EXPANSION IN THE DISTANCE PARAMETER FOR TWO VORTICES CLOSE TOGETHER

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Abstract

Static vortices close together are studied for two different models in 2-dimensional Euclidean space. In a simple model for one complex field an expansion in the parameters describing the relative position of two vortices can be given in terms of trigonometric and exponential functions. The results are then compared to those of the Ginzburg-Landau theory of a superconductor in a magnetic field at the point between type-I and type-II superconductivity. For the angular dependence a similar pattern emerges in both models. The differences for the radial functions are studied up to third order.

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1 Introduction

Ever since 't Hooft [1] and Polyakov [2] found a monopole solution in the SU(2) Yang-Mills-Higgs theory, solitons in field theories have been studied extensively. Our understanding of monopole solutions has been greatly enhanced by an existence proof for static solutions by Taubes [3] and the construction of monopole solutions started by Ward [4]. This process was not matched by quite the same progress in our understanding of the Abrikosov solutions of the Ginzburg-Landau theory, although one might have expected that the Abelian Higgs theory in 2+1 dimensions is actually simpler than the SU(2) Yang-Mills-Higgs theory in 3+1 dimensions. Again an existence proof was given by Taubes [5]. However, only superimposed vortices can be described explicitly and no explicit construction of separated vortices is known. In this paper, we want to give the solution for two vortices close together in terms of an expansion in the parameters which describe the relative location.

In Sections 2 and 3, we study a model for one complex field. Here the calculations are simpler than in the Ginzburg-Landau theory which is our second model. The first model has, however, some peculiar (unphysical) features. Assuming the most symmetric form in terms of angular dependence, only two smooth vortices can be superimposed, and when 'pulled apart', they develop a singularity at third order. In the Ginzburg-Landau model this does not happen. In fact, delicate cancellations take place to make the expansion smooth, at least up to third order. In this model the radial functions are given as solutions of certain linear ordinary differential equations. This is discussed in Section 4.

2 Vortex solutions and zero modes in a simple model

Our first model is a model [6][7] for a pair of real fields $\phi^a(\vec{x})$, a,b = 1,2, or equivalently, for a complex field $\phi = \phi_1 + i\phi_2$. The Lagrangian density of the model reads

$$\mathcal{L} = \partial_{[i}\phi^a\partial_{j]}\phi^b\partial^{[i}\phi_a\partial^{j]}\phi_b + (1-|\phi|^2)^2|\phi|^2, \tag{2.1}$$

where $a, b = 1, 2$ labels the components of the Higgs field and $i, j = 1, 2$ are the space indices. The square brackets mean antisymmetrization,

$$\partial_{[i}\phi^a \partial_{j]}\phi^b = (\partial_i \phi^a)(\partial_j \phi^b) - (\partial_j \phi^a)(\partial_i \phi^b). \tag{2.2}$$

We are working in 2-dimensional Euclidean space, i.e., the space indices can be raised and lowered without any change in the formulas. The indices which label the components of the Higgs field can also be raised and lowered without any change. In terms of the complex field ϕ the Euler-Lagrange equation reads

$$\partial_i \phi^* \partial_j (\partial^{[i}\phi \partial^{j]}\phi^*) = (1 - |\phi|^2)|\phi| \frac{\partial}{\partial \phi}(1 - |\phi|^2)|\phi|. \tag{2.3}$$

Any solution of the equation

$$2 \, det(\frac{\partial \phi^a}{\partial x^i}) = \pm(1 - |\phi|^2)|\phi| \tag{2.4}$$

solves the equation of motion (2.3). Note that Eq. (2.4) is a first order equation whereas Eq. (2.3) is of second order. So we would expect that (2.4) is somewhat easier to solve than (2.3). For different types of models, this reduction of order was first introduced by Bogomolnyi [8]. That is why we call Eq. (2.4) the Bogomolnyi equation here. Any solution of (2.4) also attains the lower bound in the following inequality,

$$A = \int_{\mathbf{R}^2} \mathcal{L} \, d^2x \geq \frac{16\pi}{15}|Q|, \tag{2.5}$$

where

$$Q = \frac{15}{8\pi} \int_{\mathbf{R}^2} i\epsilon_{ij}(1 - |\phi|^2)|\phi|(\partial^i \phi)(\partial^j \phi^*) d^2x \tag{2.6}$$

is the winding number. Finally, all finite-action solutions actually solve the Bogomolnyi equation, so we do not miss out on any by concentrating on the first order equation.

