Bell's inequalities, 50 years later

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April 2, 2014

1 Hidden variables

John Stewart Bell was an Irish particle theorist working for the latter part of his career at CERN. (For an account of his career, see [1] and [2].) However, he is best known for his work on the foundations of quantum mechanics, which he called his 'hobby'. His most famous paper [3], in which he introduced his 'Bell inequalities' was published 50 years ago, so it seems appropriate to review these inequalities and some of their consequences at this time. In fact, his research into foundations started with another paper, which was accidentally published later. This latter paper [4] was conceived much earlier after conversations with Mandel. Bell was much impressed with the work of Bohm [5] and de Broglie [6] who developed an alternative to the standard version of quantum mechanics. Their work showed that it is possible to introduce 'hidden variables' in non-relativistic quantum mechanics which determine the quantum randomness in a way similar to statistical mechanics, i.e. random variables which cannot be observed but cause the probabilistic nature of the quantum measurement results. They suggested that the trajectory of a particle is in fact deterministic, but is under the influence of a random background force which cannot be controlled. Their theory seemed to be in conflict with a theorem by Von Neumann [7] about the impossibility of hidden variables in quantum mechanics. Of course, a mathematical theorem is based on certain assumptions, and Bell rather scathingly dismissed

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Von Neumann's main assumption, namely that the sum of two observables is also observable and its expectation value is the sum of the individual expectations.

As a counter example, he introduced a simple hidden variable model for a single spin-$\frac{1}{2}$ particle, which might be worth considering here in some detail.

The algebra of observables of a spin-$\frac{1}{2}$ particle is given by

$$A = a_0 \mathbf{1} + \vec{a} \cdot \vec{\sigma},$$

where $a_0 \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^3$, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. In the state $\psi_0 = |0\rangle$ (eigenstate of σ_z with eigenvalue 1), its expectation is

$$\langle a_0 \mathbf{1} + \vec{a} \cdot \vec{\sigma} \rangle = a_0 + a \cos(\alpha),$$

where α is the angle between $\vec{a}$ and the positive z-axis and $a = |\vec{a}|$. This result can also be realised by introducing a hidden variable $\lambda \in [-\frac{1}{2}, \frac{1}{2}]$ with uniform distribution, and a map

$$f_\alpha(\lambda) = \mathrm{sgn}\left(\lambda + \frac{1}{2}\cos(\alpha)\right).$$

Then

$$\langle a_0 \mathbf{1} + \vec{a} \cdot \vec{\sigma} \rangle = \mathbb{E}\left[a_0 + a f_\alpha(\lambda)\right].$$

Actually, there is a more intuitive way of introducing a hidden variable, namely a uniform probability distribution on the unit sphere S^2. In that case, we put

$$f_\alpha(\theta, \phi) = \mathrm{sgn}\left(\theta - \alpha\right).$$

Then if $|\vec{a}| = 1$,

$$\langle \vec{a} \cdot \vec{\sigma} \rangle = \frac{1}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi\, f_\alpha(\theta, \phi) \sin(\theta).$$

A general state ψ corresponds to a unit vector $\vec{\psi}$ with angular coordinates (θ_0, ϕ_0) according to

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{-i\phi_0/2} \cos(\theta_0/2) \\ e^{i\phi_0/2} \sin(\theta_0/2) \end{pmatrix}.$$

Then

$$f_{\vec{a}}(\vec{\lambda}) = \text{sgn}\left(\arccos(\vec{\lambda} \cdot \vec{\psi}) - \arccos(\vec{a} \cdot \vec{\psi})\right). \tag{1.1}$$

This is more in line with Bell's later suggestion that the quantum state ψ should really be considered the hidden variable and the presumed hidden variable the actual state. The disturbance by the quantum state ψ then causes the measured value to differ from the actual state (here $\vec{\lambda}$, in Bohm's case the position $x(t)$). Moreover, one can introduce a time evolution as follows. A general Hamiltonian is an observable and be written as

$$H = E_0 + \vec{h} \cdot \vec{\sigma}.$$

The constant E_0 only introduces a phase in the evolution, and is therefore irrelevant. In the Heisenberg picture, the operator A above transforms according to

$$\frac{d}{dt}A(t) = i[H, A(t)] = i\sum_{i,j=1}^{3} h_i a_j(t)[\sigma_i, \sigma_j] = -2(\vec{h} \wedge \vec{a}(t)) \cdot \vec{\sigma}.$$

Thus the vector $\vec{a}(t)$ rotates (precesses) in a plane perpendicular to $\vec{h}$. Now the map f becomes time-dependent:

$$f_{\vec{a}}(\vec{\lambda}, t) = \text{sgn}\left(\arccos(\vec{\lambda} \cdot \vec{e}_z) - \arccos(\vec{a}(t) \cdot \vec{e}_z)\right).$$

Here the unit vector $\vec{\lambda}$ corresponds to the hidden variable point (θ, ϕ). Since $-2(\vec{h} \wedge \vec{a}) \cdot \vec{\psi} = 2\vec{a} \cdot (\vec{h} \wedge \vec{\psi})$, we can also consider in the Schrödinger picture, that the state vector $\vec{\psi}(t)$ rotates (in the opposite direction) with $\vec{\psi}(0) = \vec{e}_z$. Then we write

$$f_{\vec{a}}(\vec{\lambda}, t) = \text{sgn}\left(\arccos(\vec{\lambda}(t) \cdot \vec{\psi}(t)) - \arccos(\vec{a} \cdot \vec{\psi}(t))\right),$$

where $\vec{\lambda}(t)$ precesses in the same way as $\vec{u}\psi(t)$. In this picture we can therefore interpret the variable $\vec{\lambda}(t)$ as the 'real' spin rotating in a deterministic way, while the measurement result is random. This is analogous to the de Broglie-Bohm view of non-relativistic quantum mechanics of a spinless particle.

