A GENERAL VANISHING THEOREM

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ABSTRACT. Let E be a vector bundle and L be a line bundle over a smooth projective variety X. In this article, we give a condition for the vanishing of Dolbeault cohomology groups of the form $H^{p,q}(X, \mathcal{S}^\alpha E \otimes \wedge^\beta E \otimes L)$ when $S^{\alpha+\beta} E \otimes L$ is ample. This condition is shown to be invariant under the interchange of p and q. The optimality of this condition is discussed for some parameter values.

1. Introduction

Throughout this paper X will denote a smooth projective variety of dimension n over the field of complex numbers, E a vector bundle of rank e, and L a line bundle on X.

For any non-negative integers α, β we denote by $S^\alpha E$, $\wedge^\beta E$ the symmetric product and the exterior product of E. $H^{p,q}(X, S^\alpha E \otimes \wedge^\beta E \otimes L)$ will denote the Dolbeault cohomology group

$$H^q(X, S^\alpha E \otimes \wedge^\beta E \otimes L \otimes \Omega_X^p),$$

where Ω_X^p is the bundle of exterior differential forms of degree p on X.

We start with some definitions.

Definition 1.1. The function $\delta : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{N}$ is the one which satisfies

$$\delta(x) = m \Longleftrightarrow \binom{m}{2} \leq x < \binom{m+1}{2}.$$

The last two inequalities imply

$$\delta(x) = [\frac{\sqrt{8x+1}+1}{2}],$$

where the symbol [ ] denotes the integral part.

i.e., $\delta(0) = 1,\ \delta(1) = \delta(2) = 2, \delta(3) = \delta(4) = \delta(5) = 3,$ $\delta(6) = \delta(7) = \delta(8) = \delta(9) = 4,\ \ldots$

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Theorem 1.2. *Let $\alpha, \beta \in \mathbb{N}$. If $S^{\alpha+\beta}E \otimes L$ is ample , then*

$$H^{p,q}(X, S^{\alpha}E \otimes \wedge^{\beta}E \otimes L) = 0$$

$$\text{for} \quad q + p - n > (r_0 + \alpha)(e + \alpha - \beta) - \alpha(\alpha + 1),$$

$$\text{where} \quad r_0 = \min\{\beta, \delta(\mathrm{n-p}), \delta(\mathrm{n-q})\}.$$

Corollary 1.3. *Let β be a positive integer. If $S^{\beta}E \otimes L$ is ample, then*

$$H^{p,q}(X, \wedge^{\beta}E \otimes L) = 0,$$

$$\text{for} \quad q + p - n > r_0(e - \beta).$$

$$\text{where} \quad r_0 = \min\{\beta, \delta(\mathrm{n-p}), \delta(\mathrm{n-q})\}.$$

This Corollary improve the result of Manivel "theorem 1. p.91" in [13].

Corollary 1.4. *Assume $S^{\alpha}E \otimes L$ is ample. Then*

$$H^{p,q}(X, S^{\alpha}E \otimes L) = 0,$$

$$\text{for} \quad q + p - n > \alpha(e - 1).$$

This article is the final version of several attempts [16], [11]. The result of these latest were used by Chaput in [3] and by Laytimi-Nagaraj in [7].

In [15] Manivel studied the vanishing of Dolbeault cohomology of a product of vector bundles tensored with certain power of their determinant. The presence of the latest allowed to deal with the problem by more direct method.

2. The Schur Functor Version of the theorem

Our main result is a consequence of a Schur functor version of the theorem, but before giving this version, we need to recall some definitions and results:

We start by some preparation on partitions and Schur functors (for a definition see [5]).

A partition $u = (u_1, u_2, \ldots, u_r)$ is a sequence of non increasing positive integers u_i. Its length is r and its weight is $|u| = \sum_{i=1}^{r} u_i$. For $i > r$ we put $u_i = 0$. The zero-partition is the one where all u_i are zero.

For any partition u the corresponding Schur functor is denoted by $\mathcal{S}_u$.

Let V be a vector space of dimension d. To each partition u corresponds an irreducible $Gl(V)$-module $\mathcal{S}_u(V)$ which vanishes iff $u_{d+1} > 0$.

