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FREE RESOLUTIONS VIA GRÖBNER BASES

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A . In many different settings (associative algebras, commutative algebras, operads, dioperads), it is possible to develop the machinery of Gröbner bases; it allows to find a "monomial replacement" for every object in the corresponding category. The main goal of this article is to demonstrate how this machinery can be used for the purposes of homological algebra. More precisely, we define combinatorial resolutions in the monomial case and then show how they can be adjusted to be used in the general homogeneous case. We also discuss a way to make our monomial resolutions minimal. For associative algebras, we recover a well known construction due to Anick. Various applications of these results are presented, including a new proof of Hoffbeck's PBW criterion, a proof of Koszulness for a class of operads coming from commutative algebras, and a homology computation for the operad of Batalin–Vilkovisky algebras.

1. I

1.1. **Description of results.** One of important practical results provided (in many different frameworks) by Gröbner bases is that when dealing with various linear algebra information (bases, dimensions etc.) one can replace an object with complicated relations by an object with monomial relations without losing any information of that sort. When it comes to questions of homological algebra, things become more subtle, since (co)homology may "jump up" for a monomial replacement of an object. However, the idea of applying Gröbner bases to problems of homological algebra is far from hopeless. It turns out that for monomial objects it is often possible to construct very neat resolutions that can be used for various computations; furthermore, the data computed by these resolutions can be used to obtain results in the general (not necessarily monomial) case.

Our main motivating example is the case of (symmetric) operads. In [10], we introduced new monoids based on nonsymmetric collections, shuffle operads, to develop the machinery of Gröbner bases for symmetric operads. Basically, there exists a monoidal structure on nonsymmetric collections (shuffle composition) for which the forgetful functor from symmetric collections is monoidal, and this makes all constructions and results of [10] possible. The main goal of this paper is to show how questions of homological algebra for symmetric operads can be approached in the shuffle category. We restrict ourselves to the case of homology with coefficients in the trivial module, that is, homology of the bar complex for an operad. In

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the case of operads with operations of arity 1, that is associative algebras, this question was addressed in the celebrated paper of Anick [1], where in the case of monomial relations a minimal resolution was computed, and an explicit way to deform the differential was presented to handle the general homogeneous case. Later, Anick resolution was generalized to the case of categories in the paper [25], where a question of extension this work to the case of operads was raised. Here, we obtain an answer to this question in the realm of shuffle operads. We present a slightly different resolution which is sometimes larger than the one of Anick, but has the advantage of treating algebras and operads in the same way. Based on the inclusion–exclusion principle, our approach in monomial case is in a sense a generalisation of the cluster method of enumeration due to Goulden and Jackson [18]. Even though our resolutions might not be minimal even in the monomial case, they are small enough to enable us to both prove some general results and perform computations in particular cases.

Our construction has some immediate applications. Two interesting theoretical applications are a new short proof of Hoffbeck's PBW criterion [20], and a theorem stating that quadratic operads obtained from Koszul quadratic algebras are Koszul. An interesting concrete example where all steps of our construction can be completed is the case of the operad BV of Batalin–Vilkovisky algebras. Using our methods, we were able to compute its bar homology and relate it to the gravity operad of Getzler [16]. While preparing our paper, we learned that these results on BV were announced earlier this year by Drummond-Cole and Vallette (see, for example, the extended abstract [8] and the slides [38]; details to appear in the forthcoming paper [9]). Their approach is based on theorems of Galvez-Carillo, Tonks and Vallette [14] who studied the operad BV as an operad with nonhomogeneous (quadratic–linear) relations. Our methods appear to be completely different: we treat BV as an operad with homogeneous relations of degrees 2, 3, and 4. We believe that this approach is also of independent interest because of the generality of methods used.

1.2. **Outline of the paper.** This paper is organised as follows. In Section 2, we present our construction of a free resolution for graded associative algebras and shuffle operads; we start from the case of monomial relations and continue by explaining how to use results obtained to deal with the general case of a homogeneous Gröbner basis, reminding relevant definitions when necessary. In Section 3, we give an example of computations, treating carefully low levels of our resolution for the anti-associative operad. In Section 4, we exhibit applications of our result outlined above (a new proof of the PBW criterion, a proof of Koszulness for operads coming from Koszul commutative algebras, and a computation of the bar homology for the operad BV).

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2. R

2.1. **Operads, homology, and homotopical algebra: a summary.** The goal of this section is to remind the reader some concepts and ideas of homological and homotopical algebra that we use in this paper. All of them can be proved similarly to many statements proved in the literature, and we omit the proofs, referring the reader to various research papers and monographs instead.

All vector spaces and (co)chain complexes throughout this work are defined over an arbitrary field $\Bbbk$ of zero characteristic. For information on symmetric and nonsymmetric operads, we refer the reader to the monograph [30], for information on shuffle operads and Gröbner bases for operads — to our paper [10]. Gröbner bases and Anick resolution for associative algebras are discussed in detail in [36]. Associative algebras fit into our general framework, since they may be considered as operads with all operations of arity 1. However, we shall first treat the case of algebras separately, in order to enable the reader to understand our construction better in a more familiar setting. Throughout this paper by an operad we mean a shuffle operad, unless otherwise specified.

For the monoidal category of shuffle operads, it is possible to define the bar complex of an augmented operad $\mathscr{O}$. The bar complex $\mathbf{B}^\bullet(\mathscr{O})$ is a dg-cooperad freely generated by the degree shift $\mathscr{O}_+[1]$ of the augmentation ideal of $\mathscr{O}$; the differential comes from operadic compositions in $\mathscr{O}$. Similarly, for a cooperad $\mathscr{Q}$, it is possible to define the cobar complex $\Omega^\bullet(\mathscr{Q})$, which is a dg-operad freely generated by $\mathscr{Q}_+[-1]$, with the appropriate differential. The bar-cobar construction $\Omega^\bullet(\mathbf{B}^\bullet(\mathscr{O}))$ gives a free resolution of $\mathscr{O}$. This can be proved in a rather standard way, similarly to known proofs in the case of operads, PROPs etc. [17, 12, 37].

According to the general philosophy of homotopical algebra [32, 33], instead of computing the homology of the bar complex $B^\bullet(\mathscr{O})$ directly, one can find a free resolution $(\mathscr{F}, d)$ of $\mathscr{O}$ different from the bar-cobar resolution, compute its abelianization $\mathscr{F}_{ab}$, that is the space of indecomposable elements, and compute the homology of $(\mathscr{F}_{ab}, d_0)$, where d_0 is the differential induced on indecomposables by d. Also, the same homology is equal to the homology of the complex $\mathscr{G}^\bullet \circ_{\mathscr{O}} \Bbbk$, where $\Bbbk$ is the trivial bimodule over $\mathscr{O}$, and $\mathscr{G}^\bullet$ is a resolution of $\Bbbk$ by free right $\mathscr{O}$-modules. Various checks and justifications needed here are also quite standard; we refer the reader to [3, 13, 19, 28, 34, 35] where similar statements are handled for symmetric operads.

It is important to recall here that the forgetful functor ${}^f\colon \mathscr{P} \to \mathscr{P}^f$ from the category of symmetric operads to the category of shuffle operads is monoidal [10], which easily implies that for a symmetric operad $\mathscr{P}$, we have

$$\mathbf{B}^\bullet(\mathscr{P})^f \simeq \mathbf{B}^\bullet(\mathscr{P}^f),$$

that is the (symmetric) bar complex of $\mathscr{P}$ is naturally identified, as a shuffle dg-cooperad, with the (shuffle) bar complex of $\mathscr{P}^f$. Thus, our approach would enable us to compute the homology even in the symmetric case, only without any information on the symmetric groups action.

The overall message of this section is that to compute the bar homology of an operad, it is useful to have a free resolution of this operad that is substantially smaller than the bar-cobar resolution. Our goal in the next three sections is to explain in detail how Gröbner bases lead to resolutions which are often small enough to approach the bar homology.

2.2. **Monomial associative algebras.** In this section, we discuss the case of associative algebras with monomial relations. We start with an algebra $R = \Bbbk\langle x_1, \ldots, x_n\rangle/(g_1, \ldots, g_m)$ with n generators and m relations, each of which is a monomial in the given generators. We work under the assumption that $G = \{g_1, \ldots, g_m\}$ is an antichain in the set of all monomials (with the partial order given by the divisibility relation); this, for example, is the case when G is the set of leading monomials of a reduced Gröbner basis of some algebra for which R is a monomial replacement.

Let us denote by $A(p, q)$ the vector space whose basis is formed by elements of the form $x_I \otimes S_1 \wedge S_2 \wedge \ldots \wedge S_q$, where $I = (i_1, \ldots, i_p) \in [n]^p$, $x_I = x_{i_1} \otimes \ldots \otimes x_{i_p}$ is the corresponding monomial, and $S_1, \ldots, S_q$ are (in one-to-one correspondence with) certain divisors $x_{i_r} x_{i_{r+1}} \ldots x_{i_s}$ of this monomial. Each S_i is thought of as a symbol of homological degree 1, with the appropriate Koszul sign rule for wedge products (here $q \geq 0$, so the wedge product might be empty). As we shall see later, in the classical approach for associative algebras there is no need in wedge products because there exists a natural linear ordering on all divisors of the given monomial. However, we introduce the wedge notation here as it becomes crucial for the case of operads.