Notice that the map f_α is highly nonlinear so that Von Neumann's argument does not apply. In the mean time another argument had been put

forward by Jauch and Piron [9]. They assume the logical structure axioms of quantum mechanics as they were developed by Von Neumann and Birkhoff [8]. These axioms concern yes-no measurements (corresponding to projections in quantum mechanics) and are as follows:

1. The set $\mathcal{L}$ of yes-no measurements has a 'lattice structure', i.e. there is a partial order $\leq$ on $\mathcal{L}$ such that for all $a, b \in \mathcal{L}$, there exists a least upper bound $a \cup b$ and a largest lower bound $a \cap b$.

2. For every $a \in \mathcal{L}$ there is a complement $a' \in \mathcal{L}$ such that

   (a) $(a')' = a$ for all $a \in \mathcal{L}$,

   (b) $a \cap a' = 0$ and $a \cup a' = 1$ for all $a \in \mathcal{L}$, where 0 and 1 are the trivial measurements yielding no resp. yes with certainty;

   (c) $a \leq b \implies b' \leq a'$ for all $a, b \in \mathcal{L}$.

Jauch and Piron define a **state** on $\mathcal{L}$ to be a map $p : \mathcal{L} \rightarrow [0, 1]$ such that

1. $p(0) = 0$ and $p(1) = 1$;

2. For every sequence of *disjoint* propositions $(a_n)_{n \in \mathbb{N}}$, i.e. such that $a_n \leq a'_m$ for $n \neq m$,
$$\sum_{n=1}^{\infty} p(a_n) = p\left(\bigcup_{n \in \mathbb{N}} a_n\right);$$

3. If for a sequence $(a_n)_{n \in \mathbb{N}}$, $p(a_n) = 1$ for all $n \in \mathbb{N}$, then
$$p\left(\bigcap_{n \in \mathbb{N}}\right) = 1.$$

Moreover, they assume that if $a \neq b$ then there exists a state p such that $p(a) \neq p(b)$.

They define a *dispersion-free* state as a state such that $p(a) = 0$ or $p(a) = 1$ for all $a \in \mathcal{L}$, and they then say that $\mathcal{L}$ admits hidden variables if every state is an average of dispersion-free states, i.e. of the form
$$p(a) = \int_{\Omega} p_{\lambda}(a)\, \mu(d\lambda) \tag{1.2}$$

for some probability measure μ on Ω and a family of dispersion-free states p_λ.

Lemma 1.1 *If a proposition system $\mathcal{L}$ admits hidden variables, then for all $a, b \in \mathcal{L}$,*

$$p(a) + p(b) = p(a \cap b) + p(a \cup b).$$

Two propositions a and b are said to be **compatible** if they generate a Boolean lattice, i.e. a lattice in which the distributive law holds. With the additional assumption that if $a \leq b$ then a and b are compatible, one can show that arbitrary $a, b \in \mathcal{L}$ are compatible if and only if

$$(a \cap b') \cup b = (a' \cap b) \cup b'. \tag{1.3}$$

We now write

$$
\begin{aligned}
p((a \cap b') \cup b) &= p(a \cap b') + p(b) \\
&= p(a) + p(b') - p(a \cup b') + p(b) \\
&= p(a) + 1 - p(a \cup b') \\
&= p(a) + p(a' \cap b) = p(a \cup (a' \cap b)).
\end{aligned}
$$

By the assumption that the states separate the propositions, we conclude that (1.3) holds.

Bell objects that this is no argument for rejecting hidden variables in a wider sense. Namely, in the example above, if a and b are given by 1-dimensional projections $\frac{1}{2}(\mathbf{1} + \vec{a} \cdot \vec{\sigma})$ and $\frac{1}{2}(\mathbf{1} + \vec{b} \cdot \vec{\sigma})$, and $\vec{b} \neq \vec{a}$, then $a \cap b = 0$, so one should have $p_\lambda(a \cap b) = 0$. But in the example, $p_\lambda(a) = \frac{1}{2}(1 + f_\alpha(\lambda))$ and $p_\lambda(b) = \frac{1}{2}(1 + f_\beta(\lambda))$ both equal 1 at the same time for certain values of λ.

2   Bell inequalities and the EPR paradox

Einstein, Podolsky and Rosen (EPR) [10] proposed a famous 'Gedankenexperiment' to argue that quantum mechanics cannot be a complete theory.

Although they used momentum and position operators for two particles, it is now usually presented in terms of spin-coordinates of two spin-$\frac{1}{2}$ particles, as suggested by Aharonov and Bohm [11]. In this formulation one considers an entangled state of the two particles, e.g. the singlet state

$$\psi_s = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

(Here the first index labels the state of one particle, the second that of the other, and $|0\rangle$ and $|1\rangle$ are the eigenstates of σ_z.) The particles can be arbitrarily far apart. Measuring the spin of one particle, e.g. with $\sigma_z \otimes \mathbf{1}$, collapses the state to $|01\rangle$ or $|10\rangle$, thus also determining the spin of the other. EPR found this problematic because in the standard interpretation of quantum mechanics the state of the other particle was indeterminate before the measurement, which seemed to imply action at a distance. They suggested that this means that quantum mechanics is incomplete: there should be 'hidden variables' which in fact determine the state of the two particles. The situation would then be analogous to a coin having been cut in half so that one half is heads, the other tails, and the two halves given to two people ('Alice' and 'Bob' in modern parlance) in closed boxes. Then once Alice opens her box, the content of Bob's box is instantaneously known. The difference is that in this case the contents of the boxes is in fact predetermined, even if they are unknown to Alice and Bob.