For example, $\mathcal{S}_{(k)}V = S^kV$. By functoriality the definition of Schur functors carries over to vector bundles E on X.

By abuse of language we say that $\mathcal{S}_u$ has a certain property, if u has this property. For example we will say S^k has weight k.

Definition 2.1. The Young diagram $Y(u)$ of a partition u is given by

$$Y(u) = \{(i,j) \in \mathbb{N}^2 \mid j \leq u_i\}.$$

The transposed partition $\tilde{u}$ is defined by

$$Y(\tilde{u}) = \{(i,j) \in \mathbb{N}^2 \mid (j,i) \in Y(u)\}.$$

We use the notation $\wedge_u = \mathcal{S}_{\tilde{u}}$.

Definition 2.2. The rank of a partition u, is

$$rk(u) = max\{\rho \mid (\rho,\rho) \in Y(u)\}.$$

If $rk(u) = 1$, then u is called a hook.

Notation 2.3. If u is a hook with $u_1 = \alpha + 1$ and $|u| = k$, we write

$$\mathcal{S}_u = \Gamma_k^\alpha.$$

In particular, $\Gamma_k^0 = \wedge^k$ and $\Gamma_k^{k-1} = S^k$.

Recall that

$$S^\alpha E \otimes \wedge^\beta E = \Gamma_{\alpha+\beta}^\alpha E \oplus \Gamma_{\alpha+\beta}^{\alpha-1} E.$$

Definition 2.4. For partitions u, v of the same weight, the dominance partial ordering is defined by

$$u \succeq v, \quad \text{iff} \quad \sum_{i=1}^{j} u_i \;\geq\; \sum_{i=1}^{j} v_i \;\; \text{for all} \;\; j.$$

This partial ordering can be extended to a pre-ordering of the set of all non-zero partitions of arbitrary weight u, v with $|u| = n, |v| = m$, by comparing as above the partitions of the same weight mu and nv, where the multiplication

$$mu = m\ (u_1, u_2, \ldots, u_r) = (mu_1, \ldots, mu_r)\ \forall\ m \in \mathbb{N}.$$

More precisely $\quad u \succeq v \quad \text{iff} \quad mu \succeq nv.$

We write $\quad u \simeq v \quad \text{iff} \quad u \succeq v \;\text{ and }\; v \preceq u.$

When it is more convenient we will write $\mathcal{S}_u \succeq \mathcal{S}_v$ instead of $u \succeq v$. For example, $\wedge^r \succ \wedge^{r+1}$, and $S^\alpha \simeq S^1$ for any $\alpha \in \mathbb{N}$.

Lemma 2.5. (Dominance Lemma) ([8] "theorem 3.7")

For any partition u and v.
If $u \succeq v$, then $\mathcal{S}_u E$ ample $\Longrightarrow \mathcal{S}_v E$ ample.

For example: If $\wedge^2 E$ is ample, then $\wedge^3 E$ is ample.

Now we give the Schur presentation of the main theorem under which the main theorem will be shown. With the notation 2.3 we have:

Theorem 2.6. *Let $k \in \mathbb{N}$. If $S^k E \otimes L$ is ample , then*

$$H^{p,q}(X, \Gamma_k^\alpha E \otimes L) = 0,$$
$$\text{for } \; q + p - n > (r_0 + \alpha)(e - k + 2\alpha) - \alpha(\alpha + 1),$$
$$\text{where } r_0 = \min\{\beta, \delta(\mathrm{n} - \mathrm{p}), \delta(\mathrm{n} - \mathrm{q})\}.$$

Proposition 2.7. *Theorem 2.6 is equivalent to Theorem 1.2*

Proof: Since
$$\mathcal{S}^\alpha E \otimes \wedge^{k-\alpha} E = \Gamma_k^\alpha E \oplus \Gamma_k^{\alpha-1} E,$$
we have only to show that for $1 \leq \alpha \leq k-1$ the conditions of Theorem 1.2 imply the vanishing of $H^{p,q}(X, \Gamma_k^{\alpha-1} E)$, but this is clear since the function $(r_0 + \alpha)(e - k + 2\alpha) - \alpha(\alpha + 1)$ is increasing in α.

