

The $(2, 5)$ minimal model on degenerating genus two surfaces

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Abstract

In the $(2, 5)$ minimal model, the partition function for genus $g = 2$ Riemann surfaces is expected to be given by a quintuplet of Siegel modular forms that extend the Rogers-Ramanujan functions on the torus. Their expansions around the $g = 2$ boundary components of the moduli space are obtained in terms of standard modular forms. In the case where a handle of the $g = 2$ surface is pinched, our method requires knowledge of the 2-point function of the fundamental lowest-weight vector in the non-vacuum representation of the Virasoro algebra, for which we derive a third order ODE.

1 Introduction

1.1 Motivation and outline

Two-dimensional conformal field theories (CFTs) are naturally defined on compact Riemann surfaces. Every such theory is characterised by its partition function, which defines a function on the moduli space of such surfaces. Its restriction to genus $g = 1$ is given by classical modular functions. For the $(2, 5)$ minimal model, one obtains the sum of the squared norms of the well-known Rogers-Ramanujan functions. These 0-point functions satisfy a second order ODE in the modulus. For $g = 2$, a corresponding system of differential equations has been established in [6]. The method relies on the description of the Riemann surface Σ as a double covering of the Riemann sphere,

$$\Sigma : y^2 = p(x), \quad (1)$$

where p is a polynomial of degree 3 (for genus $g = 1$) resp. 5 (for $g = 2$).

A different method for computing N -point functions of CFTs on higher genus Riemann surfaces due to [8] is available, by sewing pairs of lower genus Riemann surfaces [9]. The case of interest to us in this paper is $N = 0$.

For $i = 1, 2$, let (Σ_i, P_i) with $P_i \in \Sigma_i$ be a non-singular Riemann surface of genus g_i with puncture P_i . Let z_i be a local coordinate vanishing at P_i . We allow arbitrary

complex coordinate choices. Now excise sufficiently small discs $\{|z_1| < r\}$ and $\{|z_2| < r\}$ from Σ_1 and Σ_2 , respectively, and sew the two remaining surfaces by the condition

$$z_1 z_2 = r^2 \quad (2)$$

on tubular neighbourhoods of the circles $\{|z_i| = r\}$. This operation yields a non-singular Riemann surface of genus $g_1 + g_2$ with no punctures.

Instead of sewing two one-punctured surfaces, we may self-sew a single Riemann surface (the case $\Sigma_1 = \Sigma_2$) with two different punctures. This procedure results in a Riemann surface with one new handle attached to it.

Thus we consider the inverse procedure by which the genus $g = 2$ surface degenerates. Such singular surfaces are boundary points of the the moduli space with Deligne-Mumford compactification. In the limit where $r^2 \searrow 0$, a cycle on the surface is pinched. When the cycle is homologous to zero (case discussed in Section 2.1), the squeezing results in two separate tori with a single puncture on each. In the algebraic description by eq. (1), three ramification points run together. In the case where the cycle is non-homologous to zero (addressed in Section 2.2), the above mentioned limit describes the cut through a handle. In this case two ramification points run together, yielding a single torus with two punctures. To distinguish the two cases, following [7], we shall refer to the first and second case as the ε and the ρ formalism, respectively.

Using methods from vertex operator algebras, T. Gilroy and M. Tuite have derived the first terms of the corresponding expansion for the ε formalism [2]. In this paper, we give an expansion in terms of modular forms which in particular includes these earlier results.

One purpose of this paper is to built a bridge between the two approaches, and to make the subject better accessible to researchers interested in Siegel modular forms.

1.2 Quasi-primary and derivative fields

Let F be the space of holomorphic fields, (equivalently, the space of holomorphic states). A distinguished element in F is the Virasoro field

$$T(z) = \sum_{n \in \mathbb{Z}} z^{n-2} L_n .$$

The constant field 1 corresponds to the vacuum state v , the Virasoro field to $L_2 v$. The Laurent coefficients define the Virasoro algebra

$$[L_n, L_m] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} , \quad (3)$$

where $c \in \mathbb{R}$ is the central charge. (Note the unusual index convention, which is chosen so that $L_1 = \frac{\partial}{\partial z}$). The kernel of L_1 is spanned by the vacuum vector v . L_0 defines a grading on F , called the conformal weight. Holomorphic fields in the image of L_1 will be referred to as derivative fields, whose space we denote by F_{der} . The Shapovalov metric defines a sesquilinear form on F . For the latter we have $L_{-1} = L_1^*$. The space of quasi-primary fields is the orthogonal complement of F_{der} w.r.t. that metric, thus the kernel of L_{-1} . A holomorphic field ψ is primary iff $L_n \psi = 0$ for $n < 0$. Suppose in some minimal model, W is an irreducible representation of the Virasoro algebra (3). Then there exists $w \in W$ with $L_0 w = h w$ and h is minimal in W . W is spanned by vectors of

the form $L_{n_k} \dots L_{n_1} w$ with $n_k \in \mathbb{Z}, k \geq 0$. The vacuum representation is characterised by

$$L_{-1}v = 0, \quad L_0v = 0, \quad L_1v = 0.$$

The generating function of W is the character

$$\chi_W := \text{tr}_{F_W} q^{L_0}.$$

Let \tilde{F}_W be the space of quasi-primary fields in the representation W . If $w = v$, the generating function of \tilde{F}_W is

$$\tilde{\chi}_W = (1 - q)(\chi_W - 1).$$

For other vectors one has

$$\tilde{\chi}_W = (1 - q)\chi_W.$$

1.3 The (2, 5) minimal model

For every minimal model and for every irreducible representation of the Virasoro algebra, there are two fundamental linear relations between states in that representation. In the (2, 5) minimal model, the Virasoro algebra has two irreducible representations, the vacuum representation V (with vacuum vector v) and a non-vacuum representation which we denote by W . The lowest weight vector w in W has conformal weight $h = -1/5$. The fundamental identities in V are