Bell realised that as far as measurements of the z-component of the spins is concerned, the EPR experiment is in fact classical and one cannot objectively decide about the existence of hidden variables. To do this, it is necessary to consider more general measurements. For hidden variables to be a genuine possibility, they should be able to explain more general measurements. Assuming the existence of a general hidden variable in the form of a probability measure he proceeded to derive an inequality regarding general measurements, which is not satisfied for quantum states, and hence provides a possible experimental test of the existence of hidden variables. Here we derive the slightly more general inequalities due to Clauser et al. [12]. Suppose there is a probability measure μ on a space Ω of hidden variables determining the results A and B of measurements of the spin components of the two

particles in directions $\vec{a}$ and $\vec{b}$ respectively, i.e. $A(\vec{a}, \lambda)$ and $B(\vec{b}, \lambda)$. Here the crucial assumption is that of *locality*, i.e. the outcome $A(\vec{a}, \lambda)$ does not depend on $\vec{b}$ and vice-versa $B(\vec{b}, \lambda)$ does not depend on $\vec{a}$. We know that each measurement results in one of the values ± 1. Consider the correlation given by

$$E(\vec{a}, \vec{b}) := \int_\Omega A(\vec{a}, \lambda) B(\vec{b}, \lambda) \mu(d\lambda).$$

(In fact, the measuring instruments could also have hidden variables. We then need to replace A and B by averages over these instrument variables and $|A|, |B| \leq 1$ rather than $= \pm 1$.) Now, varying the instrument settings, we have

$$
\begin{aligned}
E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}') &= \int_\Omega A(\vec{a}, \lambda)[B(\vec{b}, \lambda) - B(\vec{b}', \lambda)] \mu(d\lambda) \\
&= \int_\Omega A(\vec{a}, \lambda) B(\vec{b}, \lambda)[1 \pm A(\vec{a}', \lambda) B(\vec{b}', \lambda)] \mu(d\lambda) \\
&\quad - \int_\Omega A(\vec{a}, \lambda) B(\vec{b}', \lambda)[1 \pm A(\vec{a}', \lambda) B(\vec{b}, \lambda)] \mu(d\lambda).
\end{aligned}
$$

Using $|A|, |B| \leq 1$, we get

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| \leq 2 \pm \int_\Omega [A(\vec{a}', \lambda) B(\vec{b}', \lambda) + A(\vec{a}', \lambda) B(\vec{b}, \lambda)] \mu(d\lambda)$$

or

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| + |E(\vec{a}', \vec{b}) + E(\vec{a}', \vec{b}')| \leq 2. \tag{2.4}$$

On the other hand, consider the quantum expectation of $AB = (\vec{a} \cdot \vec{\sigma}) \otimes (\vec{b} \cdot \vec{\sigma})$ in the state $\psi_0 = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. A simple calculation shows that

$$\langle \sigma_i \otimes \sigma_j \rangle = -\delta_{i,j} \text{ for } i, j = x, y, z.$$

Hence

$$\langle (\vec{a} \cdot \vec{\sigma}) \otimes (\vec{b} \cdot \vec{\sigma}) \rangle = -\vec{a} \cdot \vec{b}. \tag{2.5}$$

Thus

$$
\begin{aligned}
&|\langle (\vec{a} \cdot \vec{\sigma}) \otimes (\vec{b} \cdot \vec{\sigma}) - (\vec{a} \cdot \vec{\sigma}) \otimes (\vec{b}' \cdot \vec{\sigma}) \rangle| \\
&+ |\langle (\vec{a}' \cdot \vec{\sigma}) \otimes (\vec{b} \cdot \vec{\sigma}) + (\vec{a}' \cdot \vec{\sigma}) \otimes (\vec{b}' \cdot \vec{\sigma}) \rangle| = |\vec{a} \cdot (\vec{b} - \vec{b}')| + |\vec{a}' \cdot (\vec{b} + \vec{b}')|.
\end{aligned}
\tag{2.6}
$$

This is clearly maximal if $\vec{a}$ is in the direction of $\vec{b} - \vec{b}'$ and $\vec{a}'$ in the direction of $\vec{b} + \vec{b}'$, in which case it equals $|\vec{b} - \vec{b}'| + |\vec{b} + \vec{b}'| = \sqrt{2 - 2\cos\beta} + \sqrt{2 + 2\cos\beta}$, where β is the angle between $\vec{b}$ and $\vec{b}'$. This in turn is maximal when $\beta = \pi/2$ and the maximum value is $2\sqrt{2} > 2$. In this optimal case, therefore, the above inequality is violated.

In the mean time many experiments have confirmed with increasing confidence that the Bell inequality is not satisfied and and in some cases that the quantum mechanical bound is closely approximated. Most experiments have been done with photons, see e.g. [13, 14, 16, 17, 18, 19, 20]. Notice that in order to properly test the *nonlocality*, in the above experiments the directions of polarisation analysers were changed while the photons were in flight. Initially, in [13], the measurement directions were fixed beforehand, then in [14] this was done in a periodic manner, whereas in later experiments it was done at random. Experiments have also been done with other particles, e.g. neutrons: see [21]. These experiments are very difficult and the challenge posed by Bell's inequality has thus strongly stimulated the advancement of experimental techniques.

Remark 1. It is easy to see that $\mathbb{P}[A(\vec{a}) = s, B(\vec{b}) = s']$ only depends on ss' and hence

$$\mathbb{P}[A(\vec{a}) = s,\ B(\vec{b}) = s'] = \frac{1}{4}(1 - s\,s'\vec{a} \cdot \vec{b}).$$

Thus, the measurement of $(\vec{a} \cdot \vec{\sigma}) \otimes (\vec{b} \cdot \vec{\sigma})$ is essentially a measurement of the spin of one particle w.r.t. a state determined by the measurement direction of the other. Nonlocality seems quite obvious from this point of view.

