

Numerical Evidence for a $p_x - ip_y$ Paired Fractional Quantum Hall State at $\nu = 12/5$

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We provide numerical evidence supporting a Bonderson–Slingerland (BS) non-Abelian hierarchy state as a candidate for the observed $\nu = 12/5$ quantum Hall plateau. We confirm the existence of a gapped incompressible $\nu = 12/5$ quantum Hall state with shift $S = 2$ matching that of the BS state. The exact ground-state of the Coulomb interaction state on the sphere is shown to have large overlap with the BS ground-state trial wavefunction. The analysis of the BS states is extended to hierarchical descendants of general paired states in the weak-pairing phase at $\nu = 5/2$.

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Fractional quantum Hall (FQH) physics in the lowest Landau level is well understood in terms of the Laughlin states [1] and the Haldane–Halperin (HH) hierarchy states [2, 3], which can equivalently be described using Jain’s composite fermion (CF) approach [4, 5]. The first appearance of an even-denominator fractional plateau at $\nu = 5/2$ made it clear that the physics of the second Landau level (2LL) could be even more interesting. Numerical studies [6, 7, 8, 9] support the non-Abelian $p_x - ip_y$ paired Moore–Read (MR) state [10] (and its particle-hole conjugate, $\overline{\text{MR}}$) as the correct description of the $\nu = 5/2$ FQH state. At first, it seemed that this exceptional filling fraction was just an anomaly that appeared amongst other “standard” odd-denominator FQH states at $\nu = 7/3, 8/3$, and $14/5$ [11, 12]. Later, a $\nu = 12/5$ plateau also emerged [13], and it was numerically shown that in addition to the Abelian HH state, the particle-hole conjugate of the non-Abelian 3-clustered Read–Rezayi (RR) state [14, 15] is also a viable candidate for this filling fraction. In fact, it has been shown numerically that pairing/clustering is generally relevant in the $7/3 \leq \nu \leq 8/3$ range [16]. Recently, a non-Abelian hierarchy of states constructed over the $\nu = 5/2$ MR state was proposed to describe all the 2LL FQH states [17]. These Bonderson–Slingerland (BS) states exhibit the same pairing as the parent MR state, thus suggesting that the physics of the $\nu = 5/2$ “anomaly” could in fact be representative of all 2LL states. There has been much recent interest in non-Abelian FQH states due to their potential use for topologically protected quantum computation [18, 19]. While the $\overline{\text{RR}}$ state can provide computationally universal gates from braiding alone, the BS states cannot, requiring at least one supplemental unprotected gate. Hence, the HH, BS, and $\overline{\text{RR}}$ $\nu = 12/5$ candidate states have vastly different levels of utility for quantum computation, and discovering which of these actually occur in experiments will be quite significant. In this Letter, we provide numerical evidence establishing the BS state as a competitive candidate at $\nu = 12/5$.

The BS hierarchy states [17] built over the MR state

are constructed by successively condensing minimal charge Abelian quasiparticles and projecting them into new FQH states. They can be succinctly described as $\text{Ising} \times \text{U}(1)_K|_{\mathcal{C}}$, where the coupling constant K -matrix has $K_{00} = 2$ corresponding to the MR parent state and \mathcal{C} is the topological charge spectrum. Some of these states can also be described using an equivalent CF type formulation [17]. Among these is the BS state with $K = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, which is a candidate for $\nu = 12/5$. It has the CF type ground-state wavefunction [31]

$$\begin{aligned} \Psi_{\frac{5}{2}}^{(\text{BS})} &= \mathcal{P}_{LLL} \left\{ \text{Pf} \left[\frac{1}{z_i - z_j} \right] \chi_1^3 \chi_{-2} \right\} \\ &= \Psi_1^{(\text{MR})} \Psi_{\frac{2}{3}}^{(\text{CF})}, \end{aligned} \quad (1)$$

where \mathcal{P}_{LLL} is lowest Landau level projection, χ_n is the wavefunction of n filled Landau levels ($n < 0$ corresponding to negative flux), $\Psi_1^{(\text{MR})}$ is the bosonic $\nu = 1$ MR ground-state wavefunction, and $\Psi_{\frac{2}{3}}^{(\text{CF})}$ is the standard $\nu = 2/3$ CF ground-state wavefunction. This BS state has shift $S = 2$ on the sphere, where

$$N_{\phi} = \nu^{-1} N_e - S \quad (3)$$

is the relation between the number of flux quanta N_{ϕ} and the number of electrons N_e . The HH and $\overline{\text{RR}}$ states at $\nu = 12/5$, respectively, have $S = 4$ and -2 on the sphere. In order to study the validity of the BS state and to compare it with these other candidates for $\nu = 12/5$, we used a combination of powerful numerical techniques on the sphere: exact diagonalization, variational Monte Carlo, and the density matrix renormalization group (DMRG) method.

