An algebraic approach to minimal models in CFTs

Marianne Leitner*
Dublin Institute for Advanced Studies,
School of Theoretical Physics,
10 Burlington Road, Dublin 4, Ireland
*leitner@stp.dias.ie

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Abstract

CFTs are naturally defined on Riemann surfaces. The rational ones can be solved using methods from algebraic geometry. One particular feature is the covariance of the partition function under the mapping class group. In genus $g = 1$, one can apply the standard theory of modular forms, which can be linked to ordinary differential equations of hypergeometric type.

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1 Introduction

This is the second in a sequence of three foundational papers on a mathematical approach to Conformal Field Theory (CFT) on compact Riemann surfaces, and it covers the second part of the author's PhD thesis in Mathematics [11]. In the first part of the thesis, a working definition of rational CFTs on general Riemann surfaces has been given. For the $(2,5)$ minimal model over compact Riemann surfaces, explicit formulae for computing N-point functions $\langle\phi_1\ldots\phi_N\rangle$ of holomorphic fields have been established for small positive values of N. N-point functions for higher N are obtained by recursion. For $N=0$, one has the identity field $\mathbf{1}$ and the partition function $\langle\mathbf{1}\rangle$ whose computation requires different methods. There is no dependence on position, but it depends on the conformal structure of the surface. Indeed, it satisfies a system of differential equations w.r.t. the moduli of the Riemann surface. For the minimal models, the vector space of solutions is finite dimensional.

The present paper is devoted to compact Riemann surfaces of genus $g=1$. Such surface can be described as a quotient $\mathbb{C}/\Lambda$, with a lattice Λ generated over $\mathbb{Z}$ by 1 and τ with $\tau\in\mathbb{H}^+$, the upper half plane. The latter is the universal cover of the moduli space $\mathcal{M}_1$ of all possible conformal structures on the $g=1$ surface, which is known as the Teichmüller space. One has $\mathcal{M}_1 = SL(2,\mathbb{Z})\setminus\mathbb{H}^+$. Meromorphic functions on finite covers of $\mathcal{M}_1$ are called (weakly) modular. They can be described as functions on $\mathbb{H}^+$ which are invariant under a subgroup of $SL(2,\mathbb{Z})$ of finite index.

Maps in the full modular group $SL(2,\mathbb{Z})$ preserve the standard lattice $\mathbb{Z}^2$ together with its orientation and so descend to self-homeomorphisms of the torus. Inversely, every self-homeomorphism of the torus is isotopic to such a map. A modular function is a function on the space $\mathcal{L}$ of all lattices in $\mathbb{C}$ satisfying [17]

$$f(\lambda\Lambda) = f(\Lambda)\,,\quad \forall\Lambda\in\mathcal{L},\ \lambda\in\mathbb{C}^*\,.$$

$\mathcal{L}$ can be viewed as the space of all tori with a flat metric.

Conformal field theories on the torus provide many interesting modular functions, and modular forms. (The latter transform as $f(\lambda\Lambda)=\lambda^{-k}f(\Lambda)$ for some $k\in\mathbb{Z}$ which is specific to f, called the weight of f.)

For the $(2,5)$ minimal model, we shall derive the second order ordinary differential equation for the $g=1$ partition function that allows to compute all N-point functions of holomorphic fields. It is shown that our approach reproduces the known result.

Much of the mathematical foundations of rational CFT will be provided by the joint paper with W. Nahm, whose main feature are the ODEs for the higher genus partition functions.

2 Notations and conventions

Let $\mathbb{H}^+ := \{z\in\mathbb{C}|\ \Im(z)>0\}$ be the complex upper half plane. $\mathbb{H}^+$ is acted upon by the *full modular group* $\Gamma_1 = SL(2,\mathbb{Z})$ with fundamental domain

$$\mathcal{F} := \left\{z\in\mathbb{H}^+\middle|\ |z|>1,\ |\Re(z)|<\frac{1}{2}\right\}\,.$$

The operation of Γ_1 on $\mathbb{H}^+$ is not faithful whence we shall also consider the *modular group* $\overline{\Gamma}_1 := \Gamma_1/\{\pm\boldsymbol{I}_2\} = PSL(2,\mathbb{Z})$, (here $\boldsymbol{I}_2\in GL(2,\mathbb{Z})$ is the identity matrix). We refer to S, T as the generators of Γ_1 (or of $\overline{\Gamma}_1$) given by the transformations

$$\begin{aligned} S &: z \mapsto -1/z \\ T &: z \mapsto z + 1 \,. \end{aligned}$$

We shall use the convention [17]

$$G_{2k}(z) = \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^{2k}} + \frac{1}{2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^{2k}} \,,$$

and define E_{2k} by $G_k(z) = \zeta(k)E_k(z)$ for $\zeta(k) = \sum_{n\geq 1}\frac{1}{n^k}$, so e.g.

$$\begin{aligned} G_2(z) &= \frac{\pi^2}{6}E_2(z) \,, \\ G_4(z) &= \frac{\pi^4}{90}E_4(z) \,, \\ G_6(z) &= \frac{\pi^6}{945}E_6(z) \,. \end{aligned}$$

Let $(q)_n := \prod_{k=1}^{n}(1-q^k)$ be the q-Pochhammer symbol. The *Dedekind η function* is

$$\eta(z) := q^{\frac{1}{24}}(q)_\infty = q^{\frac{1}{24}}\left(1 - q + q^2 + q^5 + q^7 + \ldots\right) \,, \quad q = e^{2\pi i\, z} \,.$$

$\langle 1\rangle$, $\langle T\rangle$ (or A_1) are parameters of central importance to this exposition. For better readibility, they appear in bold print ($\langle\mathbf{1}\rangle$ and $\langle\mathbf{T}\rangle$, or $\mathbf{A}_1$) throughout.

The central charge is a number $c \in \mathbb{R}$. In the $(2,5)$ minimal model, $c = -\frac{22}{5}$.

3 Introduction to modular dependence

Let

$$\Sigma_1 := \{z \in \mathbb{C} |\, |q| \leq z \leq 1\}/\{z \sim qz\} \,,$$

where $q = e^{2\pi i\,\tau}$ and $\tau \in \mathbb{H}^+$. Σ_1 is a torus. A *character* on Σ_1 is given by

$$\langle\mathbf{1}\rangle_{\Sigma_1} = \sum_{\substack{\varphi_j \\ \{\varphi_j\}_j \text{ basis of } F}} q^{h(\varphi_j)} \,.$$

Here F is the fiber of the bundle of holomorphic fields $\mathcal{F}$ in a rational CFT on Σ_1, as discussed in Part I of the thesis. By the fact that Part I lists necessary conditions for a CFT on a hyperelliptic Riemann surface, $\langle\mathbf{1}\rangle_{\Sigma_1}$ is in particular a 0-point function $\langle\mathbf{1}\rangle$ in the sense of Part I. On the other hand, $\langle\mathbf{1}\rangle_{\Sigma_1}$ is known to be a modular function of τ ([14], [19]). A *modular function on* a discrete subgroup Γ of $\Gamma_1 = SL(2,\mathbb{Z})$ is a Γ-invariant meromorphic function $f : \mathbb{H}^+ \to \mathbb{C}$ with at most exponential growth towards the boundary [17]. For $N \geq 1$, the *principal conguence subgroup* is the group $\Gamma(N)$ such that the short sequence

$$1 \;\to\; \Gamma(N) \hookrightarrow \Gamma_1 \xrightarrow{\pi_N} SL(2,\mathbb{Z}/N\mathbb{Z}) \;\to\; 1$$

is exact, where π_N is map given by reduction modulo N. A function that is modular on $\Gamma(N)$ is said to be *of level* N. Let $\zeta_N = e^{\frac{2\pi i}{N}}$ be the N-th root of unity with cyclotomic field $\mathbb{Q}(\zeta_N)$. Let F_N be the field of modular functions f of level N which have a Fourier expansion

$$f(\tau) = \sum_{n \geq -n_0} a_n q^{\frac{n}{N}} , \quad q = e^{2\pi i \tau} , \tag{1}$$

with $a_n \in \mathbb{Q}(\zeta_N)$, $\forall n$. The Ramanujan continued fraction

$$r(\tau) := q^{1/5} \frac{1}{1 + \frac{q}{1+\frac{q^2}{1+\dots}}} \tag{2}$$

which converges for $\tau \in \mathbb{H}^+$, is an element (and actually a generator) of F_5 [18]. r is algebraic over F_1 which is generated over $\mathbb{Q}$ by the *modular j-function*,

$$j(\tau) = 12^3 \frac{g_2^3}{g_2^3 - 27 g_3^2} .$$

j is associated to the elliptic curve with the affine equation

$$\Sigma_1 : \quad y^2 = 4x^3 - g_2 x - g_3 , \quad \text{with} \quad g_2^3 - 27 g_3^2 \neq 0 .$$

Here g_k for $k = 2, 3$ are (specific) modular forms of weight $2k$,[1] so that j is indeed a function of the respective modulus only (the quotient $\tau = \omega_2/\omega_1$ for the lattice $\Lambda = \mathbb{Z}.\omega_1 + \mathbb{Z}.\omega_2$), or rather its orbit under Γ_1 (since we are free to change the basis (ω_1, ω_2) for Λ). In terms of the modulus, a *modular form of weight $2k$ on Γ* is a holomorphic function $g : \mathbb{H}^+ \to \mathbb{C}$ with subexponential growth towards the boundary [17] such that $g(\tau)\,(d\tau)^{2k}$ is Γ-invariant [15]. A modular form on Γ_1 allows a Fourier expansion of the form (1) with $n_0 \geq 0$.

Another way to approach modular functions is in terms of the differential equations they satisfy. The derivative of a modular function is a modular form of weight two, and higher derivatives give rise to *quasi-modular forms*, which we shall also deal with though they are not themselves of primary interest to us.

Geometrically, the conformal structure on the surface

$$\Sigma_1 : \quad y^2 = 4(x - X_1)(x - X_2)(x - X_3) , \quad x \in \mathbb{P}^1_{\mathbb{C}} , \tag{3}$$

is determined by the quadrupel (X_1, X_2, X_3, ∞) of its ramification points, and we can change this structure by varying the position of X_1, X_2, X_3 infinitesimally. In this picture, the boundary of the moduli space is approached by letting two ramification points in the quadrupel run together [6].

When changing positions we may keep track of the branch points to obtain a simply connected space [4]. Thus a third way to describe modularity of the characters is by means of a subgroup of the braid group B_3 of 3 strands. The latter is the universal central extension of the quotient group $\overline{\Gamma}_1 = \Gamma_1/\{\pm \boldsymbol{I}_2\}$, so that we come full circle.

Suppose $\Sigma_1 = \mathbb{C}/\Lambda$ where $\Lambda = (\mathbb{Z}.1 + \mathbb{Z}.i\beta)$ with $\beta \in \mathbb{R}$. Thus the fundamental domain is a rectangle in the (x^0, x^1) plane with length $\Delta x^0 = 1$ and width $\Delta x^1 = \beta$. The dependence of $\langle \mathbf{1} \rangle_{\Sigma_1}$ on the modulus $i\beta$ follows from the identity

$$\langle \mathbf{1} \rangle_{\Sigma_1} = \operatorname{tr} e^{-H\beta} , \quad H = \int T^{00} dx^0,$$

[1] As mentioned earlier, a modular form of weight $2k$ transforms as $f(\lambda\Lambda) = \lambda^{-2k} f(\Lambda)$ for any $\lambda \in \mathbb{C}^*$.

where T^{00} is a real component of the Virasoro field.[2] As mentioned above, we may regard $\langle\mathbf{1}\rangle_{\Sigma_1}$ as the 0-point function $\langle\mathbf{1}\rangle$ w.r.t. a state $\langle\ \rangle$ on Σ_1. Note that the same argument applies to N-point functions for $N > 0$.