$$\begin{aligned} L_1v &= 0 \\ (L_2L_2 - \frac{3}{5}L_4)v &= 0. \end{aligned}$$

Equivalently, the operator product expansion (OPE) of $T(z) \otimes T(0)$ has the form

$$T(z) \otimes T(0) \mapsto \frac{c/2}{z^4} \cdot 1 + \frac{1}{z^2} \{T(z) + T(0)\} + \frac{3}{10} T''(0) + O(z), \quad (4)$$

where $c = -22/5$. The two fundamental identities in W are

$$(2L_2 - 5L_1L_1)w = 0 \quad (5)$$

$$(L_3 - 5L_2L_1)w = 0. \quad (6)$$

To w corresponds a non-holomorphic field Φ . For suitable pairs (z, \bar{z}) of a holomorphic and an antiholomorphic local coordinate the field's local representative admits a splitting $\Phi(z, \bar{z}) = \varphi_{\text{hol}}(z) \otimes \varphi_{\text{hol}}(\bar{z})$ into holomorphic and antiholomorphic part. The individual holomorphic part $\varphi = \varphi_{\text{hol}}$ is a local primary field of conformal weight $h = -1/5$. Thus eqs (5) and (6) are equivalent to the OPE

$$T(z) \otimes \varphi(0) \mapsto \frac{h}{z^2} \varphi(0) + \frac{1}{z} \varphi'(0) + \frac{5}{2} \varphi''(0) + \frac{25}{12} z \varphi^{(3)}(0) + O(z^2), \quad (7)$$

where $h = -1/5$. The space of all fields factorises as

$$F = F_V \otimes \overline{F_V} \oplus F_W \otimes \overline{F_W},$$

where F_V and F_W denote the space of holomorphic fields that correspond to states in V and W , respectively, and the bar marks the corresponding spaces of antiholomorphic fields.

For the (2, 5) minimal model, the generating function for the number of holomorphic fields of a given weight in F_V and in F_W is the character

$$\begin{aligned}\chi_V &= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} \\ &= 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + 3q^8 + 3q^9 + 4q^{10} + 4q^{11} + 6q^{12} + \dots, \\ \chi_W &= q^{-\frac{1}{5}} \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} \\ &= q^{-\frac{1}{5}} \left(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + 6q^{10} + 7q^{11} + 9q^{12} + \dots \right),\end{aligned}$$

respectively. (Here $(;)_n$ is the q -Pochhammer symbol.)

Propos. 1. *To every conformal weight $h \leq 10$, there exists at most one quasi-primary field in F_V , up to normalisation. For $h \leq 8$, their respective weight and squared Shapovalov norm are given by the following table:*

weight	quasi-primary field	squared norm
2	$L_2 v$	$c/2$
4	-	-
6	$(7L_4 L_2 - 2L_6) v$	$217c$
8	$(6L_6 L_2 + \frac{21}{5} L_4 L_4 - 7L_8) v$	$-\frac{8952}{5} c$

Here $c = -22/5$.

Proof. The number of quasi-primary fields of conformal weight h in F_V is given by the coefficient of q^h in the series

$$\tilde{\chi}_V - 1 = (1 - q)(\chi_V - 1) = q^2 + q^6 + q^8 + q^{10} + 2q^{12} + q^{15} + \dots$$

The fields and their respective weight are obtained by direct computation. \square

Propos. 2. *To every conformal weight $h \leq 11$, there exists at most one quasi-primary field in F_W , up to normalisation. For $h < 6$, their respective weight and squared Shapovalov norm are given by the following table:*

weight	quasi-primary field	squared norm
$-\frac{1}{5}$	+2 -	-
	+4 $(52L_4 - 25L_1 L_3) w$	$\frac{5928}{5} \ w\ ^2$
	+6 $(4L_1 L_5 + 3L_3 L_3 - \frac{684}{35} L_6) w$	$\frac{6539268}{6125} \ w\ ^2$

Proof. The number of quasi-primary fields of conformal weight h in F_W is given by the coefficient of q^h in the series

$$\tilde{\chi}_W = (1 - q)\chi_W = q^{-\frac{1}{5}} \left(1 + q^4 + q^6 + q^8 + q^9 + q^{10} + q^{11} + 2q^{12} + \dots \right)$$

The fields and their respective weight are obtained by direct computation. \square

Now we specialise to $g = 1$. The 0-point functions differ from the corresponding characters by a factor of $q^{-\frac{c}{24}}$, where q is identified with the nome $e^{2\pi i \tau}$. For the (2, 5)

minimal model on the torus, these are the so-called Rogers-Ramanujan functions

$$\begin{aligned}\langle 1 \rangle_1^{g=1}(q) &= H(q) := q^{\frac{11}{60}} \sum_{n \geq 0} \frac{q^{n^2+n}}{(q; q)_n}, \\ \langle 1 \rangle_2^{g=1}(q) &= G(q) := q^{-\frac{1}{60}} \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n}.\end{aligned}$$

The modular invariant partition function is given by

$$Z^{g=1}(q) = |H(q)|^2 + |G(q)|^2.$$

The Virasoro field generates changes of τ , so that ([1], or [5] for a direct proof)

$$\mathcal{D}_0 \langle 1 \rangle = \frac{1}{(2\pi i)^2} \langle T \rangle. \quad (8)$$

As an aside, the OPE (4) yields in addition

$$\mathcal{D}_2 \langle T \rangle = \frac{11}{3600} (2\pi i)^2 E_4(q) \langle 1 \rangle.$$

Here for $\ell \in \mathbb{R}$,

$$\mathcal{D}_\ell = q \frac{\partial}{\partial q} - \frac{\ell}{12} E_2(q)$$

is the Serre-derivative operator, (defined on modular forms of weight ℓ).