A necessary signature of a FQH state is the existence of a charge excitation gap

$$\Delta(N_{\phi}) = E_{N_{\phi}+1} + E_{N_{\phi}-1} - 2E_{N_{\phi}} \quad (4)$$

at the corresponding N_{ϕ} given in Eq. (3), where $E_{N_{\phi}}$ is the ground-state energy (in units of e^2/ℓ_0 , where $\ell_0 = \sqrt{\hbar c/eB}$)

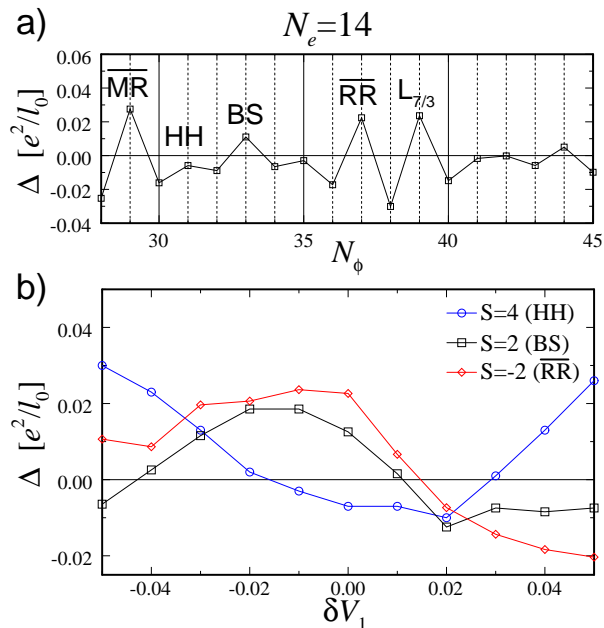


FIG. 1: The charge excitation gaps for a system of $N_e = 14$ electrons. a) A scan of gaps as a function of magnetic flux. Shifts corresponding to candidate states are labeled. b) A variation of the pseudopotential δV_1 around the Coulomb point at fluxes corresponding to $\nu = 12/5$ at shifts $S = 4, 2$, and -2 .

is the magnetic length) for the given value of N_ϕ fluxes. Δ/n is the energy gap of a quasihole-quasielectron pair, where n is the number of fundamental quasiholes produced per flux. ($n = 2$ for the HH, BS, and \overline{RR} states at $\nu = 12/5$.) As the $\nu = 12/5$ candidate states that are being considered all have distinct shifts, the existence of charge gaps can be used to help identify which states are competitive. It is, however, also important to remember that finite systems can run into the aliasing problem, where two states with different filling fractions share the same value of N_ϕ for a given N_e .

In a recent numerical study [20] with finite layer widths, a $\nu = 12/5$ state with $S = 2$ was clearly identified, with charge gaps given for up to $N_e = 14$. Second Landau level flux scans were only performed for $N_e = 10$ and 12 in Ref. [20], and, unfortunately, at these system sizes there are aliasing conflicts between $\nu = 12/5$ states with $S = 4$ and $S = -2$ and the $\nu = 5/2$ \overline{MR} and $\nu = 7/3$ Laughlin ($L_{7/3}$) states, respectively. In order to overcome these aliasing difficulties, we used the DMRG technique of Ref. [8] (see also [21]) to study system sizes of up to $N_e = 18$ electrons.