Stretching $\beta \mapsto (1+\epsilon)\beta$ changes the Euclidean metric $G_{\mu\nu}$ ($\mu, \nu = 0, 1$) according to

$$(ds)^2 \ \mapsto\ (ds)^2 + 2\epsilon (dx^1)^2 + O(\epsilon^2)\,.$$

Thus $dG_{11} = 2\frac{d\beta}{\beta}$, and

$$\begin{aligned} d\langle\mathbf{1}\rangle = -\mathrm{tr}(H d\beta\, e^{-H\beta}) &= -\frac{dG_{11}}{2}\left(\int \langle T^{00}\rangle dx^0\right)\beta \\ &= -\frac{dG_{11}}{2}\iint \langle T^{00}\rangle dx^0 dx^1\,. \end{aligned} \tag{4}$$

The fact that $\int\langle T^{00}\rangle dx^0$ does not depend on x^1 follows from the conservation law $\partial_\mu T^{\mu\nu} = 0$:

$$\frac{d}{dx^1}\oint \langle T^{00}\rangle\, dx^0 = \oint \partial_1\langle T^{00}\rangle\, dx^0 = -\oint \partial_0 \langle T^{10}\rangle\, dx^0 = 0\,,$$

using Stokes' Theorem.

We argue that on $S^1 \times S^1_{\beta/(2\pi)}$ (where $S^1_{\beta/(2\pi)}$ is the circle of perimeter β), states (in the sense of [10]) are thermal states on the VOA.

When $g > 1$, equation (4) generalises to

$$d\langle\mathbf{1}\rangle = -\frac{1}{2}\iint dG_{\mu\nu}\,\langle T^{\mu\nu}\rangle\ \sqrt{G}\, dx^0 \wedge dx^1\,.$$

Here $G := |\det G_{\mu\nu}|$, and $dvol_2 = \sqrt{G}\, dx^0 \wedge dx^1$ is the volume form which is invariant under base change.[3] The normalisation is in agreement with eq. (4) (see also [2], eq. (5.140) on p. 139).

Methods that make use of the flat metric do not carry over to surfaces of higher genus. We may choose a specific metric of prescribed constant curvature to obtain mathematically correct but cumbersome formulae. Alternatively, we consider quotients of N-point functions over $\langle\mathbf{1}\rangle$ only (as done in [5]) so that the dependence on the specific metric drops out. Yet we suggest to use a singular metric that is adapted to the specific problem [12]. On Σ_1, this metric is the lift of a polyhedral metric on $\mathbb{P}^1_{\mathbb{C}}$ which equals

$$|dz|^2 \quad \text{on } \mathbb{P}^1_{\mathbb{C}} \setminus \{X_1, X_2, X_3\}\,,$$

and has all curvature concentrated in the ramification points. The 0-point function on this metric surface is obtained through a regularisation procedure and will be denoted $\langle\mathbf{1}\rangle_{\text{sing.}}$ to distinguish it from the 0-point function on the flat torus $(\Sigma_1, |dz|^2)$, which we denote by $\langle\mathbf{1}\rangle_{\text{flat}}$.

[2] Any dynamical quantum field theory has an energy-momentum tensor $T_{\mu\nu}$ s.t. $T_{\mu\nu}dx^\mu dx^\nu$ defines a quadratic differential, by which we mean in particular that it transforms homogeneously under coordinate changes. For coordinates $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$, we have [1]

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11})\,.$$

For a discussion of the relation with the Virasoro field $T(z)$ addressed below, cf. [12].

[3] The change to complex coordinates is a more intricate, however: We have $dx^0 \wedge dx^1 = iG_{z\bar{z}}\, dz \wedge d\bar{z}$ with $G_{z\bar{z}} = \frac{1}{2}$, as can be seen by setting $z = x^0 + ix^1$.

Theorem 1. *Let Σ_1 be defined by eq. (3). We equip Σ_1 with the metric which is the lift of the polyhedral metric on $\mathbb{P}^1_{\mathbb{C}}$. Let $\langle\ \rangle_{sing}$ be a corresponding state on Σ_1. Define a deformation of the conformal structure by*

$$\xi_j = dX_j \quad \textit{for} \quad j = 1, 2, 3 .$$

Let $\varphi, \ldots$ be holomorphic fields on Σ_1. For $j = 1, 2, 3$, let (U_j, z) be a chart on Σ_1 containing the point X_j but no position of one of $\varphi, \ldots$. We have

$$d\langle\varphi \ldots\rangle_{sing} = \sum_{j=1}^{n} \left(\frac{1}{2\pi i} \oint_{\gamma_j} \langle T(z)\varphi \ldots\rangle_{sing}\, dz \right) \xi_j ,$$

where γ_j is a closed path around X_j contained in U_j.

4 Differential equations for characters in $(2, \nu)$-minimal models

4.1 Review of the differential equation for the characters of the $(2, 5)$ minimal model

The character $\langle \mathbf{1} \rangle$ of any CFT on the torus Σ_1 solves the ODE [5]

$$\frac{d}{d\tau}\langle \mathbf{1} \rangle = \frac{1}{2\pi i} \oint \langle T(z) \rangle\, dz = \frac{1}{2\pi i}\langle \mathbf{T} \rangle . \tag{5}$$

Here the contour integral is along the real period, and $\oint dz = 1$. $\langle \mathbf{T} \rangle$, while constant in position, is a modular form of weight two in the modulus. The Virasoro field generates the variation of the conformal structure [5]. In the $(2, 5)$ minimal model, we find

$$2\pi i \frac{d}{d\tau}\langle \mathbf{T} \rangle = \oint \langle T(w)T(z) \rangle\, dz = -4\langle \mathbf{T} \rangle G_2 + \frac{22}{5} G_4 \langle \mathbf{1} \rangle . \tag{6}$$

Here G_2 is the quasimodular Eisenstein series of weight 2, which enters the equation by means of the identity

$$\int_0^1 \wp(z - w|\tau)\, dz = -2G_2(\tau).$$

In terms of the *Serre derivative*

$$\mathfrak{D}_{2\ell} := \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{\ell}{6} E_2 , \tag{7}$$

the first order ODEs (5) and (6) combine to give the second order ODE [13, 9]

$$\mathfrak{D}_2 \circ \mathfrak{D}_0 \langle \mathbf{1} \rangle = \frac{11}{3600} E_4 \langle \mathbf{1} \rangle .$$

The two solutions are the well-known Rogers-Ramanujan partition functions [2]

$$\langle \mathbf{1} \rangle_1 = q^{\frac{11}{60}} \sum_{n \geq 0} \frac{q^{n^2+n}}{(q)_n} = q^{\frac{11}{60}} \left(1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + \ldots \right) ,$$

$$\langle \mathbf{1} \rangle_2 = q^{-\frac{1}{60}} \sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = q^{-\frac{1}{60}} \left(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \ldots \right) .$$

($q = e^{2\pi i\,\tau}$) which are named after the famous Rogers-Ramanujan identities

$$q^{-\frac{11}{60}}\langle\mathbf{1}\rangle_1 = \prod_{n=\pm 2 \bmod 5}(1-q^n)^{-1}\,, \quad q^{\frac{1}{60}}\langle\mathbf{1}\rangle_2 = \prod_{n=\pm 1 \bmod 5}(1-q^n)^{-1}\,.$$

Mnemotechnically, the distribution of indices seems somewhat unfortunate. In general, however, the characters of the $(2,\nu)$ minimal model, of which there are

$$M = \frac{\nu-1}{2} \tag{8}$$

(ν odd) many, are ordered by their conformal weight, which is the lowest for the respective *vacuum character* $\langle\mathbf{1}\rangle_1$, having weight zero.

The Rogers-Ramanujan identity for $q^{-\frac{11}{60}}\langle\mathbf{1}\rangle_1$ provides the generating function for the partition which to a given holomorphic dimension $h \geq 0$ returns the number of linearly independent holomorphic fields present in the $(2,5)$ minimal model. This number is subject to the constraint $\partial^2 T \propto N_0(T,T)$

h	0	1	2	3	4	5	6
basis of $F(h)$	1	–	T	∂T	$\partial^2 T$	$\partial^3 T$	$\partial^4 T$ $N_0(T,\partial^2 T)$
$\dim F(h)$	1	0	1	1	1	1	2

Holomorphic fields of dimension h in the $(2,5)$ minimal model

There is a similar combinatorical interpretation for the second Rogers-Ramanujan identity. It involves non-holomorphic fields, however, which we disregard in this paper.

4.2 Review the algebraic equation for the characters of the $(2,5)$ minimal model

Besides the analytic approach, there is an algebraic approach to the characters. This is due to the fact that $\langle\mathbf{1}\rangle_1, \langle\mathbf{1}\rangle_2$, rather than being modular on the full modular group, are modular on a subgroup of Γ_1: For the generators S, T of Γ_1 we have [18]

$$T\langle\mathbf{1}\rangle_1 = {\zeta_{60}}^{11}\langle\mathbf{1}\rangle_1\,, \quad T\langle\mathbf{1}\rangle_2 = {\zeta_{60}}^{-1}\langle\mathbf{1}\rangle_2\,,$$

while under the operation of S, $\langle\mathbf{1}\rangle_1, \langle\mathbf{1}\rangle_2$ transform into linear combinations of one another [18],

$$S\begin{pmatrix}\langle\mathbf{1}\rangle_1\\ \langle\mathbf{1}\rangle_2\end{pmatrix} = \frac{2}{\sqrt{5}}\begin{pmatrix}\sin\frac{\pi}{5} & -\sin\frac{2\pi}{5}\\ \sin\frac{2\pi}{5} & \sin\frac{\pi}{5}\end{pmatrix}\begin{pmatrix}\langle\mathbf{1}\rangle_1\\ \langle\mathbf{1}\rangle_2\end{pmatrix}.$$

However, $\langle\mathbf{1}\rangle_1, \langle\mathbf{1}\rangle_2$ are modular under a subgroup of Γ_1 of finite index. Its fundamental domain is therefore a finite union of copies of the fundamental domain $\mathcal{F}$ of Γ_1 in $\mathbb{C}$. More specifically, if the subgroup is Γ with index $[\Gamma_1 : \Gamma]$, and if $\gamma_1, \ldots, \gamma_{[\Gamma_1:\Gamma]} \in \Gamma_1$ are the coset representatives so that $\Gamma_1 = \Gamma\gamma_1 \cup \ldots \cup \Gamma\gamma_{[\Gamma_1:\Gamma]}$, then we have

$$\mathcal{F}_\Gamma = \gamma_1\mathcal{F} \cup \ldots \cup \gamma_{[\Gamma_1:\Gamma]}\mathcal{F}\,,$$

[7]. Thus $\langle\mathbf{1}\rangle_1$ and $\langle\mathbf{1}\rangle_2$ define meromorphic functions on a finite covering of the moduli space $\mathcal{M}_1 = \Gamma_1 \setminus \mathbb{H}^+$ and are algebraic. We can write [18]

$$\langle\mathbf{1}\rangle_1 = \frac{\theta_{5,2}}{\eta}\,, \quad \langle\mathbf{1}\rangle_2 = \frac{\theta_{5,1}}{\eta}\,,$$

where the functions $\eta, \theta_{5,1}, \theta_{5,2}$ on the r.h.s. are specific theta functions (e.g. [2])

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} f(n) , \quad f(n) \sim q^{n^2} , \quad q = e^{2\pi i \tau} .$$

The characters' common denominator is the Dedekind η function. Using the Poisson transformation formula, one finds that $\eta, \theta_{5,1}, \theta_{5,2}$ are all modular forms of weight $\frac{1}{2}$ ([17], Propos. 9, p. 25). For the quotient $\langle\mathbf{1}\rangle_1/\langle\mathbf{1}\rangle_2$ and $\tau \in \mathbb{H}^+$, we find [18],

$$\frac{\langle\mathbf{1}\rangle_1}{\langle\mathbf{1}\rangle_2} = \frac{\theta_{5,2}}{\theta_{5,1}} = q^{\frac{1}{5}} \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{n}{5}\right)} = r(\tau) ,$$

where $r(\tau)$ is the Ramanujan continued fraction introduced in eq. (2). (Here $(n/5) = 1, -1, 0$ for $n = \pm 1, \pm 2, 0 \pmod 5$, respectively, is the Legendre symbol.)