□

3. Some Technical Lemmas

We start with some proprieties of the function δ defined in 1.1.

Lemma 3.1. *For $\mu \in \mathbb{N}, x \in \mathbb{N}$ such that $(x + \mu\delta(x), x - \mu\delta(x)) \in \mathbb{N} \times \mathbb{N}$, we have*

$$1) \quad \delta(x + \delta(x)) = \delta(x) + 1$$
$$2) \quad \delta(x + \mu\delta(x)) \leq \delta(x) + \mu$$
$$3) \quad \delta(x - \mu\delta(x)) \leq \delta(x) - \mu.$$

Proof: The first assertion and the case $\mu = 1$ in 2) and 3) are obvious. For both remaining assertions we use induction on μ.

For 2)
$\delta(x + \mu\delta(x)) = \delta(x + \delta(x) + (\mu - 1)\delta(x))$, since
$\delta(x) \leq \delta(x + \delta(x)) = \delta(x) + 1$, we have
$\delta(x + \delta(x) + (\mu - 1)\delta(x)) \leq \delta(x + \delta(x) + (\mu - 1)\delta(x + \delta(x))$.

Now induction hypothesis gives
$\delta(x + \delta(x) + (\mu - 1)\delta(x + \delta(x)) \leq \delta(x + \delta(x)) + \mu - 1 = \delta(x) + \mu$.

For 3)
$\delta(x-\mu\delta(x)) = \delta(x-\delta(x)-(\mu-1)\delta(x))$,
$\delta(x-\delta(x)-(\mu-1)\delta(x)) \leq \delta(x-\delta(x)-(\mu-1)\delta(x-\delta(x))$.
Induction hypothesis gives
$\delta(x-\delta(x)+(\mu-1)\delta(x-\delta(x)) \leq \delta(x-\delta(x))-(\mu-1)$.
Now since it is true for $\mu=1$, we get $\delta(x-\delta(x))-(\mu-1) \leq \delta(x)-\mu$.
□

Definition 3.2. Let $\phi : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ the following injection $\phi(x,\alpha) = (\phi_1(x,\alpha), \phi_2(x,\alpha), \phi_3(x,\alpha))$, where

$$\begin{aligned}\phi_1(x,\alpha) &= \delta(x)+\alpha\\ \phi_2(x,\alpha) &= \ x-\binom{\delta(x)}{2}\\ \phi_3(x,\alpha) &= \alpha\end{aligned}$$

We define an order on the pairs $(x,\alpha)\in\mathbb{N}\times\mathbb{N}$ by the lexicographic order on $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ induced by ϕ, we denote this order by

$$(x',\alpha') \leq_\phi (x,\alpha)$$

The set $\mathbb{N}\times\mathbb{N}$ endowed with the above order will be denoted:

$$\{\mathbb{N}\times\mathbb{N},\ \ \leq_\phi\} := \mathfrak{U} \tag{3.1}$$

Lemma 3.3. *For* $\mu\in\mathbb{Z}-\{0\}$ *and* $(x+\mu\delta(x),\alpha-\mu)\in\mathbb{N}\times\mathbb{N}$, *then*

$$(x+\mu\delta(x),\alpha-\mu) \leq_\phi (x,\alpha).$$

where the order $\leq_\phi$ *is given in Definition 3.2.*

Proof: By Lemma 3.1 $\phi_1(x+\mu\delta(x),\alpha-\mu) \leq \alpha+\delta(x)$.

If $\delta(x+\mu\delta(x)) = \mu+\delta(x)$, then

$$\phi_2(x+\mu\delta(x),\alpha-\mu) = x-\binom{\delta(x)}{2}-\binom{\mu}{2} \leq x-\binom{\delta(x)}{2}.$$

If $\binom{\mu}{2}=0$, which means $\mu=1$, then

$$\phi_3(x+\mu\delta(x),\alpha-\mu) = \alpha-1 < \alpha.$$

□

We need to use these following results

Lemma 3.4. *Let* E *an ample vector bundle and* G *an arbitrary vector bundle on a projective variety* X. *Then for sufficiently large enough* n $S^nE\otimes G$ *is ample.*

