A "quantum spherical model" with transverse magnetic field.

I. Lyberg

School of Theoretical Physics, Dublin Institute for Advanced Studies

ilyberg@stp.dias.ie

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1 Introduction

The Quantum Ising Model with a transverse magnetic field is well known [1] [2]. In one dimension it has the Hamiltonian

$$\mathcal{H}_N = -J \sum_{n=1}^{N} \sigma_n^x \sigma_{n+1}^x + B \sum_{n=1}^{N} \sigma_n^z + H \sum_{n=1}^{N} \sigma_n^x, \tag{1}$$

where $J > 0$ is the coupling constant and $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices. B and H are transverse and longitudinal magnetic fields, respectively. The partition function is

$$Z_N = \mathrm{tr}\ e^{-\beta \mathcal{H}_N} \tag{2}$$

where β is the inverse temperature. In the case where $H = 0$ this model has been exactly solved [1] [3]. The free energy is [2]

$$\begin{aligned} f(\beta, J, B) &= -\lim_{N\to\infty} \frac{1}{\beta N} \log Z_N \\ &= -\frac{1}{2\pi\beta} \int_0^{2\pi} \log 2 \cosh \beta \Delta(x)\ dx \end{aligned} \tag{3}$$

where

$$\Delta(x) = \sqrt{J^2 + B^2 - 2BJ \cos x}. \tag{4}$$

In particular the ground state energy is given by

$$f_\infty(J,B) = \lim_{\beta\to\infty} f(\beta,J,B) = -\frac{1}{2\pi}\int_0^{2\pi} \Delta(x)\; dx. \tag{5}$$

In this limit there is a critical point in B at $B = J$. The correlation function

$$\langle \sigma_j^x \sigma_k^x \rangle = \lim_{N\to\infty} \frac{\mathrm{tr}\; \sigma_j^x \sigma_k^x e^{-\beta \mathcal{H}_N}}{Z_N} \tag{6}$$

can be written as a Toeplitz determinant of size $|j-k|$ just as the correlation function of the two dimensional classical Ising model [4], but only in the limit $\beta \to \infty$. In fact the correlation function $\lim_{\beta\to\infty}\langle \sigma_j^x \sigma_k^x \rangle$ is the same as the diagonal correlation function $\langle \sigma_{jj}\sigma_{kk}\rangle$ of the two dimensioanl classical Ising lattice for $T < T_c$, the ratio B/J in the one dimensional quantum model corresponding to $(\sinh 2E_1/k_\mathrm{B}T \sinh 2E_2/k_\mathrm{B}T)^{-1}$ in the two dimensional classical model. (Here E_1 and E_2 are the coupling constants in the horizontal and vertical directions, respectively). In particular the limit of infinite separation is given by [3]

$$\lim_{|j-k|\to\infty} \lim_{\beta\to\infty} \langle \sigma_j^x \sigma_k^x \rangle = \begin{cases} \{1-(B/J)^2\}^{1/4} & \text{if } B < J, \\ 0 & \text{if } B \geq J, \end{cases} \tag{7}$$

which is most easily proved using Szegö's theorem [5] [6].

2 The quantum spherical model

In analogy with (1) we define a partition function of a (d-dimensional) isotropic quantum spherical model on a lattice Λ as follows:

$$\begin{aligned} Z_N &= \int_{[0,\infty)^N}\int_{[0,2\pi)^N}\int_{[0,\pi]^N} e^{\sum_{j,k\in\Lambda\;:\;\langle jk\rangle} \beta J r_j \cos\theta_j r_k \cos\theta_k} \\ &\quad e^{\sum_{j\in\Lambda} \beta(B r_j \sin\theta_j \cos\varphi_j + H r_j \cos\theta_j)} \\ &\quad \prod_{l=1}^N r_l^2 \sin\theta_l\; d^N\theta\; d^N\varphi\; \delta\left(\sum_{m=1}^N r_m^2 - N\right) d^N r \\ &= \int_{\mathbf{R}^{3N}} e^{\sum_{\langle jk\rangle} \beta J z_j z_k + \sum_j \beta(Bx_j + Hz_j)} \delta\left(\sum_{k=1}^N (x_k^2+y_k^2+z_k^2) - N\right) d^{3N}\mathbf{x} \end{aligned} \tag{8}$$