We now seek to attain a smooth finite-action solution of Eq. (2.4). For

$$\phi = f(r)e^{in\theta} \tag{2.7}$$

Eq. (2.4) reduces to

$$\frac{nf(r)f'(r)}{r} = \frac{1}{2}(1-f^2)f. \tag{2.8}$$

Since $f \to 0$ as $r \to 0$ (otherwise ϕ in (2.7) is not defined at the origin), we have

$$f = \tanh\frac{r^2}{4n}. \tag{2.9}$$

The solution ϕ in (2.7) with f(r) given by (2.9) is defined in the whole of $\mathbf{R}^2$ and is clearly a C^∞ function in $\mathbf{R}^2 \setminus \{0\}$. Since

$$f \approx 1 - 2\exp\frac{r^2}{2n} \quad \text{as} \quad r \to \infty, \tag{2.10}$$

ϕ has the right asymptotic behaviour for a solution with winding number n. We still have to ensure that ϕ is C^∞ at the origin. There we use the Taylor expansion of f,

$$f = \sum_{K=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}(\frac{r^2}{4n})^{2k-1} = \frac{r^2}{4n} - \frac{1}{3}(\frac{r^2}{4n})^3 + ... \tag{2.11}$$

where B_k is the k^{th} Bernoulli number. We see that for $n = 2$ and only for $n = 2$, ϕ is a polynomial in x^i. In this model, we have the (somewhat peculiar) situation that within the most natural ansatz (2.7) smooth finite action solutions exist only for $n = 2$, i.e., we only have a solution of the form (2.7) for 2 vortices.

We have found the solution for two vortices sitting on top of each other, which we now denote by $\hat{\phi}$. To extend our study to two vortices slightly apart we consider $\phi = \hat{\phi} + \gamma$, where γ is very small, and we solve the Bogomolyni equation, linearized in γ. Equation (2.4) becomes

$$(f'\cos\theta\cos 2\theta + \frac{2}{r}f\sin\theta\sin 2\theta)\frac{\partial\gamma^2}{\partial x^2} + (f'\sin\theta\sin 2\theta + \frac{2}{r}f\cos\theta\cos 2\theta)\frac{\partial\gamma^1}{\partial x^1}$$

$$-(f'\sin\theta\cos 2\theta - \frac{2}{r}f\cos\theta\sin 2\theta)\frac{\partial\gamma^2}{\partial x^1} - (f'\cos\theta\sin 2\theta - \frac{2}{r}f\sin\theta\cos 2\theta)\frac{\partial\gamma^1}{\partial x^2}$$

$$= \frac{1}{2}(1-3f^2)\ (\gamma^1\cos 2\theta + \gamma^2\sin 2\theta) \tag{2.12}$$

We find a 2-parameter family of zero modes,

$$\gamma(r) = [\alpha + \beta + \imath(\alpha - \beta)]h(r) \quad \text{with} \quad h(r) = \frac{\sinh \frac{r^2}{8}}{\cosh^3 \frac{r^2}{8}}. \tag{2.13}$$

These zero modes are C^∞ functions which vanish exponentially at infinity. By a rotation, one of the parameters could be removed and the vortices could be positioned on the x-axis, say. Since this does not simplify the calculations significantly, we will retain both parameters. Retaining the two parameters would also be necessary for a study of vortex scattering in the slow-motion approximation. This study is not done in this paper.

3 The quadratic and cubic terms

We now consider $\phi = \hat{\phi} + \gamma + \delta$, and equate the second order terms in the Bogomolyni equation (2.4). This leads to the equation

$$\begin{aligned}
&\frac{2}{r}(f' \cos 2\theta \frac{\partial \delta^2}{\partial \theta} + 2f \sin 2\theta \frac{\partial \delta^2}{\partial r} - f' \sin 2\theta \frac{\partial \delta^1}{\partial \theta} + 2f \cos 2\theta \frac{\partial \delta^1}{\partial r}) \\
&= (\alpha^2 + \beta^2) f h^2 (\frac{1}{f^2} - 3) - \frac{1}{2} f h^2 (3 + \frac{1}{f^2}) \, [\alpha^2 (\cos 2\theta + \sin 2\theta)^2 \\
&\qquad + 2\alpha\beta(\cos^2 2\theta - \sin^2 2\theta) + \beta^2 (\cos 2\theta - \sin 2\theta)^2] \\
&\qquad + (1 - 3f^2)(\delta^1 \cos 2\theta + \delta^2 \sin 2\theta)
\end{aligned} \tag{3.1}$$

with $f(r)$ given in (2.9) and $h(r)$ given in (2.13).