The DMRG technique belongs to the family of variational methods, and includes ingredients of exact diagonalization and numerical renormalization group. However, no *a priori* assumptions about the physics of the variational wavefunction are made. The algorithm relies on a truncation of the Hilbert space in such a way that the loss of information is minimized. The resulting variational wavefunction is the best approxima-

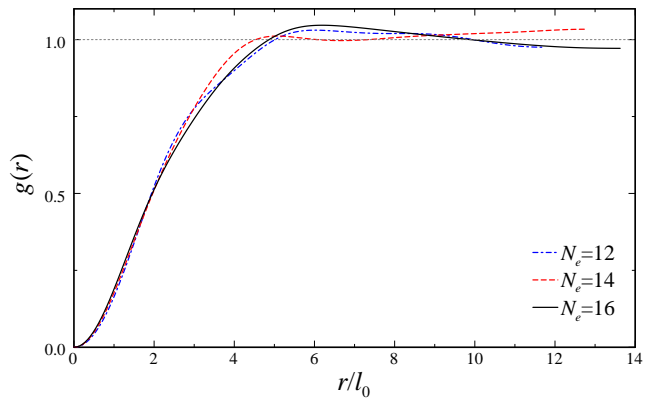


FIG. 2: The pair correlation function for the $\nu = 12/5$ state with $S = 2$ at the Coulomb point for $N_e = 12, 14$, and 16 .

tion to the actual ground-state in the form of a matrix product state. The accuracy is completely under control and it is dictated by the number of DMRG states m kept in the truncation. In this work, we use up to $m = 4000$ states for the larger system sizes, giving estimated errors in the energies per electron on the order of 5×10^{-5} in the worst cases.

In Fig. 1a), we show a scan of the charge gap as a function of magnetic flux, for $N_e = 14$ at the Coulomb point. We can identify different states according to their shift, labeling the $\nu = 5/2$ \overline{MR} state, the $\nu = 12/5$ HH, BS, and \overline{RR} states, and the $\nu = 7/3$ $L_{7/3}$ state. We find gaps for $\nu = 12/5$ states at $S = 2$ (as referenced earlier in [17]) and $S = -2$, but not at $S = 4$. In Fig. 1b), we show the behavior of the charge gap as a function of the pseudopotential V_1 varied around the Coulomb point, for $N_e = 14$. This exhibits the generally observed behavior where increasing V_1 destroys the non-Abelian clustered states (BS and \overline{RR}) and stabilizes the Abelian state (HH). We note that the $S = 2$ and -2 states both show a strong gap in the same range of δV_1 , including at the Coulomb point ($\delta V_1 = 0$). These three $\nu = 12/5$ states satisfy the $L^2 = 0$ condition of FQH ground-states when their gaps are positive, until $\delta V_1 \lesssim -0.02$.

To further investigate the characteristics of the $\nu = 12/5$ state with $S = 2$, we calculate the pair correlation function $g(r)$ obtained from exact diagonalization. The results at the Coulomb point are displayed in Fig. 2. These exhibit strongly damped long-distance oscillations indicative of an incompressible state, further corroborating that this is indeed a good FQH state. We also see a slight “shoulder” at small r , which becomes more pronounced as δV_1 decreases. Such shoulders are present for the MR and RR states [14, 22], and are considered a characteristic of non-Abelian clustered states.

The preceding discussion of the spectral properties of the Coulomb Hamiltonian in the 2LL reveals clear evidence of the existence of an incompressible state at $\nu = 12/5$ with $S = 2$, consistent with the BS state. However, finding such a state at the same filling fraction and shift as a proposed candidate

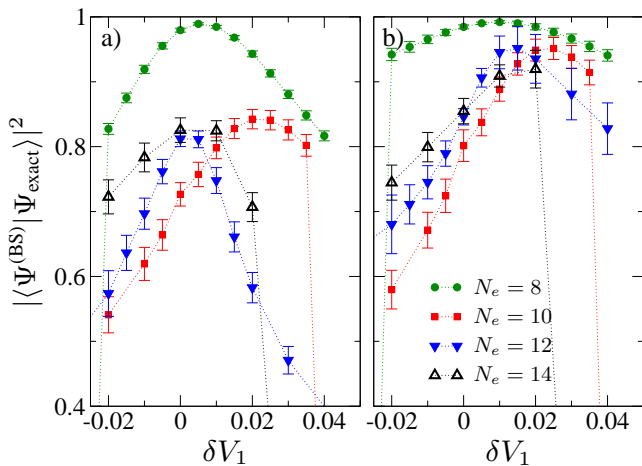


FIG. 3: Squared overlaps for $N_e = 8, 10, 12,$ and 14 between the exact diagonalization ground-state and: a) the BS ground-state wavefunction of Eq. (2), and b) the BS ground-state with optimized pair-wavefunction of Eq. (5). Error bars represent the statistical uncertainty of the Monte-Carlo sampling of the overlap integral.