Remark 2. Notice also that if we admit *signed* measures, then we can realise these probabilities as

$$\mathbb{P}[A(\vec{a}) = s,\ B(\vec{b}) = s'] = \int 1_{A(\vec{a},\lambda)=s} 1_{B(\vec{b},\lambda)=s'}\, \mu(d\lambda). \qquad (2.7)$$

Indeed, by the above remark, it suffices if

$$\langle (\vec{a} \cdot \vec{\sigma}) \otimes (\vec{b} \cdot \vec{\sigma}) \rangle = \int A(\vec{a}, \lambda) B(\vec{b}, \lambda)\, \mu(d\lambda).$$

Let us put $\lambda = (\vec{\lambda}_1, \vec{\lambda}_2)$ and define

$$A(\vec{a}, \vec{\lambda}) = \mathrm{sgn}\,(\vec{\lambda} \cdot \vec{a}) \text{ and } B(\vec{b}, \vec{\lambda}) = \mathrm{sgn}\,(\vec{\lambda} \cdot \vec{b}).$$

Then we compute

$$\int_{S^2 \times S^2} \mathrm{sgn}\,(\vec{\lambda}_1 \cdot \vec{a}) \mathrm{sgn}\,(\vec{\lambda}_2 \cdot \vec{b})\, \vec{\lambda}_1 \cdot \vec{\lambda}_2 d\vec{\lambda}_1 d\vec{\lambda}_2,$$

where $d\vec{\lambda}$ denotes normalised Lebesgue measure. This is clearly rotation-invariant, so we can take $\vec{a} = \vec{e}_z$ and $\vec{b} = \sin\gamma\,\vec{e}_x + \cos\gamma\,\vec{e}_z$. Changing variables to

$$\vec{\lambda}_2' = \begin{pmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{pmatrix} \vec{\lambda}_2$$

we have

$$\begin{aligned} \vec{\lambda}_1 \cdot \vec{\lambda}_2 \;=\;& \cos\phi_1 \sin\theta_1 (\cos\gamma \cos\phi_2' \sin\theta_2' + \sin\gamma \cos\theta_2') \\ &+ \sin\phi_1 \sin\theta_1 \sin\phi_2' \sin\theta_2' \\ &+ \cos\theta_1(-\sin\gamma \cos\phi_2' \sin\theta_2' + \cos\gamma \cos\theta_2'). \end{aligned}$$

W.r.t. these variables

$$A(\vec{a}, \vec{\lambda}_1) = \mathrm{sgn}\,(\tfrac{\pi}{2} - \theta_1) \text{ and } B(\vec{b}, \vec{\lambda}_2) = \mathrm{sgn}\,(\tfrac{\pi}{2} - \theta_2').$$

As this is independent of ϕ_1 and ϕ_2', the integrals over ϕ_1 and ϕ_2' of the terms involving $\cos\phi_1$ or $\cos\phi_2'$ vanish. Hence

$$\begin{aligned} &\int_{S^2 \times S^2} \mathrm{sgn}\,(\vec{\lambda}_1 \cdot \vec{a}) \mathrm{sgn}\,(\vec{\lambda}_2 \cdot \vec{b})\, \vec{\lambda}_1 \cdot \vec{\lambda}_2 \, d\vec{\lambda}_1 d\vec{\lambda}_2 \\ &= \frac{1}{4} \int_0^\pi d\theta_1 \int_0^\pi d\theta_2' \mathrm{sgn}\,(\tfrac{\pi}{2} - \theta_1) \mathrm{sgn}\,(\tfrac{\pi}{2} - \theta_2') \cos\gamma \cos\theta_1 \cos\theta_2' \sin\theta_1 \sin\theta_2' \\ &= \frac{1}{4} \cos\gamma. \end{aligned}$$

The measure

$$\mu(d\lambda) = -4\vec{\lambda}_1 \cdot \vec{\lambda}_2 \, d\vec{\lambda}_1 d\vec{\lambda}_2$$

therefore satisfies (2.7).

This is somewhat reminiscent of Feynman's integral, which is also a complex-valued measure in the finite-dimensional case: see [22].

3 Quantum information

3.1 Entanglement

Of course, the essential feature of the singlet state ψ_s is that it is *entangled*: it cannot be written as a tensor product. This crucial feature of general quantum states was highlighted (and named) by Schrödinger in two papers, one in German [23] and one in English [24], in reaction to the EPR paper. He reasoned that entanglement is in fact the crucial distinguishing feature of quantum mechanics and is also at the root of the nature of measurement. In order to illustrate the absurdity of the situation, he introduced his famous cat.

In fact, it is easy to see that any entangled state violates the Bell inequality. Namely, an arbitrary state on $\mathbb{C}^2 \otimes \mathbb{C}^2$ can be written in the form

$$\psi = \lambda_1 |0\rangle \otimes |0\rangle' + \lambda_2 |1\rangle \otimes |1\rangle',$$

where $\lambda_1, \lambda_2 \geq 0$, $\lambda_1^2 + \lambda_2^2 = 1$ and $|0\rangle, |1\rangle$ and $|0\rangle', |1\rangle'$ are orthogonal bases. This is obviously entangled unless $\lambda_1 \lambda_2 = 0$. Considering the expectation value

$$E(\vec{a}, \vec{b}) = \langle \psi | (\vec{a} \cdot \vec{\sigma} \otimes \vec{b} \cdot \vec{\sigma}' | \psi \rangle,$$

where $\vec{\sigma}'$ represent the Pauli matrices on the basis $\{|0\rangle', |1\rangle'\}$, we have

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| + |E(\vec{a}', \vec{b}) + E(\vec{a}', \vec{b}')|$$
$$= |a_z(b_z - b_z') + 2\lambda_1\lambda_2(a_x(b_x - b_x') - a_y(b_y - b_y'))|$$
$$+ |a_z(b_z + b_z') + 2\lambda_1\lambda_2(a_x(b_x + b_x') - a_y(b_y + b_y'))|.$$

Maximising over $\vec{a}$ and $\vec{a}'$ we get

$$\sqrt{(b_z - b_z')^2 + 4\lambda_1^2\lambda_2^2((b_x - b_x')^2 + (b_y - b_y')^2)}$$
$$+ \sqrt{(b_z + b_z')^2 + 4\lambda_1^2\lambda_2^2((b_x + b_x')^2 + (b_y + b_y')^2)}.$$

Taking for example $\vec{b} = \vec{e}_x$ and $\vec{b}' = \vec{e}_z$, this is $2\sqrt{1 + 4\lambda_1^2\lambda_2^2} > 2$ unless $\lambda_1\lambda_2 = 0$.