With δ of the form

$$\delta = \alpha^2 F(r, \theta) + 2\alpha\beta G(r, \theta) + \beta^2 H(r, \theta), \tag{3.2}$$

we obtain the following equation for $F(r, \theta)$,

$$\begin{aligned}
&\frac{2}{r}(f' \cos 2\theta \frac{\partial F^2}{\partial \theta} + 2f \sin 2\theta \frac{\partial F^2}{\partial r} - f' \sin 2\theta \frac{\partial F^1}{\partial \theta} + 2f \cos 2\theta \frac{\partial F^1}{\partial r}) \\
&= h^2 (\frac{1}{f} - 3f) - \frac{h^2}{2} (3f + \frac{1}{f})(\cos 2\theta + \sin 2\theta)^2 \\
&\qquad + (1 - 3f^2)(F^1 \cos 2\theta + F^2 \sin 2\theta).
\end{aligned} \tag{3.3}$$

To solve this equation we seek a solution of the form

$$F = f_1(r)\exp^{\imath 2\theta} - \imath f_2(r)\exp^{-\imath 2\theta}. \tag{3.4}$$

The ansatz (3.4) leads to two decoupled equations for f_1 and f_2. In terms of the variable $\xi = r^2/8$, they read

$$\frac{df_1}{d\xi} + \frac{1}{f}(3f^2 - 1 - \frac{df}{d\xi})f_1 = \frac{h^2}{2f^2}(1 - 9f^2), \tag{3.5}$$

$$\frac{df_2}{d\xi} + \frac{1}{f}(3f^2 - 1 + \frac{df}{d\xi})f_2 = -\frac{h^2}{2f}(1 + 3f^2). \tag{3.6}$$

The general solutions to equation (3.5) is

$$f_1 = \frac{1}{\cosh^2\xi}(\frac{3\sinh\xi}{2\cosh^3\xi} - \frac{\sinh\xi}{\cosh\xi} + C_1) \tag{3.7}$$

The function f_1 is a C^∞ function for $0 < \xi < \infty$. For $\xi \to 0$, $f_1 \to C_1$ holds. This implies that $C_1 = 0$; otherwise F in (3.4) is not defined at the origin. Therefore, f_1 reads

$$f_1 = \frac{3\sinh\xi}{2\cosh^5\xi} - \frac{\sinh\xi}{\cosh^3\xi}. \tag{3.8}$$

The expansion of f_1 near the origin is of the form

$$f_1 = \sum_{k=1}^{\infty} a_k\xi^k = \sum_{k=1}^{\infty} a_k(\frac{r^2}{8})^k. \tag{3.9}$$

Hence, the first term in (3.4) is a C^∞ function of x^1 and x^2 at the origin. We also see that f_1 vanishes exponentially at infinity. So its contribution to ϕ does not change the winding number (2.6) which is a multiple of the action.

A similar calculation yields a one parameter family of solutions to Eq. (3.6), namely

$$f_2 = \frac{\sinh\xi}{2\cosh^3\xi} - \frac{3\sinh^3\xi}{2\cosh^5\xi} + C_2\frac{\sinh^2\xi}{\cosh^4\xi}. \tag{3.10}$$

In contrast to f_1, all the solutions f_2 are acceptable. In fact, for all C_2, f_2 is of the form

$$f_2 = \sum_{k=1}^{\infty} b_k\xi^k = \sum_{k=1}^{\infty} b_k(\frac{r^2}{8})^k \tag{3.11}$$

near the origin, and therefore the second term in (3.4) is in $C^\infty(\mathbf{R}^2)$. The winding number and the action are also not altered because f_2 decays exponentially at infinity.

The functions G and H in (3.2) can be found in the same way. If we put all results together, we obtain the second order terms,

$$\delta = (\alpha^2+\beta^2) f_1(r)\exp^{i2\theta} + i(\alpha - i\beta)^2 f_2(r)\exp^{-i2\theta}, \tag{3.12}$$

where f_1 and f_2 are given by (3.8) and (3.10), respectively.