$r(\tau)$ is modular on $\Gamma(5)$ with index $[\Gamma_1 : \Gamma(5)] = 120$ [8]. The quotient $\Gamma(5) \setminus \mathbb{H}^+$ can be compactified and made into a Riemann surface, which is referred to as the modular curve

$$\Sigma(5) = \Gamma(5) \setminus \mathbb{H}^* .$$

Here $\mathbb{H}^* := \mathbb{H}^+ \cup \mathbb{Q} \cup \{\infty\}$ is the extended complex upper half plane. $\Sigma(5)$ has genus zero and the symmetry of an icosahedron. The rotation group of the sphere leaving an inscribed icosahedron invariant is A_5, the alternating group of order 60. By means of a stereographic projection, the notion of edge center, face center and vertex are induced on the extended complex plane [3]. They are acted upon by the icosahedral group $G_{60} \subset PSL(2, \mathbb{C})$. The face centers and finite vertices define the simple roots of two monic polynomials $F(z)$ and $V(z)$ of degree 20 and 11, respectively, which transform in such a way under G_{60} that

$$J(z) := \frac{F^3(z)}{V^5(z)}$$

is invariant. It turns out that $J(r(\tau))$ for $\tau \in \mathbb{H}^+$ is $\Gamma(1)$-invariant, and in fact that $J(r(\tau)) = j(\tau)$. Thus $r(\tau)$ satisfies

$$F^3(z) - j(\tau)V^5(z) = 0$$

(for the same value of τ), which is equivalent to $r^5(\tau)$ solving the icosahedral equation

$$(X^4 - 228X^3 + 494X^2 + 228X + 1)^3 + j(\tau)X(X^2 + 11X - 1)^5 = 0 .$$

This is actually the minimal polynomial of r^5 over $\mathbb{Q}(j)$, so that $\mathbb{Q}(r)$ defines a function field extension of degree 60 over $\mathbb{Q}(j)$.

This construction which goes back to F. Klein, doesn't make use of a metric. In order to determine the centroid of a face (or of the image of its projection onto the sphere) only the conformal structure on S^2 is required. Indeed, the centroid of a regular polygone is its center of rotations, thus a fixed point under an operation of $Aut(S^2) = SL(2, \mathbb{C})$.

4.3 Modular ODEs for the characters in $(2, \nu)$ minimal models

Sorting out the algebraic equations to describe the characters of the $(2, \nu)$ minimal model becomes tedious for $\nu > 5$. In contrast, the Serre derivative is a managable tool for encoding them in a compact way [13]. Since the characters are algebraic, the corresponding differential equations can not be solved numerically only, but actually analytically. We are interested in the fact that the coefficient of the respective highest order derivative can be normalised to one and all other coefficients are holomorphic in the modulus.

To the $(2, \nu)$ minimal model, where $\nu \geq 3$ is odd, we associate [2]

- the number $M = \frac{\nu-1}{2}$ introduced in eq. (8), which counts the characters,
- the sequence

$$\kappa_s = \frac{(\nu - 2s)^2}{8\nu} - \frac{1}{24}\,, \quad s = 1, \ldots, M\,, \tag{9}$$

  which parametrises the characters of the $(2, \nu)$ minimal model,
- the rank $r = \frac{\nu-3}{2}$.

The character corresponding to κ_s is

$$\langle \mathbf{1} \rangle_s = f_{A,B,s} := q^{\kappa_s} \sum_{\mathbf{n} \in (\mathbb{N}_0)^r} \frac{q^{\mathbf{n}^t A \mathbf{n} + \mathbf{B}^t \mathbf{n}}}{(q)_{\mathbf{n}}}\,,$$

where

$$A = C(T_r)^{-1} \in \mathbb{Q}^{r \times r}, \quad \mathbf{B} \in \mathbb{Q}^r\,,$$

C being a Cartan matrix. The tadpole diagram of T_r is obtained from the diagram of A_{2r} by folding according to its $\mathbb{Z}_2$ symmetry.

It turns out that $\langle \mathbf{1} \rangle_s$ satisfies an Mth order ODE [13]. Given M differentiable functions $f_1, \ldots, f_M$ there always exists an ODE having these as solutions. Consider the Wronskian determinant

$$\det \begin{pmatrix} f & \mathfrak{D}^1 f & \ldots & \mathfrak{D}^M f \\ f_1 & \mathfrak{D}^1 f_1 & \ldots & \mathfrak{D}^M f_1 \\ \ldots & \ldots & \ldots & \ldots \\ f_M & \mathfrak{D}^1 f_M & \ldots & \mathfrak{D}^M f_M \end{pmatrix} =: \sum_{i=0}^{M} w_i\, \mathfrak{D}^i f\,.$$

Here for $m \geq 1$,

$$\mathfrak{D}^m := \mathfrak{D}_{2(m-1)} \circ \cdots \circ \mathfrak{D}_2 \circ \mathfrak{D}_0$$

is the order m differential operator which maps a modular function into a modular form of weight $2m$. ($\mathfrak{D}_k$ is the first order Serre differential operator introduced in eq. (7).) For $m = 0$ we set $\mathfrak{D}^0 = 1$.

Whenever f equals one of the f_i, $1 \leq i \leq M$, the determinant is zero, so we obtain an ODE in f whose coefficients are Wronskian minors containing $f_1, \ldots, f_M$ and their derivatives only. These are modular when the $f_1, \ldots, f_M$ and their derivatives are or when under modular transformation, they transform into linear combinations of one another (as the characters do).

Lemma 2. *Let* $3 \leq \nu \leq 13$*,* ν *odd. The characters of the* $(2, \nu)$ *minimal model satisfy*

$$D^{(2,\nu)} \langle \mathbf{1} \rangle = 0\,, \tag{10}$$

where $D^{(2,\nu)}$ is the differential operator

$$D^{(2,\nu)} := \mathfrak{D}^M + \sum_{m=0}^{M-2} \sum_{\Omega_{2(M-m)}} \Omega_{2(M-m)} \mathfrak{D}^m$$

$$\Omega_{2(M-m)} := \alpha_m E_{2(M-m)} , \qquad 2 \leq M - m \leq 5 ,$$

$$\Omega_{12} := \alpha_0 E_{12} + \alpha_0^{(cusp)} \Delta .$$

Here $\Delta = \eta^{24}$ is the modular discriminant function, E_{2k} is the holomorphic Eisenstein series of weight $2k$, and the nonzero numbers α_m and $\alpha_0^{(cusp)}$ are given by the table below:

$(2,\nu)$	$(2,3)$	$(2,5)$	$(2,7)$	$(2,9)$	$(2,11)$	$(2,13)$
M	1	2	3	4	5	6
κ_M	0	$-\frac{1}{60}$	$-\frac{1}{42}$	$-\frac{1}{36}$	$-\frac{1}{33}$	$-\frac{5}{156}$
α_M	1	1	1	1	1	1
α_{M-2}		$-\frac{11}{60^2}$	$-\frac{5\cdot7}{42^2}$	$-\frac{2\cdot3\cdot13}{36^2}$	$-\frac{11\cdot53}{2^2\cdot33^2}$	$-\frac{7\cdot13\cdot67}{156^2}$
α_{M-3}			$\frac{5\cdot17}{42^3}$	$\frac{2^3\cdot53}{36^3}$	$\frac{3\cdot5\cdot11\cdot59}{2^3\cdot33^3}$	$\frac{2^3\cdot13\cdot17\cdot193}{156^3}$
α_{M-4}				$-\frac{3\cdot11\cdot23}{36^4}$	$-\frac{11\cdot6151}{2^4\cdot33^4}$	$-\frac{5\cdot11\cdot13\cdot89\cdot127}{156^4}$
α_{M-5}					$\frac{2^4\cdot17\cdot29}{33^5}$	$\frac{2^3\cdot3\cdot5\cdot13\cdot31\cdot2437}{156^5}$
α_{M-6}						$-\frac{5^4\cdot7^2\cdot23\cdot31\cdot67}{156^6}$
$\alpha_{M-6}^{(\mathrm{cusp})}$						$\frac{5^2\cdot7\cdot11\cdot23^2\cdot167}{2^5\cdot3^2\cdot13^4\cdot691}$

The nonzero coefficients in the order M differential operator in the $(2,\nu)$ minimal model. κ_M is displayed to explain the standard denominators of the α_m.

Remark 1. *1. In the $(2,\nu)$ minimal model, we have $\kappa_M = (3-\nu)/(24\nu)$, where $\nu|(3-\nu) \Leftrightarrow \nu = 3$. Thus for $\nu > 3$, κ_M has a factor of ν in the denominator.*

2. *The numerators $n_m^{(2,\nu)}$ of α_m in the $(2,\nu)$ minimal model have mostly few factors in the sense that*
$$n_m^{(2,\nu)} \approx rad(n_m^{(2,\nu)}) ,$$
where the r.h.s. is the radical of $n_m^{(2,\nu)}$.

3. *The prime 691 displayed in the denominator of $\alpha_{M-6}^{(cusp)}$ suggests that Bernoulli numbers are involved in the computations. This is an artefact of the choice of basis, however. Using the identity [17]*
$$E_{12} = \frac{1}{691}(441E_4^3 + 250E_6^2) ,$$
we can write
$$\Omega_{12} = -\frac{5^2 \cdot 7 \cdot 23}{2^7 \cdot 3^5 \cdot 13^6}\left(\frac{53 \cdot 1069}{2^5}E_4^3 + \frac{6047}{3}E_6^2\right) .$$

The leading coefficient can be read off from the equation for the singular vector (Lemma 4.3 in [16]) and only the specific value of the remaining coefficients in eq. (10) seem to be new. Rather than setting up a closed formula for α_m, we shall outline the algorithm to determine these numbers, and leave the actual computation as an easy numerical exercise.

Sketch of the Proof. For every κ_s in the list (9) and for $0 \le m \le M-1$, we have
$$\mathfrak{D}^m\langle\mathbf{1}\rangle_s \propto q^{\kappa_s}(1 + O(q)) ,$$
so $D^{(2,\nu)}\langle\mathbf{1}\rangle_s$ is a power series of order $\geq \kappa_s$ in q. The coefficient of q^{κ_s} is a monic degree M polynomial in κ_s, and we have
$$[D^{(2,\nu)}]_0 q^\kappa = q^\kappa \prod_{s=1}^{M}(\kappa - \kappa_s) , \tag{11}$$
since by assumption $\langle\mathbf{1}\rangle_{\kappa_s} \in \ker D^{(2,\nu)}$ for $s = 1, \dots M$. (Here $[D^{(2,\nu)}]_0$ denotes the cut-off of the differential operator $D^{(2,\nu)}$ at power zero in q.) For $2 \le k \le 5$, the space of modular forms of weight $2k$ is spanned by the Eisenstein series E_{2k}, while for $k = 6$, the space is two dimensional and spanned by E_{12} and Δ. However, only the Eisenstein series have a constant term, so that actually all coefficients α_m are determined by eq. (11). Note that vanishing of α_{M-1} (the coefficient of $\mathfrak{D}^{M-1}$ in $D^{(2,\nu)}$) implies the equality
$$-\sum_{s=1}^{M}\kappa_s = \sum_{\ell=1}^{M}\frac{1-\ell}{6} . \tag{12}$$
Indeed, the l.h.s. of eq. (12) equals the coefficient of κ^{M-1} in the polynomial
$$q^{-\kappa}[D^{(2,\nu)}]_0 q^\kappa$$
in eq. (11), while the r.h.s. equals the coefficient of κ^{M-1} in
$$q^{-\kappa}[\mathfrak{D}^M]_0 q^\kappa ,$$

where for $0 \leq i \leq M-1$,

$$q^{-\kappa}[\mathfrak{D}^{M-i}]_0 q^{\kappa} = \prod_{\ell=0}^{M-i-1}(\kappa - \frac{\ell}{6})\,.$$

Equality (12) thus states that $q^{-\kappa}[\mathfrak{D}^{M-1}]_0 q^{\kappa}$ (with leading term κ^{M-1}) does not contribute, and so is equivalent to $\alpha_{M-1} = 0$.

