Dedicated to the memory of John T. Lewis

Lowest Lyapunov Exponents for the Armchair Nanotube

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Abstract

We compute sum of the two the lowest Lyapunov exponents $\gamma_{2N-1} + \gamma_{2N}$ of a tight-binding model for an single-wall armchair carbon nanotube with point impurities to lowest (second) order in the disorder parameter λ. The result is that $\gamma_{2N-1} + \gamma_{2N} \sim \lambda^2 N^{-1}$, where N is the number of hexagons around the perimeter. This is similar to the result of Schulz-Baldes [20] for the standard Anderson model on a strip, but because there are only two conducting channels near the Fermi level (centre of the spectral band), this implies that the scattering length is proportional to the diameter of the tube as predicted by Todorov and White [10].

Keywords: Anderson localization, Carbon nanotube, Ballistic transport, Tight-binding model.

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1 Introduction

Carbon nanotubes are single molecules of carbon consisting of a hexagonal graphite-like lattice wound into a cylinder micrometres long and nanometres in diameter. They have great promise for applications in many areas, for example in scanning-tunnelling microscopes, as nanoscale transistors, and as lighting elements [1].

It is well-known that single-walled carbon nanotubes can have different configurations depending on the way they are wound into a cylinder (their helicity). The two extremal cases are called the *armchair* configuration and the *zig-zag* configuration. These two configurations have markedly different electronic properties [2]: whereas the former are metallic conductors, the latter can be metallic or semiconducting depending on their diameter. More specifically, an (n, m) nanotube is metallic if $n - m$ is a multiple if 3. This can be explained in terms of their band structure. The band structure of graphite was first computed by Wallace[3] in a tight-binding approximation. His calculation was modified by several groups to account for the periodic boundary conditions of carbon nanotubes[4, 5, 6, 7].

In the present paper we consider only the armchair configuration. It was argued by Todorov and White[10] that the conductivity of these nanotubes has another interesting feature, which is already suggested by the particular structure of the dispersion relations.They made a rough calculation of the mean free path of electrons using Fermi's Golden Rule to show that it is unusually large for electrons near the Fermi level. They argued that this effect is due to an averaging of the impurity distribution over the circumference of the nanotube. Their interesting prediction was verified experimentally by Liang et al.[11] using a Fabry-Perot electron interferometer.

In a normal metal wire, the conductance (inverse resistance) is proportional to the cross sectional area and inversely proportional to the length of the wire (Ohm's law):

$$G = \sigma \frac{A}{L}. \tag{1.1}$$

The conductivity σ is an intrinsic property of the metal. It is proportional to the mean free path ℓ_m of the electrons in the wire. The scattering of electrons is *diffusive*, i.e. the coherence length is much smaller than the mean free path. In that case Ohm's law holds and the conductivity satisfies the Drude behaviour:

$$\sigma = \frac{ne^2\tau}{m} = \frac{2me^2}{3\pi^2\hbar^3}\epsilon_F \ell_m, \tag{1.2}$$

where τ is the mean free time an the mean free path $\ell_m = v_F\tau$ is independent of the cross sectional area. At low temperatures it is dominated by impurity scattering and depends only on the number of impurities per unit volume.

In long thin mesoscopic conductors the coherence length is long compared to the mean free path. In that case the theory predicts a transition as the length of the conductor increases, from a region of *ballistic transport* to a *localised regime*, where the conductance is exponentially small[13]. This transition is determined by the *localisation length* ξ which is proportional to the number of conducting channels N_C and the scattering length ℓ. In the armchair nanotube, the number of available energy levels for transport near the Fermi level is two, i.e. $N_C = 2$, corresponding to the two branches of the dispersion relation crossing the

Fermi level. Todorov and White argue that the scattering length in that case is proportional to the circumference. This is therefore nearly *ballistic* transport. The conductance in the ballistic regime is given by Landauer's formula[14, 15, 16]:

$$G = \frac{2e^2}{h} \sum_{i,j=1}^{N_C} |t_{ij}|^2, \tag{1.3}$$

where t_{ij} are the transmission coefficients.

In this paper we compute the lowest Lyapunov exponents in a tight-binding model of the nanotube similar to the Anderson model[23] to second order in the strength of the impurities, i.e. the standard deviation of the probability distribution, assuming independent, identically distributed random impurities on all sites. The method used was invented by Figotin and Pastur[19] for the one-line Anderson model, and extended in a nontrivial way by Schulz-Baldes[20] to the quasi one-dimensional case of many linked chains. We show that for the nanotube these exponents are of order $\lambda^2 N^{-1}$, where λ is the strength of the impurities and N is the circumference of the tube, i.e. the number of elementary hexagons in the transverse direction. This result is similar to that of [20] for the standard Anderson model on a strip. Since the localisation length is the inverse of the Lyapunov exponent, we see that the scattering length is also proportional to N as asserted by Todorov and White.

For general background, we note that the Anderson model has been studied extensively, see e.g. the books by Figotin and Pastur[19] and by Carmona and Lacroix[22]. We mention the main features. In his seminal paper[23], Anderson argued that in 3 dimensions, a tight binding model with random impurities should have a so-called *mobility edge*, a critical energy above which all eigenstates are *localised*, and do not contribute to the conductivity. This claim has in fact still not been proven mathematically. However, in 1961, Mott and Twose[24] argued that in one dimension all eigenstates should be localised. This was proven in 1976 by Pastur et al.[25]. It was extended to the case of many linked chains by Lacroix[26, 27]. These proofs rely on the transfer matrix formalism, and assume that the chains are infinite. In 1985 it was proved by Fröhlich et al.[29] and by Delyon et al.[30], based on earlier work by Fröhlich and Spencer[28], that in higher dimensions there is indeed localisation at high energies or large disorder. Various results about the smoothness of the density of states have also been proven. In the one-dimensional case, the invariant measure was investigated by Bovier and Klein[35] after initial approximate calculations by Kappus and Wegner[33] and Derrida and Gardner[34]. The latter showed that there is an anomaly in the invariant measure at $\lambda \to 0$ in the sense that the measure is not continuous at the band centre ($E = 0$) as $\lambda \to 0$, and has non-analytic singularities at other energies. It was finally proved by Campanino and Klein[36] that there is an asymptotic expansion for the invariant measure at $E = 0$ (and the other anomalous energies) in powers of λ. In [38] the invariant measure for the case of two linked chains was considered. In a generic case, it could be computed exactly, in others only a differential equation could be derived. It was found that there are anomalies at $E = 0$ as well as at other band edges. Notice that Schulz-Baldes[20, 21] also find singularities in the lowest Lyapunov exponent at these energies.

The paper is organised as follows. In Section 2 we describe the tight-binding model for the armchair nanotube, compute the dispersion relations and the density of states for the model. In Section 3 we introduce the transfer matrix for the model and compute its spectrum and eigenfunctions in the case of no disorder. This leads to an identification of the

channels and a suitable change of basis. In Section 4, the two lowest Lyapunov exponents are evaluated to lowest order in the disorder parameter λ using a generalisation of the method of Figotin and Pastur[19] elaborated by Schulz-Baldes[20] in the case of the Anderson model on a strip. Some of the more detailed calculations are deferred to appendices in Sections 5 and 6.

2 The Model

The hexagonal lattice is a regular Bravais lattice with translation vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ and a basis of two points as in Figure 1 below. Choose a black point as the origin and let $\mathbf{b} = \frac{1}{3}(\mathbf{a}_1 + \mathbf{a}_2)$. The black points are of the form $n_1\mathbf{a}_1 + n_2\mathbf{a}_2$ and the white points $\mathbf{b} + n_1\mathbf{a}_1 + n_2\mathbf{a}_2$ with $n_1, n_2 \in \mathbb{Z}$.

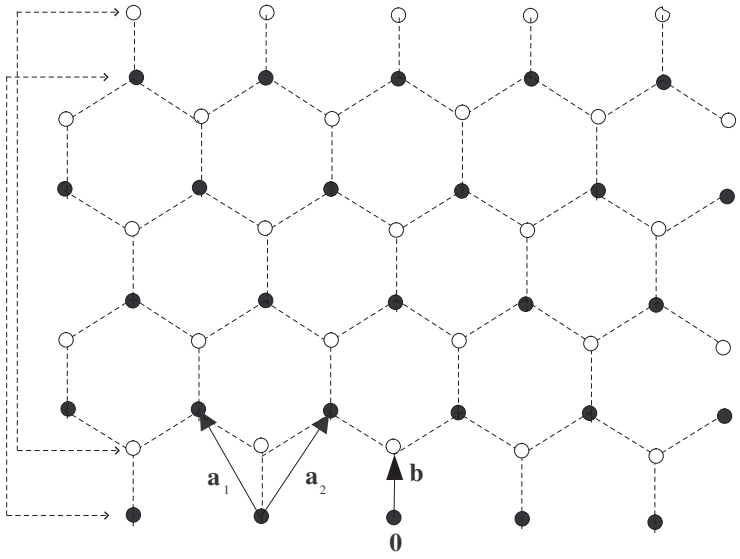

Figure 1: *Armchair nanotube with* $N = 2$

The *armchair* nanotube is obtained by imposing periodic boundary conditions such that points $\mathbf{x}$ are identified with $\mathbf{x} + N\mathbf{a}_1 + N\mathbf{a}_2$. Denote the armchair nanotube by Λ. We define a tight-binding Hamiltonian on $\mathcal{H}_\Lambda = l^2(\Lambda)$ in the usual way:

$$(H_\lambda \psi)(\mathbf{x}) = - \sum_{\mathbf{y} \text{ nearest neighbour of } \mathbf{x}} \psi(\mathbf{y}) + \lambda V(\mathbf{x})\psi(\mathbf{x}). \tag{2.1}$$

where the real numbers $V(\mathbf{x})$ are some realization of a set of bounded, centered, independent random variables with common variance $\mathbb{E}(V^2(\mathbf{x})) = \sigma^2$.