To find the cubic terms, we consider $\phi = \hat{\phi} + \gamma + \delta + \epsilon$, with γ given in (2.13) and δ given by (3.12). We set $\beta = 0$ and concentrate on

$$\epsilon = \alpha^3 I(r,\theta). \tag{3.13}$$

For the Bogomolnyi equation to hold, I must satisfy

$$\begin{aligned}
&\frac{2}{r}(f'\cos 2\theta\frac{\partial I^2}{\partial\theta} + 2f\sin 2\theta\frac{\partial I^2}{\partial r} - f'\sin 2\theta\frac{\partial I^1}{\partial\theta} + 2f\cos 2\theta\frac{\partial I^1}{\partial r})\\
&+h'(2f_1\cos 2\theta + 2f_2\sin 2\theta) + h'(2f_1\sin 2\theta + 2f_2\cos 2\theta)\\
&= -3f^2(I^1\cos 2\theta + I^2\sin 2\theta) - f_2(\cos 2\theta + \sin 2\theta)\\
&-3fh(\cos 2\theta+\sin 2\theta)(f_1 - 2f_2\cos 2\theta\sin 2\theta) - 3(\cos 2\theta + \sin 2\theta)h^2\\
&+(I^1\cos 2\theta + I^2\sin 2\theta) + \frac{h}{2}[f_1(\cos 2\theta+\sin 2\theta) - f_2(\cos 2\theta+\sin 2\theta)]\\
&+\frac{h^3}{2}(\cos 2\theta + \sin 2\theta)^3 - \frac{h^3}{f^2}(\cos 2\theta+\sin 2\theta)\\
&-\frac{1}{f}(\cos 2\theta+\sin 2\theta)(f_1 - 2f_2\cos 2\theta\sin 2\theta) + \frac{1}{2f^2}(I^1\cos 2\theta + I^2\sin 2\theta)^3
\end{aligned} \tag{3.14}$$

To solve equation (3.14) we seek a solution of the form

$$\begin{aligned}
I^1 &= g_1(\xi) + g_2(\xi)(\cos 4\theta - \sin 4\theta),\\
I^2 &= g_1(\xi) - g_2(\xi)(\cos 4\theta + \sin 4\theta).
\end{aligned} \tag{3.15}$$

This implies that g_1 and g_2 must satisfy the equations

$$\frac{dg_1}{d\xi} + (3f - \frac{1}{f})g_1 = -\frac{f_1+f_2}{f}\frac{dh}{d\xi} - 6hf_1 + \frac{9}{2}hf_2 - \frac{hf_2}{2f^2} - \frac{h^3}{4f^3} - \frac{9h^3}{4f}, \tag{3.16}$$

$$\frac{dg_2}{d\xi} - (\frac{1}{f} - 3f + \frac{2}{f}\frac{df}{d\xi})g_2 = -\frac{hf_2}{2f^2} - \frac{3hf_2}{2} - \frac{h^3}{4f^3} - \frac{h^3}{4f}. \quad (3.17)$$

The general solution to Eq. (3.17) is

$$g_2 = \frac{\sinh\xi}{4\cosh^5\xi} - \frac{5\sinh^3\xi}{4\cosh^7\xi} + C_2\left(\frac{\sinh^2\xi}{2\cosh^4\xi} - \frac{3\sinh^4\xi}{2\cosh^6\xi}\right) + C_3\frac{\sinh^3\xi}{\cosh^5\xi}. \quad (3.18)$$

All solutions (3.18) decay exponentially at infinity. For $r \to 0$, however,

$$g_2(r) = \frac{1}{24}r^2 + \ldots \quad (3.19)$$

Hence, I in (3.15) is not a C^∞ function on $\mathbf{R}^2$. Our expansion gets singular at third order for the ansatz (3.15). In the next section we will discuss a realistic model in which a similar pattern emerges but no singularities occur.