$\alpha_0^{(\mathrm{cusp})}$ is determined by considering the next highest order $[D^{(2,\nu)}\langle\mathbf{1}\rangle]_{\kappa+1}$ for some character. (Since modular transformations permute the characters only and have no effect on $D^{(2,\nu)}$, it is sufficient to do the computation for the vacuum character $\langle\mathbf{1}\rangle_1 = q^{\kappa_1}(1 + O(q^2))$). □

Remark 2. *The characters' differential equations can be deduced differently within the VOA framework by using the representation of the singular vector as Virasoro decendent of the vacuum* $\langle\mathbf{1}\rangle$ *and the fact that every such descendent is the image of a character under a linear differential operator in the modulus [19]. On the other hand, the singular vector of a minimal model is zero.*

4.4 Explicit results for the (2, 5) minimal model

Throughout this section, $\Sigma_1 : y^2 = p$ is the genus 1 Riemann surface defined by

$$p(x) = 4(x - X_1)(x - X_2)(x - X_3)\,, \tag{13}$$

where we assume that

$$\sum_{i=1}^{3} X_i = 0\,. \tag{14}$$

We shall use the following notation: Let $m(X_1, \xi_1, \ldots, X_n, \xi_n)$ be a monomial. We denote by

$$\overline{m(X_1, \xi_1, \ldots, X_n, \xi_n)}$$

the sum over all distinct monomials $m(X_{\sigma(1)}, \xi_{\sigma(1)}, \ldots, X_{\sigma(n)}, \xi_{\sigma(n)})$, where σ is a permutation of $\{1, \ldots, n\}$. E.g. eq. (14) reads $\overline{X_1} = 0$, and

$$\overline{X_1 X_2} = \sum_{\substack{i,j=1\\ i<j}}^{3} X_i X_j\,, = X_1X_2 + X_1X_3 + X_2X_3\,,$$

(for $n = 3$). For any state $\langle\ \rangle$ on Σ_1, the Virasoro 1-point function on Σ is given by

$$\langle T(x)\rangle = \frac{c}{32}\frac{[p']^2}{p^2}\langle\mathbf{1}\rangle + \frac{\Theta(x)}{4p}\,, \tag{15}$$

[10, 11, where $\Theta(x) = \Theta^{[1]}(x)$ since the polynomial $\Theta^{[y]}$ is absent], with

$$\Theta(x) = A_0 x + \mathbf{A}_1\,, \quad A_0 x = -4c\,. \tag{16}$$

Here $c = -22/5$, and $\mathbf{A}_1 \propto \langle\mathbf{1}\rangle$ is constant in x. For degree reasons, the undetermined polynomial in the formula for the connected Virasoro 2-point function w.r.t. a state $\langle\ \rangle$ on Σ_1 [10, 11] is constant in position,

$$P(x_1, x_2, y_1, y_2) = P^{[1]}\,, \tag{17}$$

but depends on $\langle \mathbf{1} \rangle$ and $\mathbf{A}_1$. For the 1-forms $\xi_j = dX_j$ ($j = 1,2,3$) we introduce the matrices

$$\Xi_{3,0} := \begin{pmatrix} X_1 & X_2 & X_3 \\ 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix}, \quad \Xi_{3,1} := \begin{pmatrix} X_1 & X_2 & X_3 \\ 1 & 1 & 1 \\ \xi_1 X_1 & \xi_2 X_2 & \xi_3 X_3 \end{pmatrix},$$

and the 3×3 Vandermonde matrix

$$V_3 := \begin{pmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ 1 & X_3 & X_3^2 \end{pmatrix}.$$

For later use, we note that

$$\begin{aligned} \det V_3 &= \prod_{1 \le i < j \le 3} (X_j - X_i) \\ &= (X_1 - X_2)(X_2 - X_3)(X_3 - X_1)\,, \end{aligned}$$

$$\frac{\det \Xi_{3,0}}{\det V_3} = \frac{\xi_1}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} = -4 \sum_{s=1}^{3} \frac{\xi_s}{p'(X_s)}\,, \tag{18}$$

$$\frac{\det \Xi_{3,1}}{\det V_3} = \frac{\xi_1 X_1}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} = -4 \sum_{s=1}^{3} \frac{\xi_s X_s}{p'(X_s)}\,. \tag{19}$$

It shall be convenient to work with the 1-form

$$\omega := d \log \det V_3\,. \tag{20}$$

A simple calculation using eq. (14) shows that

$$d \det V_3 = -3X_1(dX_1)(X_2 - X_3) + \text{cyclic} = -3 \det \Xi_{3,1}\,,$$

so that

$$\omega = -3\, \frac{\det \Xi_{3,1}}{\det V_3} = \frac{\xi_1 - \xi_2}{X_1 - X_2} + \text{cyclic}\,. \tag{21}$$

We also introduce

$$\Delta_0 := (\det V_3)^2\,.$$

and note that

$$\omega = \frac{1}{2} d \log \Delta_0\,. \tag{22}$$

Lemma 3. *Let $\Sigma_1 : y^2 = p$ be the Riemann surface defined by eq. (13), where we assume condition (14) to hold. Define a deformation of Σ_1 by*

$$\xi_j = dX_j\,, \quad j = 1,2,3.$$

In terms of the modulus τ and the scaling parameter λ (the inverse length) of the real period, we have

$$\omega = \pi i\, E_2\, d\tau - 6 d \log \lambda\,.$$

Proof. By assumption (14), we can write

$$p(x) = 4(x^3 + ax + b)\,,$$

where on the one hand,

$$a = \overline{X_1 X_2}\,, \qquad b = -X_1 X_2 X_3\,,$$

and [15]

$$\Delta_0 = -4a^3 - 27b^2\,. \tag{23}$$

On the other hand, [15]

$$a = -\frac{\pi^4}{3}\lambda^4 E_4\,, \qquad b = -\frac{2\pi^6}{27}\lambda^6 E_6\,, \tag{24}$$

so

$$\Delta_0 = \frac{4\pi^{12}}{27}\lambda^{12}(E_4^3 - E_6^2)\,, \tag{25}$$

We expand the fraction defining ω in eq. (20) by $\det V_3$ and show that for a, b introduced above, we have

$$\det(\Xi_{3,1} V_3) = 2a^2\,da + 9b\,db\,. \tag{26}$$

We first establish eq. (26) under the additional assumption that $\xi \propto X$. In this case both sides of eq. (26) are proportional to Δ_0, with the same proportionality factor: On the l.h.s.,

$$\det \Xi_{3,1}|_{\xi=X} \det V_3 \propto -\det\begin{pmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ 1 & X_3 & X_3^2 \end{pmatrix}^2 = -\Delta_0\,.$$

On the r.h.s.,

$$\begin{aligned} da &= \overline{\xi_1 X_2} \propto 2\overline{X_1 X_2} = 2a\,, \\ db &= -\overline{\xi_1 X_2 X_3} \propto -3\overline{X_1 X_2 X_3} = 3b\,. \end{aligned}$$

From this and eq. (23) follows eq. (26). Using (24), (25), and

$$\mathfrak{D}_4 E_4 = -\frac{E_6}{3}\,, \qquad \mathfrak{D}_6 E_6 = -\frac{E_4^2}{2} \tag{27}$$

([17], Proposition 15, p. 49), where $\mathfrak{D}_{2\ell}$ is the Serre derivative (7), we find

$$2\,a^2 \frac{\partial}{\partial\tau} a + 9\,b\,\frac{\partial}{\partial\tau} b = -\frac{i\,\pi}{3} E_2 \Delta_0\,.$$

For the λ derivative, we use the description of ω by eq. (20). From eq. (25) follows

$$\frac{\partial}{\partial\lambda}\log\Delta_0 = \frac{12}{\lambda}\,.$$

The last two equations prove the lemma under the assumption $\xi \propto X$. For the general case we refer to Appendix A. □

Under variation of the ramification points, the modulus changes according to

Lemma 4. *Under the conditions of Lemma 3, we have*

$$d\tau = -i\pi\lambda^2 \frac{\det \Xi_{3,0}}{\det V_3} . \tag{28}$$

Note that proportionality between the differentials on either side of eq. (28) can be seen as follows: Under the action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, both $d\tau$ and λ^2 transform by a factor of $(c\tau + d)^{-2}$. Moreover, both differentials have a simple pole at the boundary of the moduli space: $d\tau$ is singular at $\tau = i\infty$, while $\frac{\det \Xi_{3,0}}{\det V_3}$ has a pole when two X_i coincide. Thus up to a multiplicative constant they must be equal.

Proof of Lemma 4. We first show that for

$$p(x) = 4(x^3 + ax + b) ,$$

we have

$$\det(\Xi_{3,0} V_3) = 9b\, da - 6a\, db . \tag{29}$$

Indeed, suppose first

$$\xi_i \propto X_i^2 - \xi_0 , \quad \xi_0 := \frac{1}{3}\left(\sum_{i=1}^{3} X_i^2\right) = \frac{1}{3}\,\overline{X_1^2} \tag{30}$$

then the condition (14) continues to hold, and both sides of eq. (29) are proportional to Δ_0, with the same proportionality factor: On the l.h.s.,

$$\det \Xi_{3,0}|_{\xi = X^2 - \xi_0} \det V_3 \propto \det \begin{pmatrix} \overline{\xi_1} & \overline{\xi_1 X_1} & \overline{\xi_1 X_1^2} \\ \overline{X_1} & \overline{X_1^2} & \overline{X_1^3} \\ 3 & \overline{X_1} & \overline{X_1^2} \end{pmatrix} = -\Delta_0 ,$$

since

$$\det \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ X_1 & X_2 & X_3 \\ 1 & 1 & 1 \end{pmatrix} \propto \det \begin{pmatrix} X_1^2 & X_2^2 & X_3^2 \\ X_1 & X_2 & X_3 \\ 1 & 1 & 1 \end{pmatrix} - \det \begin{pmatrix} \xi_0 & \xi_0 & \xi_0 \\ X_1 & X_2 & X_3 \\ 1 & 1 & 1 \end{pmatrix} ,$$

where for the present choice of ξ, the latter determinant is zero. On the r.h.s., by the fact that $\overline{X_1} = 0$,

$$\begin{aligned} \xi_0 &= \frac{1}{3}\,\overline{X_1^2} = -\frac{2}{3}\,\overline{X_1 X_2} = -\frac{2a}{3} , \\ \overline{X_1^3} &= -3\overline{X_1^2 X_2} - 6b , \\ \overline{X_1^2 X_2} &= \overline{X_1 X_2 (X_1 + X_2)} = -3b , \end{aligned}$$

so

$$\begin{aligned} da &= -\overline{\xi_1 X_1} \propto -\overline{X_1^3} + \xi_0 \overline{X_1} = -\overline{X_1^3} = -3b , \\ db &= -\overline{\xi_1 X_2 X_3} \propto -\overline{X_1^2 X_2 X_3} + \xi_0 \overline{X_1 X_2} = b\overline{X_1} + \xi_0 a = \xi_0 a = -\frac{2}{3}\, a^2 . \end{aligned}$$

From this and eq. (23) follows eq. (29). Now by eqs (24), (25), and (27),

$$9\,b\,\frac{\partial}{\partial\tau}a - 6\,a\,\frac{\partial}{\partial\tau}b = 2\pi i\,(9\,b\,\mathfrak{D}_4 a - 6\,a\,\mathfrak{D}_6 b) = \frac{i}{\pi\lambda^2}\Delta_0\,.$$