Let Λ_b correspond to the black points of Λ and $\tilde{\mathcal{H}}_\Lambda = l^2(\Lambda_b) \otimes \mathbb{C}^2$. We identify $\mathcal{H}_\Lambda$ with $\tilde{\mathcal{H}}_\Lambda$ through the map $\psi \mapsto \Psi$ where

$$\Psi(\mathbf{n}) := \begin{pmatrix} \psi_1(\mathbf{n}) \\ \psi_2(\mathbf{n}) \end{pmatrix} := \begin{pmatrix} \psi(\mathbf{n}) \\ \psi(\mathbf{n}') \end{pmatrix} \tag{2.2}$$

with the identification $\Psi(n_1+N, n_2+N) = \Psi(n_1, n_2)$. On $\tilde{\mathcal{H}}_\Lambda$, H_λ becomes $\tilde{H}_\lambda$ where

$$(\tilde{H}_\lambda \Psi)(\mathbf{n}) = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1(\mathbf{n}) + \psi_1(n_1+1, n_2) + \psi_1(n_1, n_2+1) \\ \psi_2(\mathbf{n}) + \psi_2(n_1-1, n_2) + \psi_1(n_1, n_2-1) \end{pmatrix} + \lambda \begin{pmatrix} V_1(\mathbf{n}) & 0 \\ 0 & V_2(\mathbf{n}) \end{pmatrix} \begin{pmatrix} \psi_1(\mathbf{n}) \\ \psi_2(\mathbf{n}) \end{pmatrix} \tag{2.3}$$

where $V_1(\mathbf{n}) = V(\mathbf{n})$ and $V_2(\mathbf{n}) = V(\mathbf{n}')$.

It is more convenient to straighten out the nanotube by taking $\Lambda = \mathbb{Z} \times \{0, 1, \ldots, 2N-1\}$ as in Figure 2. Let (n, m) with $n \in \mathbb{Z}$ and $m \in \{0, 1, \ldots, 2N-1\}$ be the coordinates of the lattice points. Then the black points correspond to $n - m$ even and the white points to $n - m$ odd. H_λ is then given by

$$(H_\lambda \psi)(n, m) = -\psi(n+1, m) - \psi(n-1, m) - \psi(n, m + (-1)^{n-m}) + \lambda V(n, m)\psi(n, m) \tag{2.4}$$

where $\psi(n, m+2N) = \psi(n, m)$.

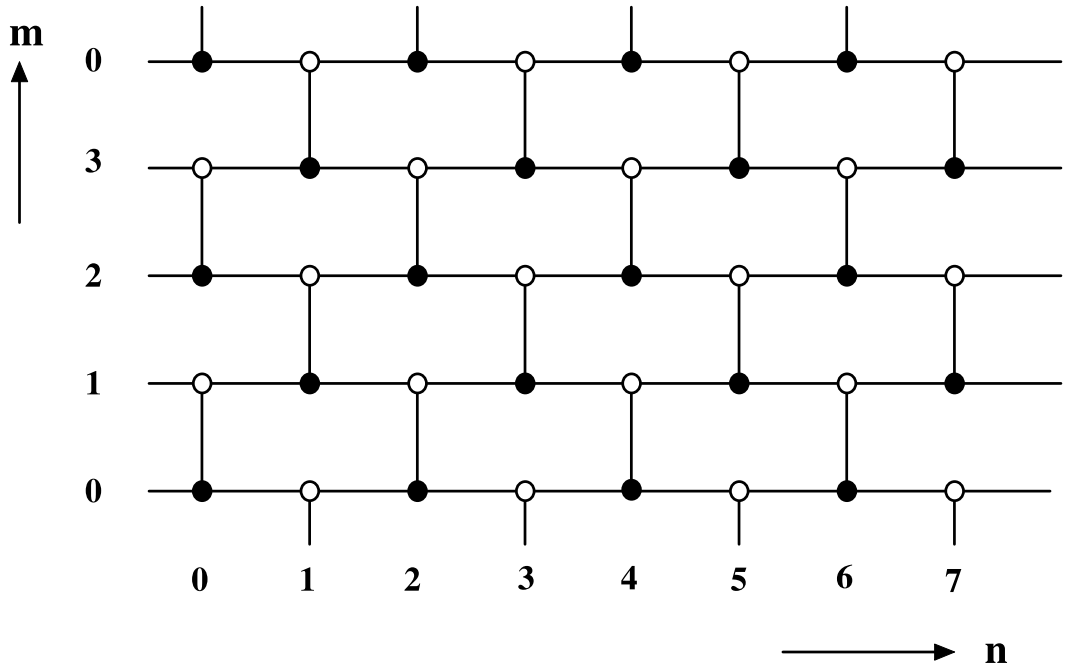

Figure 2 : *Straightened nanotube with $N = 2$: dark lines indicate bonds*

For the case $\lambda = 0$ the spectrum and the density of states are easily computed. Define the Fourier Transform of Ψ, $\hat{\Psi} \in \hat{\mathcal{H}}_N := L^2(0, 2\pi) \otimes \mathbb{C}^{2N} \otimes \mathbb{C}^2$ by

$$\hat{\Psi}(k, q) = \frac{1}{\sqrt{2\pi N}} \sum_{n \in \mathbb{Z}} \sum_{m = 0 // n-m \,\text{even}}^{2N-1} e^{ikn} e^{\pi i \frac{qm}{N}} \begin{pmatrix} \psi(n, m) \\ \psi(n, m+1) \end{pmatrix}. \tag{2.5}$$

Then for $n - m$ even,

$$\begin{pmatrix} \psi(n, m) \\ \psi(n, m+1) \end{pmatrix} = \frac{1}{\sqrt{2\pi N}} \sum_{q=0}^{2N-1} \int_{-\pi}^{\pi} dk\, e^{-ikn} e^{-\pi i \frac{qm}{N}} \hat{\Psi}(k, q). \tag{2.6}$$

On $\hat{\mathcal{H}}$, $\tilde{H}_0$ becomes $\hat{H}_0$ where

$$(\hat{H}_0\hat{\Psi})(k,q) = A(k,q)\hat{\Psi}(k,q), \tag{2.7}$$

where

$$A(k,q) = -\begin{pmatrix} 0 & 1+2e^{\pi i\frac{q}{N}}\cos k \\ 1+2e^{-\pi i\frac{q}{N}}\cos k & 0 \end{pmatrix}. \tag{2.8}$$

The spectrum is therefore described by the bands :

$$\{\pm E(k,q) \mid k \in (-\pi,\pi),\ q = 0,\dots,N\} \tag{2.9}$$

where, introducing the notation $\alpha_q = \frac{q\pi}{N}$,

$$E(k,q) = (1 + 4\cos\alpha_q\cos k + 4\cos^2 k)^{1/2}. \tag{2.10}$$

Note that $E(k,q)$ has a minimum equal to $\sin\alpha_q$ and therefore in the interval $(\sin\frac{\pi}{N}, -\sin\frac{\pi}{N})$ there are only the bands corresponding to $q = 0$ and $q = N$ which become zero at $\pm\frac{2\pi}{3}$ and $\pm\frac{\pi}{3}$ respectively.

