition of a balance between gaining (ρ_c) and losing $(1-q(\infty))$ one particle at each step of a large avalanche.

The considered model is a directed version of the model of activated random walks introduced in [13] to see how a conservative dynamical system with the sandpile toppling rules approaches criticality. It has been shown in [14] that the relaxation time τ and correlation length ξ diverge as $\tau\sim|\rho_c-\rho|^{-\nu_1}$ and $\xi\sim|\rho_c-\rho|^{-\nu_2}$. The exponents ν_1 and ν_2 have been determined numerically for several kinds of toppling rules. In the directed case, ξ coinsides with τ and is proportional to the average size of avalanches $\langle s\rangle$. On the other hand, $\langle s\rangle=v$ so we have from (37) $\langle s\rangle\sim(\rho_c-\rho)^{-2}$ and $\nu_1=\nu_2=2$. An extension of this result to the symmetrical case [19] is a very interesting and difficult problem.

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