For each p the graded vector space $A(p) = \bigoplus_q A(p, q)$ is a chain complex with the differential given by the usual formula with omitted factors:

$$d(x_I \otimes S_1 \wedge \ldots \wedge S_q) = \sum_l (-1)^{l-1} x_I \otimes S_1 \wedge \ldots \wedge \hat{S}_l \wedge \ldots \wedge S_q. \tag{1}$$

Moreover, there exists a natural algebra structure on $A = \bigoplus_{p,q} A(p, q)$:

$$\begin{aligned}(x_I \otimes S_1 \wedge \ldots \wedge S_q) \cdot (x_J \otimes T_1 \wedge \ldots \wedge T_{q'}) \\ = (x_I x_J) \otimes S_1 \wedge \ldots \wedge S_q \wedge T_1 \wedge \ldots \wedge T_{q'}.\end{aligned} \tag{2}$$

The differential d makes A into an associative dg-algebra.

So far we did not use the relations of our algebra. Let us incorporate relations in the picture. We denote by $G = \{g_1, \ldots, g_m\}$ the set of relations of our algebra, and by A_G the subspace of A spanned by all elements $x_I \otimes S_1 \wedge \ldots \wedge S_q$ for which the divisor corresponding to S_j coincides, for every j, with one of the relations g_t. This subspace is stable under product and differential, i.e. is a dg-subalgebra of A.

Theorem 1. *The dg-algebra (A_G, d) is a free resolution of the corresponding algebra with monomial relations $\Bbbk\langle x_1, \ldots, x_n\rangle/(g_1, \ldots, g_m)$.*

Proof. Let us call a collection of divisors $S_1, \ldots, S_q$ *indecomposable*, if each product $x_{i_k} x_{i_{k+1}}$ is contained in at least one of them. Then it is easy to see that A is freely generated by elements $x_k \otimes 1$ and $x_I \otimes S_1 \wedge \ldots \wedge S_q$, where $S_1, \ldots, S_q$ is an indecomposable collection. Similarly, A_G is freely generated by its basis elements $x_k \otimes 1$ and all elements $x_I \otimes S_1 \wedge \ldots \wedge S_q$ where $S_1, \ldots, S_q$ is an indecomposable collection of divisors, each of which is a relation of R.

Let us prove that A_G provides a resolution for R. Since the differential d does not change the monomial x_I, the chain complex A_G is isomorphic to the direct sum of chain complexes A_G^f indexed by monomials $f \in \Bbbk\langle x_1, \ldots, x_n\rangle$. If f is not divisible by any relation, the complex A_G^f is concentrated in degree 0 and is spanned by $f \otimes 1$. Thus, to prove the theorem, we should show that A_G^f is acyclic whenever f is divisible by some relation g_i.

Let $(S_1, \ldots, S_k)$ be a complete unordered list of all divisors of f which are relations of R. We immediately see that the complex A_G^f is isomorphic to the inclusion–exclusion complex for the set $[k]$

$$0 \leftarrow \varnothing \leftarrow \bigoplus_{i=1}^{k} \{i\} \leftarrow \bigoplus_{1\le i<j\le k} \{i, j\} \leftarrow \ldots \leftarrow \{1, \ldots, k\} \leftarrow 0 \tag{3}$$

(with the usual differential omitting elements). The latter one is acyclic whenever $k > 0$, which completes the proof. □

As we know, this result can be used to compute the bar homology of R: it is the homology of the differential induced on the space of generators $(A_G)_+/(A_G)_+^2$, which, as we already mentioned, in our case is spanned by $x_k \otimes 1$ and all elements $x_I \otimes S_1 \wedge \ldots \wedge S_q$ where $S_1, \ldots, S_q$ is an indecomposable collection of intervals, each correspoding to a divisor of x_I which is a relation. This homology coincides with the homology $\mathrm{Tor}^R_\bullet(\Bbbk, \Bbbk)$ of A, which can also be computed from a free resolution of $\Bbbk$ by free R-modules. Let us recall the remarkable Anick resolution that exists in this case, and then construct explicitly a resolution of $\Bbbk$ by free right R-modules where the space of generators is $(A_G)_+/(A_G)_+^2$ which we then compare with the Anick resolution.

Let us denote by C_0 the linear span of all $x_k \otimes 1$, and by C_q, $q > 0$, the linear span of all *q-chains* which are monomials of the free algebra $\Bbbk\langle x_1, \ldots, x_n\rangle$; q-chains and tails of q-chains are defined inductively as follows:

- each generator x_i is a 0-chain; it coincides with its tail;
- each q-chain is a monomial m equal to a product nst, where t is the tail of m, and ns is a $(q-1)$-chain whose tail is s;
- in the above decomposition, the product st has exactly one divisor which is a relation of R; this divisor is a right divisor of st.

In other words, a q-chain is a monomial formed by linking one after another q relations so that only neighbouring relations are linked, the first $(q-1)$ of them form a $(q-1)$-chain, and no proper left divisor is a q-chain. In our notation above, such a monomial m corresponds to the generator $m \otimes S_1 \wedge \ldots \wedge S_q$, where $S_1, \ldots, S_q$ are the relations we linked.

Theorem 2 (Anick[1]). *There exists an exact sequence of free right modules*

$$\ldots \to C_q \otimes R \to C_{q-1} \otimes R \to \ldots \to C_1 \otimes R \to C_0 \otimes R \to R \to \Bbbk \to 0, \tag{4}$$

where all boundary maps have positive degree. Consequently, $\mathrm{Tor}^R_q(\Bbbk, \Bbbk) \simeq C_{q-1}$.

Now let us get back to our approach. We denote by V_0 the linear span of all $x_k \otimes 1$, and by $V_q, q > 0$, the linear span of all elements $x_I \otimes S_1 \wedge \ldots \wedge S_q$ as above. We shall construct a free resolution of the form

$$(5)\qquad \ldots \to V_q \otimes R \to V_{q-1} \otimes R \to \ldots \to V_1 \otimes R \to V_0 \otimes R \to R \to \Bbbk \to 0.$$

It is enough to define boundary maps on the free module generators V_q, since boundary maps are morphisms of R-modules. First of all, we let $d_0 \colon V_0 \otimes R \to R$ be defined as $d_0(x_k \otimes 1) = x_k$. Assume that $q > 0$ and let $x_I \otimes S_1 \wedge \ldots \wedge S_q \in V_q$. In the free algebra A_G, the differential d maps this element to a sum of elements corresponding to all possible omissions of S_j. If after the omission of S_j we still have an indecomposable covering, this summand survives in the differential. If the covering is decomposable, the corresponding summand disappears in all cases but the following one: if $x_I = x_J x_K$, and $S_1, \ldots, \hat{S}_j, \ldots, S_q$ form an indecomposable covering of x_J, then the corresponding summand of the differential becomes $(x_J \otimes S_1 \wedge \ldots \wedge \hat{S}_j \wedge \ldots \wedge S_q) \otimes x_K \in V_{q-1} \otimes R$.

The following proposition is quite easy to prove, we omit the details.

Proposition 1. *The construction above provides a resolution of the trivial module by free R-modules.*

Readers familiar with the machinery of twisting cochains [5] may see our construction of the free right module resolution from the free dg-algebra resolution as a variation of the twisting cochain construction. More precisely, the differential of a generator in our free algebra resolution is a sum of products of generators; this provides the space of generators with a structure of an ∞-coalgebra, and the twisting cochain method should be adapted for this case.

It is easy to see that when we compute the homology of d restricted to the space of generators $(A_G)_+/((A_G)_+)^2$, Anick's q-chains, as defined above, represent the cokernel of d on the space of all generators $x_I \otimes S_1 \wedge \ldots \wedge S_q$ which become decomposable after removing S_j, for each $1 \leq j \leq q$. Anick's theorem implies that this cokernel gives a basis for the homology groups.

Remark 1. A similar construction works for commutative algebras as well, producing the corresponding homology groups. We shall not discuss it in detail here; the main idea is that one can make the symmetric groups act on the free algebras A and A_G from this section in such a way that the subalgebra of invariants is a free (super)commutative dg-algebra whose cohomology is the given monomial commutative algebra (acyclicity of the corresponding resolution can be derived from the acyclicity in the general associative case, subcomplexes of invariants of symmetric groups acting on the acyclic complexes have to be acyclic by the Maschke's theorem). It would be interesting to compare these resolutions with resolutions for monomial commutative algebras obtained by Berglund [4].

2.3. **Monomial operads.** It turns out that our construction from the previous section works perfectly fine in the case of (shuffle) operads as well. Let us give the appropriate definitions. First, we shall give a brief reminder of tree combinatorics used in the shuffle operads. See [10] for more details.

Basis elements of the free operad are represented by (decorated) trees. A (rooted) *tree* is a non-empty connected directed graph T of genus 0 for which each vertex has at least one incoming edge and exactly one outgoing edge. Some edges of a tree might be bounded by a vertex at one end only. Such edges are called *external*. Each tree should have exactly one outgoing external edge, its *output*. The endpoint of this edge which is a vertex of our tree is called the *root* of the tree. The endpoints of incoming external edges which are not vertices of our tree are called *leaves*.