The partial derivatives are actually ordinary derivatives since from eqs (24) follows

$$9\,b\,\frac{\partial}{\partial\lambda}a - 6\,a\,\frac{\partial}{\partial\lambda}b = 0\,.$$

Factoring out $d\tau$ in eq. (29) and dividing both sides by $\Delta_0/(-i\pi\lambda^2)$ yields the claimed formula. The general case without the assumption (30) is proved in Appendix B. □

Theorem 5. *Let $\Sigma_1 : y^2 = p$ be the Riemann surface defined by eq. (13), where we assume condition (14) to hold. We equip Σ_1 with the lift of the polyhedral metric on $\mathbb{P}^1_{\mathbb{C}}$. Let $\langle\ \rangle_{sing}$ be a state on Σ_1 w.r.t. this metric. Define a deformation of Σ_1 by*

$$\xi_j = dX_j\,,\quad j = 1,2,3\,.$$

We have the following system of linear differential equations

$$\begin{aligned}(d + \frac{c}{24}\,\omega)\,\langle\mathbf{1}\rangle_{sing.} &= -\frac{1}{8}(\mathbf{A}_1)_{sing.}\,\frac{\det\Xi_{3,0}}{\det V_3}\,, \\ (d + \frac{c-8}{24}\omega)\,(\mathbf{A}_1)_{sing.} &= C_{sing.}\,\frac{\det\Xi_{3,0}}{\det V_3}\,,\end{aligned} \tag{31}$$

where ω is be the 1-form defined by eq. (21), and

$$C_{sing.} := -2P^{[1]} - \frac{1}{8}\langle\mathbf{1}\rangle_{sing.}^{-1}(\mathbf{A}_1)_{sing.}^2 - \frac{8c}{3}a\langle\mathbf{1}\rangle_{sing.}\,.$$

Here $P^{[1]}$ is defined by eq. (17), and $a = \overline{X_1X_2}$. In particular, in the $(2,5)$-minimal model,

$$C_{sing.} = \frac{22}{75}\,a\langle\mathbf{1}\rangle_{sing.}\,.$$

In general, $C_{\text{sing.}}$ is a function of $\langle\mathbf{1}\rangle_{\text{sing.}}$ and $(\mathbf{A}_1)_{\text{sing.}}$. Note that the occurrence of a term $\sim(\mathbf{A}_1)^2_{\text{sing.}}$ in the definition of $C_{\text{sing.}}$ is an artefact of our presentation since $P^{[1]}$ has been defined by means of the *connected* Virasoro 2-point function.

Remark 3. *In contrast to the ODE (5) for $\langle\mathbf{1}\rangle_{flat}$, the corresponding differential equation (31) for $\langle\mathbf{1}\rangle_{sing.}$ w.r.t the singular metric comes with a covariant derivative. Define*

$$(\mathbf{A}_1)_{flat} := 4\lambda^2\langle\mathbf{T}\rangle_{flat}$$

and let

$$(\mathbf{A}_1)_{flat} =: \alpha_{flat}\langle\mathbf{1}\rangle_{flat}\,,\quad (\mathbf{A}_1)_{sing.} =: \alpha_{sing.}\langle\mathbf{1}\rangle_{sing.}\,.$$

By eqs (5), (28) and (31),

$$d\log\frac{\langle\mathbf{1}\rangle_{sing.}}{\langle\mathbf{1}\rangle_{flat}} = -\frac{c}{24}\omega + \frac{1}{8\pi i\lambda^2}\left(\alpha_{sing.} - \alpha_{flat}\right)d\tau\,.$$

By eq. (22), this yields

$$\langle\mathbf{1}\rangle_{sing.} \propto \Delta_0^{-\frac{c}{48}}\,\langle\mathbf{1}\rangle_{flat}\,. \tag{32}$$

The proportionality factor is actually equal to one [12]. In particular, $\langle\mathbf{1}\rangle_{sing.}$ is not a modular function.

Proof of the Theorem. Notations: All state-dependent objects are understood to refer to the singular metric on Σ_1. The following two identities will be useful:

$$\frac{dp}{p} = -\sum_{s=1}^{3} \frac{\xi_s}{x - X_s}\,, \tag{33}$$

$$d\left(\frac{p'}{p}\right) = \sum_{s=1}^{3} \frac{\xi_s}{(x - X_s)^2}\,. \tag{34}$$

For $j = 1, 2, 3$, let γ_j be a closed path enclosing $X_j \in \mathbb{P}^1_{\mathbb{C}}$ and no other zero of p. x does not define a coordinate close to X_j, however y does. On the ramified covering, a closed path winds around X_j by an angle of 4π. We shall be working with the x coordinate, and mark the double circulation along γ_j in $\mathbb{P}^1_{\mathbb{C}}$ by a symbolic $2 \times \gamma_j$ under the integral. Thus for $j = 1$ we have

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{2\times\gamma_1} \langle T(x)\rangle \; dx &= 2 \lim_{x\to X_1} (x - X_1)\langle T(x)\rangle \\
&= \frac{1}{8}\left(\frac{c\langle\mathbf{1}\rangle}{X_1 - X_2} + \frac{c\langle\mathbf{1}\rangle}{X_1 - X_3} + \frac{\Theta(X_1)}{(X_1 - X_2)(X_1 - X_3)}\right) \\
&= \frac{1}{8}\,\frac{c(-2X_1 + X_2 + X_3)\langle\mathbf{1}\rangle - \mathrm{A}_0 X_1 - \mathbf{A}_1}{(X_1 - X_2)(X_3 - X_1)} \\
&= -\left(\frac{c}{4}\langle\mathbf{1}\rangle + \frac{\mathrm{A}_0}{8}\right)\frac{X_1\langle\mathbf{1}\rangle}{(X_1 - X_2)(X_3 - X_1)} \\
&\quad - \frac{1}{8}\,\frac{\mathbf{A}_1}{(X_1 - X_2)(X_3 - X_1)} + \frac{c}{8}\langle\mathbf{1}\rangle\frac{X_2 + X_3}{(X_1 - X_2)(X_3 - X_1)}\,.
\end{aligned}$$

So

$$\begin{aligned}
d\langle\mathbf{1}\rangle = \sum_{i=1}^{3}\left(\frac{1}{2\pi i}\oint_{2\times\gamma_i}\langle T(x)\rangle\, dx\right) dX_i &= -\left(\frac{c}{4}\langle\mathbf{1}\rangle + \frac{\mathrm{A}_0}{8}\right)\frac{\det \Xi_{3,1}}{\det V_3} - \frac{1}{8}\mathbf{A}_1\frac{\det \Xi_{3,0}}{\det V_3} \\
&\quad + \frac{c}{8}\langle\mathbf{1}\rangle\left(\frac{\xi_1(X_2 + X_3)}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic}\right),
\end{aligned}$$

using eqs (18) and (19). When (14) is imposed and $\mathrm{A}_0 = -4c\langle\mathbf{1}\rangle$ is used, we obtain the differential equation (31) for $\langle\mathbf{1}\rangle$. When $\langle T(x)\rangle$ is varied by changing all ramifications points X_1, X_2, X_3 simultaneously, we must require the position x not to lie on or be enclosed by any of the corresponding three curves γ_1, γ_2 and γ_3. Then we have

$$\begin{aligned}
d\langle T(x)\rangle &= \sum_{j=1}^{3}\left(\frac{1}{2\pi i}\oint_{2\times\gamma_j}\langle T(x')T(x)\rangle\, dx'\right) dX_j \\
&= \sum_{j=1}^{3}\left(\frac{\langle\mathbf{1}\rangle}{2\pi i}\oint_{2\times\gamma_j}\langle T(x')T(x)\rangle_c\, dx'\right) dX_j + \langle\mathbf{1}\rangle^{-1}\langle T(x)\rangle\, d\langle\mathbf{1}\rangle.
\end{aligned}$$

Here $\langle T(x)\rangle$ is given by formula (15). A formula for $\langle T(x)T(x')\rangle_c$ is given in [10, 11]. The terms $\propto yy'$ (with $y'^2 = p(x')$) do not contribute: As $X_j \in \mathbb{P}^1_{\mathbb{C}}$ is wound around twice along the closed curve γ_j, the square root y' changes sign after one tour, so the corresponding terms cancel. Thus for $j = 1$ we have, using eq. (16) for $\Theta(x')$,

$$\frac{\langle \mathbf{1} \rangle}{2\pi i} \oint_{2\times\gamma_1} \langle T(x')T(x)\rangle_c \, dx' \tag{35}$$

$$= 2 \lim_{x'\to X_1} (x' - X_1)\Big\{\frac{c}{4}\frac{\langle \mathbf{1} \rangle}{(x'-x)^4} + \frac{c}{32}\frac{p'(x')p'\langle \mathbf{1} \rangle}{(x'-x)^2 p(x')p} + \frac{1}{8}\frac{p(x')\Theta + p\Theta(x')}{(x'-x)^2 p(x')p}$$
$$+ \frac{P^{[1]}}{p(x')p} - \frac{a_0}{8}\frac{x'\Theta + x\Theta(x')}{p(x')p} - \frac{a_0^2 c}{8}\frac{x' x\langle \mathbf{1} \rangle}{p(x')p}\Big\}$$

$$= \frac{c}{16}\frac{\langle \mathbf{1} \rangle}{(X_1 - x)^2}\frac{p'}{p} + \frac{1}{4}\frac{\Theta(X_1)}{(X_1 - x)^2 p'(X_1)} \tag{36}$$
$$+ \frac{2P^{[1]}}{p'(X_1)p} - \frac{a_0}{4}\frac{X_1 \mathbf{A}_1}{p'(X_1)p} - \frac{a_0}{4}\frac{x\Theta(X_1)}{p'(X_1)p} .$$

Multiplying the first term on the r.h.s. of eq. (36) by ξ_1 and adding the corresponding terms as j takes the values $2, 3$ yields, by eq. (34),

$$\frac{c}{32}\langle \mathbf{1} \rangle d\left(\frac{p'}{p}\right)^2 .$$

The cyclic symmetrisation of the remaining four terms on the r.h.s. of eq. (36) gives

$$d\left(\frac{\Theta}{4p}\right) - \frac{\Theta}{4p}\, d\log\langle \mathbf{1} \rangle .$$

We deduce the differential equation for $\mathbf{A}_1$. Firstly,

$$d\Theta = 4p\, d\left(\frac{\Theta}{4p}\right) + \Theta\,\frac{dp}{p} .$$

By the above, using $p'(X_1) = -a_0(X_1 - X_2)(X_3 - X_1)$ with $a_0 = 4$,

$$4p\, d\left(\frac{\Theta}{4p}\right)|_x = -\frac{p(x)}{4}\Big(\frac{1}{(x - X_1)^2}\frac{\xi_1\Theta(X_1)}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic}\Big)$$
$$+ x\left(\frac{\xi_1\Theta(X_1)}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic}\right)$$
$$- 2P^{[1]}\,\frac{\det \Xi_{3,0}}{\det V_3} + \mathbf{A}_1\frac{\det \Xi_{3,1}}{\det V_3} + \Theta(x)\, d\log\langle \mathbf{1} \rangle . \tag{37}$$

Secondly, by partial fraction decomposition,

$$\frac{\Theta}{p} = -\frac{1}{(x - X_1)}\,\frac{\Theta(X_1)}{4(X_1 - X_2)(X_3 - X_1)} + \text{cyclic}.$$

Solving for Θ and using eq. (33) yields

$$\Theta\,\frac{dp}{p}|_x = \frac{p(x)}{4}\left(\frac{\Theta(X_1)}{(x - X_1)(X_1 - X_2)(X_3 - X_1)} + \text{cyclic}\right)\sum_{j=1}^{3}\frac{\xi_j}{(x - X_j)} . \tag{38}$$