Lemma 3.5. Bloch-Gieseker [2] *Let L be a line bundle on a projective variety X and d be a positive integer. Then there exist a projective variety Y, a finite surjective morphism $f : Y \to X$, and a line bundle M on Y, such that $f^*L \simeq M^d$.*

Lemma 3.6. *Let $p, q, n, f_1, \ldots, f_r$ be fixed positive integers and $\alpha^1, \ldots, \alpha^r$ be fixed non-zero partitions. If $H^{p,q}(X, \otimes_{i=1}^r \mathcal{S}_{\alpha^i} F_i) = 0$ for all smooth projective varieties X of dimension n and all ample vector bundles $F_1, \ldots, F_r$ of ranks $f_1, \ldots, f_r$ on X, then this vanishing statement remains true if one of the F_i is ample and the others are nef.*

Proof: We can reorder the F_i such that F_1 is ample. Let $E = F_1$ and $\alpha = \alpha^1$. Let N be a sufficiently large number such that $S^N E \otimes \det E^*$ is ample (for the existence of such N see Lemma 3.4, and let $a = \sum_{i=2}^m |\alpha^i|$. By Lemma 3.5 we can find a finite surjective morphism $f : Y \to X$, and a line bundle M on Y, such that $f^*(\det E) = M^{Na}$. Then $E_a = f^*E \otimes (M^*)^a$ is ample since $S^N E_a$ is. We have

$$f^*(\mathcal{S}_\alpha E \otimes_{i=2}^m \mathcal{S}_{\alpha^i} F_i) = \mathcal{S}_\alpha E_a \otimes_{i=2}^m \mathcal{S}_{\alpha^i} F_i',$$

where $F_i' = M^{|\alpha|} \otimes f^*F_i$ for $i = 2, \ldots, m$. All F_i' are ample. To finish the proof, we use "lemma 10 in [14] which says, For any vector bundle $\mathcal{F}$ on X and any finite surjective morphism $f : Y \to X$, the vanishing of $H^{p,q}(Y, f^*\mathcal{F})$ implies the vanishing of $H^{p,q}(X, \mathcal{F})$.

□

Lemma 3.7. *Fix $n, p, q, k, \alpha \in \mathbb{N}$ and $t \in \mathbb{Z}$. Assume that*

$$H^{p,q}(X, \Gamma_k^\alpha E)$$

vanishes for all smooth projective varieties X of dimension n and all ample vector bundles E of rank $e = k + t$ on X. Let $\alpha < k' < k$. Then $H^{p,q}(X, \Gamma_{k'}^\alpha E')$ vanishes for all ample vector bundles E' of rank $e' = k' + t$ on X.

Proof: For given E', put $E = E' \oplus L^{\oplus(k-k')}$, where L is any ample line bundle. Since $\Gamma_{k'}^\alpha E' \otimes L^{k-k'}$ is a direct summand of $\Gamma_k^\alpha E$, we have

$$H^{p,q}(X, \Gamma_{k'}^\alpha E' \otimes L^{k-k'}) = 0$$

for ample vector bundle E' of rank e' and ample line bundle L. By Lemma 3.6 , this vanishing result remains true, when L is replaced by the trivial line bundle. □

Corollary 3.8. *Assume that there is an integer k_0 such that*

$$H^{p,q}(X, \Gamma_k^\alpha E) = 0 \quad \textit{if} \quad k > k_0,$$

for any projective smooth variety X of dimension n and any ample vector bundle E of rank e, under the condition $C(n,p,q,\alpha,e-k)$. Then under this same condition the vanishing remains true for all k.