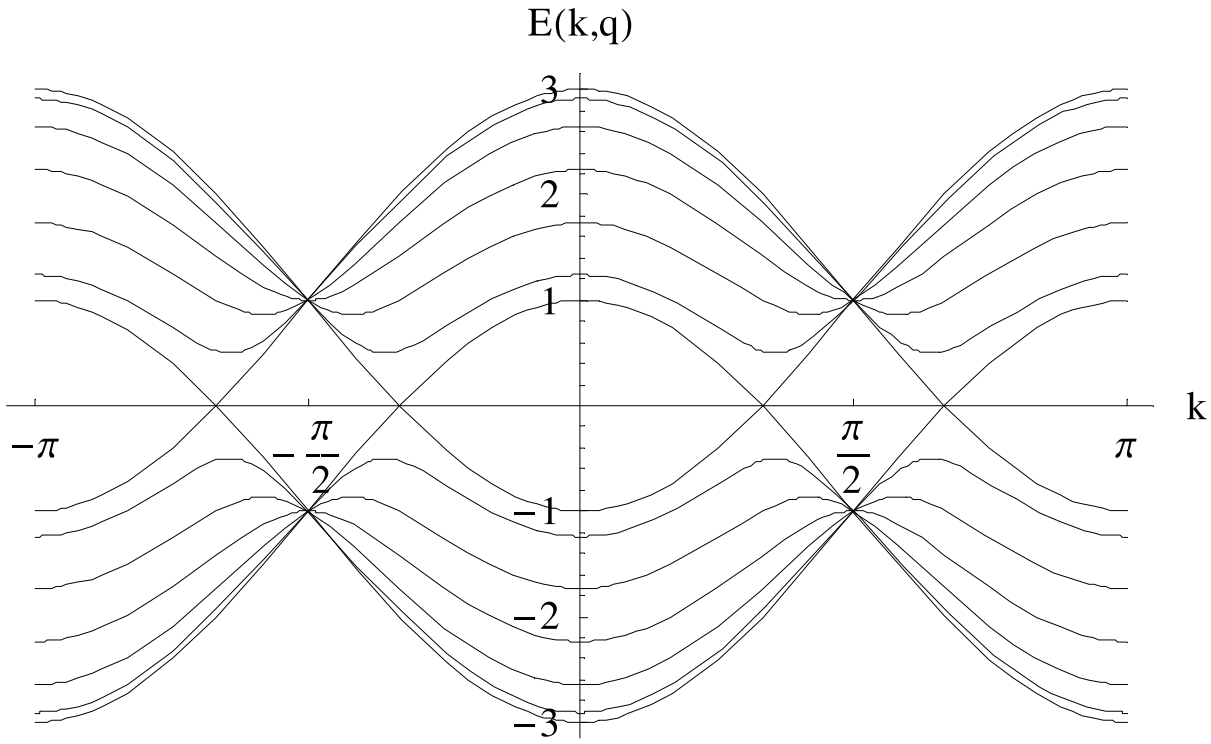

Figure 3 : *dispersion law for* $\lambda = 0$ *with* $N = 6$

The (generalized) eigenstates of the hamiltonian at energy $E = \pm E(k_0,q_0)$ in Fourier space read :

$$\hat{\Psi}_{\pm E(k_0,q_0)}(k,q) = \delta(k-k_0)\delta(q-q_0)\begin{pmatrix} 1 \\ \mp e^{-i\alpha(k_0,q_0)} \end{pmatrix} \tag{2.11}$$

where $\alpha(k_0,q_0) = \arg(1 + 2e^{i\alpha_{q_0}}\cos(k_0))$. In real space, the eigenstates are given by :

$$\begin{pmatrix} \psi_{\pm E(k_0,q_0)}(n,m) \\ \psi_{\pm E(k_0,q_0)}(n,m+1) \end{pmatrix} = e^{-ik_0 n}e^{-im\alpha_{q_0}}\begin{pmatrix} 1 \\ \mp e^{-i\alpha(k_0,q_0)} \end{pmatrix} \tag{2.12}$$

where $n-m$ is even.

It is no surprise to see that this corresponds to two plane waves with the same wave vector and a global phase shift. One of them is supported by the black sublattice and the other by the white sublattice. To compute the density of states one has to be a bit careful. The bands have the symmetry $E(k_0,q_0)=E(-k_0,q_0)=E(k_0,2N-q_0)$. Moreover, if $q\leq N$, one has $E(\pm|k_0|,q_0)=E(\pm(|k_0|-\pi),N-q_0)$. A direct computation using (2.12) then shows that for $q_0<N$:

$$\Psi_{\pm E(\pm|k_0|,q_0)}(n,m)=\Psi_{\pm E(\pm(|k_0|-\pi),N+q_0)}(n,m) \tag{2.13}$$

Hence, only the bands with $q\in\{0,\ldots,N-1\}$ have to be taken into account for the computation of the density of states. The other bands are redundant because they give the same eigenstates. The density of states, $\rho(E)$, is thus given by :

$$\rho(E)=\frac{1}{N}\sum_{q=0}^{N-1}\rho_q(E), \tag{2.14}$$

where the density of states for the q-branch is given by

$$\begin{aligned}\rho_q(E) &= \frac{1}{2\pi}\sum_{k_0\in(-\pi,\pi):\pm E(k_0,q)=E}\left|\frac{\mathrm{d}k}{\mathrm{d}E(k,q)}\right|_{k=k_0} &(2.15)\\ &= \frac{1}{2\pi}\left(\frac{|E|}{2\sqrt{E^2-\sin^2\alpha_q}\sqrt{1-c_+(E)^2}}\mathbf{1}_{(\sin^2\alpha_q,\ 5-4\cos\alpha_q)}(E^2)\right.\\ &\qquad\left.+\frac{|E|}{2\sqrt{E^2-\sin^2\alpha_q}\sqrt{1-c_-(E)^2}}\mathbf{1}_{(\sin^2\alpha_q,\ 5+4\cos\alpha_q)}(E^2)\right) &(2.16)\end{aligned}$$

where

$$c_\pm(E)=-\frac{1}{2}\left[\cos\alpha_q\pm\sqrt{E^2-\sin^2\alpha_q}\right]. \tag{2.17}$$

Note that, as a consequence of the symmetry $E(\pm|k|,q)=E(\pm(|k|-\pi),q)$, one has $\rho_{N-q}(E)=\rho_q(E)$. The density of states can thus also be written :

$$\rho(E)=\frac{1}{N}\sum_{q=0}^{\frac{N}{2}}\nu_q\rho_q(E) \tag{2.18}$$

where $\nu_q=1$ if $q\in\{0,\frac{N}{2}\}$, and $\nu_q=2$ otherwise.

As for the Anderson model we now identify $\mathcal{H}_\Lambda$ with $\mathcal{H}_N=l^2(\mathbb{Z})\otimes\mathbb{C}^{2N}$ writing $\Psi_k(n)=\psi(n,k-1)$, $k=1,2,\ldots,2N$. With this definition, the components of $\Psi(n)$ correspond to the values taken by the original wave function ψ of the straightened nanotube at the points of the n^{th} vertical line written from the bottom up, as in Figure 2. If one defines the three

$2N \times 2N$ matrices:

$$W = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots & \dots & 0 & 0 \\ -1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & \dots & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & -1 \\ 0 & 0 & 0 & 0 & \dots & \dots & -1 & 0 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

and

$$V(n) = \begin{pmatrix} V(n,0) & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & V(n,1) & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & V(n,2) & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & V(n,3) & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \dots & V(n,2N-2) & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & V(n,2N-1) \end{pmatrix},$$

then the Hamiltonian on $\mathcal{H}_N$ becomes $\tilde{H}_\lambda$, where:

$$(\tilde{H}_\lambda \Psi)(n) = W(n)\Psi(n) - \Psi(n-1) - \Psi(n+1) + \lambda V(n)\Psi(n)$$

where:

$$W(n) = \begin{cases} W_{\mathrm{e}} \equiv W & \text{if } p(n) = 1, \\ W_{\mathrm{o}} \equiv SWS^{-1} & \text{if } p(n) = -1. \end{cases} \tag{2.19}$$

3 The transfer matrix and its spectrum

The transfer matrix for this model is the $4N \times 4N$ matrix

$$\begin{pmatrix} W_\lambda(n) - E & -\mathbb{I}_{2N} \\ \mathbb{I}_{2N} & 0 \end{pmatrix}$$

where $W_\lambda(n) = W(n) + \lambda V(n)$. Since $W(n)$ depends on the parity of n it is convenient to introduce the two-step transfer matrix $T_\lambda(n)$:

$$T_\lambda(n) = \begin{pmatrix} W_\lambda(2n) - E & -\mathbb{I}_{2N} \\ \mathbb{I}_{2N} & 0 \end{pmatrix} \begin{pmatrix} W_\lambda(2n-1) - E & -\mathbb{I}_{2N} \\ \mathbb{I}_{2N} & 0 \end{pmatrix}.$$

We can write

$$T_\lambda(n) = T_0 + \lambda A_\lambda(n) \tag{3.1}$$

where, using the notation $W_e^- \equiv W_e - E$, $W_o^- \equiv W_o - E$ we have :

$$T_0(E) = \begin{pmatrix} W_e^- W_o^- - \mathbb{I}_{2N} & -W_e^- \\ W_o^- & -\mathbb{I}_{2N} \end{pmatrix} \tag{3.2}$$

and

$$\begin{aligned} A_\lambda(n) = A(n) + \lambda B(n) \;\; = \;\; & \begin{pmatrix} V(2n)W_{\rm o}^- + W_{\rm e}^- V(2n-1) & -V(2n) \\ V(2n-1) & \mathbb{O}_{2N} \end{pmatrix} \\ & + \lambda \begin{pmatrix} V(2n)V(2n-1) & \mathbb{O}_{2N} \\ \mathbb{O}_{2N} & \mathbb{O}_{2N} \end{pmatrix}. \end{aligned} \tag{3.3}$$

Since $W_\lambda(n)$ is a symmetric matrix for any n and λ, it is easy to check that the transfer matrix is symplectic. That is, if we define the matrix:

$$J \equiv \begin{pmatrix} 0 & -\mathbb{I}_{2N} \\ \mathbb{I}_{2N} & 0 \end{pmatrix},$$

then the transfer matrix satisfies the equation:

$$T_\lambda(n,E) J T_\lambda^{\rm T}(n,E) = J$$

In the remainder of this section we shall study the spectrum of the free transfer matrix T_0.