Note that three terms in the sum on the r.h.s. of eq. (38) are equal but opposite to the first term on the r.h.s. of eq. (37). Since $\overline{\xi_1} = 0$, we have for the remaining sum

$$\frac{p(x)}{4}\left(\frac{\Theta(X_1)}{(x - X_1)(X_1 - X_2)(X_3 - X_1)}\sum_{j\neq 1}\frac{\xi_j}{(x - X_j)} + \text{cyclic}\right)$$
$$= -\left(\frac{\Theta(X_1)(\xi_2 X_3 + \xi_3 X_2)}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic}\right) - x\left(\frac{\xi_1\Theta(X_1)}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic}\right) ,$$

where the second term on the r.h.s. is equal but opposite to the one before last on the r.h.s. of eq. (37). For the first term we have (cf. Appendix C)

$$-\frac{\Theta(X_1)(\xi_2 X_3+\xi_3 X_2)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic} = -\frac{8c}{3}a\langle\mathbf{1}\rangle\frac{\det\Xi_{3,0}}{\det V_3}-2\mathbf{A}_1\frac{\det\Xi_{3,1}}{\det V_3}\,.$$

Using $\Theta(X_1)=-4cX_1\langle\mathbf{1}\rangle+\mathbf{A}_1$, we conclude that

$$d\mathbf{A}_1 = -\mathbf{A}_1\frac{\det\Xi_{3,1}}{\det V_3}-\left(2P^{[1]}+\frac{8c}{3}\,a\langle\mathbf{1}\rangle\right)\frac{\det\Xi_{3,0}}{\det V_3}+\mathbf{A}_1\,d\log\langle\mathbf{1}\rangle\,.$$

Plugging in to eq. (31) yields the claimed formula. To determine the constant in the $(2,5)$-minimal model, we write

$$p(x)=4x^3+a_1x^2+a_2x+a_3\,.$$

By Lemma 5 in [10], or Lemma 16 in [11], using $c=-\frac{22}{5}$, we find

$$P^{[1]} = -\frac{77}{400}a_1^2\langle\mathbf{1}\rangle+\frac{1}{10}a_1\mathbf{A}_1+\frac{143}{100}a_2\langle\mathbf{1}\rangle-\frac{1}{16}\langle\mathbf{1}\rangle^{-1}\mathbf{A}_1^2\,.$$

□

The formulation of the differential equations using determinants relies on the permutation symmetry of the equations' constituent parts. This symmetry will continue to be present as the number of ramification points increases. With the genus, however, also the degree of the polynomial Θ will grow and give rise to additional terms having no lower genus counterpart.

5 Alternative formulations of the system of differential equations

5.1 Comparison with the analytic approach of Section 4.1

We provide a rough check that the system of linear differential equations obtained from Theorem 5 for the $(2,5)$ minimal model is consistent with the system discussed in Section 4.1. By formula (32), we have

$$\langle\mathbf{1}\rangle_{\text{sing.}}=\Delta_0{}^{-\frac{c}{48}}f\,,\quad (\mathbf{A}_1)_{\text{sing.}}=\Delta_0{}^{-\frac{c}{48}}g\,, \tag{39}$$

for some functions f,g of τ, with $f,g\propto\langle\mathbf{1}\rangle_{\text{flat}}$, where [17]

$$\Delta_0=\prod_{i<j}(X_i-X_j)^2\sim\Delta=\eta^{24}=q-24q^2+O(q^3)\,.$$

Close to the boundary of the moduli space where $X_1\approx X_2$, we have

$$(X_1-X_2)\sim q^{\frac{1}{2}}=e^{\pi i\tau}\,. \tag{40}$$

Since in this region only the difference X_1-X_2 matters, we may w.l.o.g. suppose that

$$X_2=\text{const.}$$

($\xi_2 = 0$). As before, we shall work with assumption (14). In view of (40) on the one hand, and the series expansion of the Rogers-Ramanujan partition functions $\langle \mathbf{1} \rangle_{\text{flat}}$ on the other, we have to show that

$$f \sim (X_1 - X_2)^{-\frac{1}{30}} \,, \quad \text{or} \quad f \sim (X_1 - X_2)^{\frac{11}{30}} \,. \tag{41}$$

Eqs (39) and (22) yield

$$d\langle \mathbf{1} \rangle_{\text{sing.}} = \Delta_0^{-\frac{c}{48}} \left(df - \frac{c}{24} \omega f \right) ,$$

and a similar equation is obtained for $d\langle \mathbf{A}_1 \rangle_{\text{sing.}}$. So by Theorem 5,

$$\begin{aligned} df &= -\frac{1}{8}\, g\, \frac{\det \Xi_{3,0}}{\det V_3} \,, \\ \left(d - \frac{\omega}{3} \right) g &= \frac{22a}{75}\, f\, \frac{\det \Xi_{3,0}}{\det V_3} \,. \end{aligned} \tag{42}$$

Since $f \sim (X_1 - X_2)^{\alpha}$ for some $\alpha \in \mathbb{R}$,

$$df \sim \frac{\xi_1 \alpha}{X_1 - X_2} f \,. \tag{43}$$

On the r.h.s. of eq. (42), we have by the assumption (14),

$$\frac{\det \Xi_{3,0}}{\det V_3} = \frac{\xi_1}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} \sim \frac{\xi_1}{(X_1 - X_2)(-3X_2)} \sim \frac{\omega}{(-3X_2)}$$

since $X_1 \approx X_2$, and we have omitted the regular terms. Eq. (42) thus yields

$$g \approx 24 X_2 \alpha f \,.$$

Using the differential equation for g,

$$24 X_2 \alpha \left(d - \frac{1}{3}\omega \right) f \sim \frac{22a}{75} f\, \frac{\omega}{(-3X_2)}$$

which by the approximate eq. (43) and by the fact that $a \sim -3X_2^2$ reduces to the quadratic equation

$$\alpha \left(\alpha - \frac{1}{3} \right) \sim \frac{11}{900} \,.$$

This is solved by $\alpha = -\frac{1}{30}$ and $\frac{11}{30}$, yielding (41), so the check works.

5.2 Equivalent systems of ODEs

Let ϑ be the field defined by [10, 11]

$$T(x)\, p(x) = \vartheta(x) + \frac{c}{32} \frac{[p'(x)]^2}{p(x)}.1 \,,$$

that is,

$$\langle \vartheta(x) \rangle = \frac{\Theta(x)}{4} \,,$$

where Θ is the polynomial of eq. (16).

Claim 1. *Let $\Sigma_1 : y^2 = p$ be the Riemann surface defined by eq. (13), where we assume condition (14) to hold. In the* $(2, 5)$ *minimal model, the system*

$$
\begin{aligned}
(d + \frac{c}{24}\,\omega)\langle\mathbf{1}\rangle &= -\frac{1}{8}\mathbf{A}_1\,\frac{\det\Xi_{3,0}}{\det V_3} \\
(d + \frac{c-8}{24}\omega)\,\mathbf{A}_1 &= -\frac{c}{15}\,a\langle\mathbf{1}\rangle\,\frac{\det\Xi_{3,0}}{\det V_3}
\end{aligned}
$$

of Theorem 5 is equivalent to the system [12]

$$
\begin{aligned}
\left(d - \frac{c}{8}\omega\right)\langle\mathbf{1}\rangle &= 2\sum_{s=1}^{3}\frac{\xi_s}{p'(X_s)}\langle\vartheta(X_s)\rangle\,, \\
\left(d - \frac{c}{8}\,\omega\right)\langle\vartheta(x)\rangle &= 2\sum_{s=1}^{3}\frac{\xi_s}{p'(X_s)}\langle\vartheta(X_s)\vartheta(x)\rangle - \langle\vartheta(x)\rangle\frac{dp}{p}|_x - \frac{c}{16}p'\,d\left(\frac{p'}{p}\right)|_x\langle\mathbf{1}\rangle\,.
\end{aligned}
$$

Proof. We show equivalence of the eqs for $\langle\mathbf{1}\rangle$:

$$
\begin{aligned}
2\sum_{s=1}^{3}\frac{\xi_s}{p'(X_s)}\langle\vartheta(X_s)\rangle &= 2\sum_{s=1}^{3}\frac{\xi_s}{p'(X_s)}\left(-cX_s\langle\mathbf{1}\rangle + \frac{\mathbf{A}_1}{4}\right) \\
&= -\frac{c}{6}\omega\langle\mathbf{1}\rangle - \frac{\mathbf{A}_1}{8}\frac{\det\Xi_{3,0}}{\det V_3}\,,
\end{aligned}
$$

by eqs (18) and (19). We address the eq. for $\langle\vartheta(x)\rangle$. As $x \to \infty$,

$$
\vartheta(x) = -cx.1 + O(1)\,,
$$

so for large x,

$$
\begin{aligned}
\langle\vartheta(x)\vartheta(X_s)\rangle &= -cx\,\langle\vartheta(X_s)\rangle + O(1) \\
&= -c\,x\left(-cX_s\langle\mathbf{1}\rangle + \frac{\mathbf{A}_1}{4}\right) + O(1)\,.
\end{aligned}
\tag{44}
$$

This way the differential equation for $\langle\vartheta(x)\rangle$ reduces to one for $\langle\mathbf{1}\rangle$. These two equations are compatible since they are derived from the same general formula in [12, Lemma 6]. Thus by the first step the differential equation for $\langle\vartheta(x)\rangle$ in the region where x is large is equivalent to the differential equation for $\langle\mathbf{1}\rangle$ in the first system. It remains to check the differential equation for the x-independent terms in $\langle\vartheta(x)\rangle$.

$$
\begin{aligned}
\left(d - \frac{c}{8}\,\omega\right)\langle\vartheta(x)\rangle &= -cx\left(d - \frac{c}{8}\omega\right)\langle\mathbf{1}\rangle + \frac{1}{4}\left(d - \frac{c}{8}\,\omega\right)\mathbf{A}_1 \\
&= \left(cx\frac{\mathbf{A}_1}{8} - \frac{c}{60}\,a\langle\mathbf{1}\rangle\right)\frac{\det\Xi_{3,0}}{\det V_3} \\
&\quad + \frac{1}{6}\left(c^2x\langle\mathbf{1}\rangle - \frac{1}{2}\left(\frac{c}{2} - 1\right)\mathbf{A}_1\right)\omega\,.
\end{aligned}
$$

By eqs (18) and (19), and for $c = -22/5$,

$$
\begin{aligned}
&\left(d - \frac{c}{8}\,\omega\right)\langle\vartheta(x)\rangle \\
&\qquad = c\left(-\frac{\mathbf{A}_1}{2}\,x + \frac{a}{15}\langle\mathbf{1}\rangle\right)\sum_{s=1}^{3}\frac{\xi_s}{p'(X_s)} + \left(\frac{16}{5}\mathbf{A}_1 + 2c^2x\langle\mathbf{1}\rangle\right)\sum_{s=1}^{3}\frac{\xi_s X_s}{p'(X_s)}\,.
\end{aligned}
\tag{45}
$$

On the other hand, by [12, Lemma 6], using eqs (33), (34) and (44),

$$\begin{aligned}\left(d - \frac{c}{8}\,\omega\right)\langle\vartheta(x)\rangle &= c\sum_{s=1}^{3}\frac{\xi_s X_s^2}{p'(X_s)}\langle\mathbf{1}\rangle \qquad (46)\\ &+ 16\left(\frac{c^2}{8}\langle\mathbf{1}\rangle\, x + \frac{\mathbf{A}_1}{5}\right)\sum_{s=1}^{3}\frac{\xi_s X_s}{p'(X_s)}\\ &+ 2c\left(-\frac{\mathbf{A}_1}{4}\, x + \frac{a}{5}\langle\mathbf{1}\rangle + O(x^{-1})\right)\sum_{s=1}^{3}\frac{\xi_s}{p'(X_s)} \qquad (47)\\ &- \left(-cx\langle\mathbf{1}\rangle + \frac{\mathbf{A}_1}{4}\right)\sum_{s=1}^{3}\frac{\xi_s}{x - X_s}\\ &- \frac{c}{4}\left(3x^2 + a\right)\langle\mathbf{1}\rangle\,\sum_{s=1}^{3}\frac{\xi_s}{(x - X_s)^2}\,.\end{aligned}$$

Comparison yields the claim: Since $\sum_{s=1}^{3}\xi_s = 0$, we have

$$\sum_{s=1}^{3}\frac{\xi_s}{(x - X_s)^m} = O(x^{-(m+1)})\,,$$

so the last two lines in the second equation are $O(x^{-1})$. Moreover, by eqs (58) and (59),

$$\sum_{s=1}^{3}\frac{\xi_s X_s^2}{p'(X_s)} \;=\; -\,\frac{1}{3}a\,\sum_{s=1}^{3}\frac{\xi_s}{p'(X_s)}\,,$$

so the term $\propto a$ in line (45) equals the sum of such terms in lines (46) and (47). So the formulas are equivalent up to $O(1)$ terms. We have already checked that the r.h.s. of either differential equation for $\mathbf{A}_1$ resp. $\langle\vartheta(x)\rangle$ has the correct singularities, so the remaining $O(x^{-1})$ terms must be zero. □

5.3 The hypergeometric equation

The hypergeometric differential equation in z

$$z(1 - z)\frac{d^2w}{dz^2} + [C - (A + B + 1)z]\frac{dw}{dz} - ABw = 0 \qquad (48)$$

is an ODE with regular singularities when z assumes one of the values $0, 1, \infty$. Every second-order ODE with at most three regular singular points can be transformed into the hypergeometric differential equation.