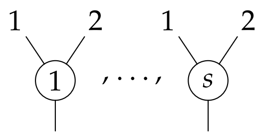
Each tree with n leaves should be (bijectively) labelled by $[n]$. For each vertex v of a tree, the edges going in and out of v will be referred to as inputs and outputs at v. A tree with a single vertex is called a *corolla*. There is also a tree with a single input and no vertices called the *degenerate* tree. Trees are originally considered as abstract graphs but to work with them we would need some particular representatives that we now going to describe.

For a tree with labelled leaves, its canonical planar representative is defined as follows. In general, an embedding of a (rooted) tree in the plane is determined by an ordering of inputs for each vertex. To compare two inputs of a vertex v, we find the minimal leaves that one can reach from v via the corresponding input. The input for which the minimal leaf is smaller is considered to be less than the other one. Note that this choice of a representative is essentially the same one as we already made when we identified symmetric compositions with shuffle compositions.

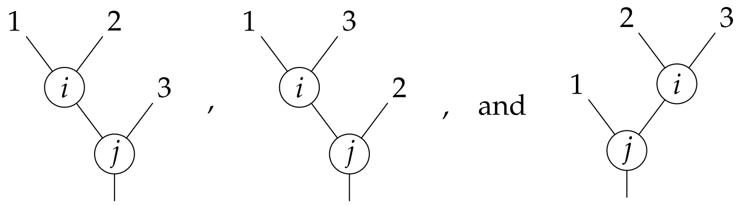
Let us introduce an explicit realisation of the free operad generated by a collection $\mathscr{M}$. The basis of this operad will be indexed by planar representative of trees with decorations of all vertices. First of all, the simplest possible tree is the degenerate tree; it corresponds to the unit of our operad. The second simplest type of trees is given by corollas. We shall fix a basis $B^{\mathscr{M}}$ of $\mathscr{M}$ and decorate the vertex of each corolla with a basis element; for a corolla with n inputs, the corresponding element should belong to the basis of $\mathscr{V}(n)$. The basis for whole free operad consists of all planar representatives of trees built from these corollas (explicitly, one starts with this collection of corollas, defines compositions of trees in terms of grafting, and then considers all trees obtained from corollas by iterated shuffle compositions). We shall refer to elements of this basis as *tree monomials*.

There are two standard ways to think of elements of an operad defined by generators and relations: using either tree monomials or operations. Our approach is somewhere in the middle: we prefer (and strongly encourage the reader) to think of tree monomials, but to write formulas required for definitions and proofs we prefer the language of operations since it makes things more compact.

Let us give an example of how to translate between these two languages. Let $\mathscr{O} = \mathscr{F}_{\mathscr{M}}$ be the free operad for which the only nonzero component of $\mathscr{M}$ is $\mathscr{M}(2)$, and the basis of $\mathscr{M}(2)$ is given by



Then the basis of $\mathscr{F}_{\mathscr{M}}(3)$ is given by the tree monomials



with $1 \leq i, j \leq s$. If we assume that the j^{th} corolla corresponds to the operation $\mu_j \colon a, b \mapsto \mu_j(a, b)$, then the above tree monomials correspond to operations

$$\mu_j(\mu_i(a_1, a_2), a_3), \quad \mu_j(\mu_i(a_1, a_3), a_2), \quad \text{and} \quad \mu_j(a_1, \mu_i(a_2, a_3))$$

respectively.

Take a tree monomial $\alpha \in \mathscr{F}_{\mathscr{M}}$. If we forget the labels of its vertices and its leaves, we get a planar tree. We shall refer to this planar tree as *the underlying tree of* α. Divisors of α in the free operad correspond to a special kind of subgraphs of its underlying tree. Allowed subgraphs contain, together with each vertex, all its incoming and outgoing edges (but not necessarily other endpoints of these edges). Throughout this paper we consider only this kind of subgraphs, and we refer to them as subtrees hoping that it does not lead to any confusion. Clearly, a subtree T' of every tree T is a tree itself. Let us define the tree monomial α' corresponding to T'. To label vertices of T', we recall the labels of its vertices in α. We immediately observe that these labels match the restriction labels of a tree monomial should have: each vertex has the same number of inputs as it had in the original tree, so for a vertex with n inputs its label does belong to the basis of $\mathscr{M}(n)$. To label leaves of T', note that each such leaf is either a leaf of T, or is an output of some vertex of T. This allows us to assign to each leaf l' of T' a leaf l of T: if l' is a leaf of T, put $l = l'$, otherwise let l be the smallest leaf of T that can be reached through l'. We then number the leaves according to these "smallest descendants": the leaf with the smallest possible descendant gets the label 1, the second smallest — the label 2 etc.

For two tree monomials α, β in the free operad $\mathscr{F}_{\mathscr{M}}$, we say that α *is divisible by* β, if there exists a subtree of the underlying tree of α for which the corresponding tree monomial α' is equal to β.

Let us construct a free resolution for an arbitrary operad with monomial relations. Assume that $\mathscr{O} = \mathscr{F}_{\mathscr{M}}/(g_1, \ldots, g_m)$ is an operad generated by a collection of finite sets $\mathscr{M} = \{\mathscr{M}(n)\}$, with m monomial relations $g_1, \ldots, g_m$ (this means that every tree monomial divisible by any of relations is equal to zero). We denote by $\mathscr{A}(T, q)$ the vector space with the basis consisting of all elements $T \otimes S_1 \wedge S_2 \wedge \ldots \wedge S_q$, where T is a tree monomial from the free shuffle operad $\mathscr{F}_M$, and $S_1, \ldots, S_q$, $q \geq 0$, are tree divisors of T.

The differential d with

$$d(T \otimes S_1 \wedge \ldots \wedge S_q) = \sum_l (-1)^{l-1} T \otimes S_1 \wedge \ldots \wedge \hat{S}_l \wedge \ldots \wedge S_q \tag{6}$$

makes the graded vector space

$$\mathscr{A}(n) = \bigoplus_{T \text{ with } n \text{ leaves}} \bigoplus_{q} \mathscr{A}(T,q) \tag{7}$$

into a chain complex. There is also a natural operad structure on the collection $\mathscr{A} = \{\mathscr{A}(n)\}$ the operadic composition composes the trees, and computes the wedge product of divisors. Overall, we defined a dg-shuffle operad.

Let $\mathscr{G} = \{g_1, \dots, g_m\}$ be the set of relations of our operad. The dg-operad $(\mathscr{A}_{\mathscr{G}}, d)$ is spanned by the elements $T \otimes S_1 \wedge \dots \wedge S_k$, where for each j the divisor of T corresponding to S_j is a relation. The differential d is the restriction of the differential defined above. Informally, an element of the operad $\mathscr{A}_{\mathscr{G}}$ is a tree with some distinguished divisors that are relations from the given set.

Theorem 3. *The dg-operad $(\mathscr{F}_{\mathscr{G}}, d)$ is a free resolution of the corresponding operad with monomial relations $\mathscr{O} = \mathscr{F}_{\mathscr{M}}/(\mathscr{G})$.*

Proof. The proof is unsurprisingly close to the proof in the case of associative algebras. Let us call a collection of divisors $S_1, \dots, S_q$ of T *indecomposable*, if each internal edge of T is contained in at least one of them. It is easy to see that the operad $\mathscr{A}$ is freely generated by elements $c \otimes 1$, where c is a corolla labelled by one of the generators from the collection $\mathscr{M}$, and $T \otimes S_1 \wedge \dots \wedge S_q$ where S_k form an indecomposable collection. Similarly, the operad $\mathscr{A}_{\mathscr{G}}$ is freely generated by the elements $c \otimes 1$ and $T \otimes S_1 \wedge \dots \wedge S_q$ where S_k form an indecomposable collection of divisors, each of which belongs to $\mathscr{G}$. Now that the definition of generators is given, the proof is all about the inclusion-exclusion principle. □

Remark 2. Results of the previous sections can be formulated in a very general setting: for a $\Bbbk$-linear monoidal category where free monoids have combinatorial bases which are represented uniquely (up to the associativity relation) as products of generators, one can produce free resolutions for an arbitrary monoid with monoidal relations. This can be applied both to various algebraic structures like dialgebras and to monoids representing some higher structures, like dioperads, $\frac{1}{2}$PROPs etc. We shall discuss details elsewhere.

Let us conclude this section with a conjecture. As we saw in the case of associative algebras, our resolution is potentially much bigger than the minimal resolution of the original algebra, unlike the Anick resolution. We expect that it is possible to minimize our resolution in the operadic case accordingly.

Conjecture 1. *The bar homology of $\mathscr{O}$, that is the homology of d restricted to the space of generators $(\mathscr{A}_{\mathscr{G}})_+/((\mathscr{A}_{\mathscr{G}})_+)^2$ is equal to the cokernel of d on the space of all generators $T \otimes S_1 \wedge \dots \wedge S_q$ which become decomposable after removing S_j, for each $1 \le j \le q$.*

2.4. **General homogeneous case.** In this section, we shall explain the machinery that transforms our resolution for a monomial replacement of the given operad (or augmented algebra) into a resolution for the original operad (algebra).