is still only circumstantial evidence, and more direct evidence is necessary to establish the BS state as an accurate description. We therefore consider the overlap of the ground-state wavefunction of Eq. (2) with the exact ground-state wavefunction obtained for the same shift on the sphere. As shown in Fig. 3a), these overlaps reach as high as 0.989(2) for $N_e = 8$ and remain as large as 0.83(2) for the largest system considered ($N_e = 14$) at $\delta V_1 = 0$. Again, we manipulated the first pseudopotential coefficient V_1 around the Coulomb potential of a thin 2DEG in 2LL to obtain a simple parametrization of the relevant interactions. The largest values of the overlap are obtained at slightly positive values of $\delta V_1 \simeq 0.01$. The numerical evaluation of the overlap integral was undertaken by a Monte-Carlo sampling of $\mathcal{O} = \int d(z_1, \dots, z_N) \Psi_{\text{exact}}^* \Psi_{\text{trial}}$ in position space. The composite fermion part $\Psi_{\frac{2}{3}}^{(\text{CF})}$ of the trial state in Eq. (2) was generated as a Slater determinant of individually projected CF orbitals [23] at negative effective flux [24]. The rate-limiting step is the evaluation of the exact wavefunction, which requires calculating a number of Slater determinants equal to the dimension of the Hilbert-subspace $D_{L_z=0}$ projected onto fixed $L_z = 0$. For our largest system, $N_e = 14$, we have $D_{L_z=0} \simeq 1.9 \times 10^7$.

The MR state may be regarded as one representative of an entire family of weakly paired CF states [9, 25]. Similarly, this holds true for the BS states that are derived from the MR state by condensation of quasiparticles. Other representatives in either class of paired states can be obtained explicitly by varying the pair wavefunction [9]. In this variational approach, we introduce a number of parameters g_k to replace the pair-wavefunction as follows:

$$\text{Pf} \left[\frac{1}{z_i - z_j} \right] \longrightarrow \text{Pf} \left[\sum_{\mathbf{k}} g_{\mathbf{k}} \tilde{\phi}_{\mathbf{k}}(z_i) \tilde{\phi}_{-\mathbf{k}}(z_j) \right], \quad (5)$$

where $\tilde{\phi}_{\mathbf{k}}(z_i) = J_i^{-1} \mathcal{P}_{\text{LLL}}[\phi_{\mathbf{k}}(z_i) J_i]$ are the projected CF orbitals, and $J_i = \prod_{k \neq i} (z_i - z_k)$. To obtain a paired state at the shift of the BS state, an effective field with -1 flux quanta is required for these CFs. The pair-wavefunction then acquires a phase of -2π when two CFs are braided, which we denote as $p_x - ip_y$, or *negative p-wave pairing*. Our simulations are undertaken on the sphere, where the expansion in Eq. (5) involves monopole harmonics (for details, see [26], App. A). As for the paired state at $\nu = 5/2$, the number of relevant parameters, g_k , on the sphere is small [9] – only up to 5 for the system sizes considered.

Fig. 3b) shows results for the overlaps of BS states with pair wavefunctions optimized such as to yield maximum overlap with the exact ground-state at $\nu = 12/5$ and shift $S = 2$. Comparing to the results in Fig. 3a), we find that the region of large overlaps with the exact ground-state becomes wider, while the overlap peaks increase significantly and shift to slightly higher δV_1 . The overlaps now reach as high as 0.990(2) for $N_e = 8$ and climb to 0.92(3) for our largest system ($N_e = 14$) at $\delta V_1 = 0.02$.

For $\nu = 5/2$, the weakly paired states are continuously connected [9] to a CF Fermi-liquid state (similar to the one at $\nu = 1/2$) at large δV_1 , where the CF formulation becomes virtually exact. At $\nu = 12/5$, the HH state occurs at a different shift, so it comes as no surprise that the overlap of the BS state decreases for large δV_1 . We find a discontinuous drop to zero of the overlap in some cases ($N_e = 10$ and 14), indicating level crossings in the ground-state.