Let us check against our system of ODEs for $n = 3$ ($g = 1$) and

$$p(x) = a_0(x - X_1)(x - X_2)(x - X_3)\,.$$

Claim 2. *Suppose $n = 3$, $X_0 = 0, X_1 = 1$, and $X_2 = z \in \mathbb{C}$ is a free parameter. Our system of ODEs gives rise to a pair of differential equations parametrised by $k = -\frac{7}{10}, -\frac{11}{10}$,*

$$\left[\frac{d^2}{dz^2} - \left(\frac{4}{5} + 2k\right)\left(\frac{1}{z - 1} + \frac{1}{z}\right)\frac{d}{dz} + \left(\frac{13c}{40} - 2k\right)\frac{1}{z\,(z - 1)}\right]w_k(z) = 0\,. \qquad (49)$$

The ODEs given by eq. (49) are hypergeometric for the following choice of parameters in eq. (48):

$$A, B = \begin{cases} \frac{3}{10}, -\frac{1}{10} & \text{for } k = -\frac{7}{10} \\ \frac{7}{10}, \frac{11}{10} & \text{for } k = -\frac{11}{10} \end{cases} .$$

and

$$C = \begin{cases} \frac{3}{5} & \text{for } k = -\frac{7}{10} \\ \frac{7}{5} & \text{for } k = -\frac{11}{10} \end{cases}$$

Moreover, for either value of k,

$$\begin{aligned} w_k(X_2) = & \ [X_2(X_2 - 1)]^{-\frac{11}{20}+k} \langle \mathbf{1} \rangle \\ & = e^{(-\frac{11}{20}+k)\pi i} \ [X_2(1 - X_2)]^{-\frac{11}{20}+k} \langle \mathbf{1} \rangle \end{aligned}$$

defines a solution to the hypergeometric differential equation (49) to that value of k iff $\langle \mathbf{1} \rangle$ *defines a solution of the second order ODE associated to the system of first order ODEs in Claim 1 for* $s = 2$ *and with* $X_0 = 0, X_1 = 1$.

Remark 4. *1. Claim 2 reads as predicting four partition functions* $\langle \mathbf{1} \rangle$*, since for every k, (49) is a second order ODE, and there are two values of k. To given k, however, the two hypergeometric solutions* $w_k(X_2)$ *(one about* $X_2 = 0$*, the other about* $X_2 = 1$*) are actually the same.*

2. The discriminant is defined only up to a root of unity, but the really meaningful quantity is the absolute value of $\langle \mathbf{1} \rangle$*, or*

$$|\langle \mathbf{1} \rangle_1|^2 + |\langle \mathbf{1} \rangle_2|^2 .$$

Proof. When $F(z) = \int f(z)\, dz$ then we have

$$\frac{d}{dz} + f(z) = e^{-F(z)} \frac{d}{dz} e^{F(z)} .$$

Thus by going over to the functions $\langle \mathbf{1} \rangle^* := e^{F(X_2)} \langle \mathbf{1} \rangle$ and $\langle \vartheta(X_2) \rangle^* := e^{F(X_2)} \langle \vartheta(X_2) \rangle$, where

$$f(X_2) = -\frac{c}{8} \frac{\omega_2}{dX_2} = -\frac{c}{8} \left\{ \frac{1}{X_2} + \frac{1}{X_2 - 1} \right\} ,$$

and so

$$e^{F(X_2)} = [X_2(X_2 - 1)]^{-\frac{c}{8}} ,$$

we can transform the equations for the covariant derivative on $\langle \mathbf{1} \rangle$ and $\langle \vartheta(X_2) \rangle$ w.r.t. X_2 into equations involving the ordinary derivative $\frac{d}{dX_2}$ only. Set

$$g(x) := x(x - 1) .$$

Then

$$\begin{aligned} a_0 g(X_2) &= p'(X_2) = a_0(X_2 - X_0)(X_2 - X_1) \\ 2a_0 g'(X_2) &= p''(X_2) = 2a_0[(X_2 - X_0) + (X_2 - X_1)] \\ 3a_0 g''(X_2) &= p^{(3)}(X_2) = 6a_0 . \end{aligned}$$

So

$$\frac{p''(X_2)}{p'(X_2)} = \frac{2g'(X_2)}{g(X_2)}\,, \qquad\qquad \frac{p^{(3)}}{p'(X_2)} = \frac{6}{g(X_2)}$$

and the 2nd order ODEs now reads, for $s = 1$,

$$\begin{aligned} \frac{d^2}{dX_2^2}\langle \mathbf{1}\rangle^* &= 2\frac{d}{dX_2}\frac{\langle \vartheta(X_2)\rangle^*}{p'(X_2)} \\ &= \left[\frac{7c}{40}\left(\frac{g'(X_2)}{g(X_2)}\right)^2 - \frac{13c}{40}\frac{1}{g(X_2)}\right]\langle \mathbf{1}\rangle^* + \frac{4}{5}\frac{g'(X_2)}{g(X_2)}\frac{d}{dX_2}\langle \mathbf{1}\rangle^* \,. \end{aligned} \tag{50}$$

Now let

$$\tilde{f}(X_2) = -k\frac{g'(X_2)}{g(X_2)}$$

Then for $\tilde{F}(X_2) = \int f(X_2)dX_2$, we have

$$e^{\tilde{F}(X_2)} = g(X_2)^{-k}\,,$$

and

$$\begin{aligned} g^k(X_2)\,\frac{d}{dX_2}\,g^{-k}(X_2) &= \frac{d}{dX_2} - k\frac{g'(X_2)}{g(X_2)} \\ g^k(X_2)\,\frac{d^2}{dX_2^2}\,g^{-k}(X_2) &= \frac{d^2}{dX_2^2} - 2k\frac{g'(X_2)}{g(X_2)}\frac{d}{dX_2} - k\frac{g''(X_2)}{g(X_2)} + k(k+1)\left(\frac{g'(X_2)}{g(X_2)}\right)^2 \,. \end{aligned}$$

Now suppose $\langle \mathbf{1}\rangle^*$ satisfies eq. (50). Then

$$\begin{aligned} w_k(X_2) &:= e^{\tilde{F}(X_2)}\langle \mathbf{1}\rangle^* \\ &= e^{\tilde{F}(X_2)}e^{F(X_2)}\langle \mathbf{1}\rangle = \prod_{i\neq 2}(X_2 - X_i)^{-\frac{c}{8}+k}\langle \mathbf{1}\rangle = \prod_{i\neq 2}(X_2 - X_i)^{-\frac{11}{20}+k}\langle \mathbf{1}\rangle \end{aligned}$$

satisfies

$$\left[\frac{d^2}{dX_2^2} - \left(\frac{4}{5}+2k\right)\frac{g'(X_2)}{g(X_2)}\frac{d}{dX_2} + \left(\frac{4k}{5} - \frac{7c}{40} + k(k+1)\right)\left(\frac{g'(X_2)}{g(X_2)}\right)^2 + \frac{13c}{40}\frac{1}{g(X_2)} - k\frac{g''(X_2)}{g(X_2)}\right]w_k(X_2)$$
$$= 0\,.$$

Only those values of k are allowed for which the second order poles drop out:

$$k(k+1) = \frac{7c}{40} - \frac{4k}{5} \quad\Leftrightarrow\quad k^2 + \frac{9}{5}k - \frac{7c}{40} = 0\,.$$

So for

$$k_{1/2} = -\frac{9}{10} \pm \sqrt{\frac{81}{100} - \frac{77}{100}} = -\frac{9}{10} \pm \frac{2}{10} = -\frac{7}{10}, -\frac{11}{10}\,,$$

the equation reduces to

$$\left[\frac{d^2}{dX_2^2} - \left(\frac{4}{5}+2k\right)\frac{g'(X_2)}{g(X_2)}\frac{d}{dX_2} + \left(\frac{13c}{40} - 2k\right)\frac{1}{g(X_2)}\right]w_k(X_2) = 0\,,$$

or to eq. (49) for the choice of $X_0 = 0, X_1 = 1$. For either of the two values of k, the equation has two solutions. We compare this result with the hypergeometric differential equation (48) when $z = X_2$. Using

$$\frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}\,,$$

eq. (48) can be written as

$$\frac{d^2w}{dz^2} + \left[\frac{C}{z} + \frac{C}{1-z} - \frac{A+B+1}{1-z}\right]\frac{dw}{dz} - \left[\frac{AB}{z(1-z)}\right]w = 0\,.$$

Thus

$$-AB = \frac{13c}{40} - 2k = -\frac{143}{100} - 2k = \begin{cases} -\frac{3}{100} & \text{for } k = -\frac{7}{10} \\ \frac{77}{100} & \text{for } k = -\frac{11}{10} \end{cases},$$

$$-C = \frac{4}{5} + 2k = \begin{cases} -\frac{3}{5} & \text{for } k = -\frac{7}{10} \\ -\frac{7}{5} & \text{for } k = -\frac{11}{10} \end{cases}.$$

Moreover,

$$A + B = 2C - 1 = \begin{cases} \frac{1}{5} & \text{for } k = -\frac{7}{10} \\ \frac{9}{5} & \text{for } k = -\frac{11}{10} \end{cases}$$

so

$$A, B = \begin{cases} \frac{3}{10}, -\frac{1}{10} & \text{for } k = -\frac{7}{10} \\ \frac{7}{10}, \frac{11}{10} & \text{for } k = -\frac{11}{10}\,. \end{cases}$$

We conclude that

$$w(X_2) = [X_2(X_2 - 1)]^{-\frac{3}{20}} \langle \mathbf{1} \rangle$$

(for $k = -\frac{7}{10}$) solves the hypergeometric equation

$$\frac{d^2w}{dz^2} + \frac{3}{5}\frac{1}{z\,(z-1)}\frac{dw}{dz} + \frac{3}{100}\frac{1}{z\,(z-1)}w = 0\,,$$

resp.

$$w(X_2) = [X_2(X_2 - 1)]^{-\frac{11}{20}} \langle \mathbf{1} \rangle$$

(for $k = -\frac{11}{10}$) solves the hypergeometric equation

$$\frac{d^2w}{dz^2} + \frac{7}{5}\frac{1}{z\,(z-1)}\frac{dw}{dz} - \frac{77}{100}\frac{1}{z\,(z-1)}w = 0\,,$$

iff $\langle \mathbf{1} \rangle^*$ solves the second order ODE (50). For either value of k, solves the hypergeometric equation (48) for the above mentioned values of k, A, B and C. $\square$