3.1 Reduction of the problem

Suppose that $\Phi_\kappa \in \mathbb{C}^{4N}$ is an eigenvector of $T_0(E)$ with eigenvalue κ. We write Φ_κ as

$$\Phi_\kappa = \begin{pmatrix} \tilde{\Phi}_\kappa(1) \\ \tilde{\Phi}_\kappa(0) \end{pmatrix}$$

with $\tilde{\Phi}_\kappa(1)$ as well as $\tilde{\Phi}_\kappa(0)$ belonging to $\mathbb{C}^{2N}$, The eigenvalue equation for Φ_κ then reads:

$$\begin{cases} (W_e^- W_o^- - 1)\tilde{\Phi}_\kappa(1) - W_e^- \tilde{\Phi}_\kappa(0) = \kappa \tilde{\Phi}_\kappa(1) \\ W_o^- \tilde{\Phi}_\kappa(1) - \tilde{\Phi}_\kappa(0) = \kappa \tilde{\Phi}_\kappa(0) \end{cases}$$

which gives by inserting the second equation into the first one:

$$\begin{cases} W_e^- W_o^- \tilde{\Phi}_\kappa(1) = \frac{(1+\kappa)^2}{\kappa} \tilde{\Phi}_\kappa(1) \\ W_o^- \tilde{\Phi}_\kappa(1) = (1+\kappa)\tilde{\Phi}_\kappa(0) \end{cases}$$

Multiplying the first equation by W_o^-, and then inserting the second equation now gives:

$$\begin{cases} W_e^- W_o^- \tilde{\Phi}_\kappa(1) = \frac{(1+\kappa)^2}{\kappa} \tilde{\Phi}_\kappa(1) \\ W_o^- W_e^- \tilde{\Phi}_\kappa(0) = \frac{(1+\kappa)^2}{\kappa} \tilde{\Phi}_\kappa(0) \end{cases}$$

That is, $\tilde{\Phi}_\kappa(1)$ is an eigenvector of $W_e^- W_o^-$ with eigenvalue $\mu = \frac{(1+\kappa)^2}{\kappa}$ and $\tilde{\Phi}_\kappa(0)$ is an eigenvector of $W_o^- W_e^- = (W_e^- W_o^-)^{\mathrm{T}}$ with the same eigenvalue μ. Note that the second condition is satisfied by $W_o^- \tilde{\Phi}_\kappa(1)$ and that the two conjugate eigenvalues κ and κ^{-1} of the symplectic matrix $T_0(E)$ give rise to the same eigenvalue μ for $W_e^- W_o^-$.

Conversely, let $\tilde{\Phi}_\mu$ be an eigenvector of $W_e^- W_o^-$ with eigenvalue μ and let $\kappa_\pm(\mu) = \frac{(\mu-2)\pm\sqrt{\mu^2-4\mu}}{2}$, where the square root is taken on the first branch. It is then easy to check that the two vectors $\Phi_{\kappa_\pm(\mu)} \in \mathbb{C}^{4N}$ given by :

$$\Phi_{\kappa_\pm(\mu)} \equiv \begin{pmatrix} \tilde{\Phi}_\mu \\ \frac{1}{1+\kappa_\pm(\mu)} W_o^- \tilde{\Phi}_\mu \end{pmatrix}$$

are eigenvectors of $T_0(E)$ with eigenvalues $\kappa_+(\mu)$ (resp. $\kappa_-(\mu)$). The problem of finding the spectrum of $T_0(E)$ reduces thus to finding the spectrum of $W_e^- W_o^-$.

3.2 The spectrum of $W_e^- W_o^-$

3.2.1 The $E = 0$ case

In order to determine the spectrum of $W_e^- W_o^-$, we will first focus on the case when $E = 0$, and then extrapolate to other values of E. When $E = 0$, we have $W_e^- W_o^- = W_e W_o$ and this matrix takes the simple form:

$$W_e W_o = \begin{pmatrix} 0_2 & P & 0_2 & 0_2 & \dots & \dots & 0_2 & \overline{P} \\ \overline{P} & 0_2 & P & 0_2 & \dots & \dots & 0_2 & 0_2 \\ 0_2 & \overline{P} & 0_2 & P & \dots & \dots & 0_2 & 0_2 \\ 0_2 & 0_2 & \overline{P} & 0_2 & \dots & \dots & 0_2 & 0_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ 0_2 & 0_2 & 0_2 & 0_2 & \dots & \dots & 0_2 & P \\ P & 0_2 & 0_2 & 0_2 & \dots & \dots & \overline{P} & 0_2 \end{pmatrix}$$

where P, $\overline{P}$, and 0_2 are the 2×2 matrices given by:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \; ; \; \overline{P} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \; ; \; 0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Obviously the relations $P^2 = P$ and $\overline{P}^2 = \overline{P} = \mathbb{I}_2 - P$ are satisfied. P and $\overline{P}$ are thus orthogonal projections, and they commute. The eigenvalues μ of $W_e W_o$ are given by the characteristic equation:

$$\det_{2N}(W_e W_o - \mu \mathbb{I}_{2N}) = 0$$

In order to compute the latter determinant, let us introduce the set $\mathcal{M}_2(N)$ of $N \times N$ block matrices, with each block being a 2×2 matrix. This is just the set of $N \times N$ matrices where the numbers have been replaced by 2×2 matrices. There is an obvious bijection between the set of $2N \times 2N$ matrices and $\mathcal{M}_2(N)$ given by the function $F : \; M \to \tilde{M}$, where:

$$\tilde{M}_{k,j} = \begin{pmatrix} M_{2k-1,2j-1} & M_{2k-1,2j} \\ M_{2k,2j-1} & M_{2k,2j} \end{pmatrix} \quad k, j \in \{1, \dots, N\}$$

Hence, we will use these two notions interchangeably in the sequel. We define the multiplication on the left of block matrices $\tilde{M} \in \mathcal{M}_2(N)$ by 2×2 matrices A via the formula:

$$(A\tilde{M})_{i,j} = A\tilde{M}_{i,j} \quad i,j \in \{1,\ldots,N\}$$

With these notations, we have the identity:

$$W_e W_o = PS^2 + \overline{P}S^{-2}$$

For $N \times N$ block matrices $\tilde{M} \in \mathcal{M}_2(N)$ with pairwise commuting blocks:

$$[\tilde{M}_{i,j}, \tilde{M}_{k,l}] = 0 \quad i,j,k,l \in \{1,\ldots,N\}$$

it is a known result that:

$$\det_{2N}(\tilde{M}) = \det_2(\tilde{\det}_N(\tilde{M}))$$

where $\det_{2N}$ (resp. $\det_2$) denotes the usual determinant of a $2N \times 2N$ (resp. 2×2) matrix, and $\tilde{\det}_N(\tilde{M})$ is the 2×2 matrix obtained from $\tilde{M}$ via the usual determinant formula for an $N \times N$ matrix with the numbers replaced by the building blocks of $\tilde{M}$.

The matrix $W_e W_o - \mu \mathbb{I}_{2N}$, viewed as an element of $\mathcal{M}_2(N)$, belongs to this category, and the formula:

$$\det_{2N}(W_e W_o - \mu I_{2N}) = \det_2(\tilde{\det}_N(PS^2 + \overline{P}S^{-2} - \mu \mathbb{I}_{2N}))$$

thus holds. Moreover, because of the identity $P\overline{P} = \overline{P}P = 0$, the cross terms containing both P and $\overline{P}$ in $\tilde{\det}_N$ are moreover vanishing so that we obtain:

$$\tilde{\det}_N(PS^2 + \overline{P}S^{-2} - \mu \mathbb{I}_{2N}) = \tilde{\det}_N(PS^2 - \mu \mathbb{I}_{2N}) + \tilde{\det}_N(\overline{P}S^{-2} - \mu \mathbb{I}_{2N}) - \tilde{\det}_N(-\mu \mathbb{I}_{2N}),$$

where the last term on the right hand side compensates for the fact that the diagonal term has been counted twice in the first part of the sum. An easy computation then shows that:

$$\begin{cases} \tilde{\det}_N(PS^2 - \mu \mathbb{I}_{2N}) = (-\mu)^N \mathbb{I}_2 + (-1)^{N-1} P \\ \tilde{\det}_N(\overline{P}S^{-2} - \mu \mathbb{I}_{2N}) = (-\mu)^N \mathbb{I}_2 + (-1)^{N-1} \overline{P} \\ \tilde{\det}_N(-\mu \mathbb{I}_{2N}) = (-\mu)^N \mathbb{I}_2 \end{cases}$$

So that:

$$\det_{2N}(W_e W_o - \mu I_{2N}) = \det_2((-1)^{N-1}(1-\mu^N)\mathbb{I}_2) = (1-\mu^N)^2$$

Hence, $W_e W_o$ has exactly N eigenvalues given by $\mu_q = \mathrm{e}^{i2\alpha_q}$ with $q \in \{0,1,\ldots,N-1\}$ each of which has multiplicity two. Note that the eigenvalues come in complex conjugate pairs since μ_0 is real and $\mu_{N-q} = \overline{\mu_q}$ for q greater than 1.