The large overlaps between the BS ground-state and the exact ground-state at $\nu = 12/5$ with $S = 2$, together with the evidence for a gapped, incompressible non-Abelian FQH state at this filling factor and shift, clearly establish the BS state as a strong candidate for the observed $\nu = 12/5$ FQH state [13], joining the ranks of HH and $\overline{\text{RR}}$ as the primary contenders. Naturally, we would like to pit these states against each other to see which emerges victorious. However, this is not so easily accomplished with numerics. For example, since these states have different shifts on the sphere, one cannot directly compare energetics, e.g. it would not be valid to claim the larger gap exhibited in Fig. 1b) favors $\overline{\text{RR}}$ over BS. In order to compare the energetics in a somewhat meaningful way, we attempt a finite size scaling to the thermodynamic limit, where the shift becomes irrelevant. When comparing states at different shifts, we use the rescaled magnetic length $\ell'_0 = \sqrt{N_e/\nu} \ell_0$ and units of energy e^2/ℓ'_0 , which compensates for finite size effects in spherical systems [27].

In Fig. 4, we plot the rescaled ground-state energies per electron at the Coulomb point for the shifts corresponding to the candidate $\nu = 12/5$ states, and use a least-squares fit to linearly extrapolate to the thermodynamic limit. Even though the $\overline{\text{RR}}$ state has lower energy in finite systems, the ground-state energies per electron in the thermodynamic limit, $E/N_e = -0.3416(5), -0.342(3),$ and $-0.3421(5)$ for $S = 4, 2,$ and -2 , respectively, are very close and within extrapolation errors of each other [32], so there does not appear to be a clear preference between them. Hence, it is likely that

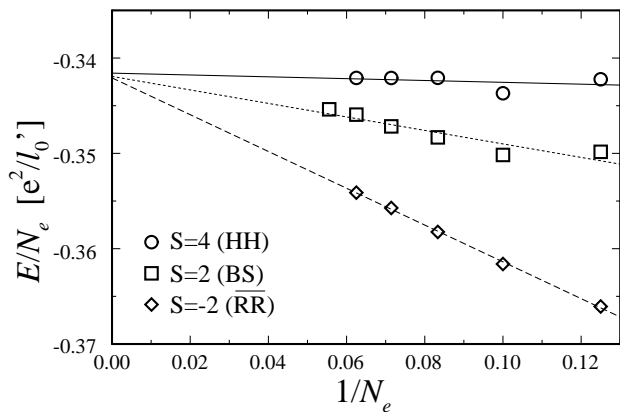


FIG. 4: Finite-size scaling of ground-state energies per electron as a function of $1/N_e$ for $\nu = 12/5$ states with $S = 4, 2,$ and -2 at the Coulomb point.

the physical effects of finite layer thickness and Landau level mixing will play an important role in determining which state is actually favored and experimentally realized, and a more thorough analysis of such factors is certainly warranted.

Another way to more directly compare different states of the same filling fraction is to examine them on the torus, where all states trivially have zero shift. The numerical work in Ref. [15] examined a particle-hole symmetric system at $\nu = 13/5$ for $N_e = 15$ and 18. The results exhibited ground-state degeneracy on the torus that agrees with the \overline{HH} state for most of the parameter space, and best agrees with the \overline{RR} state in a small region near the Coulomb point. However, the gap is not so large in this region, and close inspection of the numerics also reveals low lying states that may be identified as \overline{BS} ground-states [28]. This again indicates it is likely that the inclusion of important physical effects will be significant in determining which state is actually energetically favored. Furthermore, no scaling analysis was carried out in Ref. [15], so, as we have shown on the sphere, it is still unclear which will be favored in the thermodynamic limit.

It will be very interesting to see which state experiments support as the correct description of the $\nu = 12/5$ FQH plateau (or the so far unobserved $\nu = 13/5$ plateau). Indeed, it may even turn out that more than one of these states can be experimentally obtained by realizing different physical regimes. Experiments that measure the electric charge of the fundamental quasihole will not distinguish between \overline{HH} , \overline{BS} , and \overline{RR} , since these all have $e/5$ charged fundamental quasiholes. Experiments that probe scaling behavior or thermal conductance may potentially be able to distinguish between these states [29], but will likely be obfuscated by non-universal effects. Interference experiments, however, should be able to unambiguously distinguish between these possibilities [30].

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- [31] Eqs. (1) and (2) are not strictly equal as wavefunctions since the lowest Landau level projection is applied in different ways, but they represent the same universality class.
- [32] Extrapolation errors are estimated by comparing with fits to only the $N_e \geq 12$ points.