Let $\widetilde{\mathscr{O}} = \mathscr{F}_{\mathscr{M}}/(\widetilde{\mathscr{G}})$ be an operad (or algebra), and let $\mathscr{O} = \mathscr{F}_{\mathscr{M}}/(\mathscr{G})$ be its monomial version, that is, $\widetilde{\mathscr{G}}$ is a Gröbner basis of relations, and $\mathscr{G}$ consists of all leading monomials of $\widetilde{\mathscr{G}}$. In the previous section, we defined a free resolution $(\mathscr{A}_{\mathscr{G}}, d)$ for $\mathscr{O}$, so that $H_\bullet(\mathscr{A}_{\mathscr{G}}, d) \simeq \mathscr{O}$. Let π (resp., $\widetilde{\pi}$) be the algebra homomorphism from $\mathscr{A}_{\mathscr{G}}$ to $\to \mathscr{O}$ (resp., $\widetilde{\mathscr{O}}$) that kills all generators of positive homological degree, and on elements of homological degree 0 is the canonical projection from $\mathscr{F}_{\mathscr{M}}$ to its quotient. Denote by h the contracting homotopy for this resolution, so that $(dh)|_{\ker d} = \mathrm{Id} - \pi$.

Theorem 4. *There exists a "deformed" differential D on $\mathscr{A}_{\mathscr{G}}$ and a homotopy $H\colon \ker D \to \mathscr{A}_{\mathscr{G}}$ such that $H_\bullet(\mathscr{A}_{\mathscr{G}}, D) \simeq \widetilde{\mathscr{O}}$, and $(DH)|_{\ker D} = \mathrm{Id} - \widetilde{\pi}$.*

Proof. We shall construct D and H simultaneously by induction. Let us introduce the following partial ordering of monomials in $\mathscr{A}_{\mathscr{G}}$: $T \otimes S_1 \wedge \ldots \wedge S_q$ is, by definition, less than $\prec T' \otimes S'_1 \wedge \ldots \wedge S'_{q'}$ if the tree monomial T is less than T' in the free operad $\mathscr{F}_{\mathscr{M}}$. This partial ordering suggests the following definition: for an element $u \in \mathscr{A}_{\mathscr{G}}$, its leading term $\hat{u}$ is the part of the expansion of u as a combination of basis elements where we keep only basis elements $T \otimes S_1 \wedge \ldots \wedge S_q$ with maximal possible T.

If L is a homogeneous linear operator on $\mathscr{A}_{\mathscr{G}}$ of some fixed (homological) degree of homogeneity (like D, H, d, h), we denote by L_k the operator L acting on elements of homological degree k. We shall define the operators D and H by induction: we define the pair (D_{k+1}, H_k) assuming that all previous pairs are defined. At each step, we shall also be proving that

$$D(x) = d(\hat{x}) + \text{lower terms}, \quad H(x) = h(\hat{x}) + \text{lower terms},$$

where the words "lower terms" mean in each case a linear combination of basis elements whose underlying tree is smaller than the underlying tree of $\hat{x}$.

Basis of induction: $k = 0$, so we have to define D_1 and H_0 (note that $D_0 = 0$ because there are no elements of negative homological degrees). In general, to define D_l, we should only consider the case when our element is a generator of $\mathscr{A}_{\mathscr{G}}$, since in a dg-operad the differential is defined by images of generators. For $l = 1$, this means that we should consider the case where our generator corresponds to a leading monomial $T = \mathrm{lt}(g)$ of some relation g, and is of the form $T \otimes S$, where S corresponds to the only divisor of m which is a leading term, that is T itself. We put

$$D_1(T \otimes S) = \frac{1}{c_g} g,$$

where c_g is the leading coefficient of g. We see that

$$D_1(T \otimes S) = T + \text{lower terms},$$

as required. To define H_0, we use a yet another inductive argument, decreasing tree monomials on which we want to define H_0. First of all, if a tree monomial T is not divisible by any of the leading terms of relations, we put $H_0(T) = 0$. Assume that T is divisible by some leading terms of relations, and $S_1, \ldots, S_p$ are the corresponding divisors. Then on $\mathscr{A}_{\mathscr{G}}^T$ we can use $S_1 \wedge \cdot$ as a homotopy, so $h_0(T) = T \otimes S_1$. We put

$$H_0(T) = h_0(T) + H_0(T - D_1 h_0(T)).$$

Here the leading term of $T - D_1h_0(T)$ is smaller than T (since we already know that the leading term of $D_1h_0(T)$ is $d_1h_0(T) = T$), so induction on the leading term applies. Note that by induction the leading term of $H_0(T)$ is $h_0(T)$.

Suppose that $k > 0$, that we know the pairs (D_{l+1}, H_l) for all $l < k$, and that in these degrees

$$D(x) = d(\hat{x}) + \text{lower terms}, \quad H(x) = h(\hat{x}) + \text{lower terms}.$$

To define D_{k+1}, we should, as above, only consider the case of generators. In this case, we put

$$D_{k+1}(x) = d_{k+1}(x) - H_{k-1}D_kd_{k+1}(x).$$

The property $D_{k+1}(x) = d_{k+1}(\hat{x}) + \text{lower terms}$ now easily follows by induction. To define H_k, we proceed in a way very similar to what we did for the induction basis. Assume that $u \in \ker D_k$, and that we know H_k on all elements of $\ker D_k$ whose leading term is less than $\hat{u}$. Since $D_k(u) = d_k(\hat{u}) + \text{lower terms}$, we see that $u \in \ker D_k$ implies $\hat{u} \in \ker d_k$. Then $h_k(\hat{u})$ is defined, and we put

$$H_k(u) = h_k(\hat{u}) + H_k(u - D_{k+1}h_k(\hat{u})).$$

Here $u - D_{k+1}h_k(\hat{u}) \in \ker D_k$ and its leading term is smaller than $\hat{u}$, so induction on the leading term applies (and it is easy to check that by induction $H_{k+1}(x) = h_{k+1}(\hat{x}) + \text{lower terms}$).

The proof is completed by the following

Lemma 1. *The mappings D and H defined by these formulas satisfy, for each $k > 0$, $D_kD_{k+1} = 0$ and $(D_{k+1}H_k)|_{\ker D_k} = \mathrm{Id} - \widetilde{\pi}$.*

Proof. The structure of this proof is somewhat similar to the way D and H were constructed. Let us prove both statements simultaneously by induction. If $k = 0$, the first statement is obvious. Let us prove the second one and establish that $D_1H_0(m) = (\mathrm{Id} - \widetilde{\pi})(m)$ for each tree monomial m. Slightly rephrasing that, we shall prove that for each tree monomial T we have $D_1H_0(T) = T - \overline{T}$, where $\overline{T}$ is the residue of T modulo $\mathscr{G}$ [10]. We shall prove this statement by induction on T. If the monomial T is not divisible by any leading terms of relations, we have $H_0(T) = 0 = T - \overline{T}$. Let T be divisible by leading terms $T_1, \ldots, T_p$, and let $S_1, \ldots, S_p$ be the corresponding divisors. We have $H_0(T) = h_0(T) + H_0(T - D_1h_0(T))$, so

$$D_1H_0(T) = D_1h_0(T) + D_1H_0(T - D_1h_0(T)).$$

By induction, we may assume that

$$D_1H_0(T - D_1h_0(T)) = T - D_1h_0(T) - \overline{(T - D_1h_0(T))}.$$

Also,

$$D_1h_0(T) = D_1(T \otimes S_1) = \frac{1}{c_g}T',$$

where g is the relation with the leading monomial T_1, and

$$\frac{1}{c_g}T' = \frac{1}{c_g}m_{T,T_1}(f) = T - r_g(T)$$

is the (normalized) result of substitution of g into T in the place described by S_1. Consequently,

$$D_1H_0(T) = T - r_g(T) + \Big((T - D_1h_0(T)) - \overline{(T - D_1h_0(T))}\Big) = \\ = T - r_g(T) + (r_g(T) - \overline{r_g(T)}) = T - \overline{r_g(T)} = T - \overline{T},$$

since the residue does not depend on a particular choice of reductions.

Assume that $k > 0$, and that our statement is true for all $l < k$. We have

$$D_kD_{k+1}(x) = 0$$

since

$$D_kD_{k+1}(x) = D_k(d_{k+1}(x) - H_{k-1}D_kd_{k+1}(x)) = \\ = D_kd_{k+1}(x) - D_kH_{k-1}D_kd_{k+1}(x) = D_kd_{k+1}(x) - D_kd_{k+1}(x) = 0,$$

because $D_kd_{k+1}k \in \ker D_{k-1}$, and so $D_kH_{k-1}(D_k(y)) = D_k(y)$ by induction. Also, for $u \in \ker D_k$ we have

$$D_{k+1}H_k(u) = D_{k+1}h_k(\hat{u}) + D_{k+1}H_k(u - D_{k+1}h_k(\hat{u})),$$

and by the induction on $\hat{u}$ we may assume that

$$D_{k+1}H_k(u - D_{k+1}h_k(\hat{u})) = u - D_{k+1}h_k(\hat{u})$$

(on elements of positive homological degree, $\widetilde{\pi} = 0$), so

$$D_{k+1}H_k(u) = D_{k+1}h_k(\hat{u}) + u - D_{k+1}h_k(\hat{u}) = u.$$

□

□

This construction works without any change for the case of free resolution of the trivial module over the given operad (or algebra). For the case of algebras, it was described by Anick [1] and Kobayashi [22] (see also Lambe [23]), the case of operads is treated analogously.