We now turn to the problem of determining the corresponding eigenvectors. Let $\Phi_{\mu_q} \in \mathbb{C}^{2N}$ be an arbitrary eigenvector of $W_e W_o$ with eigenvalue μ_q, and write it as:

$$\Phi_{\mu_q} = \begin{pmatrix} \Phi_{\mu_q}(1) \\ \Phi_{\mu_q}(2) \\ \cdots \\ \cdots \\ \Phi_{\mu_q}(N) \end{pmatrix}$$

where the components $\Phi_{\mu_q}(1), \ldots, \Phi_{\mu_q}(N)$ of Φ_{μ_q} are all in $\mathbb{C}^2$. Since the $2N \times 2N$ matrix S^2 performs a cyclic shift of the components of Φ_{μ_q}:

$$S^2\Phi_{\mu_q} = \begin{pmatrix} \Phi_{\mu_q}(2) \\ \Phi_{\mu_q}(3) \\ \cdots \\ \cdots \\ \Phi_{\mu_q}(1) \end{pmatrix}$$

the eigenvalue equation for Φ_{μ_q} reads:

$$\mu_q\Phi_{\mu_q}(r) = P\Phi_{\mu_q}(r+1) + \overline{P}\Phi_{\mu_q}(r-1) \quad ; \; r \in \{1, \ldots, N\}$$

With the identification $\Phi_{\mu_q}(r \pm N) = \Phi_{\mu_q}(r)$. Multiplying the latter equation by P (resp. $\overline{P}$), we get the two linearly independent sets of equations:

$$P\Phi_{\mu_q}(r) = \mu_q^{r-1}P\Phi_{\mu_q}(1) \;\text{ and }\; \overline{P}\Phi_{\mu_q}(r) = \overline{\mu}_q^{r-1}\overline{P}\Phi_{\mu_q}(1) \;\; ; \;\; r \in \{1, \ldots, N\}$$

These equations imply that for each eigenvalue μ_q of W_eW_o, an orthonormal basis for the corresponding eigenspace $\mathcal{H}_{\mu_q} \subset \mathbb{C}^{2N}$ is provided by the two vectors $\Phi^u_{\mu_q}$ and $\Phi^l_{\mu_q}$ given by:

$$\Phi^u_{\mu_q} = \frac{1}{\sqrt{N}} \begin{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \mathrm{e}^{i2\alpha_q} \\ 0 \end{bmatrix} \\ \cdots \\ \cdots \\ \begin{bmatrix} \mathrm{e}^{i2\alpha_q(N-1)} \\ 0 \end{bmatrix} \end{pmatrix} \;\text{ and }\; \Phi^l_{\mu_q} = \frac{1}{\sqrt{N}} \begin{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ \mathrm{e}^{-i2\alpha_q} \end{bmatrix} \\ \cdots \\ \cdots \\ \begin{bmatrix} 0 \\ \mathrm{e}^{-i2\alpha_q(N-1)} \end{bmatrix} \end{pmatrix}$$

For later use, let us here collect some useful relations:

$$S^{-1}\Phi^u_{\mu_q} = \Phi^l_{\overline{\mu_q}} \; ; \; S\Phi^l_{\mu_q} = \Phi^u_{\overline{\mu_q}} \; ; \; S\Phi^u_{\mu_q} = \mu_q\Phi^l_{\overline{\mu_q}} \; ; \; S^{-1}\Phi^l_{\mu_q} = \mu_q\Phi^u_{\overline{\mu_q}} \tag{3.4}$$

$$W_e\Phi^u_{\mu_q} = -\Phi^l_{\overline{\mu_q}} \; ; \; W_e\Phi^l_{\mu_q} = -\Phi^u_{\overline{\mu_q}} \; ; \; W_o\Phi^u_{\mu_q} = -\mu_q\Phi^l_{\overline{\mu}_q} \; ; \; W_o\Phi^l_{\mu_q} = -\mu_q\Phi^u_{\overline{\mu}_q}.$$

The relations remain true if one replaces μ_q with its complex conjugate $\overline{\mu}_q$.

3.2.2 The $E \neq 0$ case

Let $\mathcal{H}^{\oplus}_{\mu_q} \equiv \mathcal{H}_{\mu_q} \oplus \mathcal{H}_{\overline{\mu_q}}$ if μ_q is complex, and $\mathcal{H}^{\oplus}_{\mu_q} = \mathcal{H}_{\mu_q}$ if μ_q is real (that is: for $q = 0$, and $q = \frac{N}{2}$ if N is even). It follows directly from equations (3.4) that W_e and W_o map $\mathcal{H}_{\mu_q}$ onto $\mathcal{H}_{\overline{\mu}_q}$ and conversely. Hence, we can see that the spaces $\mathcal{H}^{\oplus}_{\mu_q}$ are globally left invariant by the action of $W_e^-W_o^-$ so that we can focus on its restrictions $W_e^-W_o^-|_{\mu_q}$ to these subspaces of $\mathbb{C}^{2N}$. If $0 < q < \frac{N}{2}$ this restriction reads:

$$W_e^-W_o^-|_{\mu_q} = \begin{pmatrix} (\mu_q + E^2)\mathbb{I}_2 & E(1+\overline{\mu}_q)T \\ E(1+\mu_q)T & (\overline{\mu}_q + E^2)\mathbb{I}_2 \end{pmatrix}$$

where T is the 2×2 matrix given by:

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Since T and $\mathbb{I}_2$ commute, and $T^2 = \mathbb{I}_2$, we conclude by using the same method as in the previous section that the eigenvalue equation for $W_e^- W_o^-|_{\mu_q}$ reads:

$$\det_4(W_e^- W_o^-|_{\mu_q} - \mu) = \Big((\mu_q + E^2 - \mu)(\overline{\mu}_q + E^2 - \mu) - |E(1+\mu_q)|^2\Big)^2 = 0$$

$$\Leftrightarrow \qquad \Big(\mu^2 - 2\mu(E^2 + \cos(2\alpha_q)) + (E^2-1)^2\Big)^2 = 0$$

Hence $W_e^- W_o^-|_{\mu_q}$ has at most two eigenvalues $\mu_q^{\pm}(E)$ given by:

$$\mu_q^{\pm}(E) = \left(\cos(\alpha_q) \pm \sqrt{E^2 - \sin^2(\alpha_q)}\right)^2 \tag{3.5}$$

The corresponding eigenvectors are given by:

$$\Phi^1_{\mu_q^{\pm}(E)} = \Phi^u_{\mu_q} + \mathrm{r}_{\pm}(E)\Phi^l_{\overline{\mu}_q} \quad \text{and} \quad \Phi^2_{\mu_q^{\pm}(E)} = \Phi^l_{\mu_q} + \mathrm{r}_{\pm}(E)\Phi^u_{\overline{\mu}_q}$$

where:

$$\mathrm{r}_{\pm}(E) = -\frac{\mu_q + E^2 - \mu_q^{\pm}(E)}{E(1+\overline{\mu}_q)}$$

An easy calculation shows that $\mathrm{r}_{\pm}(E)$ can also be written as:

$$\mathrm{r}_{\pm}(E) = \mathrm{e}^{i\alpha_q}\left(\pm \mathrm{sign}(E)\sqrt{1 - \left(\frac{\sin(\alpha_q)}{E}\right)^2} - i\left(\frac{\sin(\alpha_q)}{E}\right)\right)$$

Now let k_E be defined by:

$$\cos(k_E) = \frac{\sin(\alpha_q)}{E}$$

Here, $\mathrm{sign}(E) = 1$ if $E > 0$, and $\mathrm{sign}(E) = -1$ otherwise. Moreover, one has set:

$$k_E \in [0, \pi] \ \text{ if } \ \left|\frac{\sin(\alpha_q)}{E}\right| \leq 1$$

and:

$$k_E = ix_E + \frac{1 - \mathrm{sign}(E)}{2}\pi, \quad x_E \in [0, +\infty] \ \text{ if } \ \left|\frac{\sin(\alpha_q)}{E}\right| \geq 1$$

Then $\mathrm{r}_{\pm}(E)$ takes finally the simple form:

$$\mathrm{r}_{\pm}(E) = -i\mathrm{e}^{i(\alpha_q \pm k_E)}$$

So that the (normalized) eigenvectors take the form:

$$\Phi^1_{\mu_q^{\pm}(E)} = \frac{1}{\sqrt{1 + |\mathrm{e}^{\pm ik_E}|^2}}\left(\Phi^u_{\mu_q} + \mathrm{e}^{i(\alpha_q - \frac{\pi}{2} \pm k_E)}\Phi^l_{\overline{\mu}_q}\right)$$

and

$$\Phi^2_{\mu_q^\pm(E)} = \frac{1}{\sqrt{1+|\mathrm{e}^{\pm ik_E}|^2}}\left(\Phi^l_{\mu_q} + \mathrm{e}^{i(\alpha_q - \frac{\pi}{2} \pm k_E)}\Phi^u_{\mu_q}\right)$$