In general, quantum systems are in a mixed state, so it is interesting to wonder to what extent Bell inequalities are satisfied for mixed states. *Mixed states* are given by density matrices ρ, i.e. non-negative matrices with trace equal 1. A natural generalisation of an entangled mixed state is a non-separable state: A state ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is called *separable* if it can be written as a convex combination of product states,

$$\rho = \sum_{i=1}^{m} c_i \rho_i^{(1)} \otimes \rho_i^{(2)}.\tag{3.8}$$

Indeed, it is easy to see that such states admit a hidden-variable model for the correlations between A and B and Bell's inequalities hold. However, it was discovered by Werner [25] that there exist non-separable states which nevertheless satisfy Bell's inequalities and even admit a classical (hidden variable) model. His example is as follows:

$$\rho_W = \frac{1}{6}\begin{pmatrix} 1+q & 0 & 0 & 0 \\ 0 & 2-q & 2q-1 & 0 \\ 0 & 2q-1 & 2-q & 0 \\ 0 & 0 & 0 & 1+q \end{pmatrix},\tag{3.9}$$

where $q \in [-1, 1]$. (In fact, his construction is valid for higher dimensions, which is relevant in connection with an argument using Gleason's theorem demonstrating the impossibility of hidden variables, which is only valid for $d > 2$.) Now, $q = \mathrm{Tr}\,(V\rho_W)$, where V is the exchange operator

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that if ρ is of the form (3.8) then $\mathrm{Tr}\,(\rho V) \geq 0$, so ρ_W is not separable if $q < 0$.

A hidden variable model for this state in the form

$$\int_\Omega f_A(a, \lambda)\, f_B(b, \lambda)\, \mu(d\lambda) = \mathrm{Tr}\,(\rho_W P_a \otimes Q_b),\tag{3.10}$$

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where P_a is the eigenprojection of A for the eigenvalue a, and similarly Q_b for B, can be constructed as follows. We can assume that A and B are 1-dimensional projections, $A = \frac{1}{2}(\mathbf{1} + \vec{a} \cdot \vec{\sigma})$ and $B = \frac{1}{2}(\mathbf{1} + \vec{b} \cdot \vec{\sigma})$, and take the measure space Ω to be the unit sphere S^2 with normalised Lebesgue measure as before, and define

$$f_A(\vec{\lambda}) = \mathrm{Tr}\,(A\,P_{\vec{\lambda}})$$

and

$$f_B(\vec{\lambda}) = 1_{\{\vec{\lambda}:\,\vec{b}\cdot\vec{\lambda}<0\}}.$$

Then the left-hand side of (3.10) equals

$$\int_{S^2} \frac{1}{2}(1 + \vec{a}\cdot\vec{\lambda})\,1_{\{\vec{\lambda}:\,\vec{b}\cdot\vec{\lambda}<0\}}d\vec{\lambda}.$$

To compute this, we may assume $\vec{b} = \vec{e}_z$ and $\vec{a}$ given by polar angles (θ_a, ϕ_a) and $\vec{a}\cdot\vec{\lambda} = \cos\theta_a\cos\theta + \cos(\phi_a - \phi)\sin\theta_a\sin\theta$, so

$$\int_{S^2} \frac{1}{2}(1 + \vec{a}\cdot\vec{\lambda})\,1_{\{\vec{\lambda}:\,\vec{b}\cdot\vec{\lambda}<0\}}d\vec{\lambda}$$

$$= \frac{1}{4\pi}\int_{\pi/2}^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\,\frac{1}{2}\left\{1 + \cos\theta_a\cos\theta + \cos(\phi_a-\phi)\sin\theta_a\sin\theta\right\}$$

$$= \frac{1}{4} - \frac{1}{8}\cos\theta_a.$$

On the other hand, the right-hand side of (3.10) is

$$\mathrm{Tr}\left[\rho_W\frac{1}{2}(\mathbf{1} + \vec{a}\cdot\vec{\sigma})\otimes\frac{1}{2}(\mathbf{1} + \vec{b}\cdot\vec{\sigma})\right] = \frac{1}{4}\left(1 + \frac{2}{3}(q - \frac{1}{2})\vec{a}\cdot\vec{b}\right).$$

It follows that $q = -\frac{1}{4}$ and the state is not separable. Since it admits a classical model (3.10) Bell's inequalities (and generalisations) are satisfied, and these inequalities are therefore insufficient to conclude that a state is classically correlated (separable).

Remark. This representation is not entirely satisfactory since the function $f_A(\lambda)$ is not an indicator function, i.e. it does not take values in $\sigma(A)$. However, this can be remedied by writing $\frac{1}{2}(1 + \vec{a}\cdot\vec{\lambda})$ in the form (1.1):

$$\frac{1}{2}(1 + \vec{a}\cdot\vec{\lambda}) = \int 1_{\{\vec{\lambda}':\,(\vec{\lambda}'-\vec{\lambda})\cdot\vec{a}<0\}}d\vec{\lambda}'.$$

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Moreover, replacing f_B by

$$f_B(\vec{\lambda}) = 1_{\{\vec{\lambda}:\, u < \vec{b} \cdot \vec{\lambda} < u+1\}}$$

we obtain

$$\int_{S^2} \frac{1}{2}(1 + \vec{a} \cdot \vec{\lambda})\, 1_{\{\vec{\lambda}:\, \vec{b} \cdot \vec{\lambda} \in (u, u+1)\}}\, d\vec{\lambda}$$
$$= \frac{1}{4}\left(1 + (u + \frac{1}{2})\cos\theta_a\right).$$

This covers the range $q \in [-\frac{1}{4}, 0]$ when $u \in [-\frac{1}{2}, -1]$.

In fact, in the case of a pair of spin-$\frac{1}{2}$ particles, a necessary and sufficient condition for a state to be of the form (3.8) was introduced by Peres [26]. Introducing the *partial transpose* ρ^{T2} by

$$\langle ik \,|\, \rho^{T2} \,|\, jl \rangle = \langle il \,|\, \rho \,|\, jk \rangle, \qquad (3.11)$$

we say that ρ is positive under partial transposition if ρ^{T2} is also a positive definite matrix. It is clear that this is a necessary condition for a state to be separable, i.e. of the form (3.8). It was shown by Horodecki et al. [27] that for the case of spin-$\frac{1}{2}$ particles, it is also sufficient. However, this is not so for higher-dimensional cases.