Remark 3. There is another way to define a free resolution due to Brown [6] (see also the paper of Cohen [7], where the main construction of [6] is made impressively transparent under some technical assumptions). Brown's idea is to start from the bar resolution, select there candidates that we want to be the generators of a smaller free resolution, and construct a contraction of the bar complex on that subcomplex explicitly by induction. From the computational point of view, this approach is sometimes very useful, since in our approach to compute D and H in the homological degree k we have to know them in all smaller homological degrees while in Brown's approach (under assumptions of [7]) one only uses the information about d and h in homological degrees k and $k-1$.

3. T - :

The main character of this section is the anti-associative operad studied by Markl and Remm in [29]. More precisely, it is the nonsymmetric operad $\mathscr{A}$ with one generator $f(\text{-},\text{-}) \in \mathscr{A}(2)$ and one relation

$$f(f(\text{-},\text{-}),\text{-}) + f(\text{-},f(\text{-},\text{-})) = 0. \tag{8}$$

For the path-lexicographic ordering, the element $f(f(f(\text{-},\text{-}),\text{-}),\text{-})$ is a small common multiple of the leading monomial with itself, and the corresponding S-polynomial is equal to $2f(\text{-},f(\text{-},f(\text{-},\text{-})))$. These relations together already imply that $\mathscr{A}(k) = 0$ for $k \geq 4$, so they form a Gröbner basis. The corresponding monomial operad is defined by relations

$$f(f(\text{-},\text{-}),\text{-}) \quad \text{and} \quad f(\text{-},f(\text{-},f(\text{-},\text{-}))),$$

and has a monomial basis

$$\{\mathrm{id}, f, g := f(\text{-},f(\text{-},\text{-}))\}.$$

Let us compute some low degree maps of our resolution of the trivial module by free right modules, thus showing how the general recipes from Section 2.4 work on the example of the low-dimensional homology. Let us denote the modules of our resolution by

$$\ldots \to V_2 \circ \mathscr{A} \to V_1 \circ \mathscr{A} \to V_0 \circ \mathscr{A} \to \mathscr{A} \to \Bbbk \to 0,$$

and use the following notation for the low degree basis elements: $\alpha \in V_0(2)$ is the element corresponding to f, $\beta \in V_1(3)$ and $\gamma \in V_1(4)$ are the elements corresponding to $f(f(\text{-},\text{-}),\text{-})$ and $f(\text{-},f(\text{-},f(\text{-},\text{-})))$ respectively, and $\omega \in V_2(4)$ corresponds to the small common multiple we discussed above (the overlap of two copies of $f(f(\text{-},\text{-}),\text{-})$); there are other elements in V_2, but we shall use only this one in our example.

Let us first write down the results for the monomial case:

Proposition 2. *We have*

$$\begin{gathered}
d_0(\alpha(\text{-},\text{-})) = f(\text{-},\text{-}),\\
h_0(f(\text{-},\text{-})) = \alpha(\text{-},\text{-}),\\
d_1(\beta(\text{-},\text{-},\text{-})) = \alpha(f(\text{-},\text{-}),\text{-}),\\
d_1(\gamma(\text{-},\text{-},\text{-},\text{-})) = \alpha(\text{-},g(\text{-},\text{-},\text{-})),\\
h_1(\alpha(\text{-},g(\text{-},\text{-},\text{-}))) = \gamma(\text{-},\text{-},\text{-},\text{-}),\\
h_1(\alpha(f(\text{-},\text{-}),f(\text{-},\text{-}))) =\\
= \beta(\text{-},\text{-},f(\text{-},\text{-})),\\
h_1(\alpha(g(\text{-},\text{-},\text{-}),\text{-})) = \beta(\text{-},f(\text{-},\text{-}),\text{-}),\\
d_2(\omega(\text{-},\text{-},\text{-},\text{-})) = \beta(f(\text{-},\text{-}),\text{-},\text{-}).
\end{gathered}$$

These results can be used to compute our maps in the deformed case.

Proposition 3. *We have*

$$
\begin{gathered}
D_0(\alpha(\text{-},\text{-})) = f(\text{-},\text{-}),\\
H_0(f(\text{-},\text{-})) = \alpha(\text{-},\text{-}),\\
D_1(\beta(\text{-},\text{-},\text{-})) = \alpha(f(\text{-},\text{-}),\text{-}) + \alpha(\text{-},f(\text{-},\text{-})),\\
D_1(\gamma(\text{-},\text{-},\text{-},\text{-})) = \alpha(\text{-},g(\text{-},\text{-},\text{-})),\\
H_1(\alpha(\text{-},g(\text{-},\text{-},\text{-}))) = \gamma(\text{-},\text{-},\text{-},\text{-}),\\
H_1(\alpha(f(\text{-},\text{-}),f(\text{-},\text{-}))) = \beta(\text{-},\text{-},f(\text{-},\text{-})) - \gamma(\text{-},\text{-},\text{-},\text{-}),\\
H_1(\alpha(g(\text{-},\text{-},\text{-}),\text{-})) = \beta(\text{-},f(\text{-},\text{-}),\text{-}) + \gamma(\text{-},\text{-},\text{-},\text{-}),\\
D_2(\omega(\text{-},\text{-},\text{-},\text{-})) = \beta(f(\text{-},\text{-}),\text{-},\text{-}) + \beta(\text{-},f(\text{-},\text{-}),\text{-}) - \beta(\text{-},\text{-},f(\text{-},\text{-})) + 2\gamma(\text{-},\text{-},\text{-},\text{-}).
\end{gathered}
$$

Proof. Formulas for D_0 and H_0 are obvious. For D_1 and H_1, the computation goes as follows:

$$
\begin{aligned}
D_1(\beta(\text{-},\text{-},\text{-})) &= \alpha(f(\text{-},\text{-}),\text{-}) - H_0D_0(\alpha(f(\text{-},\text{-}),\text{-})) =\\
&= \alpha(f(\text{-},\text{-}),\text{-}) - H_0(f(f(\text{-},\text{-}),\text{-})) = \alpha(f(\text{-},\text{-}),\text{-}) + H_0(f(\text{-},f(\text{-},\text{-}))) =\\
&= \alpha(f(\text{-},\text{-}),\text{-}) + \alpha(\text{-},f(\text{-},\text{-})),
\end{aligned}
$$

$$
D_1(\gamma(\text{-},\text{-},\text{-},\text{-})) = \alpha(\text{-},g(\text{-},\text{-},\text{-})) - H_0D_0(\alpha(\text{-},g(\text{-},\text{-},\text{-}))) = \alpha(\text{-},g(\text{-},\text{-},\text{-})),
$$

$$
H_1(\alpha(\text{-},g(\text{-},\text{-},\text{-}))) = \gamma(\text{-},\text{-},\text{-},\text{-}) + H_1(\alpha(\text{-},g(\text{-},\text{-},\text{-})) - D_1\gamma(\text{-},\text{-},\text{-},\text{-})) = \gamma(\text{-},\text{-},\text{-},\text{-}),
$$

$$
\begin{aligned}
&H_1(\alpha(f(\text{-},\text{-}),f(\text{-},\text{-}))) =\\
&= \beta(\text{-},\text{-},f(\text{-},\text{-})) + H_1(\alpha(f(\text{-},\text{-}),f(\text{-},\text{-})) - D_1(\beta(\text{-},\text{-},f(\text{-},\text{-})))) =\\
&= \beta(\text{-},\text{-},f(\text{-},\text{-})) - H_1(\alpha(\text{-},g(\text{-},\text{-},\text{-}))) = \beta(\text{-},\text{-},f(\text{-},\text{-})) - \gamma(\text{-},\text{-},\text{-},\text{-}),
\end{aligned}
$$

$$
\begin{aligned}
H_1(\alpha(g(\text{-},\text{-},\text{-}),\text{-})) &= \beta(\text{-},f(\text{-},\text{-}),\text{-}) + H_1(\alpha(g(\text{-},\text{-},\text{-}),\text{-}) - D_1(\beta(\text{-},f(\text{-},\text{-}),\text{-}))) =\\
&= \beta(\text{-},f(\text{-},\text{-}),\text{-}) + H_1(-\alpha(\text{-},-g(\text{-},\text{-},\text{-}))) = \beta(\text{-},f(\text{-},\text{-}),\text{-}) + \gamma(\text{-},\text{-},\text{-},\text{-}).
\end{aligned}
$$

For D_2, we may use the formulas we already obtained, getting

$$
\begin{aligned}
D_2(\omega(\text{-},\text{-},\text{-},\text{-})) &= \beta(f(\text{-},\text{-}),\text{-},\text{-}) - H_1D_1(\beta(f(\text{-},\text{-}),\text{-},\text{-})) =\\
&= \beta(f(\text{-},\text{-}),\text{-},\text{-}) - H_1(\alpha(f(f(\text{-},\text{-}),\text{-}),\text{-}) + \alpha(f(\text{-},\text{-}),f(\text{-},\text{-}))) =\\
&= \beta(f(\text{-},\text{-}),\text{-},\text{-}) - H_1(-\alpha(g(\text{-},\text{-},\text{-}),\text{-}) + \alpha(f(\text{-},\text{-}),f(\text{-},\text{-}))) =\\
&= \beta(f(\text{-},\text{-}),\text{-},\text{-}) + (\beta(\text{-},f(\text{-},\text{-}),\text{-}) + \gamma(\text{-},\text{-},\text{-},\text{-})) - (\beta(\text{-},\text{-},f(\text{-},\text{-})) - \gamma(\text{-},\text{-},\text{-},\text{-})) =\\
&= \beta(f(\text{-},\text{-}),\text{-},\text{-}) + \beta(\text{-},f(\text{-},\text{-}),\text{-}) - \beta(\text{-},\text{-},f(\text{-},\text{-})) + 2\gamma(\text{-},\text{-},\text{-},\text{-}).
\end{aligned}
$$

□

In particular, when we use our resolution to compute $\mathrm{Tor}_{\bullet}^{\widetilde{\mathrm{As}}}(\Bbbk,\Bbbk)$, all summands killed by the augmentation vanish, and we get

$$d_1\beta = d_1\gamma = 0, \quad d_2\omega = 2\gamma,$$

so $\mathrm{Tor}_2^{\mathscr{A}}(\Bbbk,\Bbbk)$ is one-dimensional. This result is not surprising: the second term of the bar homology encodes relations, and in our case the space of relations is one-dimensional (and β is the leading term of that relation).