Note that:

$$\langle\Phi^1_{\mu_q^\pm(E)}|\Phi^2_{\mu_q^\pm(E)}\rangle = \langle\Phi^1_{\mu_q^\pm(E)}|\Phi^2_{\mu_q^\mp(E)}\rangle = 0$$

and:

$$\langle\Phi^1_{\mu_q^\pm(E)}|\Phi^1_{\mu_q^\pm(E)}\rangle = \langle\Phi^2_{\mu_q^\pm(E)}|\Phi^2_{\mu_q^\pm(E)}\rangle = 1$$

But:

$$\begin{cases} \langle\Phi^1_{\mu_q^+(E)}|\Phi^1_{\mu_q^-(E)}\rangle = \frac{1+\mathrm{e}^{-2ik_E}}{2} \neq 0 \ \text{ if } \ \left|\frac{\sin(\alpha_q)}{E}\right| \leq 1 \\ \langle\Phi^1_{\mu_q^+(E)}|\Phi^1_{\mu_q^-(E)}\rangle = \left|\frac{1}{\cosh(ik_E)}\right| \neq 0 \ \text{ if } \ \left|\frac{\sin(\alpha_q)}{E}\right| \geq 1 \end{cases}$$

Finally, let us mention that when $E^2 = \sin^2(\alpha_q)$, one has $\mu_q^+ = \mu_q^-$ as well as $\Phi^1_{\mu_q^+} = \Phi^1_{\mu_q^-}$ and $\Phi^2_{\mu_q^+} = \Phi^2_{\mu_q^-}$. The eigenvalue $\mu_q^+ = \cos^2(\alpha_q)$ is thus only twice degenerate in that case, and the restriction $W_e^- W_o^-|_{\mu_q}$ can not be diagonalized.

If $q = 0$, $W_e^- W_o^-\big|_{\mu_q}$ is the 2 by 2 matrix:

$$W_e^- W_o^-\big|_{\mu_0} = \begin{pmatrix} 1+E^2 & 2E \\ 2E & 1+E^2 \end{pmatrix} \tag{3.6}$$

with eigenvalues:

$$\mu_0^\pm(E) = (1 \pm |E|)^2 \tag{3.7}$$

and corresponding eigenvectors:

$$\Phi_{\mu_0^\pm(E)} = \frac{1}{\sqrt{2}}\left(\Phi^u_{\mu_0} \pm \mathrm{sign}(E)\Phi^l_{\mu_0}\right) \tag{3.8}$$

Finally, the case $q = \frac{N}{2}$ which only occurs for even N gives:

$$W_e^- W_o^-\big|_{\mu_{\frac{N}{2}}} = \begin{pmatrix} E^2-1 & 0 \\ 0 & E^2-1 \end{pmatrix} \tag{3.9}$$

with obvious double eigenvalue $\mu_{\frac{N}{2}}(E) = E^2 - 1$ and eigenvectors $\Phi^u_{\mu_{\frac{N}{2}}}$ and $\Phi^l_{\mu_{\frac{N}{2}}}$. Note that the formula (3.5) giving the value of $\mu_q^\pm(E)$ also holds for $q = 0$ and $q = \frac{N}{2}$.

3.3 The spectrum of the free transfer matrix

With the results of the previous section, we are now able to describe the spectrum of the free transfer matrix. Remember indeed from section (3.1) that to each eigenvalue $\mu_q^\pm(E)$ $0 \leq q \leq \frac{N}{2}$ of $W_e^- W_o^-$ correspond two eigenvalues of the transfer matrix given by :

$$\kappa^\pm_{q^\pm} \equiv \frac{(\mu_q^\pm(E) - 2) \pm \mathrm{sign}(E^2 - \sin^2\alpha_q)\sqrt{\mu_q^\pm(E)^2 - 4\mu_q^\pm(E)}}{2} \tag{3.10}$$

where we used the convention $\mu^+_{\frac{N}{2}} \equiv \mu^-_{\frac{N}{2}} \equiv \mu_{\frac{N}{2}}$, and $\mathrm{sign}(x) = 1$ if $x \geq 0$ and $\mathrm{sign}(x) = -1$ otherwise. Moreover, the lower superscript refers to the superscript of $\mu_q^\pm$ whereas the upper

superscript corresponds to the sign in front of the squareroot. The two eigenvalues satisfy the relations:

$$\kappa^{+}_{q^{\pm}} = \frac{1}{\kappa^{-}_{q^{\pm}}} \tag{3.11}$$

and:

$$\mu^{\pm}_{q}(E) = 2 + \kappa^{\pm}_{q^{\pm}}(E) + \frac{1}{\kappa^{\pm}_{q^{\pm}}(E)} \tag{3.12}$$

It follows from the definition and equation (3.11), that

$$|\kappa^{+}_{q^{\pm}}| \geq 1 \geq |\kappa^{-}_{q^{\pm}}| \tag{3.13}$$

They can thus be represented under the form:

$$\kappa^{\pm}_{q^{\pm}}(E) = \exp\{\pm\left(\eta_{q^{\pm}} + i\beta_{q^{\pm}}\right)\} \tag{3.14}$$

where $\eta_{q^{\pm}} \in \mathbb{R}^{+}$, $\beta_{q^{\pm}} \in (-\pi, \pi]$, and the overall sign in the exponential coincides with the superscript of κ on the left hand side. The special cases ($\eta_{q^{\pm}} = 0$, $\beta_{q^{\pm}} \notin \{0, \pi\}$), ($\eta_{q^{\pm}} \neq 0$, $\beta_{q^{\pm}} \in \{0, \pi\}$), ($\eta_{q^{\pm}} = 0$, $\beta_{q^{\pm}} \in \{0, \pi\}$) will be called elliptic, hyperbolic and parabolic respectively. The other eigenvalues will be called mixed. Notice that two conjugate eigenvalues $\kappa^{+}_{q^{\pm}}$ and $\kappa^{-}_{q^{\pm}}$ always belong to the same class.
It is clear from (3.12), that the occurrence of mixed eigenvalues is due to the fact that the operator $W_e^{-}W_o^{-}$ is not self adjoint and can have complex eigenvalues. They don't occur in the Anderson model on the strip where the spectrum of the transfer matrix is determined by the spectrum of the self adjoint operator $\Delta - E$ where Δ denotes the transverse Laplacian.

One reads off from (3.12) that $\kappa^{\pm}_{q^{\pm}}(E)$ is mixed iff $\mu^{\pm}_{q}(E)$ has a nonvanishing imaginary part. Moreover, if $\mu^{\pm}_{q}(E)$ is real, then:

$\kappa^{\pm}_{q^{\pm}}(E)$ is elliptic iff:

$$0 < \mu^{\pm}_{q}(E) < 4 \tag{3.15}$$

$\kappa^{\pm}_{q^{\pm}}(E)$ is hyperbolic iff:

$$\mu^{\pm}_{q}(E) > 4 \quad \text{or} \quad \mu^{\pm}_{q}(E) < 0 \tag{3.16}$$

$\kappa^{\pm}_{q^{\pm}}(E)$ is parabolic iff:

$$\mu^{\pm}_{q}(E) \in \{0, 4\} \tag{3.17}$$

A *channel* of the transfer matrix T_0 is the vectorspace spanned by all the eigenvectors of T_0 whose eigenvalues κ have same $|\eta|$ and $|\beta|$.

For completeness, let us describe the set of eigenvectors of the free transfer matrix with the help of sections (3.1) and (3.2.2) : If $0 < q \leq \frac{N}{2}$ each eigenvalue $\kappa^{\pm}_{q^{\pm}}$ is twice degenerate and has corresponding eigenvectors :

$$\Phi^{i}_{\kappa^{\pm}_{q^{\pm}}(E)} \equiv \begin{pmatrix} \Phi^{i}_{\mu^{\pm}_{q}(E)} \\ \frac{1}{1+\kappa^{\pm}_{q^{\pm}}(E)} W_o^{-} \Phi^{i}_{\mu^{\pm}_{q}(E)} \end{pmatrix}, \; i \in \{1, 2\} \tag{3.18}$$

if $0 < q < \frac{N}{2}$ and :

$$\Phi^1_{\kappa^{\pm}_{\mu_{\frac{N}{2}}(E)}} \equiv \begin{pmatrix} \Phi^u_{\mu_{\frac{N}{2}}} \\ \frac{1}{1+\kappa^{\pm}_{\mu_{\frac{N}{2}}(E)}} W_o^- \Phi^u_{\mu_{\frac{N}{2}}} \end{pmatrix} \; ; \; \Phi^2_{\kappa^{\pm}_{\mu_{\frac{N}{2}}(E)}} \equiv \begin{pmatrix} \Phi^l_{\mu_{\frac{N}{2}}} \\ \frac{1}{1+\kappa^{\pm}_{\mu_{\frac{N}{2}}(E)}} W_o^- \Phi^l_{\mu_{\frac{N}{2}}} \end{pmatrix} \tag{3.19}$$

if $q = \frac{N}{2}$. The four eigenvalues $\kappa^{\pm}_{0^{\pm}}$ are only once degenerate and have eigenvectors :

$$\Phi_{\kappa^{\pm}_{0^{\pm}}(E)} \equiv \begin{pmatrix} \Phi_{\mu^{\pm}_0(E)} \\ \frac{1}{1+\kappa^{\pm}_{0^{\pm}}(E)} W_o^- \Phi_{\mu^{\pm}_0(E)} \end{pmatrix} \tag{3.20}$$