Let $\wp(z|\tau)$ and $\zeta(z|\tau)$ (or $\wp(z)$ and $\zeta(z)$ when $\tau \in \mathbb{H}^+$, the upper complex half plane, is fixed) be the Weierstrass \wp -function and the Weierstrass ζ -function, respectively. For brevity, we write \wp_{ij} and ζ_{ij} in place of $\wp(z_{ij})$ and $\zeta(z_{ij})$ respectively, where for $z_i, z_j \in \mathbb{C}$, $z_{ij} := z_i - z_j$.

Now we calculate the 1-point function of the field $\varphi \in F_W$ corresponding to the lowest weight vector w in W .

Propos. 3. *We have*

$$\mathcal{D}_{-1/5} \langle \varphi \rangle = 0.$$

Proof. By the OPE (7),

$$\frac{\langle T(z)\varphi(0) \rangle}{\langle \varphi(0) \rangle} = -\frac{1}{5} \wp(z). \quad (9)$$

Indeed, the regular part must be constant, and is zero by the fact that $\langle \varphi''(0) \rangle = \partial^2 \langle \varphi(0) \rangle = 0$. Thus

$$q \frac{d}{dq} \langle \varphi(0) \rangle = \oint \langle T(z)\varphi(0) \rangle \frac{dz}{(2\pi i)^2} = -\frac{1}{5} \langle \varphi(0) \rangle \int_0^1 \wp(z) \frac{dz}{(2\pi i)^2} = -\frac{1}{60} E_2(q) \langle \varphi(0) \rangle.$$

(Here the contour integral is taken along the real period, and $\oint dz = 1$). \square

Thus we have

$$\begin{aligned}\langle \varphi \rangle &= \eta(q)^{-2/5} = q^{-\frac{1}{60}} \prod_{n \geq 1} (1 - q^n)^{-2/5} \\ &= q^{-\frac{1}{60}} \left(1 + \frac{2}{5}q + \frac{17}{25}q^2 + \frac{98}{125}q^3 + \frac{714}{625}q^4 + \frac{18,768}{15,625}q^5 + \dots \right).\end{aligned}$$

Here $\eta(q)$ is Dedekind η -function.

Corollary 4. Set $z_0 = 0$. We have

$$\frac{\langle T(z_1)T(z_2)\varphi(0) \rangle}{\langle \varphi(0) \rangle} = \frac{c}{2}\wp_{12}^2 - \frac{1}{5}\wp_{12}(\wp_{10} + \wp_{20}) + \frac{6}{25}\wp_{10}\wp_{20} + \frac{\pi^4}{45}E_4. \quad (10)$$

Proof. On the one hand, from the OPE (7) for $T(z) \otimes \varphi(0)$ and eq. (9),

$$\eta^{2/5}\langle T(z)T(w)\varphi(0) \rangle = \frac{h^2}{z^2}\wp(w) - \frac{h}{z}\wp'(w) + \text{terms that are regular for } z \rightarrow 0,$$

where the occurring even and odd negative power of z can be replaced with $\wp(z)$ and $z\wp(z)$, respectively. The latter expression is not elliptic, however we may use

$$-z\wp'(w) = \wp(z-w) - \wp(w) + O(z^2).$$

Thus we have

$$\begin{aligned} \eta^{2/5}\langle T(z)T(w)\varphi(0) \rangle \\ = h\wp(z)\wp(z-w) + (h^2 - h)\wp(z)\wp(w) + \text{terms that are regular for } z \rightarrow 0. \end{aligned} \quad (11)$$

On the other hand, by the OPE (4) for $T(z) \otimes T(w)$, using eq. (9),

$$\eta^{-2/5}\langle T(z)T(w)\varphi(0) \rangle = \frac{c/2}{(z-w)^4} + \frac{h}{(z-w)^2}\{\wp(z) + \wp(w)\} + \frac{3}{10}h\wp''(w) + O(z-w).$$

Thus

$$\begin{aligned} \eta^{2/5}\langle T(z)T(w)\varphi(0) \rangle \\ = \frac{c}{12}\wp''(z-w) + h\wp(z-w)\{\wp(z) + \wp(w)\} + (h^2 - h)\wp(z)\wp(w) + K, \end{aligned} \quad (12)$$

where K is constant in z and w . Comparison of eqs (11) and (12) yields

$$h\left(h + \frac{1}{5}\right) = 0, \quad K = -(c + 10h(h-1))\frac{\pi^4}{90}E_4 = \frac{\pi^4}{45}E_4.$$

□

2 The genus 2 partition function

2.1 Results in the ε formalism

Let $\{\psi_i\}_{i \geq 0}$ be an orthonormal basis of F_V with the Shapovalov metric, where $\psi_0 = 1$ and $L_0\psi_i = h_i\psi_i$ for $i \geq 0$. For any $\psi \in F_V$, denote by $\psi(z)$ and $\hat{\psi}(\hat{z})$ the local representative of ψ w.r.t. a chart of an affine structure [3] on the torus with modulus τ and $\hat{\tau}$, respectively. In the respective coordinate z and \hat{z} , all 1-point functions on either torus are constant in position. On a small annulus centered at $z = 0$ resp. $\hat{z} = 0$, we glue the two tori using

$$z \hat{z} = \varepsilon$$

for $\varepsilon > 0$ small. This procedure yields a $g = 2$ surface with a projective structure. Let $\tilde{z} = \hat{z}/\varepsilon$ and write $\tilde{\psi}(\tilde{z})$ accordingly. For $a, b \in \{1, 2\}$, the choice of the Roger-Ramanujan function $\langle 1 \rangle_a^{g=1}$ and $\langle 1 \rangle_b^{g=1}$ on the torus of modulus τ and $\hat{\tau}$, respectively, gives rise to the 0-point function for the index pair (a, b)

$$\langle 1 \rangle_{a,b}^{g=2}(q, \hat{q}, \varepsilon) = \sum_{i \geq 0} \langle \psi_i(z) \rangle_a^{g=1}(q) \langle \tilde{\psi}_i(\tilde{z}) \rangle_b^{g=1}(\hat{q}) \quad (13)$$

on the resulting genus $g = 2$ surface. A fifth solution $\langle 1 \rangle_{\varphi}^{g=2}(q, \hat{q}, \varepsilon)$ is obtained by choosing $\langle \varphi \rangle$ on either torus.