The Bloch-Gieseker lemma can be used in other way to generalize vanishing theorems. In particular one has

Lemma 3.9. *Fix $n,p,q,e \in \mathbb{N}$ and partitions u,v of the same weight. Assume that $H^{p,q}(X,\mathcal{S}_uE)$ vanishes for all projective varieties X of dimension n and all vector bundles E of rank e for which $\mathcal{S}_vE$ is ample. Let L be a line bundle and F a vector bundle of rank e. Then $H^{p,q}(X,\mathcal{S}_uF\otimes L)=0$, if $\mathcal{S}_vF\otimes L$ is ample.*

Proof: Let's denote $|u|=|v|=d$. By Lemma 3.5 we can find a finite surjective morphism $f:Y\to X$, and a line bundle M on Y, such that $f^*L=M^d$. Then

$$\mathcal{S}_v(f^*F\otimes M)=f^*(\mathcal{S}_vF\otimes L) \text{ is ample.} \tag{3.2}$$

Due to the analogous equation (3.2) for $\mathcal{S}_u$ one has by assumption

$$H^{p,q}(Y,f^*(\mathcal{S}_uF\otimes L))=0,$$

and the vanishing of $H^{p,q}(X,\mathcal{S}_uF\otimes L)$ follows by using "lemma 10 in [14].

□

The lemma applies for example if $\mathcal{S}_vF$ is nef and L is ample.

Corollary 3.10. To generalize vanishing of type $H^{p,q}(X,\mathcal{S}_uF\otimes L)$, from $L=\mathcal{O}_X$ to arbitrary L, it suffices to use Lemma 3.9.

We need to recall

Lemma 3.11. ([6] "lemma 1.3") Let X be a projective variety, E, F be vector bundles on X. If E is ample and F nef , then $E\otimes F$ is ample.

4. The Borel-Le Potier Spectral Sequence

To prepare the proof, we need a lemma and some properties of the Borel-Le Potier spectral sequence, which has been made a standard tool in the derivation of vanishing theorems [4].

Let E be a vector bundle over a smooth projective variety X, dim(X) = n. Let $Y=G_r(E)$ be the corresponding Grassmann bundle and Q be the canonical quotient bundle over Y.

Lemma 4.1. *Let l,r be positive integer and $k=lr$, if $\wedge^rE$ is ample. Then for $P+q>n+r(e-r)$*

$$H^{P,q}(G_r(E),\det Q^l)=0.$$

Proof: Since $\det Q = \mathcal{O}_{\mathbb{P}(\wedge^r E)}(1)|_{G_r(E)}$. Thus $\Lambda^r E$ ample implies that $\det Q$ is ample. One conclude by using Nakano-Akizuki-Kodaira vanishing theorem [1]. □

Definition 4.2. Let $\pi : Y \to X$ be a morphism of projective manifolds, P a positive integer and $\mathcal{F}$ a vector bundle over Y. The Borel-Le Potier spectral sequence ${}^P E$ given by the data $\pi, P, \mathcal{F}$ is the spectral sequence which abuts to $H^{P,q}(Y, \mathcal{F})$, it is obtained from the filtration on $\Omega_Y^P \otimes \mathcal{F}$ which is induced by the filtration

$$F^p(\Omega_Y^P) = \pi^*\Omega_X^p \wedge \Omega_Y^{P-p}$$

on the bundle Ω_Y^P of exterior differential forms of degree P.

The graded bundle which corresponds to the filtration on Ω_Y^P is given by

$$F^p(\Omega_Y^P)/F^{p+1}(\Omega_Y^P) = \pi^*\Omega_X^p \otimes \Omega_{Y/X}^{P-p},$$

where $\Omega_{Y/X}^{P-p}$ is the bundle of relative differential forms of degree $P-p$. Thus the E_1 terms of ${}^P E$ have the form

$${}^P E_1^{p,q-p} = H^q(Y, \pi^*\Omega_X^p \otimes \Omega_{Y/X}^{P-p} \otimes \mathcal{F}).$$

These E_1 terms can be calculated as limits groups of the Leray spectral sequence associated to the projection π,

$${}^{p,P}E_{2,L}^{q-j,j} = H^{p,q-j}(X, R^j\pi_*(\Omega_{Y/X}^{P-p} \otimes \mathcal{F}))$$

Now we consider the Borel-Le Potier spectral sequence which abuts to $H^{P,q}(G_r(E), \det Q^l)$.

Proposition 4.3. *Let $\pi : G_r(E) = Y \to X$, the E_1 terms of the Borel-Le Potier spectral sequence given by $\pi, P, \det Q^l$ have the form*

$${}^P E_1^{p,q-p} = \bigoplus_{u \in \sigma(P-p,r)} H^q(G_r(E), \mathcal{S}_u Q^* \otimes \det Q^l \otimes \wedge_u S \otimes \pi^*\Omega_X^p).$$

Here S is the tautological sub-bundle of π^*E over Y and $\sigma(p,r)$ is the set of partitions of weight p and length at most r.