These eigenvectors are not normalized. Namely :

$$\|\Phi_{\kappa^{\pm}_{0^{\pm}}(E)}\|^2 = 2 \tag{3.21}$$

For $0 < q < \frac{N}{2}$:

$$\|\Phi^i_{\kappa^{\pm}_{q^{\pm}}(E)}\|^2 = 1 + \mathrm{e}^{\mp\eta_{q^{\pm}}}, \; i \in \{1, 2\} \tag{3.22}$$

where the sign in the exponent on the l.h.s. is minus the upper superscript of $\kappa^{\pm}_{q^{\pm}}$. Finally, when $q = \frac{N}{2}$:

$$\|\Phi^i_{\kappa^{\pm}_{\mu_{\frac{N}{2}}(E)}}\|^2 = 1 + \mathrm{e}^{\mp\eta_{\frac{N}{2}}} \left(\frac{1+E^2}{1-E^2}\right), \; i \in \{1, 2\} \tag{3.23}$$

3.4 Ordering of the channels

In this section we want to order the eigenvalues $\{\kappa^+_{q^{\pm}}\}_{q\in\{0,\ldots,\frac{N}{2}\}}$ according to their modulus (or equivalently the corresponding $\eta_{q^{\pm}}$) from the highest to the lowest. This ordering will depend on the value E of the energy. We are only interested in cases where the free transfer matrix can be fully diagonalized, so we exclude the energies for which $E^2 = \sin^2(\alpha_q)$, $0 < q < \frac{N}{2}$. Indeed, remember from section (3.2.2) that these energies correspond to anomalies of $W_e^- W_o^-$. We also have to exclude values of the energy at which parabolic eigenvalues occur (see section (3.1)), i.e. we suppose that $E^2 \neq 5 \pm 4\cos(\alpha_q)$ for all $q \in \{0, \ldots, \frac{N}{2}\}$ (see below). Using the definitions (3.14) and (3.5), we can rewrite equation (3.12) in the form :

$$\left(\cos(\alpha_q) \pm \sqrt{E^2 - \sin^2(\alpha_q)}\right)^2 = 2(1 + \cosh(\eta_{q^{\pm}})\cos(\beta_{q^{\pm}})) + 2i\sinh(\eta_{q^{\pm}})\sin(\beta_{q^{\pm}}) \tag{3.24}$$

We can then distinguish two cases : In the first case, $E^2 \geq \sin^2(\alpha_q)$. Equating the real and imaginary parts on either side of equation (3.24), we obtain in this case :

$$\begin{cases} 2(1 + \cosh(\eta_{q^{\pm}})\cos(\beta_{q^{\pm}})) = \left(\cos(\alpha_q) \pm \sqrt{E^2 - \sin^2(\alpha_q)}\right)^2 \\ \\ 2\sinh(\eta_{q^{\pm}})\sin(\beta_{q^{\pm}}) = 0 \end{cases} \tag{3.25}$$

The second equation can be satisfied only if either $\eta_{q^{\pm}} = 0$ or $\beta^{\pm}_q \in \{0, \pi\}$. In the first case, the upper equation reads :

$$4\cos^2(\frac{\beta_{q^{\pm}}}{2}) = \left(\cos(\alpha_q) \pm \sqrt{E^2 - \sin^2(\alpha_q)}\right)^2 \tag{3.26}$$

It is easy to show that this equation is consistent only when $E^2 \leq 5-4\cos(\alpha_q)$ if one chooses the plus sign and $E^2 \leq 5+4\cos(\alpha_q)$ if one chooses the minus sign. The value of $\beta_{q^\pm}$ is moreover uniquely determined by this equation because whenever $\eta_{q^\pm} = 0$, one has $\frac{\beta_{q^\pm}}{2} \in [0, \frac{\pi}{2}]$ by equations (3.11) and (3.13).

If we suppose that $\beta_{q^\pm} = 0$, we obtain :

$$4\cosh^2(\frac{\eta_{q^\pm}}{2}) = \left(\cos(\alpha_q) \pm \sqrt{E^2 - \sin^2(\alpha_q)}\right)^2 \tag{3.27}$$

This equation in turn is consistent only when $E^2 \geq 5-4\cos(\alpha_q)$ if one chooses the plus sign and $E^2 \geq 5+4\cos(\alpha_q)$ if one chooses the minus sign. In the case of equality, both equations apply and we have a parabolic eigenvalue.
Finally, since the right-hand side of the upper equation in (3.25) is always positive, the option $\beta_{q^+} = \pi$ can only be fulfilled if $q = \frac{N}{2}$ and $E^2 = 1 = \sin^2(\alpha_{\frac{N}{2}})$, in which case $\eta_{\frac{N}{2}^+} = 0$. For the same reason, the option $\beta_{q^-} = \pi$ can only be fulfilled if $E^2 = 1$, in which case $\eta_{q^-} = 0$ for all $q \in \{0, \ldots, \frac{N}{2}\}$. At the energies $E = \pm 1$ one has thus appearance of parabolic eigenvalues.

We now turn to the case where $E^2 < \sin^2(\alpha_q)$. Equating the real and imaginary parts on either side of equation (3.24), yields this time :

$$\begin{cases} 2(1 + \cosh(\eta_{q^\pm})\cos(\beta_{q^\pm})) = \cos(2\alpha_q) + E^2 \\ \\ 2\sinh(\eta_{q^\pm})\sin(\beta_{q^\pm}) = \pm 2\cos(\alpha_q)\sqrt{\sin^2(\alpha_q) - E^2} \end{cases} \tag{3.28}$$

Squaring and adding the latter equations up yields :

$$(\cosh(\eta_{q^\pm}) + \cos(\beta_{q^\pm}))^2 = \frac{(1-E^2)^2}{4} \tag{3.29}$$

Since $\cosh(\eta_{q^\pm}) + \cos(\beta_{q^\pm})$ as well as $1 - E^2$ are positive, it follows :

$$\cos(\beta_{q^\pm}) = \frac{1-E^2}{2} - \cosh(\eta_{q^\pm}) \tag{3.30}$$

Developing the product on the left hand side of (3.29), making use of the upper equation in (3.28) and then inserting (3.30), yields :

$$2\cosh^2(\eta_{q^\pm}) - (1-E^2)\cosh(\eta_{q^\pm}) + (\cos(2\alpha_q) + E^2 - 2) = 0 \tag{3.31}$$

Solving this quadratic equation we obtain finally :

$$\cosh(\eta_{q^\pm}) = \frac{1-E^2}{4} + \frac{\sqrt{(E^2-5)^2 - 16\cos^2(\alpha_q)}}{4}. \tag{3.32}$$

It follows from (3.30) that

$$\cos(\beta_{q^\pm}) = \frac{1-E^2}{4} - \frac{\sqrt{(E^2-5)^2 - 16\cos^2(\alpha_q)}}{4} \tag{3.33}$$

The latter equation determines $\beta_{q^\pm}$ only up to a sign, but one deduces from the lower equation in (3.28) that β_{q^+} corresponds to the positive solution and β_{q^-} to the negative one. Since $\eta_{q^\pm} > 0$, η_{q^+} and η_{q^-} are equal and uniquely determined by (3.33). We thus have an eight dimensional mixed channel (except for $q = \frac{N}{2}$, where the channel is only of dimension four). Finally, let us mention that in the borderline case $E^2 = \sin^2(\alpha_q)$ the corresponding channel is elliptic, except for $q = \frac{N}{2}$ which is parabolic in this case.

Let us now turn to the description of the ordering of the various $\eta_{q^\pm}$ depending on the value of the energy. Based on equations (3.26), (3.27) and (3.32) and their domain of validity, one sees that four cases have to be distinguished :

1) $|E| < 1$

In this case, let $q_c(E)$ be the highest $q \in \{0, \dots \frac{N}{2}\}$ such that $E^2 \geq \sin^2(\alpha_q)$. Then for $0 \leq q \leq q_c$, $\eta_{q^\pm} = 0$ and the corresponding channels are elliptic, whereas for $q_c < q \leq \frac{N}{2}$, $\eta_{q^\pm}$ is given by (3.32) and an increasing function of q. The corresponding channels are mixed, except the one corresponding to $\eta_{\frac{N}{2}^\pm}$ which is hyperbolic. In increasing order, the collection of $\eta_{q^\pm}$ may thus be written : $\{\eta_{0^-}, \eta_{0^+}, \dots, \eta_{\frac{N}{2}^-}, \eta_{\frac{N}{2}^+}\}$.