4. A

4.1. **Another proof of the PBW criterion for Koszulness.** The goal of this section is to prove the following statement (which brings to the common ground the PBW criterion of Priddy [31] for associative algebras and the PBW criterion of Hoffbeck [20] for operads).

Theorem 5. *An associative algebra (commutative algebra, operad etc.) with a quadratic Gröbner basis is Koszul.*

Proof. It is enough to prove it in the monomial case, since it gives an upper bound on the homology: for the deformed differential, the cohomology may only decrease. In the monomial case the statement is obvious, since for a thick indecomposable collection of *quadratic* monomials the internal degree has to coincide with the homological degree. □

4.2. **Operads and commutative algebras.** Recall a construction of an operad from a graded commutative algebra [21].

Let A be a graded commutative algebra. Define an operad $\mathscr{O}_A$ as follows. We put $\mathscr{O}_A(n) := A_{n-1}$, and let the partial composition map

$$\circ_i \colon \mathscr{O}_A(k) \otimes \mathscr{O}_A(l) = A_{k-1} \otimes A_{l-1} \to A_{k+l-2} = \mathscr{O}_A(k+l-1)$$

be the product in A.

As we remarked in [10], a basis of the algebra A leads to a basis of the operad $\mathscr{O}_A$: product of generators of the polynomial algebra is replaced by the iterated composition of the corresponding generators of the free operad, where each composition is substitution into the last slot of an operation. Assume that we know a Gröbner basis for the algebra A. It leads to a Gröbner basis for the operad $\mathscr{O}_A$ as follows: we first impose the quadratic relations defining the operad $\mathscr{O}_{\Bbbk[x_1,\ldots,x_n]}$ coming from the polynomial algebra (stating that the result of a composition depends only on the operations composed, not on the order in which we compose operations), and then use the identification of relations in the polynomial algebra with elements of the corresponding operad, as above. Our next goal is to explain how to use the Anick resolution of the trivial module for A to construct a small resolution of the trivial module for $\mathscr{O}_A$.

Let C_q be the space of Anick's q-chains, as in Section 2.2. We define the "product" $\bullet\colon C_p \otimes C_q \to C_{p+q+1}$ as follows: $c_1 \bullet c_2 = c_3$ if we can remove one of the relations used to build c_3 so that what remains is a disjoint union of c_1 and c_2. This product descends to the spaces of homology H_p of the differential D_p for the deformed case.

Now, let $\mathscr{M}(k) = H_{k-1}$, and consider the operad $\mathscr{R}$ which is the quotient of the free operad $\mathscr{F}_{\mathscr{M}}$ by all the relations $c_1 \circ_p c_2 = c_1 \bullet c_2$, where $c_1 \in H_{p-2} = \mathscr{M}(p)$ and $c_2 \in H_{q-2} = \mathscr{M}(q)$. If we introduce a new grading on $\mathscr{R}$, putting the degree of a q-chain equal to $(q+1)$, the relations of this operad become homogeneous.

Theorem 6. *There exists a free right module resolution*

$$\mathscr{R} \circ \mathscr{O}_A \to \Bbbk \to 0$$

of the trivial $\mathscr{O}_A$-module.

Proof. This statement is obvious from our previous results. Indeed, we know how to obtain a Gröbner basis for $\mathscr{O}_A$ from a Gröbner basis of A. From that, we immediately see that for the monomial version of $\mathscr{O}_{\mathscr{A}}$ there exists a resolution of the same form, but based on the operad generated by chains, not the homology. To obtain a resolution over $\mathscr{O}_A$, let us look carefully into the general reconstruction scheme from the previous section. It recovers lower terms of differentials and homotopies by recalling lower terms of elements of the Gröbner basis. Let us do the reconstruction in two steps. At first, we shall recall all lower terms of relations except for those starting with $\alpha(\beta(\text{-},\text{-}),\text{-})$; the latter are still assumed to vanish. On the next step we shall recall all lower terms of those quadratic relations. Note that after the first step we model many copies of the associative algebra resolution and the differential there; so we can compute the homology explicitly. At the next step, a differential will be induced on this homology we computed, and we end up with a resolution of the required type. □

Sometimes the resolution we constructed gives estimates on the bar homology of $\mathscr{O}_A$ that are sufficient to compute it completely.

Recall that if the algebra A is quadratic, then the operad $\mathscr{O}_A$ is quadratic as well. In [10], we proved that if the algebra A is PBW, then the operad $\mathscr{O}_A$ is PBW as well, and hence is Koszul. Now we shall prove the following substantial generalisation of this statement (substantially simplifying the proof of this statement given in [21]).

Theorem 7. *If the algebra A is Koszul, then the operad $\mathscr{O}_A$ is Koszul as well.*

Proof. Koszulness of our algebra implies that the homology of the bar resolution is concentrated on the diagonal. Consequently, the operad $\mathscr{R}$ constructed above is automatically concentrated on the diagonal, and so is its homology, which completes the proof. □

Remark 4. The same results can be applied to dioperads. For a graded commutative algebra A, let us put $\mathscr{D}_A(m,n) := A_{m+n-2}$, and let the partial composition map

$$\begin{aligned}\circ_{i,j}\colon \mathscr{D}_A(m,n)\otimes\mathscr{D}_A(p,q) = A_{m+n-2}\otimes A_{p+q-2} \to \\ \to A_{m+n+p+q-4} = \mathscr{D}_A(m+p-1,n+q-1)\end{aligned}$$

be the product in A. The bi-collection $\{\mathscr{D}_A(m,n)\}$ forms a dioperad, which is quadratic whenever the algebra A is, and, as it turns out, is Koszul whenever the algebra A is. This can be proved similarly to how the previous theorem is proved.

4.3. **The operad BV and hypercommutative algebras.** The main goal of this section is to explain how our results can be used to study the operad of Batalin–Vilkovisky algebras. The key results below (Proposition 6 and Theorem 10) were announced by Drummond-Cole and Vallette earlier this year [8, 38]; our proofs are based on methods entirely different from theirs.

4.3.1. *The operad BV and its Gröbner basis.* Batalin–Vilkovisky algebras show up in various questions of mathematical physics. In [14], a cofibrant resolution for the corresponding operad was presented. However, that resolution is a little bit more that minimal. In this section, we present a minimal resolution for this operad in the shuffle category. The operad BV, as defined in most sources, is an operad with quadratic–linear relations; as such, it cannot have a minimal free resolution. Our main idea is to study it as an operad with homogeneous relations of degrees 2, 3, and 4. Our choice of degrees and signs is taken from [14], where it is explained how to translate between this convention and another popular definition of BV-algebras.

Definition 1 (Batalin–Vilkovisky algebras as in [14])**.** A *Batalin-Vilkovisky algebra*, or *BV-algebra* for short, is a differential graded vector space (A, d_A) endowed with

- a symmetric binary product $\bullet$ of degree 0,
- a symmetric bracket $\langle\ ,\ \rangle$ of degree +1,
- a unary operator Δ of degree +1,

such that (A, d_A, Δ) is a bicomplex, d_A is a derivation with respect to both the product and the bracket, and such that

- the product $\bullet$ is associative,
- the bracket satisfies the Jacobi identity

$$\langle\langle\, \text{-}, \text{-}\,\rangle, \text{-}\,\rangle.(1 + (123) + (132)) \;=\; 0, \tag{9}$$

- the product $\bullet$ and the bracket $\langle\ ,\ \rangle$ satisfy the Leibniz relation

$$\langle\, \text{-}, \text{-} \bullet \text{-}\,\rangle \;=\; (\langle\, \text{-}, \text{-}\,\rangle \bullet \text{-}) \;+\; (\text{-} \bullet \langle\, \text{-}, \text{-}\,\rangle).(12), \tag{10}$$

- the operator Δ satisfies $\Delta^2 = 0$,
- the bracket is the obstruction to Δ being a derivation with respect to the product $\bullet$

$$\langle\, \text{-}, \text{-}\,\rangle \;=\; \Delta \circ (\text{-} \bullet \text{-}) \;-\; (\Delta(\text{-}) \bullet \text{-}) \;-\; (\text{-} \bullet \Delta(\text{-})), \tag{11}$$

- the operator Δ is a graded derivation with respect to the bracket

$$\Delta(\langle\, \text{-}, \text{-}\,\rangle) \;+\; \langle \Delta(\text{-}), \text{-}\,\rangle \;+\; \langle\, \text{-}, \Delta(\text{-})\rangle \;=\; 0. \tag{12}$$

We shall make relations homogeneous by eliminating the redundant operation, that is the bracket. This definition of a BV-algebra as a dg-commutative algebra equipped with a square zero operator Δ satisfying one identity is far from new, see, e. g., [15].