Proof: One has $\Omega_{Y/X} = Q^* \otimes S$. Thus

$$\Omega_{Y/X}^{P-p} = \bigoplus_{u \in \sigma(P-p,r)} \mathcal{S}_u Q^* \otimes \wedge_u S.$$ □

Obviously Leray spectral sequence degenerates at the $E_{2,L}$ level.

Using the corollary 1. in ([13] page 94) of Bott formula, Manivel computes the E_1 terms under some condition on P, ([13] Proposition 3. page 96). He states his result under the supplementary condition $e \geq k$, which is not necessary for the calculation.

Proposition 4.4. [13]
Assume $P \geq n + (l-1)\binom{r+1}{2} - l(r-1)$, *and* $k = lr$. *Let*

$$\alpha(p) = \frac{(l-1)(r+1)}{2} - \frac{P-p}{2}$$
$$j(p) = (l-1)\binom{r}{2} - (r-1)\alpha(p).$$

Then the E_1 *terms of the spectral sequence have the form*

$${}^{P}E_1^{p,q} = \begin{cases} H^{p,q-j(p)}(X, \Gamma^{\alpha,k}E) & \textit{for } \ (n-p, \alpha(p)) \in \mathfrak{U} \\ 0 & \textit{otherwise,} \end{cases}$$

where the set $\mathfrak{U}$ *is defined in* (3.1).

Note that the connecting morphisms of Borel-Le Potier spectral sequence

$$d_m : \quad {}^{P}E_m^{p,q-p} \longrightarrow \quad {}^{P}E_m^{p+m,q-p+1-m}$$

all vanish, unless m is a multiple of r since under d_m the integer α goes to the integer $\alpha + \frac{m}{r}$.

5. Proof of the main theorem

Before giving the proof of the main theorem, we will first explain the case $r_0 = \beta$ in the main theorem, which corresponds to Corollary 5.2 bellow.

We need to recall these results

Theorem 5.1. [9] *Let* E_i *be vector bundles, with* $rank(E_i) = e_i$, *over a smooth projective variety* X *of dimension* n, *and let* L *be a line bundle on* X. *If* $\otimes_{i=1}^m \Lambda^{r_i} E_i \otimes L$ *is ample, then*

$$H^{p,q}(X, \otimes_{i=1}^m \Lambda^{r_i} E_i \otimes L) = 0 \ \text{ for } \ p+q-n > \sum_{i=1}^m r_i(e_i - r_i).$$

Corollary 5.2. *Let* E *be a vector bundle of rank* e, *and let* L *be a line bundle on a smooth projective variety* X *of dimension* n. *If* $S^{\alpha+\beta}E \otimes L$ *is ample, then*

$$H^{p,q}(X, S^{\alpha}E \otimes \Lambda^{\beta}E \otimes L) = 0 \quad \text{for} \quad q+p-n > \alpha(e-1) + \beta(e-\beta).$$

Proof: We will apply the Theorem 5.1 to the vector bundle $\underbrace{E \otimes E \cdots \otimes E}_{\alpha \ times} \otimes \Lambda^\beta E \otimes L$, which $S^\alpha E \otimes \Lambda^\beta E \otimes L$ is a direct summand of.

Let's first show this equivalence of ampleness

$$S^\alpha E \otimes F \simeq \underbrace{E \otimes E \cdots \otimes E}_{\alpha \ times} \otimes F \tag{5.1}$$

for any vector bundles F.

Indeed: For the first direction, Note that $S^\alpha E \otimes L$ is direct summand of $\underbrace{E \otimes E \cdots \otimes E}_{\alpha \ times} \otimes F$.

For the second direction, Littlewood-Richardson rules gives,

$$\underbrace{E \otimes E \cdots \otimes E}_{\alpha \ times} = S^\alpha E \oplus \sum_{|\lambda| = \alpha} S_\lambda E,$$

we have clearly $\alpha \succ \lambda$ in the dominance partial order. Use Remark 2.5 to conclude.