For $i > 0$, only the ψ_i that are quasi-primary contribute to the sum. Under the coordinate change $\tilde{z} \mapsto \hat{z}$, the 1-point functions transform according to

$$\langle \tilde{\psi}_i(\tilde{z}) \rangle = \varepsilon^{hi} \langle \hat{\psi}_i(\hat{z}) \rangle,$$

so eq. (13) becomes an infinite series in powers of ε . We will use the notation

$$\langle \psi_i \rangle = \langle \psi_i \rangle^{g=1}(q), \quad \widehat{\langle \psi_i \rangle} = \langle \hat{\psi}_i \rangle^{g=1}(\hat{q}).$$

We also write $E_{2k} = E_{2k}(q)$ and $\widehat{E}_{2k} = E_{2k}(\hat{q})$ and likewise for other modular forms.

Theorem 1. *For $a = 1, 2$ we have*

$$\begin{aligned} \langle 1 \rangle_{a,a}^{g=2}(q, \hat{q}, \varepsilon) &= F_0 \widehat{F}_0 + \frac{\varepsilon^2}{7920} (2\pi i)^4 F_2 \widehat{F}_2 + \frac{\varepsilon^6}{445,471,488,000} (2\pi i)^{12} F_6 \widehat{F}_6 \\ &\quad - \frac{\varepsilon^8}{125,067,317,760,000} (2\pi i)^{16} F_8 \widehat{F}_8 + O(\varepsilon^{10}), \end{aligned}$$

Here $F_{2k} = F_{2k}(q)$ and $\widehat{F}_{2k} = F_{2k}(\hat{q})$ are given by

$$\begin{aligned} F_0 &= \langle 1 \rangle_a, \quad F_2 := 60q \frac{\partial}{\partial q} F_0 = \frac{60}{(2\pi i)^2} \langle T \rangle_a, \quad a = 1, 2; \\ F_6 &:= 110E_6 F_0 + 21E_4 F_2, \\ 6F_8 &:= 1309E_8 F_0 + 235E_6 F_2. \end{aligned}$$

For $F_0 = H$, the expansions

$$\begin{aligned} F_2(q) &= q^{11/60} \left(11 + 131q^2 + 191q^3 + 251q^4 + 311q^5 + 742q^6 + 862q^7 + 1473q^9 + O(q^{10}) \right) \\ F_6(q) &= q^{11/60} \left(341 - 1,327,699q^2 - 11,366,119q^3 - 49527739q^4 - 153310159q^5 - 418324358q^6 + O(q^7) \right) \\ F_8(q) &= q^{11/60} \left(649 - 112,420q + 6,348,609q^2 + 173,671,679q^3 + 1,424,241,669q^4 + O(q^5) \right) \end{aligned}$$

(and similar expansions for $F_0 = G$) have been found previously by [2] though the coefficients have not been identified with standard modular forms.

Proof. According to Proposition 1, we have for $a, b \in \{1, 2\}$,

$$\begin{aligned} \langle 1 \rangle_{a,b}^{g=2}(q, \hat{q}, \varepsilon) &= \langle 1 \rangle_a \widehat{\langle 1 \rangle}_b - \frac{2}{c} \varepsilon^2 \langle T \rangle_a \widehat{\langle T \rangle}_b - \frac{7}{31c} \varepsilon^6 \langle L_4 L_2 1 \rangle_a \widehat{\langle L_4 L_2 1 \rangle}_b \\ &\quad - \frac{5\varepsilon^8}{8952c} \langle (6L_6 L_2 + \frac{21}{5} L_4 L_4) \nu \rangle_a \widehat{\langle (6L_6 L_2 + \frac{21}{5} L_4 L_4) \nu \rangle}_b + O(\varepsilon^{10}). \end{aligned}$$

We have

$$\langle T(z)T(0) \rangle = \sum_{n \in \mathbb{Z}} z^{n-2} \langle L_n L_2 v \rangle .$$

Comparison with

$$\langle T(z_1)T(z_2) \rangle = \frac{c}{2} \wp_{12}^2 \langle 1 \rangle + 2\wp_{12} \langle T \rangle - \frac{\pi^2}{15} E_4 c \langle 1 \rangle$$

[4, and references therein] yields:

$$\begin{aligned} \langle L_4 L_2 v \rangle &= \frac{2c}{189} \pi^6 E_6 \langle 1 \rangle + \frac{2}{15} \pi^4 E_4 \langle T \rangle \\ \langle L_6 L_2 v \rangle &= \frac{c}{270} \pi^8 E_8 \langle 1 \rangle + \frac{4}{189} \pi^6 E_6 \langle T \rangle \\ \langle L_4 L_4 v \rangle &= \frac{7c}{315} \pi^8 E_8 \langle 1 \rangle + \frac{48}{189} \pi^6 E_6 \langle T \rangle \end{aligned}$$

We conclude that for the other quasi-primary fields listed in Proposition 1, we have

$$\begin{aligned} \langle (7L_4 L_2 - 2L_6) v \rangle_1 &= (2\pi i)^6 \frac{F_6}{21600} \\ \langle (6L_6 L_2 + \frac{21}{5} L_4 L_4 - 7L_8) v \rangle_1 &= -(2\pi i)^8 \frac{1309E_8 F_0 + 235E_6 F_2}{756000} . \end{aligned}$$

□

In order to compute the higher order terms (i.e., the one-point function of quasi-primary fields of conformal weight $h \geq 12$), N -point functions for $N \geq 3$ are required.