Definition 2 (Batalin–Vilkovisky algebras with homogeneous relations)**.** A *Batalin-Vilkovisky algebra*, or *BV-algebra* for short, is a differential graded vector space (A, d_A) endowed with

- a symmetric binary product $\bullet$ of degree 0,
- a unary operator Δ of degree +1,

such that (A, d_A, Δ) is a bicomplex, d_A is a derivation with respect to the product, and such that

- the product $\bullet$ is associative,
- the operator Δ satisfies $\Delta^2 = 0$,
- the operations satisfy the cubic identity

$$\Delta(\text{-} \bullet \text{-} \bullet \text{-}) = ((\Delta(\text{-} \bullet \text{-}) \bullet \text{-}) - (\Delta(\text{-}) \bullet \text{-} \bullet \text{-})).(1 + (123) + (132)), \tag{13}$$

Let us consider the ordering of the free operad where we first compare lexicographically the operations on the paths from the root to leaves, and then the planar permutations of leaves; we assume that $\Delta > \bullet$.

Proposition 4. *The above relations together with the degree* 4 *relation*

$$(\Delta(\text{-} \bullet \Delta(\text{-} \bullet \text{-})) - \Delta(\Delta(\text{-}) \bullet \text{-} \bullet \text{-})).(1 + (123) + (132)) = 0 \tag{14}$$

form a Gröbner basis of relations for the operad of BV-algebras.

Proof. Here and below we use the language of operations, as opposed the language of tree monomials; our operations reflect the structure of the corresponding tree monomials in the free shuffle operad. For each i, the argument a_i of an operation corresponds to the leaf i of the corresponding tree monomial.

With respect to our ordering, the leading monomials of our original relations are $(a_1 \bullet a_2) \bullet a_3$, $(a_1 \bullet a_3) \bullet a_2$, $\Delta^2(a_1)$, and $\Delta(a_1 \bullet (a_2 \bullet a_3))$. The only small common multiple of $\Delta^2(a_1)$ and $\Delta(a_1 \bullet (a_2 \bullet a_3))$ gives a nontrivial S-polynomial which, is precisely the relation (14). The leading term of that relation is $\Delta(\Delta(a_1 \bullet a_2) \bullet a_3)$.

It is well known that $\dim BV(n) = 2^n n!$ [15], so to verify that our relations form a Gröbner basis, it is sufficient to show that the restrictions imposed by these leading monomials are strong enough, that is that the number of arity n tree monomials that are not divisible by any of these is equal to $2^n n!$. Moreover it is sufficient to check that for $n \leq 4$, since all S-polynomials of our relations will be elements of arity at most 4. This can be easily checked by hand, or by a computer program [11]. □

4.3.2. *Bar homology of the operad BV.* Let us denote by $\mathscr{G}$ the Gröbner basis from the previous section.

Proposition 5. *For the monomial version of BV, the resolution $\mathscr{A}_{\mathscr{G}}$ from Section 2.3 is minimal, that is the differential induced on the space of generators is zero.*

Proof. Let us describe explicitly the space of generators, that is possible indecomposable coverings of monomials by leading terms of relations (all monomials below are chosen from the basis of the free shuffle operad, so the correct ordering of subtrees is assumed). These are

- all monomials $\Delta^k(a_1)$, $k \geq 2$ (covered by several copies of $\Delta^2(a_1)$),
- all "Lie monomials"

$$\lambda = (\ldots((a_1 \bullet a_{k_2}) \bullet a_{k_3}) \bullet \ldots) \bullet a_{k_n} \tag{15}$$

  where $(k_2, \ldots, k_n)$ is a permutation of numbers $2, \ldots, n$ (only the leading terms $(a_1 \bullet a_2) \bullet a_3$ and $(a_1 \bullet a_3) \bullet a_2$ are used in the covering),
- all the monomials

$$\Delta^k(\Delta(\lambda_1 \bullet (\lambda_2 \bullet \lambda_3))) \tag{16}$$

  where $k \geq 1$, each λ_i is a Lie monomial as described above (several copies of Δ^2, the leading term of degree 3, and several Lie monomials are used),

- all monomials

$$\Delta^k(\Delta(\ldots\Delta(\Delta(\lambda_1\bullet\lambda_2)\bullet\lambda_3)\bullet\ldots)\bullet\lambda_n) \tag{17}$$

where $k\geq 0$, $n\geq 3$, and λ_i are Lie monomials (several copies of all leading terms are used, including at least one copy of the degree 4 leading term).

This is a complete list of tree monomials T for which $\mathscr{A}_{\mathscr{G}}^T$ is nonzero in positive homological degrees. It is easy to see that for each of them there exists only one indecomposable covering by relations, that is only one generator of $\mathscr{A}_{\mathscr{G}}$ of shape T. Consequently, the differential maps such a generator to $(\mathscr{A}_{\mathscr{G}})_+^2$, so the differential induced on generators is identically zero. □

The resolution of the operad BV which one can derive by our methods from this one is quite small (in particular, smaller than the one of [14]) but still not minimal. However, we now have enough information to compute the bar homology of the operad BV.

Theorem 8. *The basis of $H(\mathbf{B}(BV))$ is formed by monomials*

$$\Delta^k(a_1),\quad k\geq 1,$$

and all monomials of the form

$$\underbrace{\Delta(\ldots\Delta(\Delta(}_{n-1\ times}\lambda_1\bullet\lambda_2)\bullet\ldots)\bullet(\lambda_n\bullet a_j)),\quad n\geq 1 \tag{18}$$

from the monomial resolution discussed above. Here all λ_i are Lie monomials.

Proof. Similarly to how things work for the operad $\mathscr{A}$ in Section 3, it is easy to check that the element $\Delta(\Delta(a_1\bullet a_2)\bullet a_3)$ that corresponds to the leading term of the only contributing S-polynomial will be killed by the differential of the element $\Delta^2(a_1\bullet(a_2\bullet a_3))$ (covered by two leading terms $\Delta^2(a_1)$ and $\Delta(a_1\bullet(a_2\bullet a_3)))$ in the deformed resolution. This observation goes much further, namely we have for $k\geq 1$

$$\begin{aligned} &D(\Delta^k(\Delta(\ldots\Delta(\Delta(\lambda_1\bullet\lambda_2)\bullet\lambda_3)\bullet\ldots)\bullet(\lambda_n\bullet a_j)) = \\ &\quad = \Delta^{k-1}((\Delta(\ldots\Delta(\Delta(\lambda_1\bullet\lambda_2)\bullet\lambda_3)\bullet\ldots)\bullet\lambda_n)\bullet a_j) + \text{lower terms}\end{aligned} \tag{19}$$

in the sense of the partial ordering we discussed earlier). So, if we retain only leading terms of the differential, the resulting homology classes are represented by all the monomials of arity m

$$\Delta(\ldots\Delta(\Delta(\lambda_1\bullet\lambda_2)\bullet\ldots)\bullet\lambda_n) \tag{20}$$

with λ_n having at least two leaves. They all have the same homological degree $m-2$ in the resolution, and so there are no further cancellations. □

So far we have not been able to describe a minimal resolution of the operad BV by relatively compact closed formulas, even though in principle our proof, once processed by a version of Brown's machinery [6, 7], would clearly yield such a resolution (in the shuffle category).

4.3.3. *Operads* Hycom *and* Grav. The operads Hycom and its Koszul dual Grav were originally defined in terms of moduli spaces of curves of genus 0 with marked points $\mathscr{M}_{0,n+1}$ [16, 17]. However, we are interested in the algebraic aspects of the story, and we use the following descriptions of these operads as quadratic algebraic operads [16]. An algebra over Hycom is a chain complex A with a sequence of graded symmetric products

$$(x_1, \ldots, x_n)\colon A^{\otimes n} \to A$$

of degree $2(n-2)$, which satisfy the following relations (here $a, b, c, x_1, \ldots, x_n$, $n \geq 0$, are elements of A):

$$\sum_{S_1 \amalg S_2 = \{1,\ldots,n\}} \pm((a, b, x_{S_1}), c, x_{S_2}) = \sum_{S_1 \amalg S_2 = \{1,\ldots,n\}} \pm(a, (b, c, x_{S_1}), x_{S_2}). \tag{21}$$

Here, for a finite set $S = \{s_1, \ldots, s_k\}$, x_S denotes for $x_{s_1}, \ldots, x_{s_k}$, and $\pm$ means the Koszul sign rule.