A Completion of the Proof of Lemma 3 (Section 4.4)

It remains to show eq. (26) for general deformations $\xi_i = dX_i$, assuming that $\overline{X_1} = 0$, eq. (14). We have

$$
\begin{aligned}
a &= \overline{X_1 X_2}\,, & da &= d(\overline{X_1 X_2}) \\
& & &= \xi_1 X_2 + \xi_1 X_3 + \xi_2 X_1 + \xi_2 X_3 + \xi_3 X_1 + \xi_3 X_2 = \overline{\xi_1 X_2} \\
b &= -X_1 X_2 X_3\,, & db &= -\,d(X_1 X_2 X_3) \\
& & &= -\,\xi_1 X_2 X_3 - \xi_2 X_1 X_3 - \xi_3 X_1 X_2 = -\overline{\xi_1 X_2 X_3}\,.
\end{aligned}
$$

Let $\alpha, \beta \in \mathbb{Q}$. On the one hand, since $\overline{X_1} = 0$, we have

$$
(\overline{X_1 X_2})^2 = \overline{X_1^2 X_2^2} + 2X_1 X_2 X_3 \cdot \overline{X_1} = \overline{X_1^2 X_2^2}\,, \tag{51}
$$

so

$$
\alpha a^2 da + \beta b\, db = \alpha\, \overline{X_1^2 X_2^2} \cdot \overline{\xi_1 X_2} + \beta X_1 X_2 X_3 \cdot \overline{\xi_1 X_2 X_3}\,.
$$

On the other hand,

$$
\begin{aligned}
\det \Xi_{3,1} \det V_3 &= \det \begin{pmatrix} \xi_1 X_1 & \xi_2 X_2 & \xi_3 X_3 \\ X_1 & X_2 & X_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ 1 & X_3 & X_3^2 \end{pmatrix} \\
&= \det \begin{pmatrix} \overline{\xi_1 X_1} & \overline{\xi_1 X_1^2} & \overline{\xi_1 X_1^3} \\ 0 & \overline{X_1^2} & \overline{X_1^3} \\ 3 & 0 & \overline{X_1^2} \end{pmatrix} \\
&= 3\left(\overline{X_1^3} \cdot \overline{\xi_1 X_1^2} - \overline{X_1^2} \cdot \overline{\xi_1 X_1^3}\right) + \left(\overline{X_1^2}\right)^2 \cdot \overline{\xi_1 X_1}\,.
\end{aligned}
$$

Here

$$
\begin{aligned}
\left(\overline{X_1^2}\right)^2 &= 4\left(\overline{X_1 X_2}\right)^2 = 4\,\overline{X_1^2 X_2^2} \qquad \text{(by eq. (51))} \\
\overline{\xi_1 X_1} &= -\,\overline{\xi_1 X_2}\,,
\end{aligned} \tag{52}
$$

and

$$
\begin{aligned}
\overline{\xi_1 X_1^2} &= -\,\overline{\xi_1 X_1 X_2} \\
&= -\,\xi_1 X_1 (X_2 + X_3) + \text{cyclic} = -\overline{X_1 X_2} \cdot \overline{\xi_1} + \overline{\xi_1 X_2 X_3} = \overline{\xi_1 X_2 X_3}
\end{aligned} \tag{53}
$$

$$
\overline{X_1^3} = X_1 (X_2 + X_3)^2 + \text{cyclic} = \overline{X_1 X_2^2} + 6 X_1 X_2 X_3 = 3 X_1 X_2 X_3\,, \tag{54}
$$

since

$$
\overline{X_1 X_2^2} = -\,X_1 X_2 (X_1 + X_3) - X_1 X_2 (X_2 + X_3) + \text{cyclic} = -6 X_1 X_2 X_3 - \overline{X_1^2 X_2} = -3 X_1 X_2 X_3\,.
$$

Moreover,

$$
\begin{aligned}
\overline{\xi_1 X_1^3} &= \xi_1 X_1 (X_2 + X_3)^2 + \text{cyclic} = \overline{\xi_1 X_1 X_2^2} + 2 X_1 X_2 X_3 \cdot \overline{\xi_1} = \overline{\xi_1 X_1 X_2^2} \\
\overline{X_1^2} &= \overline{\xi_1 X_1^2 X_2^3} + \overline{\xi_1 X_1^2 X_2 X_3^2} + \overline{\xi_1 X_2^3} - X_1 (X_2 + X_3) + \text{cyclic} = -2\,\overline{X_1 X_2}\,,
\end{aligned} \tag{55}
$$

and

$$\begin{aligned}
\overline{X_1X_2}\cdot\overline{\xi_1X_1X_2^2} &= (X_1X_2+X_1X_3+X_2X_3)(\xi_1X_1X_2^2+\xi_1X_1X_3^2+\text{cyclic})\\
&= X_1^2X_2^2\cdot\xi_1X_2+X_1^2X_3^2\cdot\xi_1X_3+\text{cyclic}\\
&+ X_1X_2\cdot\xi_1X_1X_3^2+X_1X_3\cdot\xi_1X_1X_2^2+\text{cyclic}\\
&+ X_2X_3\cdot(\xi_1X_1X_2^2+\xi_1X_1X_3^2)+\text{cyclic}\\
&= \overline{X_1^2X_2^2}\cdot\overline{\xi_1X_2}+X_1X_2X_3\cdot\overline{\xi_1X_1X_2}+X_1X_2X_3\cdot\overline{\xi_1X_2^2}\\
&= \overline{X_1^2X_2^2}\cdot\overline{\xi_1X_2}\,,
\end{aligned}$$

by eq. (53) and

$$\begin{aligned}
\overline{\xi_1X_2^2} &= -\xi_1X_2(X_1+X_3)-\xi_1(X_1+X_2)X_3+\text{cyclic}\\
&= -\overline{\xi_1X_1X_2}-2\,\overline{\xi_1X_2X_3}=\overline{\xi_1X_2X_3}\,.
\end{aligned}$$

We conclude that

$$\begin{aligned}
\det\Xi_{3,1}\det V_3 &= 9X_1X_2X_3\cdot\overline{\xi_1X_2X_3}+6\overline{X_1X_2}\cdot\overline{\xi_1X_1X_2^2}-4\overline{X_1^2X_2^2}\cdot\overline{\xi_1X_2}\\
&= 9X_1X_2X_3\cdot\overline{\xi_1X_2X_3}+2\overline{X_1^2X_2^2}\cdot\overline{\xi_1X_2}\,,
\end{aligned}$$

and so $\alpha=2,\beta=9$, as required.

B Completion of the Proof of Lemma 4 (Section 4.4)

It remains to show eq. (29) for general deformations $\xi_i=dX_i$, assuming that $\overline{X_1}=0$, eq. (14).

We use the expressions for a,b,da,db listed at the beginning of Appendix A.

Let $\alpha,\beta\in\mathbb{Q}$. On the one hand,

$$\begin{aligned}
\alpha\,a\,db+\beta\,b\,da &= -\alpha\overline{X_1X_2}\cdot\overline{\xi_1X_2X_3}-\beta\,X_1X_2X_3\cdot\overline{\xi_1X_2}\\
&= -(\alpha+\beta)\,X_1X_2X_3\cdot\overline{\xi_1X_2}-\overline{\xi_1X_1^2X_2^3}+\overline{\xi_1X_1^2X_2X_3^2}+\overline{\xi_1X_2^3}\alpha\,\overline{\xi_1X_2^2X_3^2}\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\det\Xi_{3,0}\det V_3 &= \det\begin{pmatrix}\xi_1&\xi_2&\xi_3\\X_1&X_2&X_3\\1&1&1\end{pmatrix}\begin{pmatrix}1&X_1&X_1^2\\1&X_2&X_2^2\\1&X_3&X_3^2\end{pmatrix}\\
&= \det\begin{pmatrix}0&\overline{\xi_1X_1}&\overline{\xi_1X_1^2}\\0&\overline{\xi_1X_1^2X_2^3}+\overline{\xi_1X_1^2X_2X_3^2}+\overline{\xi_1X_2^3X_1^2}&\overline{X_1^3}\\3&0&\overline{X_1^2}\end{pmatrix}=3\left(\overline{X_1^3}\cdot\overline{\xi_1X_1}-\overline{X_1^2}\cdot\overline{\xi_1X_1^2}\right)\,.
\end{aligned}$$

Eqs (52), (53), (54), and (55) from Appendix A yield

$$\begin{aligned}
\det\Xi_{3,0}\det V_3 &= 3\left(-3X_1X_2X_3\cdot\overline{\xi_1X_2}+2\,\overline{X_1X_2}\cdot\overline{\xi_1X_2X_3}\right)\\
&= 3\left(-3X_1X_2X_3\cdot\overline{\xi_1X_2}+2\,\overline{\xi_1X_2^2X_3^2}+2X_1X_2X_3\cdot\overline{\xi_1X_2}\right)\\
&= -3X_1X_2X_3\cdot\overline{\xi_1X_2}+6\,\overline{\xi_1X_2^2X_3^2}\,.
\end{aligned}$$

We conclude that $\alpha=-6$, $\alpha+\beta=3$, so $\beta=9$. This completes the proof.

C Completion of the proof of Theorem 5 (Section 4.4)

It remains to show that

$$-\frac{\Theta(X_1)(\xi_2X_3+\xi_3X_2)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}=-\frac{2}{3}ca_2\langle\mathbf{1}\rangle\frac{\det\Xi_{3,0}}{\det V_3}-2\mathbf{A}_1\frac{\det\Xi_{3,1}}{\det V_3}\ .$$

We have

$$\begin{aligned}\xi_2X_3+\xi_3X_2&=(\xi_2+\xi_3)(X_2+X_3)-(\xi_2X_2+\xi_3X_3)\\&=\xi_1X_1-(\xi_2X_2+\xi_3X_3)\\&=2\xi_1X_1-\overline{\xi_1X_1}\ .\end{aligned}$$

It follows that

$$-\frac{\Theta(X_1)(\xi_2X_3+\xi_3X_2)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}=\frac{8c\langle\mathbf{1}\rangle\,\xi_1X_1^2-2\mathbf{A}_1\xi_1X_1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\ ,$$

since $\overline{\xi_1X_1}$ is symmetric and both

$$\frac{1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}=0\ ,\tag{56}$$

$$\frac{X_1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}=0\ .\tag{57}$$

Now

$$X_1^2=-X_1(X_2+X_3)=-\frac{a_2}{4}+X_2X_3\ ;\tag{58}$$

we claim that

$$\frac{\xi_1X_2X_3}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}=\frac{a_2}{6}\frac{\det\Xi_{3,0}}{\det V_3}\ .\tag{59}$$

Indeed, since $\xi_1X_2X_3+\text{cyclic}=\overline{\xi_1X_2X_3}$ is symmetric, we have by eq. (56),

$$\frac{\xi_1X_2X_3}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}=-\frac{\xi_2X_3X_1+\xi_3X_1X_2}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\ .$$

Since $\overline{\xi_1}=0$, we have

$$\begin{aligned}-\frac{\xi_2X_3X_1+\xi_3X_1X_2}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}&=\left(\frac{\xi_1(X_3X_1+X_1X_2)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right)+\left(\frac{(\xi_3X_3+\xi_2X_2)X_1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right)\\&=\frac{a_2}{4}\frac{\det\Xi_{3,0}}{\det V_3}-\left(\frac{\xi_1X_2X_3}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right)\\&\quad-\left(\frac{\xi_1X_1^2}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right)\ ,\end{aligned}$$

using symmetry of $\overline{\xi_1X_1}$ and eq. (57) again. From eq. (58) follows eq. (59), and the proof of Theorem 5 is complete.

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