3.2   Quantum teleportation

It is nowadays recognised that entanglement can in fact be a useful resource for quantum operations. An example of this is quantum teleportation. This is a scheme for moving a quantum state from one place to another using a shared entangled state, but transmitting only classical information. It assumes that quantum states can be accurately and reliably manipulated, i.e. it is possible to apply well-defined unitary evolutions. The original example due to Bennett et al. [28] is as follows:

Assume that Alice wants to send a general qubit state $\psi = \alpha|0\rangle + \beta|1\rangle$ to Bob, and they each possess one half of a singlet state ψ_s. Alice first performs

a CNOT operation on ψ and her half of the Bell state, i.e. the unitary

$$U_{CN} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The resulting combined state is:

$$U_{CN}\psi \otimes \psi_s = \frac{1}{\sqrt{2}}\left(\alpha|0\rangle \otimes (|01\rangle - |10\rangle) + \beta|1\rangle \otimes (|11\rangle - |00\rangle)\right).$$

Next she applies a Hadamard operation to the first qubit: $U_H \otimes \mathbf{1}$, with

$$U_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

This yields

$$
\begin{aligned}
& \frac{1}{2}\left(\alpha(|0\rangle \otimes +|1\rangle) \otimes (|01\rangle - |10\rangle)\right. \\
& \left. + \beta(|0\rangle - |1\rangle) \otimes (|11\rangle - |00\rangle)\right) \quad\quad (3.12) \\
= & \frac{1}{2}\left(|00\rangle \otimes (\alpha|1\rangle - \beta|0\rangle) - |01\rangle \otimes (\alpha|0\rangle - \beta|1\rangle)\right. \\
& \left. + |10\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) - |11\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)\right). \quad\quad (3.13)
\end{aligned}
$$

Finally she performs a measurement on her parts of the combined state resulting in one of the terms in brackets of (3.12). If her measurement results in $(1, 1)$ then Bob's state is just ψ. Otherwise, she needs to transmit her measurement result to Bob, who can then perform a suitable unitary transformation himself to bring the state back to ψ. For example, if the result is $(1, 0)$ the third term results and he needs to act with σ_x.

Notice that the inverse operation $U_{CN}(U_H \otimes \mathbf{1})$ maps the standard basis to the basis $\{\psi_k\}_{k=0}^3$ consisting of 'Bell states'

$$
\begin{aligned}
\psi_{0,1} &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \\
\psi_{2,3} &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).
\end{aligned}
$$

One can therefore also say that Alice simply performs a measurement w.r.t. this basis.

One can wonder if teleportation of states is possible using more general entangled states between Alice and Bob. In that case, the teleportation is obviously not going to be perfect. As for general quantum channels, one therefore introduces the concept of *fidelity* of transmission. If Alice wants to transmit a (pure) state ψ to Bob, but the state received by Bob is the (mixed) state ν, then the fidelity is defined the overlap

$$\mathcal{F}_\psi = \langle \psi \,|\, \nu \,|\, \psi \rangle = \mathrm{Tr}\,(\nu\, P_\psi).$$

Now, suppose Alice performs a measurement w.r.t. the Bell basis $\{\psi_k\}_{k=0}^3$, obtaining one of the results $k = 0, 1, 2, 3$ with probability p_k. She sends this result to Bob, as above, who performs a unitary transformation U_k to obtain the state ν_k. The expected value of the fidelity is then

$$\mathbb{E}(\mathcal{F}_\psi) = \sum_{k=0}^3 p_k \, \mathrm{Tr}\,(\nu_k\, P_\psi).$$

A measure of the efficiency of this procedure is given by the average of this quantity over possible states ψ:

$$\overline{\mathcal{F}} = \int \mathbb{E}(\mathcal{F}_\psi) d\psi = \sum_{k=0}^3 p_k \int \mathrm{Tr}\,(\nu_k\, P_\psi)\, d\psi. \tag{3.14}$$

If ρ is the shared entangled state and $P_k = |\psi_k\rangle\langle\psi_k|$ ($k = 0, 1, 2, 3$) are the projections corresponding to the measurement basis, then Bob's output state ν_k is

$$\nu_k = \frac{1}{p_k}\,\mathrm{Tr}\,_{1,2}\left[(P_k \otimes U_k)(P_\psi \otimes \rho)(P_k \otimes U_k^*)\right]$$

and the probabilities p_k are

$$p_k = \mathrm{Tr}\,\left[(P_k \otimes \mathbf{1})(P_\psi \otimes \rho)\right].$$

Following Horodecki et al. [29], we write ρ in terms of the basis of Pauli matrices:

$$\rho = \frac{1}{4}\left(\mathbf{1} + \vec{r} \cdot \vec{\sigma} \otimes \mathbf{1} + \mathbf{1} \otimes (\vec{s} \cdot \vec{\sigma}) + \sum_{n,m=1}^3 t_{nm}\sigma_n \otimes \sigma_m\right). \tag{3.15}$$

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Putting $P_\psi = \frac{1}{2}(\mathbf{1} + \vec{a} \cdot \vec{\sigma})$, we have

$$
\begin{aligned}
p_k \nu_k &= \frac{1}{8}\Bigg\{ \text{Tr}\left[P_k(\mathbf{1} + \vec{a} \cdot \vec{\sigma}) \otimes (\mathbf{1} + \vec{r} \cdot \vec{\sigma}) P_k \right] \mathbf{1} \\
&\quad + \text{Tr}\left[P_k((\mathbf{1} + \vec{a} \cdot \vec{\sigma}) \otimes \mathbf{1}) P_k \right] U_k (\vec{s} \cdot \vec{\sigma}) U_k^* \\
&\quad + \sum_{n,m=1}^{3} t_{nm} \text{Tr}\left[P_k((\mathbf{1} + \vec{a} \cdot \vec{\sigma}) \otimes \sigma_n) P_k \right] U_k \sigma_m U_k^* \Bigg\}.
\end{aligned}
$$