An algebra over Grav is a chain complex with graded antisymmetric products

$$[x_1, \ldots, x_n]\colon A^{\otimes n} \to A$$

of degree $2 - n$, which satisfy the relations:

$$\sum_{1 \leq i < j \leq k} \pm[[a_i, a_j], a_1, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_k, b_1, \ldots, b_\ell] = \tag{22}$$

$$= \begin{cases} [[a_1, \ldots, a_k], b_1, \ldots, b_l], & l > 0, \\ 0, & l = 0, \end{cases}$$

for all $k > 2$, $l \geq 0$, and $a_1, \ldots, a_k, b_1, \ldots, b_l \in A$. For example, setting $k = 3$ and $l = 0$, we obtain the Jacobi relation for $[a, b]$. (Similarly, the first relation for Hycom is the associativity of the product (a, b).)

Let us define an admissible ordering of the free operad whose quotient is Grav as follows. We introduce an additional weight grading, putting the weight of the corolla corresponding to the binary bracket equal to 0, all other weights of corollas equal to 1, and extending it to compositions by additivity of weight. To compare two monomials, we first compare their weights, then the root corollas, and then path sequences [10] according to the reverse path-lexicographic order. For both of the latter steps, we need an ordering of corollas; we assume that corollas of larger arity are smaller. Then for the relation (k, l) in (22) (written in the shuffle notation with variables in the proper order), its leading monomial is equal to the monomial in the right hand side for $l > 0$, and to the monomial $[a_1, \ldots, a_{n-2}, [a_{n-1}, a_n]]$ for $l = 0$.

The following theorem, together with the PBW criterion, implies that the operads Grav and Hycom are Koszul, the fact first proved by Getzler [15].

Theorem 9. *For our ordering, the relations of* Grav *form a Gröbner basis of relations.*

Proof. The tree monomials that are not divisible by leading terms of relations are precisely

$$[\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, a_j], \tag{23}$$

where all λ_i, $1 \leq i \leq (n-1)$ are Lie monomials as in (15) (but made from brackets, not products).

Lemma 2. *The graded character of the space of such elements of arity n is*

$$(2+t^{-1})(3+t^{-1})\ldots(n-1+t^{-1}). \tag{24}$$

Proof. To compute the number of basis elements where the top degree corolla is of arity $k+1$ (or, equivalently, degree $1-k$), $k \geq 1$, let us notice that this number is equal to the number of basis elements

$$[\lambda_1, \lambda_2, \ldots, \lambda_k]$$

where the arity of λ_k is at least 2 (a simple bijection: join λ_{n-1} and a_j into $[\lambda_{n-1}, a_j]$). The latter number is equal to

$$\sum_{\substack{m_1+\ldots+m_k=n,\\ m_i\geq 1, m_k\geq 2}} \frac{(m_1-1)!(m_2-1)!\ldots(m_k-1)!m_1m_2\cdot\ldots\cdot m_k}{(m_1+m_2+\ldots+m_k)(m_2+\ldots+m_k)\cdot\ldots\cdot m_k}\binom{m_1+\ldots+m_k}{m_1,m_2,\ldots,m_k} \tag{25}$$

where each factor $(m_i-1)!$ counts the number of Lie monomials of arity m_i, and the remaining factor is the number of shuffle permutations of the type $(m_1, \ldots, m_k)$ ([11]). This can be rewritten in the form

$$\sum_{m_1+\ldots+m_k=n, m_i\geq 1, m_k\geq 2} \frac{(m_1+\ldots+m_k-1)!}{(m_2+\ldots+m_k)(m_3+\ldots+m_k)\cdot\ldots\cdot m_k}$$

and if we introduce new variables $p_i = m_i + \ldots + m_k$, it takes the form

$$\sum_{2\leq p_{k-1}<\ldots<p_1\leq n-1} \frac{(n-1)!}{p_2\ldots p_k},$$

which clearly is the coefficient of t^{1-k} in the product

$$\begin{aligned}(n-1)!\left(1+\frac{1}{2t}\right)\left(1+\frac{1}{3t}\right)\cdot\ldots\cdot\left(1+\frac{1}{(n-1)t}\right) &= \\ &= \left(2+t^{-1}\right)\left(3+t^{-1}\right)\ldots\left(n-1+t^{-1}\right).\end{aligned} \tag{26}$$

□

Since the graded character of Grav is given by the same formula [16], we indeed see that the leading terms of defining relations give an upper bound on dimensions homogeneous components of Grav that coincides with the actual dimensions, so there is no room for further Gröbner basis elements. □

4.3.4. *BV_∞ and hypercommutative algebras.*

Proposition 6. *On the level of collections of graded vector spaces, we have*

$$H(\mathbf{B}(BV)) \simeq \mathrm{Grav}^* \oplus \delta\Bbbk[\delta], \tag{27}$$

where Grav^* *is the cooperad dual to* Grav, *and* $\delta \leftrightarrow \Delta^*$ *is an element of degree* 1.

Proof. We shall use the basis of $H(\mathbf{B}(BV))$ obtained in Theorem 8. In arity 1, our statement is almost tautological: the element δ^k corresponds to $\Delta^k(a_1)$. The case of elements of internal degree 0 (which in both cases are Lie monomials) is also obvious. For elements of degree $k-1$, let us extract from a typical monomial

$$\underbrace{\Delta(\ldots\Delta(\Delta(}_{k-1\text{ times}}\lambda_1 \bullet \lambda_2) \bullet \ldots) \bullet (\lambda_k \bullet a_j)),$$

of this degree the Lie monomials $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k, a_j$, and assign to this the element of Grav^* dual to the monomial $[\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k, a_j]$ in the gravity operad. This establishes a degree-preserving bijection between the bases of our two vector spaces. $\square$

We conclude with a theorem which is an algebraic shadow of results of Barannikov and Kontsevich ([2], see also [24, 26]) who proved in a rather indirect way that for a dgBV-algebra that satisfies the "$\partial-\overline{\partial}$-lemma", there exists a Hycom-algebra structure on its cohomology. Using their result, we shall explain that our isomorphism (27) exists not just on the level of graded vector spaces, but rather has some deep operadic structure behind it. From Theorem 9, it follows that the operads Grav and Hycom are Koszul, so $\Omega(\mathrm{Grav}^*)$ is a minimal model for Hycom. More precisely, we shall show that the differential of $BV_\infty = \Omega(H(\mathbf{B}(BV)))$ on generators Grav^* deforms the differential of $\mathrm{Hycom}_\infty = \Omega(H(\mathbf{B}(\mathrm{Hycom})))$ in the following sense. Recall that on the level of graded vector spaces we have the isomorphism $H(\mathbf{B}(BV)) \simeq \mathrm{Grav}^* \oplus \delta\Bbbk[\delta]$. Denote by D and d the differentials of BV_∞ and Hycom_∞ respectively. We can decompose $D = D_2 + D_3 + \ldots$ (respectively $d = d_2 + d_3 + \ldots$) according to the ∞-cooperad structure it provides on the space of generators $H(\mathbf{B}(BV)) \simeq \mathrm{Grav}^* \oplus \delta\Bbbk[\delta]$ (respectively, $H(\mathbf{B}(\mathrm{Hycom})) \simeq \mathrm{Grav}^*$). Finally, denote by m^* the obvious coalgebra structure on $\delta\Bbbk[\delta]$. We shall call a tree monomial in the cobar complex $\Omega(\mathrm{Grav}^* \oplus \delta\Bbbk[\delta])$ *mixed*, if it contains both corollas from Grav^* and from $\delta\Bbbk[\delta]$.

Theorem 10. *For the differential of BV_∞, we have*

$$D_2 = d_2 + m^* \tag{28}$$

and for $k \geq 3$ the co-operation D_k is zero on the generators $\delta\Bbbk[\delta]$, and maps generators from Grav^* *into linear combinations of mixed tree monomials.*

Proof. The result of Barannikov and Kontsevich [2] imply that there exists a mapping from Hycom to the homotopy quotient BV/Δ. In fact, it is an isomorphism, which can be proved in several different ways, both using Gröbner bases and geometrically; see [27] for a short geometric argument proving that. This means that the following maps exist (the vertical arrows are quasiisomorphisms between the operads and their minimal models):

$$\begin{tikzcd} BV_\infty \arrow[d] \arrow[dr, "\pi"] & & \mathrm{Hycom}_\infty \arrow[d] \\ BV & BV/\Delta & \mathrm{Hycom} \arrow[l, "\sim"'] \end{tikzcd}$$

Lifting $\pi\colon BV_\infty \to BV/\Delta \simeq \mathrm{Hycom}$ to the minimal model Hycom_∞ of Hycom, we obtain the commutative diagram

$$\begin{tikzcd}
BV_\infty \arrow[rr, two heads, "\psi"] \arrow[d] \arrow[dr, "\pi"'] & & \mathrm{Hycom}_\infty \arrow[d] \\
BV & BV/\Delta & \mathrm{Hycom} \arrow[l, "\sim"']
\end{tikzcd}$$

so there exists a map of dg-operads (and not just graded vector spaces, as it follows from our previous computations) between the cobar complexes $\Omega(H(\mathbf{B}(BV)))$ and $\Omega(H(\mathbf{B}(\mathrm{Hycom})))$. Commutativity of our diagram together with simple degree considerations completes the proof. $\square$

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