4 Abrikosov vortices

The Ginzburg-Landau theory of a superconductor in a magnetic field in direction z is given by the Lagrangian density

$$\mathcal{L} = \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}(D_i\phi)(D^i\phi)^* + \frac{\lambda}{8}(\mid\phi\mid^2 - 1)^2, \quad (4.1)$$

where ϕ is the complex Higgs field, and $D_i\phi = \partial_i\phi - iA_i\phi$ and $F_{ij} = \partial_i A_j \partial_j A_i$ in terms of the gauge potentials A_i, $i = 1, 2$. The Euler-Lagrange equations are

$$D_iD^i\phi = \frac{\lambda}{2}\phi(1 - \mid\phi\mid^2), \qquad \partial_i F^{ij} = \frac{\imath}{2}[\phi(D^j\phi)^* - \phi^* D^j\phi] \quad (4.2)$$

In the special case $\lambda = 1$ it can be shown [9] that all finite action solutions of Eq. (4.2) satisfy the first-order Bogmolnyi equations [8],

$$F_{12} = \frac{1}{2}(1 - \mid\phi\mid^2), \qquad D_1\phi = -iD_2\phi. \quad (4.3)$$

It has also been shown [9] that a $2n$-parameter family of solution of (4.3) exists with winding number

$$n = \frac{1}{2\pi}\int_{\mathbf{R}^2} F_{12}\, d^2x. \tag{4.4}$$

This family describes n vortices sitting at n position in space.

Even for n vortices sitting on top of each other, the solution is not known explicitly in terms of elementary functions. It is known [10], however, that this solution is of the form

$$\phi = f(r)e^{\imath n\theta}, \qquad A_i = -\frac{na(r)}{r^2}\varepsilon_{ij}x^j, \tag{4.5}$$

where f and a satisfy

$$rf' - n(1-a)f = (2n/r)a' + f^2 - 1 = 0 \tag{4.6}$$

and

$$f(0) = a(0) = 0, \quad \lim_{r\to\infty} f(r) = \lim_{r\to\infty} a(r) = 1. \tag{4.7}$$

In the following, we restrict our attention to $n = 2$ and use the solution (4.5) as the zero order term in an expansion in the separation parameters. The first order terms are given by the two zero modes describing the separation of the votices. These were found by Weinberg [11]. Using his results we can write, up to quadratic terms,

$$\phi = fe^{2\imath\theta} + 2(\alpha + \imath\beta)kf + \alpha^2\psi + \alpha\beta\phi + \beta^2\chi + \ldots, \tag{4.8}$$

$$A_1 + \imath A_2 = \imath\frac{2a}{r}e^{\imath\theta} - 2\imath(\alpha + \imath\beta)(k' + \frac{2k}{r})e^{-\imath\theta}$$
$$+\alpha^2(B_1 + \imath B_2) + \alpha\beta(C_1 + \imath C_2) + \beta^2(D_1 + \imath D_2) + \ldots \tag{4.9}$$

Here the radial function $k(r)$ satisfies

$$k'' + \frac{1}{r}k' - (f^2 + \frac{4}{r^2})k = 0, \tag{4.10}$$

with

$$\lim_{r\to 0} r^2 k = 1, \qquad \lim_{r\to\infty} k(r) = 0. \tag{4.11}$$

Our task is to determine $\psi, \phi, \chi, B_i, C_i, D_i$, which are functions of r and θ.

Equating the α^2-terms in the Bogomolnyi equations (4.3), we obtain

$$(\partial_1 + \imath\partial_2)\psi + \frac{2a}{r}\psi e^{\imath\theta} - \imath f(B_1 + \imath B_2)e^{2\imath\theta} = 4kf(k' + \frac{2k}{r})e^{-\imath\theta}, \qquad (4.12)$$

$$\partial_1 B_2 - \partial_2 B_1 + \frac{1}{2}(f\bar{\psi}e^{-2\imath\theta} + f\psi e^{2\imath\theta}) = -2k^2 f^2. \qquad (4.13)$$

A Fourier expansion with the minimal number of nonzero terms leads to the ansatz

$$\psi = g(r)f(r)e^{2\imath\theta} + \tilde{g}(r)e^{-2\imath\theta},$$
$$B_1 + \imath B_2 = \tilde{b}(r)e^{\imath\theta} + \imath b(r)f(r)e^{-3\imath\theta}, \qquad (4.14)$$

and to equations for $g(r), \tilde{g}(r), b(r)$ and $\tilde{b}(r)$. The equations for $\tilde{g}(r)$ and $\tilde{b}(r)$ read

$$\tilde{g} = \frac{1+2a}{r}b - b', \qquad \tilde{b} = -\imath h'. \qquad (4.15)$$