We now use

$$
\begin{aligned}
\langle \psi_{0,1} \,|\, \sigma_n \otimes \sigma_m \,|\, \psi_{0,1} \rangle &= \pm \delta_{n,1}\delta_{m,1} \mp \delta_{n,2}\delta_{m,2} + \delta_{n,3}\delta_{m,3} \\
\langle \psi_{2,3} \,|\, \sigma_n \otimes \sigma_m \,|\, \psi_{2,3} \rangle &= \pm \delta_{n,1}\delta_{m,1} \pm \delta_{n,2}\delta_{m,2} - \delta_{n,3}\delta_{m,3}.
\end{aligned}
$$

The result is:

$$
p_k \nu_k = \frac{1}{8} \left\{ (1 + \vec{a} \cdot D_k \vec{r}) \mathbf{1} + U_k \left((\vec{s} \cdot \vec{\sigma}) + \vec{a} \cdot D_k T \vec{\sigma} \right) U_k^* \right\}, \tag{3.16}
$$

where D_k are diagonal matrices: $D_0 = \text{diag}(+1, -1, +1)$, $D_1 = \text{diag}(-1, +1, +1)$, $D_2 = \text{diag}(+1, +1, -1)$, $D_3 = \text{diag}(-1, -1, -1)$. The unitary transformation U_k affects a rotation of the vector $\vec{s}$:

$$
U_k (\vec{s} \cdot \vec{\sigma}) U_k^* = (O_k \vec{s}) \cdot \vec{\sigma}.
$$

Averaging over ψ according to the uniform measure over $\vec{a} \in S^2$, we have

$$
\int_{S^2} (\vec{a} \cdot A\vec{a}) \, d\vec{a} = \frac{1}{3} \text{Tr } A
$$

and taking the trace using $\text{Tr } \sigma_i = 0$, we get

$$
\overline{\mathcal{F}} = \frac{1}{8} \sum_{k=0}^{3} p_k \left(1 + \frac{1}{3} \text{Tr } D_k T O_k \right). \tag{3.17}
$$

We need to maximise this expression over all possible choices of U_k, or equivalently O_k ($k = 0, 1, 2, 3$). Since each D_k is a reflection, each term has the same maximum and

$$
\max_{\{U_k\}} \overline{\mathcal{F}} = \max_{O} \frac{1}{2}\left(1 - \frac{1}{3} \text{Tr } TO\right).
$$

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Note that if $\rho = \rho^{(1)} \otimes \rho^{(2)}$ is a product state and we write $\rho^{(1)} = \frac{1}{2}(\mathbf{1} + \vec{r} \cdot \vec{\sigma})$, $\rho^{(2)} = \frac{1}{2}(\mathbf{1} + \vec{s} \cdot \vec{\sigma})$, where in general $|\vec{r}| \leq 1$ and $|\vec{s}| \leq 1$, then $t_{nm} = r_n s_m$ and hence

$$\overline{\mathcal{F}} = \max_O \frac{1}{2}\left(1 - \frac{1}{3}\langle \vec{s}, O\vec{r}\rangle\right) = \frac{1}{2}\left(1 + \frac{1}{3}|\vec{r}|\,|\vec{s}|\right) \leq \frac{2}{3}.$$

For separable states, therefore, the maximum is 2/3, attained for a pure product state.

In order that a general entangled state ρ improves on this, we need $\mathrm{Tr}\,(TO) < -1$. This is the case if $\det(T) < 0$ and $||T||_1 > 1$ because in that case we can define O by $-T\psi \mapsto |T|\psi$.

Horodecki et al. [30] show that the states ρ can be written as $\rho = (U_1 \otimes U_2)\tilde{\rho}(U_1 \otimes U_2)^*$, where $\tilde{\rho}$ has a diagonal matrix T belonging to the tetrahedron with corners

$$\vec{t_0} = (-1, -1, -1), \ \vec{t_1} = (-1, 1, 1), \ \vec{t_2} = (1, -1, 1), \ \vec{t_3} = (1, 1, -1).$$

It follows from this that $||T||_1 > 1$ is in fact a necessary and sufficient condition. Moreover, they also show that the diagonal matrices for separable states belong to the octahedron with corners

$$\vec{o_1}^{\pm} = (\pm 1, 0, 0), \ \vec{o_2}^{\pm} = (0, \pm 1, 0) \text{ and } \vec{o_3}^{\pm} = (0, 0, \pm 1).$$

This implies that all non-separable states are useful for state teleportation in the sense that $\overline{\mathcal{F}} > 2/3$. For example, for Werner's state, which can be written as

$$\rho_W = \frac{1}{4}\left(\mathbf{1} + \frac{2}{3}(q - \frac{1}{2})\sum_{i=1}^{3} \sigma_i \otimes \sigma_i\right),$$

$||T||_1 = |2q-1| > 1$ for all $q \in [-1, 0)$, i.e. whenever the state is not separable, even if Bell's inequalities hold. This was first remarked by Popescu [31].

3.3 Quantum channels

State teleportation is a special example of a quantum channel. Information is transmitted in the form of quantum states. This can be classical information

(bits) or quantum information (quantum states or qubits). (In the case of teleportation, the shared quantum state can be seen to be the channel, but in addition classical side-information is transmitted.)