Theorem 2. [4] *Let $S(z_1, \dots, z_N)$, $N \in \mathbb{N}$, be the set of oriented graphs with vertices z_1, \dots, z_N (which may or may not be connected), subject to the condition that every vertex has at most one ingoing and at most one outgoing line, and none is a tadpole (with the incoming line being identical to the outgoing line). We have*

$$\langle T(z_1) \dots T(z_N) \rangle^{g=1} = \sum_{\Gamma \in S(z_1, \dots, z_N)} F(\Gamma) ,$$

where for $\Gamma \in S(z_1, \dots, z_N)$,

$$F(\Gamma) := \left(\frac{c}{2} \right)^{\#\text{loops}} \prod_{(z_i, z_j) \in \Gamma} \wp_{ij} \left\langle \bigotimes_{k \in E_N^c} T(z_k) \right\rangle_r .$$

Here $(z_i, z_j) \in \Gamma$ is an oriented edge,

$$E_N := \{1 \leq j \leq N \mid \exists i \text{ such that } (z_i, z_j) \in \Gamma\} ,$$

and E_N^c denotes its complement in $\{1, \dots, N\}$. Moreover, for all $n \in \mathbb{N}$, $\langle T(z_{k_1}) \otimes \dots \otimes T(z_{k_n}) \rangle_r$ with $k_i \in E_N^c$ for $i = 1, \dots, n$, is a modular form of weight $2n$.

Example 5. For $\langle 1 \rangle = \langle 1 \rangle_a^{g=1}$ with $a \in \{1, 2\}$, and for $\langle T \rangle$ given by eq. (8), we have

$$\begin{aligned}\langle 1 \rangle_r &= \langle 1 \rangle \\ \langle T(z) \rangle_r &= \langle T \rangle \\ \langle T(z_1)T(z_2) \rangle_r &= -\frac{\pi^4}{15}E_4c\langle 1 \rangle \\ \langle T(z_1)T(z_2)T(z_3) \rangle_r &= -\frac{4\pi^6}{45}E_6c\langle 1 \rangle + \frac{14\pi^4}{25}E_4\langle T \rangle \\ \langle T(z_1)T(z_2)T(z_3)T(z_4) \rangle_r &= -\frac{1,468\pi^8}{10,125}E_4^2c\langle 1 \rangle + \frac{1,792\pi^6}{1,575}E_6\langle T \rangle.\end{aligned}$$

We discuss the fifth solution, which is characterised by properties of φ .

Theorem 3. We have

$$\begin{aligned}\langle 1 \rangle_\varphi^{g=2}(q, \hat{q}, \varepsilon) &= \varepsilon^{-1/5} (\eta \widehat{\eta})^{-2/5} \left\{ 1 + \frac{13}{8,208,000} (2\pi i)^8 \varepsilon^4 E_4 \widehat{E}_4 \right. \\ &\quad \left. + \varepsilon^6 (2\pi i)^{12} \frac{989}{33,591,075,840} E_6 \widehat{E}_6 + O(\varepsilon^8) \right\}.\end{aligned}$$

Proof. By Proposition 2,

$$\begin{aligned}\langle 1 \rangle_\varphi^{g=2}(q, \hat{q}, \varepsilon) &= \varepsilon^{-1/5} \left\{ \langle \varphi \rangle \widehat{\langle \varphi \rangle} + \frac{5 \cdot (52)^2}{5,928} \varepsilon^4 \langle L_4 w \rangle \widehat{\langle L_4 w \rangle} \right. \\ &\quad \left. + \varepsilon^6 \frac{6,125}{6,539,268} \langle (3L_3L_3 - \frac{684}{35}L_6)w \rangle \widehat{\langle (3L_3L_3 - \frac{684}{35}L_6)w \rangle} + O(\varepsilon^8) \right\}.\end{aligned}$$

We list the partial results: By eq. (9) we have

$$\begin{aligned}\langle L_4 w \rangle &= -\frac{(2\pi i)^4}{1200} E_4 \eta^{-2/5} \\ \langle L_6 w \rangle &= \frac{(2\pi i)^6}{30,240} E_6 \eta^{-2/5}.\end{aligned}$$

Sorting out the terms $\propto z_1 z_2$ in eq. (10) yields:

$$\langle L_3 L_3 w \rangle = \frac{5}{3,024} (2\pi i)^6 E_6 \eta^{-2/5}.$$

We conclude that for the quasi-primary fields listed in Proposition 2, we have

$$\begin{aligned}\langle (52L_4 - 25L_1L_3)w \rangle &= -(2\pi i)^4 \frac{13}{300} E_4 \eta^{-2/5} \\ \langle (4L_1L_5 + 3L_3L_3 - \frac{684}{35}L_6)w \rangle &= -(2\pi i)^6 \frac{989}{176,400} E_6 \eta^{-2/5}.\end{aligned}$$

□

The $g = 2$ partition function is

$$Z^{g=2} = \sum_{a,b=1,2} |\langle 1 \rangle_{a,b}^{g=2}|^2 + \lambda |\langle 1 \rangle_\varphi^{g=2}|^2,$$

where $\lambda \in \mathbb{R}$ is such that $Z^{g=2}$ is modular.