Here $J > 0$, $B \geq 0$ and $H > 0$. δ signifies the Dirac distribution. The notation $\langle jk \rangle$ means that j and k are nearest neighbors on Λ. Unlike the

Quantum Ising Model with $H = 0$, in this model the critical point is $B = 2Jd$ (in the limit $H \to 0$). In fact, it will be shown that in this limit the ground state free energy $f_{H,\infty} := -\lim_{\beta\to\infty} \lim_{N\to\infty} (N\beta)^{-1} \log Z_N$ is given by

$$f_{0,\infty} = \lim_{H\to 0} f_{H,\infty} = -\begin{cases} Jd + B^2/4Jd & \text{if } B \le 2Jd, \\ B & \text{if } B > 2Jd. \end{cases} \tag{9}$$

We shall now give a proof of (9).

2.1 The case $B > 2Jd$

We use the method of steepest descent to prove this result, following the calculation by Baxter [7]. We let $H = 0$ in (8). Clearly the integrand in (8) may be multiplied by a factor $\exp a(\sum_{k=1}^{N}(x_k^2 + y_k^2 + z_k^2) - N)$ without changing the partition function Z_N. Using the identity

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} ds, \tag{10}$$

together with (8) and letting $a > 0$, we get

$$Z_N = \frac{\pi^{N-1}}{2} \int_{\mathbf{R}^N} \int_{-\infty}^{\infty} \Big(\frac{1}{a+is}\Big)^N \exp \frac{N(\beta B)^2}{4(a+is)}$$
$$\exp \big[\sum_{\langle jk\rangle} \beta J z_j z_k + \sum_j (a+is)(1 - z_j^2)\big] \; ds \; d^N z \tag{11}$$

after integrating over $\mathbf{x}$ and $\mathbf{y}$. Let $\mathbf{V}$ be the symmetric matrix such that

$$\mathbf{z}^T \mathbf{V} \mathbf{z} = (a+is) \sum_{j=1}^{N} z_j^2 - \beta J \sum_{\langle jk\rangle}^{N} z_j z_k. \tag{12}$$

In this way (11) can be written as

$$Z_N = \frac{\pi^{N-1}}{2} \int_{\mathbf{R}^N} \int_{-\infty}^{\infty} \Big(\frac{1}{a+is}\Big)^N \exp \frac{N(\beta B)^2}{4(a+is)}$$
$$\exp\left[-\mathbf{z}^T \mathbf{V} \mathbf{z} + N(a+is)\right] \; ds \; d^N z. \tag{13}$$

We now choose the constant a so large that all the eigenvalues of $\mathbf{V}$ have positive real part. This allows us to change the order of integration, and we may now write (13) as

$$Z_N = \frac{\pi^{3N/2-1}}{2} \int_{-\infty}^{\infty} \Big(\frac{1}{a+is}\Big)^N (\det \mathbf{V})^{-1/2}$$
$$\exp\left[\frac{N(\beta B)^2}{4(a+is)} + N(a+is)\right] \; ds. \tag{14}$$

We need to calculate the eigenvalues of $\mathbf{V}$. Since $\mathbf{V}$ is cyclic, this is easily done. We let the lattice be d-dimensional hypercubic, so that

$$N = L^d \tag{15}$$

for some positive integer L. It now follows from (12) that the eigenvalues are

$$\lambda(\omega_1, ..., \omega_d) = a + is - \beta J \sum_{j=1}^{d} \cos \omega_j \tag{16}$$

where each ω_j takes the values $\{2\pi k/L\}_{k=0}^{L-1}$, and $a > \beta J d$. The determinant of $\mathbf{V}$ is the product of its eigenvalues, so

$$\log \det \mathbf{V} = \sum_{\omega_j \;:\; 1 \le j \le d} \log \lambda(\omega_1, ..., \omega_d). \tag{17}$$