Information theory, initiated by Shannon, has largely been extended to the quantum domain. In particular, there is an analogue of Shannon's theorem about the capacity of a channel [32], both for the case of classical information and for quantum information. For classical information, the result is due to Holevo and Schumacher and Westmoreland [34, 33]. (See also [35] and [36].) A quantum channel can be modelled by a completely positive map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ mapping (in general mixed) states on the input Hilbert space $\mathcal{H}$ to states on the output Hilbert space $\mathcal{K}$. In the case of a *memoryless channel* this map acts repeatedly, and a classical message is encoded by Alice into a quantum state $\rho^{(n)}$ of $\mathcal{H}^{\otimes n}$. The output state $\sigma^{(n)} = \Phi^{\otimes n}(\rho^{(n)})$ is then decoded by Bob by performing a generalised measurement. Such a measurement is given by a set of positive operators (not necessarily projections) $\{E_j^{(n)}\}$ with $\sum_j E_j^{(n)} = \mathbf{1}$. This is called a positive-operator-valued measure (POVM). The probability of outcome j is then given by $\mathrm{Tr}\,(\sigma^{(n)} E_j^{(n)})$. As in the case of Shannon's theorem, the (classical) *capacity* of the channel is then given by the maximal rate

$$\lim_{n \to \infty} \frac{1}{n} \log_2 N_n$$

at which messages can be transmitted with negligible error in the limit as $n \to \infty$. More precisely, one has:

Theorem 3.1 *Given $\epsilon > 0$, there exists n_0 such that for all $n \geq n_0$ there are at least $N_n = [2^{n(\chi(\Phi) - \epsilon)}]$ product states $\rho_1^{(n)}, \ldots, \rho_{N_n}^{(n)} \in \mathcal{B}(\mathcal{H}^{\otimes n})$ and a POVM $\{E_j^{(n)}\}_{j=1}^{N_n}$ such that $\mathrm{Tr}\left(\Phi^{\otimes n}(\rho_j^{(n)}) E_j^{(n)}\right) > 1 - \epsilon$ for all j.*

Here the quantity $\chi(\Phi)$ is the *Holevo capacity* given by

$$\chi(\Phi) = \sup_{\{\rho_j, p_j\}} \left[S\left(\sum_j p_j \Phi(\rho_j) \right) - \sum_j S(\Phi(\rho_j)) \right],$$

where S is the Von Neumann entropy $S(\rho) = -\mathrm{Tr}\,(\rho \log \rho)$, and the supremum is over ensembles of states $\rho_j \in \mathcal{B}(\mathcal{H})$ with probabilities p_j. If general

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states $\rho_j^{(n)}$ are admitted, the capacity is a limit:

$$\lim_{n\to\infty} \frac{1}{n}\chi(\Phi^{\otimes n})$$

but this quantity is obviously not easily computed. There are also extensions to channels with memory: see [37] and [38].

The quantum analogue of this theorem was proved by Devetak [39]. Here, one encodes and decodes states according to $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_P^{\otimes n})$ and $\mathcal{D} : \mathcal{B}(\mathcal{H}_Q^{\otimes n}) \to \mathcal{B}(\mathcal{H})$ and one wants to transmit an arbitrary (pure) state $\phi \in \mathcal{H}$ with near-perfect fidelity:

$$\min_{\phi \in \mathcal{H}} \mathcal{F}\left(\phi, (\mathcal{D} \circ \Phi^{\otimes n} \circ \mathcal{E})(|\phi\rangle\langle\phi|)\right) > 1 - \epsilon. \tag{3.18}$$

The analogue of Holevo's quantity is the *coherent information* $I_c(\rho, \Phi)$. It is given by

$$I_c(\rho, \Phi) = S(\Phi(\rho)) - S(\rho, \Phi)$$

where $S(\rho, \Phi)$ is the entropy exchange: see [40]. His theorem then reads as follows:

Theorem 3.2 *Given $\epsilon > 0$ there exists n_0 such that if $N_n = [2^{n(I(\Phi)-\epsilon)}]$, where*

$$I(\Phi) = \lim_{n\to\infty} \frac{1}{n} \max_{\rho^{(n)} \in \mathcal{H}_P^{\otimes n}} I_c(\rho^{(n)}, \Phi^{\otimes n}),$$

then for a Hilbert space $\mathcal{H}^{(n)}$ of dimension N_n there are encoding and decoding maps $\mathcal{E}$ and $\mathcal{D}$ such that (3.18) holds.

4 Quantum field theory

It is worth mentioning a generalisation of the Bell inequalities to quantum field theories considered by Summers and Werner [41, 42]. If $\mathcal{A}$ and $\mathcal{B}$ are commuting sub C*-algebras of a C* algebra $\mathcal{C}$ and ω is a state on $\mathcal{C}$, then whenever $A_1, A_2 \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$ satisfy $-\mathbf{1}_{\mathcal{A}} \leq A_i \leq \mathbf{1}_{\mathcal{A}}$ and $-\mathbf{1}_{\mathcal{B}} \leq B_j \leq \mathbf{1}_{\mathcal{B}}$ then

$$\chi := \frac{1}{2}|\omega(A_1(B_1 + B_2)) + \omega(A_2(B_1 - B_2))| \leq \sqrt{2}.$$

Moreover, if ω is separable then $\chi \leq 1$.

In the algebraic framework of relativistic quantum field theory, $\mathcal{A}$ and $\mathcal{B}$ can be local algebras $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ where O_1 and O_2 are space-like separated. Assuming in particular that there is a unitary representation U of the translation group which acts covariantly, i.e.

$$U(x)\mathcal{A}(O)U(x)^{-1} = \mathcal{A}(O_x) \text{ for } x \in \mathbb{R}^4$$

and a unique vacuum vector Ω, the corresponding state ϕ_0 given by $\phi_0(A) = \langle \Omega, | A\Omega \rangle$ satisfies a much more stringent bound:

$$\chi \leq 1 + 2e^{-md(O_1,O_2)},$$

where $d(O_1, O_2)$ is the maximal time-like distance between O_1 and O_2, and it is assumed that the Hamiltonian H has spectrum contained in $\{0\} \cup [m, +\infty)$ with $m > 0$. This suggests that verifying the violation of Bell's inequality is unrealistic in massive field theories. They also show, however, that in case O_1 and O_2 are complimentary 'wedges', Bell's inequality is generically maximally violated in quantum field theories, more precisely, χ approaches $\sqrt{2}$ for suitable sequences of observables with norm ≤ 1.

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