Now by Littlewood-Richardson rules

$$S^\alpha E \otimes \Lambda^\beta E = \oplus \ \mathcal{S}_\nu E, \text{ with } |\nu| = \alpha + \beta,$$

satisfying $\mathcal{S}_\nu \prec S^{\alpha+\beta}$. Thus the ampleness of $S^{\alpha+\beta} E \otimes L$ implies the ampleness of $S^\alpha E \otimes \Lambda^\beta E \otimes L$ by Remark 2.5. Use the equivalence of ampleness (5.1) to conclude.

□

Due to Remark 3.10 one can prove our main theorem without L.

We prove Theorem 1.2 by induction on $(n-p, \alpha) \in \mathfrak{U}$, where the set $\mathfrak{U}$ is given in Definition 3.2.

Assume that the result is true for all pairs (p', α') such that

$$(n - p', \alpha') \leq_\phi (n - p, \alpha),$$

with respect to the order introduced in Definition 3.2.

Choose $r = \delta(n-p)$. Let l be arbitrary if $n = p$, otherwise let $l \geq \frac{r\alpha + n - p}{r-1}$. Choose P such that $\alpha(p) = \alpha$, and consider the Borel-Le Potier spectral sequence. Then for $k = lr$

$${}^P E_1^{p,q+j(p)-p} = H^{p,q}(X, \Gamma^{\alpha,k} E).$$

When m is a multiple of r, the morphisms d_m connect ${}^P E_1^{p,q+j(p)-p}$ with ${}^P E_1^{p',q'+j(p')-p'}$ where for the terms on the right of ${}^P E_1^{p,q+j(p)-p}$

$$p' = p + \mu r, \quad q' = q + \mu(r-1) + 1, \tag{5.2}$$

and

$$(5.3)\qquad p' = p - \mu r, \quad q' = q - \mu(r-1) - 1$$

for the terms on the left. Here μ is any positive integer.

Lemma 5.3. *For any integers p' and q' of the form* (5.2) *or* (5.3)*,*

$$ {}^{P}E_1^{p',q'+j(p')-p'} = 0,$$

when $q > Q(n-p,\alpha)$, where

$$Q(n-p,\alpha) = n - p + (\delta(n-p) + \alpha)(e - k + 2\alpha) - \alpha(\alpha+1).$$

Proof: The assertion is trivially true for $\alpha(p') < 0$ or $e-k+\alpha(p') < 0$, such that we may assume $e - k + 2\alpha(p') \geq 0$.

We need to prove that the assertion $q > Q(n-p,\alpha)$ implies the assertion $q' > Q(n-p',\alpha')$.

The terms on the right of ${}^{P}E_1^{p,q+j(p)-p}$ have $\alpha' = \alpha + \mu$, and the parameters in (5.2), a straight calculation yields

$$Q(n-p,\alpha) - Q(n-p',\alpha') + \mu(\delta(n-p) - 1) + 1 =$$
$$(5.4)\quad (e - k + 2(\alpha+\mu)(\delta(n-p) - \mu - \delta(n-p-\mu\delta(n-p)) + \mu^2 + 1.$$

The terms on the left of ${}^{P}E_1^{p,q+j(p)-p}$ have $\alpha' = \alpha - \mu$, and the parameters in (5.3), the calculation yields

$$Q(n-p,\alpha) - Q(n-p',\alpha') - \mu(\delta(n-p) - 1) - 1 =$$
$$(5.5)\quad (e - k + 2(\alpha-\mu)(\delta(n-p) + \mu - \delta(n-p+\mu\delta(n-p)) + \mu^2 - 1.$$

By Lemma 3.1 both terms of (5.4) and (5.5) are non negative and positive if $\mu \neq 1$. Thus $q > Q(n-p,\alpha)$ implies $q' > Q(n-p',\alpha(p'))$.