Clearly

$$Z_N = \frac{\beta J}{2\pi i} \left(\frac{\pi}{\beta J} \right)^{3N/2} \int_{c-i\infty}^{c+i\infty} e^{N\phi(w)} dw, \tag{18}$$

where

$$\phi(w) = \beta J w - \frac{1}{2} g(w) + (\beta B)^2 / 4\beta J w, \tag{19}$$

$c = (a - \beta J d)/\beta J$ and

$$g(z) = 2 \log w + \frac{1}{N} \sum_{\omega_j} \log \left(w - \sum_j \cos \omega_j\right). \tag{20}$$

Since ϕ approaches $+\infty$ as w approaches 0 or $+\infty$ along the real line, ϕ has a minumum at some w_0, $0 < w_0 < \infty$. Thus $\Re\phi$ has a maximum at w_0 along the line $(w_0 - i\infty, w_0 + i\infty)$. Since $B > 2Jd$, we may choose $c = w_0$. We now use the method of steepest descent (see for instance Murray [8]), by letting N approach infinity. In this way, the free energy is

$$\begin{aligned} f &= -\beta^{-1} \lim_{N \to \infty} N^{-1} \log Z_N \\ &= -\frac{3}{2\beta} \ln \left(\pi/\beta J\right) - \beta^{-1} \phi(w_0). \end{aligned} \tag{21}$$

Now

$$\lim_{\beta \to \infty} w_0 = B/2J, \tag{22}$$

and thus the ground state energy is

$$\begin{aligned} \lim_{\beta \to \infty} f &= -\lim_{\beta \to \infty} \beta^{-1} \phi(w_0) \\ &= -B. \end{aligned} \tag{23}$$

2.2 The case $B \leq 2Jd$

In this case we let $H > 0$, so instead of (13) we have

$$Z_N = \frac{\pi^{N/2-1}}{2} \int_{\mathbf{R}^N} \int_{-\infty}^{\infty} \Big(\frac{1}{a+is}\Big)^N \exp \frac{N(\beta B)^2}{4(a+is)} \exp\left[-\mathbf{z}^T \mathbf{V} \mathbf{z} + \mathbf{h}^T \mathbf{z} + N(a+is)\right] \, ds \, d^N z, \tag{24}$$

where $\mathbf{h} = \beta H(1, ..., 1)$. We change variables to $\mathbf{t} = \mathbf{z} - \frac{1}{2}\mathbf{V}^{-1}\mathbf{h}$, and rotate the axes in $(t_1, ..., t_N)$ to make $\mathbf{V}$ diagonal. Thus we get

$$Z_N = \frac{\pi^{N/2-1}}{2} \int_{-\infty}^{\infty} \Big(\frac{1}{a+is}\Big)^N (\det \mathbf{V})^{-1/2} \exp \frac{N(\beta B)^2}{4(a+is)} \exp\left[\mathbf{h}^T \mathbf{V}^{-1} \mathbf{h}/4 + N(a+is)\right] \, ds. \tag{25}$$

Thus

$$Z_N = \frac{\pi^{N/2}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{N\phi(w)} \, dw, \tag{26}$$

where $a + is - \beta Jd = \beta Jw$ and

$$\begin{aligned} \phi(w) &= \beta J(w+d) + \frac{(\beta H)^2}{4\beta Jw} + \frac{(\beta B)^2}{4\beta J(w+d)} \\ &- \log \beta J(w+d) - \frac{1}{2} \sum_{\omega_j} \log\big(\beta J(w+d) - \beta J \sum_j \cos \omega_j\big). \end{aligned} \tag{27}$$

We proceed in the same way as before, taking the limit $N \to \infty$ and then $\beta \to \infty$. In this case $w_0 \to 0$ as $H \to 0$. The free energy is thus

$$f = -Jd - \frac{B^2}{4Jd}. \tag{28}$$

This ends the proof.

3 Discussion

Comparison of (5) and (9) shows that the susceptibilites of the two models at $B = 0$ are equal when $d = 1$; that is $-\partial^2 f_\infty / \partial B^2|_{B=0} = 1/2J$ and $-\partial^2 f_{0,\infty}/\partial B^2|_{B=0} = 1/2Jd$. While the Quantum Ising Model has only been exactly solved in the one dimensional case, the quantum spherical modelcan be solved in any finite dimension.

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