2.2 Results in the ρ formalism

We consider a torus with modulus τ and two punctures separated from one another by a pair of disjoint neighbourhoods with local coordinates z_1 and z_2 respectively, which vanish at the respective puncture. The torus is self-sewed by imposing the condition

$$z_1 z_2 = \rho$$

for some $\rho > 0$. For any of the two local coordinates, we define $\tilde{z} = z/\rho$. This gives rise to local representatives of a field ψ denoted by $\tilde{\psi}(\tilde{z})$. Let $\{\psi_i\}_{i \geq 0}$ be an orthonormal basis of F_V with the Shapovalov metric, where $\psi_0 = 1$ and $L_0 \psi_i = h_i \psi_i$ for $i \geq 0$. For $k \geq 0$, we have

$$\frac{2}{c} \|\partial^k T\|^2 = \frac{k!(k+3)!}{3!} = 1, \frac{1}{4}, \frac{1}{40}, \frac{1}{720}, \frac{1}{20,160}, \dots$$

The choice of a Rogers Ramanujan function $\langle 1 \rangle_a^{g=1}$ on the torus with $a \in \{1, 2\}$ gives rise to a 0-point function

$$\langle 1 \rangle_a^{g=2} = \sum_{i \geq 0} \langle \psi_i(z_1) \tilde{\psi}_i(\tilde{z}_2) \rangle_a^{g=1}. \quad (14)$$

for genus $g = 2$. Since

$$\tilde{\partial}^k \tilde{T}(\tilde{z}) = \rho^{2+k} \partial^k T(z), \quad k \geq 0,$$

where $\partial = \partial/\partial_z$ and $\tilde{\partial} = \partial/\partial_{\tilde{z}}$, eq. (14) becomes an expansion in powers of ρ ,

$$\langle 1 \rangle_a^{g=2} = \sum_{i \geq 0} \rho^{h_i} \langle \psi_i(z_1) \psi_i(z_2) \rangle_a^{g=1}. \quad (15)$$

Note that upon setting three ramification points equal to 0, 1, ∞ , respectively, for every choice of a , either side of the equation depends on three parameters. On the l.h.s. we have the remaining three ramification points for genus $g = 2$. On the r.h.s., we are free to choose the difference $z_1 - z_2$, the perturbation parameter ρ and the modulus τ (or the remaining ramification point) of the torus.

For the vacuum sector, the first non-trivial term in the series of eq. (15) occurs for weight $h = 6$.

Propos. 6. *We have*

$$\langle (7L_4 L_2 - 2L_6)(z_1) (7L_4 L_2 - 2L_6)(z_2) \rangle = 7 \{12g(z_{12})\langle T \rangle + f(z_{12})c\langle 1 \rangle\},$$

where

$$\begin{aligned} f(z_{12}) = & -151,831 \wp_{12}^6 + \frac{303,662}{5} \pi^4 E_4 \wp_{12}^4 + \frac{1,813,300}{189} \pi^6 E_6 \wp_{12}^3 - \frac{71,057}{15} \pi^8 E_4^2 \wp_{12}^2 \\ & - \frac{1,046,828}{945} \pi^{10} E_4 E_6 \wp_{12} - \frac{5,768}{135} \pi^{12} E_6^2 + \frac{1,706}{125} \pi^{12} E_4^3, \end{aligned}$$

and

$$g(z_{12}) = 5,765 \wp_{12}^5 - \frac{88,643}{45} \pi^4 E_4 \wp_{12}^3 - \frac{294,326}{945} \pi^6 E_6 \wp_{12}^2 + \frac{4,192}{45} \pi^8 E_4^2 \wp_{12} + \frac{77,542}{4725} \pi^{10} E_4 E_6.$$

Proof. Using the contour integral method and sorting out the coefficient of $z^4 w^4$ in $\langle T(z+z_1)T(w+z_2) \rangle$ yields

$$\begin{aligned} \langle L_6(z_1)L_6(z_2) \rangle = & \left\{ \frac{400}{243} \pi^{12} E_6^2 - \frac{14}{27} \pi^{12} E_4^3 \right. \\ & + \frac{380}{9} \pi^{10} E_4 E_6 \wp_{12} + \frac{539}{3} \pi^8 E_4^2 \wp_{12}^2 - \frac{1,100}{3} \pi^6 E_6 \wp_{12}^3 - 2,310 \pi^4 E_4 \wp_{12}^4 + 5,775 \wp_{12}^6 \left. \right\} c\langle 1 \rangle \\ & + \left\{ \frac{88}{27} \pi^{10} E_4 E_6 + \frac{56}{3} \pi^8 E_4^2 \wp_{12} - \frac{200}{3} \pi^6 E_6 \wp_{12}^3 - 420 \pi^4 E_4 \wp_{12}^3 + 1,260 \wp_{12}^5 \right\} \langle T \rangle. \end{aligned}$$

Sorting out the coefficient of $(z-z_1)^2$ in $\langle T(z)T(z_1)T^{(4)}(z_2) \rangle$ yields

$$\begin{aligned} \langle L_4 L_2(z_1) L_6(z_2) \rangle = & \left\{ \frac{896}{81} \pi^{12} E_6^2 - \frac{32}{9} \pi^{12} E_4^3 \right. \\ & + 288 \pi^{10} E_4 E_6 \wp_{12} + 1,232 \pi^8 E_4^2 \wp_{12}^2 - \frac{7,520}{3} \pi^6 E_6 \wp_{12}^3 - 15,840 \pi^4 E_4 \wp_{12}^4 + 39,600 \wp_{12}^6 \left. \right\} c\langle 1 \rangle \\ & + \left\{ -\frac{448}{9} \pi^{10} E_4 E_6 - 288 \pi^8 E_4^2 \wp_{12} + 960 \pi^6 E_6 \wp_{12}^2 + 6,144 \pi^4 E_4 \wp_{12}^3 - 18,144 \wp_{12}^5 \right\} \langle T \rangle. \end{aligned}$$