The functions $g(r)$ and $b(r)$ must satisfy the equations

$$g'' + \frac{1}{r}g' - f^2 g = 2k^2 f^2, \qquad (4.16)$$

$$b'' + \frac{1}{r}b' - (\frac{1+f^2}{2} + \frac{1+4a+4a^2}{r^2})b = -4kf(k' + \frac{2k}{r}). \qquad (4.17)$$

Equating the $\alpha\beta$-terms and the β^2-terms in the Bogonolnyi equation (4.3), we obtain equations for ϕ and C_i, and for χ and D_i respectively. These equations, which are very similar to equations (4.12) and (4.13), can again be solved by functions with the same θ-dependence as in (4.14) but with slightly different radial functions. Collecting all results, we can write the solution, up to quadratic terms, in the form

$$\phi = fe^{2\imath\theta} + 2(\alpha + \imath\beta)kf$$
$$+(\alpha^2 + \beta^2)gfe^{2\imath\theta} + (\alpha + \imath\beta)^2(\frac{1+2a}{r}b - b')e^{-2\imath\theta} + \ldots$$
$$A_1 + \imath A_2 = \imath\frac{2a}{r}e^{\imath\theta} - 2\imath(\alpha + \imath\beta)(k' + \frac{2k}{r})e^{-\imath\theta}$$
$$-\imath(\alpha^2 + \beta^2)g'e^{\imath\theta} + \imath(\alpha + \imath\beta)^2 bfe^{-3\imath\theta} + \ldots \qquad (4.18)$$

It remains to be shown that the quadratic terms in (4.18) are C^∞ functions on $\mathbf{R}^2$ which do not change the action (and the winding number). To this end we use the asymptotic expansions of f, a and k at zero [12],

$$f(r) = f_1 r^2 + \frac{1}{8} f_1 r^4 + \ldots, \quad a(r) = \frac{1}{8} r^2 - \frac{1}{24} f_1^2 r^6 + \ldots, \quad k(r) = r^{-2} + k_1 r^2 + \ldots, \tag{4.19}$$

where $f_1 = .236$ and $k_1 = -.025$ from the numerical analysis. We find that the solutions of (4.16) and (4.17) have the following expansions at the origin,

$$g(r) = g_{-1} \log r + g_1 + \frac{1}{2} f_1^2 r^2 + \ldots$$

$$b(r) = b_{-1} r^{-1} + b_1 r + (\frac{1}{8} b_1 - 2 f_1 k_1) r^3 + \ldots \tag{4.20}$$

The higher order terms in $g(r)$ are even powers of r, whereas the higher order term in $b(r)$ are odd powers of r. Hence, the quadratic terms in (4.18) are C^∞ near the origin if and only if $h_{-1} = b_{-1} = 0$. So far the constants g_1 and b_1 are arbitrary.

For large r the functions f, a, and k have the following asymptotic behavior [12]:

$$\begin{aligned} f(r) &= 1 + \tilde{f}_1(r) e^{-r} + \ldots, \\ a(r) &= 1 + \tilde{a}_1(r) e^{-r} + \ldots, \\ k(r) &= \tilde{k}_1(r) e^{-r} + \ldots, \end{aligned} \tag{4.21}$$

with coefficient functions which are polynomially bounded. This leads to the existence of exponentially decaying solutions which asymptotically are of the form

$$g(r) = \tilde{g}_1(r) e^{-r} + \ldots, \qquad b(r) = \tilde{b}_1(r) e^{-r} + \ldots \tag{4.22}$$

Here $\tilde{g}_1$ and $\tilde{b}_1$ are polynomially bounded.

By numerical integration, the coefficients g_1 and b_1 which lead to an exponential fall-off at infinity, are found to be $g_1 = -.144$ and $b_1 = -.026$. The existence of such functions can be explained analytically as follows: Equation (4.16) shows that for positive g_1, g cannot have a maximum for any r. So the function diverges exponentially. For very small g_1, the term on the right-hand side of (4.16) will force the function to cross the r-axis, and then, as before, diverge exponentially. For very large negative g_1, the third term in (4.16) will force g to go through a maximum for large r. After that, the function cannot have a minimum and must go to minus infinity. Because of the continuous dependence on the initial data, we have an open set of data for which g crosses the r-axis, and an open set of data for which g goes through a maximum below the r-axis. Therefore, we have at least one value of g_1 for which the function does neither. This function must converge and does so to zero, exponentially.