By Lemma 3.3 $(n-p',\alpha(p')) \leq_\phi (n-p,\alpha)$, such that the groups ${}^{P}E_1^{p',q'+j(p')-p'}$ vanish by induction hypothesis. Thus all co-bordant morphisms of ${}^{P}E_1^{p,q+j(p)-p}$ vanish. This implies that ${}^{P}E_1^{p,q+j(p)-p}$ is a sub-factor of $H^{P,q+j(p)-p}(Y,F)$, where $F = \det(Q)^l$.

Recall that $P = p + (l-1)\binom{r+1}{2} - \alpha r$ and $\mathrm{dimY} = \mathrm{n + r(e - r)}$. Thus the condition $q > Q(n-p,\alpha)$ is equivalent to

$$P + q + j(p) - \mathrm{dimY} > \alpha(\mathrm{e - k} + \alpha).$$

When the right hand side is non-negative, $H^{P,q+j(p)-p}(Y,F) = 0$ by Nakano-Kodaira-Akizuki vanishing theorem. Thereby

$$H^{p,q}(X,\Gamma_k^\alpha E) = 0 \ \text{ for } \ q > Q(n-p,\alpha).$$

Remember that this proof was under the condition $k = rl$ see Proposition 4.4, but this condition can be removed by Corollary 3.8.

To get $r_0 = \delta(n-q)$ in our theorem, we interchange the role of p and q at every stage of the proof, in particular we use $r = \delta(n-q)$.

6. Optimality

Proposition 6.1. *Let $G = Gr_{(r,d)}$ be the Grassmannian of all co-dimensional r subspaces of a vector space V of dimension $d = f + r$. Let Q be the universal sub-bundle of rank r on G*, $\dim G = n = fr$.

Then, for $q = n - f$, $\alpha = f - 1$

$$H^q(G, S^{\alpha} Q \otimes Q \otimes \det Q \otimes K_X) \neq 0$$

Proof: Since $S^{\alpha+1}Q$ is direct summand of $S^{\alpha}Q \otimes Q$, it's enough to show

$$H^q(G, S^{\alpha+1} Q \otimes \det Q \otimes K_X) \neq 0.$$

For the universal sub-bundle S on G, we have $K_G = ((\det Q)^*)^{\otimes d} = \det S^{\otimes d}$.

Thus since $\alpha = f - 1$

$$H^q(G, S^f Q \otimes \det Q \otimes K_X) = H^q(G, S^f Q \otimes \det S^{\otimes (d-1)}.$$

Now by Bott formula (see corollary 1. page 94 of [13])

$$H^q(G, S^f Q \otimes \det S^{\otimes (d-1)} = \delta_{q,i((a,b)-c(d))} \mathcal{S}_{\psi(a,b)} V,$$

where

$$a = (f, \underbrace{0, \cdots, 0}_{r-1 \text{ times}}), \quad b = (\underbrace{d-1, \cdots, d-1}_{d-r \text{ times}}).$$

For any sequence $v = (v_1, v_2, \ldots)$

$$i(v) = \mathrm{card}\{(\mathrm{i},\mathrm{j}) \ / \ \mathrm{i} < \mathrm{j}, \ \mathrm{v_i} < \mathrm{v_j}\},$$

where

$$\psi(v) = (v - c(d))^{\geq} + c(d),$$

$$c(d) = (1, 2, \ldots, d),$$

and $(v)^{\geq}$ is the partition obtained by ordering the terms of v in non increasing order.

$$(a, b) = (f, \underbrace{0, \cdots, 0}_{r-1 \text{ times}}), \underbrace{d-1, \cdots, d-1}_{d-r \text{ times}}).$$

$$((a, b) - c(d)) = (f-1, -2, -3, \ldots, -r, f-2, f-3, \ldots, 0, -1),$$

we get $i((a,b)-c(d)) = f(r-1) = n-f$, and

$$\psi(a,b) = (\underbrace{f, f, \ldots, f}_{d\ times}).$$

Thus $\mathcal{S}_{\psi((a,b)}V = (\det V)^{\otimes f}$.

Note that the non-vanishing example of the above proposition happens for the limit condition

$$q+p-n = (r_0+\alpha)(e+\alpha-\beta)-\alpha(\alpha+1),$$
$$\text{where}\quad r_0 = \min\{\beta, \delta(\mathrm{n-p}), \delta(\mathrm{n-q})\}.$$

□

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