Sorting out the coefficient of $(z-z_1)^2(w-z_2)^2$ in $\langle T(z)T(z_1)T(w)T(z_2) \rangle$ yields

$$\begin{aligned} \langle L_4 L_2(z_1) L_4 L_2(z_2) \rangle = & \left\{ \frac{4,936}{59,535} \pi^{12} E_6^2 - \frac{134}{3,375} \pi^{12} E_4^3 \right. \\ & + \frac{388}{135} \pi^{10} E_4 E_6 \wp_{12} + \frac{19}{15} \pi^8 E_4^2 \wp_{12}^2 - \frac{860}{27} \pi^6 E_6 \wp_{12}^3 - \frac{934}{5} \pi^4 E_4 \wp_{12}^4 + 467 \wp_{12}^6 \left. \right\} c\langle 1 \rangle \\ & + \left\{ \frac{2,728}{4,725} \pi^{10} E_4 E_6 + \frac{32}{5} \pi^8 E_4^2 \wp_{12} + \frac{904}{45} \pi^6 E_6 \wp_{12}^2 + \frac{2,524}{15} \pi^4 E_4 \wp_{12}^3 - 588 \wp_{12}^5 \right\} \langle T \rangle. \end{aligned}$$

From this follows the claimed equation. \square

We list the first few terms in eq. (15).

k	coefficient of ρ^k/c
0	$c\langle 1 \rangle$
2	$4P_1\langle T \rangle + (P_2 - \frac{1}{90}E_4\pi^4)c\langle 1 \rangle$
3	$-6P_2\langle T \rangle - 5P_3c\langle 1 \rangle$
4	$12P_3\langle T \rangle + 21P_4c\langle 1 \rangle$
5	$28P_4\langle T \rangle - 84P_5c\langle 1 \rangle$
6	$\frac{1}{217}\langle (7L_4L_2 - 2L_6)(z_1)(7L_4L_2 - 2L_6)(z_2) \rangle + 72P_5\langle T \rangle + 330P_6c\langle 1 \rangle$

where $\langle 1 \rangle = \langle 1 \rangle_a^{g=1}$, $a \in \{1, 2\}$, and $\langle T \rangle$ is given by eq. (8). The P_i are polynomials in $\wp = \wp_{12}$ defined by

$$\begin{aligned} P_1 &= \wp \\ P_2 &= \wp^2 - \frac{1}{9}E_4\pi^4 \\ P_3 &= \wp^3 - \frac{1}{5}E_4\pi^4\wp - \frac{4}{135}E_6\pi^6 \\ P_4 &= \wp^4 - \frac{4}{15}E_4\pi^4\wp^2 - \frac{8}{189}E_6\pi^6\wp + \frac{1}{135}E_4^2\pi^8 \\ P_5 &= \wp^5 - \frac{1}{3}E_4\pi^4\wp^3 - \frac{10}{189}E_6\pi^6\wp^2 + \frac{2}{135}E_4^2\pi^8\wp + \frac{22}{8,505}E_4E_6\pi^{10} \\ P_6 &= \wp^6 - \frac{2}{5}E_4\pi^4\wp^4 - \frac{4}{63}E_6\pi^6\wp^3 + \frac{11}{495}E_4^2\pi^8\wp^2 + \frac{76}{10,395}E_4E_6\pi^{10}\wp \\ &\quad - \frac{2}{22,275}E_4^3\pi^{12} + \frac{16}{56,133}E_6^2\pi^{12}. \end{aligned}$$

For $N \geq 2$, N -point functions involving $\varphi(z)$ can be properly defined for $\Phi(z, \bar{z})$ only. On the torus, they may fail to be elliptic in z . In order to deal with this problem, we assume that z takes on a fixed value, or varies little about a fixed value. We show below that $\langle \varphi(z)\varphi(0) \rangle$ satisfies a third order ODE in z , so $\Phi(z, \bar{z})$ defines a 3-dimensional representation of the lattice translation group. In order to continue eq. (15) to $a = 3, 4, 5$, we must assume that $\langle \varphi(z_1)\varphi(z_2) \rangle_a$ is translationally invariant. In particular, $\langle \varphi(z)\varphi(0) \rangle_a$ is an even function of z .

Propos. 7. *Let $z_0 = 0$ and let z_2 be fixed. We have*

$$\begin{aligned} \langle T(z_1)\varphi(z_2)\varphi(0) \rangle^{g=1} &= h\{\wp_{12} + \wp_{10} - \wp_{20}\}\langle \varphi(z_2)\varphi(0) \rangle \\ &+ \{\zeta_{01} + \zeta_{12} + \zeta_{20}\}\langle \varphi'(z_2)\varphi(0) \rangle + \frac{5}{2}\langle \varphi''(z_2)\varphi(0) \rangle. \end{aligned}$$

Proof. By the OPE of $T(u) \otimes \varphi(z)$ and $T(u) \otimes \varphi(0)$, respectively,

$$\begin{aligned} \langle T(u)\varphi(z)\varphi(0) \rangle &= h\wp(u-z)\langle \varphi(z)\varphi(0) \rangle + \zeta(u-z)\langle \varphi'(z)\varphi(0) \rangle + \text{regular for } u \rightarrow z \\ &= h\wp(u)\langle \varphi(z)\varphi(0) \rangle + \zeta(u)\langle \varphi(z)\varphi'(0) \rangle + \text{regular for } u \rightarrow 0 \end{aligned}$$

The Weierstrass zeta function fails to be periodic w.r.t. the torus periods ω_1, ω_2 but satisfies

$$\zeta(z + m\omega_1 + n\omega_2) - \zeta(z) = 2m\zeta\left(\frac{\omega_1}{2}\right) + 2n\zeta\left(\frac{\omega_2}{2}\right), \quad m, n \in \mathbb{Z}.$$

Thus the difference $\zeta(u-z) - \zeta(u)$ defines an elliptic function of u , while the sum does not. It follows that we necessarily have¹

$$\langle \varphi(z)\varphi'(0) \rangle = -\langle \varphi'(z)\varphi(0) \rangle.$$

So

$$\begin{aligned} \langle T(u)\varphi(z)\varphi(0) \rangle &= h\{\wp(z-u) + \wp(u)\}\langle \varphi(z)\varphi(0) \rangle + \{\zeta(u-z) - \zeta(u)\}\langle \varphi'(z)\varphi(0) \rangle \quad (16) \\ &+ \text{terms that are regular in } u. \end{aligned}$$