A similar argument explains the existence of an acceptable solution $b(r)$ to Eq. (4.17). The right-hand side of that equation is positive. So again b cannot have a maximum above the r-axis. Also, for very small negative b_1, the right-hand side will force b to go through a minimum and then cross the r-axis. For very large negative b_1, the third term in (4.17) prevents b from going through a minimum. In between these two possibilities we find the desired solution which goes through a minimum but does not cross the r-axis. Such a solution must decay exponentially.

The cubic terms can be calculated in the same manner. We find, at third order,

$$\phi = \ldots + (\alpha + \imath\beta)(\alpha^2 + \beta^2) f h + (\alpha + \imath\beta)^3 (-c' + \frac{3 + 2a}{r} c) e^{-4\imath\theta} + \ldots,$$

$$A_1 + \imath A_2 = \ldots$$

$$+\imath(\alpha+\imath\beta)(\alpha^2+\beta^2)[-h' - \frac{2}{r}h + 2g(k' + \frac{2k}{r}) + 2kg']e^{-\imath\theta} + \imath(\alpha+\imath\beta)^3 f c e^{-5\imath\theta} + \ldots \quad (4.23)$$

The new radial functions, $h(r)$ and $c(r)$, satisfy the equations,

$$h'' + \frac{1}{r}h' - (f^2 + \frac{4}{r^2})h = 4k'g' + 2fk(2fk^2 + 3fg + \frac{1 + 2a}{r} b - b'), \quad (4.24)$$

$$c'' + \frac{1}{r}c' - (\frac{1 + f^2}{2} + \frac{9 + 12a + 4a^2}{r^2})c = 2kf^2 b - 2(k' + \frac{2k}{r})(\frac{1 + 2a}{r} b - b'). \quad (4.25)$$

Near the origin, Eq. (4.25) has a series solution in powers of r^2 of the form

$$h(r) = f_1^2 + h_1 r^2 + h_2 r^4 + \ldots \quad (4.26)$$

The constant term is given in terms of the coefficient f_1 of the leading term in the expansion (4.19) of $f(r)$. The form of this term leads to the cancellation of the r^{-1}-terms in the radial function multiplying $e^{-\imath\theta}$ in (4.24), and thus ensures that this term in (4.23) is C^∞ on $\mathbf{R}^2$. The series in odd powers of r for $c(r)$ which solves Eq. (4.25) near the origin, is

$$c(r) = c_1 r^3 + c_2 r^5 + \dots \tag{4.27}$$

The form of the series solutions at the origin guarantees that the cubic terms in (4.23) are C^∞ functions on $\mathbf{R}^2$. For large r, Eqs (4.24) and (4.25) have exponentially decaying solutions.

5 Conclusions

Our expansions show a simple θ-dependence in terms of trigonometric functions. In both models, the expansion of ϕ exhibits the following pattern:

$$\begin{array}{ccccccc}
 & & & & & & e^{2\imath\theta} \\
 & & & & & e^{0\imath\theta} & \\
 & & & & e^{-2\imath\theta} & & e^{2\imath\theta} \\
 & & & e^{-4\imath\theta} & & e^{0\imath\theta} & \\
 & & e^{-6\imath\theta} & & e^{-2\imath\theta} & & e^{2\imath\theta} \\
 & e^{-8\imath\theta} & & e^{-4\imath\theta} & & e^{0\imath\theta} & \\
\dots & & \dots & & \dots & & \dots
\end{array}$$

Here the first line gives the θ dependence of the zero order term; the second line gives the first order term, and so on. We get a similar triangular pattern for the θ dependence of $A_1 + \imath A_2$ at any order. For the radial functions we find differences between the two models. In the model for one complex field, the radial functions can be given explicitly in terms of exponential functions. However, for the angular dependence (3.15), a singularity occurs at the origin. (We have found no solution to (3.14) which is not of the form (3.15); we have no proof that there is none.)

For the Ginzburg-Landau theory on the other hand, the expansion is smooth, at least up to the order to which we carried out our calculations. In this model the radial functions are not given in terms of well-known functions. Having used the technique to calculate the terms up to third order, it is quite clear how to proceed to any order, and also how to proceed in the case of more than two vortices. We expect these expansions to converge for small separation parameters in the physical Ginzburg-Landau model. However, we do not have an estimate of the radius of convergence.

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