In order for $\langle T(u)\varphi(z)\varphi(0) \rangle$ to be elliptic in u , the terms regular in u must actually be constant. Comparison of the u^0 terms in line (16) with the OPE (7) for $T(u) \otimes \varphi(0)$ shows that the terms constant in u are equal to

$$\frac{5}{2}\langle \varphi(z)\varphi''(0) \rangle - h\wp(z)\langle \varphi(z)\varphi(0) \rangle + \zeta(z)\langle \varphi'(z)\varphi(0) \rangle$$

(ζ is an odd function). Comparison with the terms constant in u which are obtained from the OPE for $T(u) \otimes \varphi(z)$ shows that

$$\langle \varphi''(z)\varphi(0) \rangle = \langle \varphi(z)\varphi''(0) \rangle.$$

□

Corollary 8. *Let $z_0 = 0$ and $z_1 = z$. The two-point function of φ satisfies the ODE*

$$\frac{25}{12} \frac{d^3}{dz^3} \langle \varphi(z)\varphi(0) \rangle^{g=1} = h\wp'_{10}\langle \varphi(z)\varphi(0) \rangle + \wp_{10}\langle \varphi'(z)\varphi(0) \rangle, \quad (17)$$

where $h = -1/5$.

¹Alternatively, this follows from the assumption that $\langle \varphi(z)\varphi(0) \rangle$ is translationally invariant.

Proof. This follows from comparing the terms in line (16) which are linear in u with the OPE (7) for $T(u) \otimes \varphi(0)$, using that $\zeta'(z) = -\varphi(z)$ for $z \in \mathbb{C}$, and the fact that $\langle \varphi^{(3)}(z)\varphi(0) \rangle = -\langle \varphi^{(3)}(0)\varphi(z) \rangle$. \square

Using that for $k \geq 0$,

$$\frac{\|\partial^k \varphi\|^2}{\|\varphi\|^2} = k! \prod_{n=0}^{k-1} \left(k - n - \frac{7}{5}\right) \in \left\{1, -\frac{5}{2}, -\frac{25}{12}, -\frac{125}{288}, \dots\right\},$$

solving eq. (17) will allow to compute the coefficients of $\rho^{k-1/5} / \|\varphi\|^2$ in eq. (15). For example, $\langle L_4 \varphi(z_1) L_1 L_3 \varphi(z_2) \rangle$ sorts out the coefficient proportional to $(z - z_1)^2 (u - z_2)^{-1} (v - z_2)$ in $\langle T(z) T(u) T(v) \varphi(z_1) \varphi(z_2) \rangle$.

2.3 Outlook

Using the Frobenius Ansatz $\langle \varphi(z)\varphi(0) \rangle \sim z^\alpha$, the differential equation (17) imposes the condition

$$\frac{25}{12} \alpha(\alpha - 1)(\alpha - 2) = \frac{2}{5} + \alpha.$$

on α , which produces the values $1/5, 2/5$ and $12/5$. The obvious solutions to the ODE are, to leading order,

$$z^{1/5} \langle \varphi \rangle, \quad z^{2/5} \langle 1 \rangle,$$

but the third exponent remains to be understood.

Remark 9. *Solving the ODE (17) is equivalent to solving the ODE*

$$y^{4/5} \left(p(x) \frac{d^3}{dx^3} + f(x) \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x) \right) \Psi(x) = 0,$$

where

$$p(x) = 4 \left(x^3 - \frac{\pi^4}{3} E_4 x - \frac{2}{27} \pi^6 E_6 \right),$$

and

$$\begin{aligned} f &= \frac{6}{5} p' \\ g &= \frac{3}{100} \frac{[p']^2}{p} + \frac{9}{50} p'' \\ h &= -\frac{33}{500} \frac{[p']^3}{p^2} + \frac{33}{250} \frac{p' p''}{p} - \frac{288}{125}. \end{aligned}$$

In particular, the ODE has simple poles at the four ramification points.

Proof of the Remark. We change to the algebraic coordinates $x = \varphi(z)$ and $y = \varphi'(z)$ with $y^2 = p(x)$. Let $\check{\varphi}(x)$ be the local representative of φ and let $\Psi(x) = \langle \check{\varphi}(x)\varphi(0) \rangle$. By the ODE (17), $\langle \varphi(z)\varphi(0) \rangle = y^{-1/5} \Psi(x)$ lies in the kernel of the operator

$$L = y \left(p \frac{d^3}{dx^3} + \frac{3}{2} p' \frac{d^2}{dx^2} + \frac{12}{25} p'' \frac{d}{dx} + \frac{12}{125} \right),$$

since $\frac{d}{dz} = \wp' \frac{d}{d\wp}$. Moreover, $y^{-1/5}$ lies in the kernel of the three operators

$$\frac{d}{dx} + \frac{1}{10} \frac{p'}{p}, \quad \frac{d^2}{dx^2} - \frac{1}{10} \left(\frac{11}{10} \left[\frac{p'}{p} \right]^2 - \frac{p''}{p} \right), \quad \frac{d^3}{dx^3} + \frac{1}{10} \left(\frac{231}{100} \left[\frac{p'}{p} \right]^3 - \frac{33}{10} \frac{p' p''}{p^2} + \frac{p'''}{p} \right),$$

respectively. So $L(y^{-1/5}\Psi(x)) = y^{-1/5}(L - \frac{3}{10}L_1)\Psi(x)$, where

$$L_1 = y \left(p' \frac{d^2}{dx^2} + \left(p'' - \frac{1}{10} \frac{[p']^2}{p} \right) \frac{d}{dx} + \frac{1}{3} \frac{p'''}{p} + \frac{11}{50} \frac{[p']^3}{p^2} - \frac{11}{25} \frac{p' p''}{p} \right).$$

□

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