2) $1 < |E| < \sqrt{5}$

In this case, one has $\eta_{q^-} = 0$ for all $q \in \{0, \dots \frac{N}{2}\}$ and the corresponding channels are elliptic. Let moreover q_c be the highest $q \in \{0, \dots \frac{N}{2}\}$ such that $E^2 > 5 - 4\cos(\alpha_q)$. Then, for $0 \leq q \leq q_c$, η_{q^+} is given by (3.27) and a decreasing function of q, whereas for $q_c < q \leq \frac{N}{2}$, $\eta_{q^+} = 0$ and the corresponding channels are elliptic. In increasing order, the collection of $\eta_{q^\pm}$ may thus be written : $\{\eta_{0^-} \dots, \eta_{\frac{N}{2}^-}, \eta_{\frac{N}{2}^+}, \dots, \eta_{0^+}\}$.

3) $\sqrt{5} < |E| < 3$

In this case, η_{q^+} is given by (3.27) for all $q \in \{0, \dots \frac{N}{2}\}$ and is a decreasing function of q. The corresponding channels are hyperbolic. Let moreover q_c be the highest $q \in \{0, \dots \frac{N}{2}\}$ such that $E^2 < 5 + 4\cos(\alpha_q)$. Then, for $0 \leq q \leq q_c$, $\eta_{q^-} = 0$ and the corresponding channels are elliptic, whereas for $q_c < q \leq \frac{N}{2}$, η_{q^-} is given by (3.27) and an increasing function of q. The corresponding channels are hyperbolic. Note moreover that $\eta_{0^+} > \eta_{\frac{N}{2}^-}$ so that for any q and $\tilde{q} \in \{0, \dots, \frac{N}{2}\}$, $\eta_{q^+} > \eta_{\tilde{q}^-}$ holds. In increasing order, the collection of $\eta_{q^\pm}$ may thus again be written : $\{\eta_{0^-} \dots, \eta_{\frac{N}{2}^-}, \eta_{\frac{N}{2}^+}, \dots, \eta_{0^+}\}$.

4) $3 < |E|$

In this case, η_{q^+} is given by (3.27) for all $q \in \{0, \dots \frac{N}{2}\}$ and is a decreasing function of q. The corresponding channels are hyperbolic. Also η_{q^-} is given by (3.27) and an increasing function of q. The corresponding channels are also hyperbolic. Once again, the collection of $\eta_{q^\pm}$ may thus be written in increasing order as : $\{\eta_{0^-} \dots, \eta_{\frac{N}{2}^-}, \eta_{\frac{N}{2}^+}, \dots, \eta_{0^+}\}$.

3.5 Diagonal and real symplectic form of the (free) transfer matrix in the band centre

3.5.1 Diagonalization

As it will be needed below, we are now going to diagonalize the free transfer matrix T_0 when the energy E lies in the band center, i.e. when $E^2 < \sin^2(\frac{\pi}{N})$. For simplicity, we will moreover suppose that N is even and that $E > 0$. According to the previous section, the structure of the free transfer matrix is as follows with these assumptions : there are $N_m \equiv \frac{N}{2} - 1$ mixed channels of dimension eight with exponent $\eta_q \equiv \eta_{q^+} = \eta_{q^-}$ given by equation (3.32) and phase factor $\beta_q \equiv \beta_{q^+} = -\beta_{q^-},\ q \in \{1, \ldots, \frac{N}{2} - 1\}$ given by (3.33), one hyperbolic channel of dimension four with exponent $\eta_{\frac{N}{2}} \equiv \eta_{\frac{N}{2}^+} = \eta_{\frac{N}{2}^-}$ given by (3.32) and phase factor $\beta_{\frac{N}{2}} \equiv \beta_{\frac{N}{2}^+} = \pi$, and two elliptic channels (i.e. with exponent $\eta_0 \equiv 0$) of dimension two with distinct phase factor β_{0^+} and β_{0^-} respectively, given by (3.26). With these conventions, η_q is an increasing function of q, and all the β_q's are positive numbers. For $q \in \{0, \ldots \frac{N}{2}\}$, let us introduce the set of 2×2 matrices $\kappa_q^+ = \mathrm{diag}(\kappa_{q^+}^+, \kappa_{q^-}^+)$. More explicitly, for $q \neq 0$:

$$\kappa_q^+ = \begin{pmatrix} \mathrm{e}^{\eta_q + i\beta_q} & 0 \\ 0 & \mathrm{e}^{\eta_q - i\beta_q} \end{pmatrix} \tag{3.34}$$

and for $q = 0$:

$$\kappa_0^+ = \begin{pmatrix} \mathrm{e}^{i\beta_{0^+}} & 0 \\ 0 & \mathrm{e}^{i\beta_{0^-}} \end{pmatrix} \tag{3.35}$$

We then define the $2N \times 2N$ matrix κ_+ given by :

$$\kappa_+ = \begin{pmatrix} \kappa^+_{\frac{N}{2}} & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \kappa^+_{\frac{N}{2}-1} & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \kappa^+_{\frac{N}{2}-1} & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \kappa_1^+ & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & \kappa_1^+ & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & \kappa_0^+ \end{pmatrix} \tag{3.36}$$

and let $\kappa_- = \frac{1}{\kappa_+}$ be its inverse.

The matrix $F \equiv (\Phi^u_{\mu_{\frac{N}{2}}}, \Phi^l_{\mu_{\frac{N}{2}}}, \Phi^1_{\mu^+_{\frac{N}{2}-1}(E)}, \Phi^1_{\mu^-_{\frac{N}{2}-1}(E)}, \Phi^2_{\mu^+_{\frac{N}{2}-1}(E)}, \Phi^2_{\mu^-_{\frac{N}{2}-1}(E)}, \ldots, \Phi_{\mu_0^+(E)}, \Phi_{\mu_0^-(E)})$ (conf. Section 3.2.2) is an adequate $2N \times 2N$ base change matrix to diagonalize $W_e^- W_o^-$:

$$F^{-1} W_e^- W_o^- F = (\mathbb{I}_{2N} + \kappa_+)(\mathbb{I}_{2N} + \kappa_-) \equiv \mu \tag{3.37}$$

Let moreover :

$$\begin{aligned} N_\pm \;=\; \mathrm{diag}\Bigg(& \frac{1}{\|\Phi^1_{\kappa^\pm_{\frac{N}{2}}}\|}, \frac{1}{\|\Phi^2_{\kappa^\pm_{\frac{N}{2}}}\|}, \frac{1}{\|\Phi^1_{\kappa^\pm_{(\frac{N}{2}-1)^+}}\|}, \frac{1}{\|\Phi^1_{\kappa^\pm_{(\frac{N}{2}-1)^-}}\|}, \\ & \frac{1}{\|\Phi^2_{\kappa^\pm_{(\frac{N}{2}-1)^+}}\|}, \frac{1}{\|\Phi^2_{\kappa^\pm_{(\frac{N}{2}-1)^-}}\|}, \ldots, \frac{1}{\|\Phi_{\kappa^\pm_{0^+}}\|}, \frac{1}{\|\Phi_{\kappa^\pm_{0^-}}\|} \Bigg) \end{aligned} \tag{3.38}$$

Then, the $4N \times 4N$ base change matrix :

$$R = \begin{pmatrix} FN_+ & FN_- \\ W_o^- F \frac{1}{1+\kappa_+} N_+ & W_o^- F \frac{1}{1+\kappa_-} N_- \end{pmatrix} \tag{3.39}$$

with inverse :

$$R^{-1} = \begin{pmatrix} \frac{1}{N_+}\frac{1}{1-\kappa_-} F^{-1} & \frac{1}{N_+}\frac{1+\kappa_+}{1-\kappa_+} F^{-1}(W_0^-)^{-1} \\ \frac{1}{N_-}\frac{1}{1-\kappa_+} F^{-1} & \frac{1}{N_-}\frac{1+\kappa_-}{1-\kappa_-} F^{-1}(W_o^-)^{-1} \end{pmatrix} \tag{3.40}$$

takes T_0 to its diagonal form $\hat{T}_0$:

$$\hat{T}_0 = R^{-1} T_0 R = \begin{pmatrix} \kappa_+ & 0 \\ 0 & \kappa_- \end{pmatrix} \tag{3.41}$$

Let $\hat{M} = R^{-1} M R$ denote the expression of some matrix M in the basis associated with R. Then :

$$\hat{T}_\lambda(n) = \hat{T}_0 + \lambda \hat{A}_\lambda(n) = \hat{T}_0 + \lambda \hat{A}(n) + \lambda^2 \hat{B}(n) \tag{3.42}$$

The part of the perturbation that is linear in λ reads:

$$A(n) = \begin{pmatrix} V_e(n) W_o^- & -V_e(n) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} W_e^- V_o(n) & 0 \\ V_o(n) & 0 \end{pmatrix} \equiv A_e(n) + A_o(n) \tag{3.43}$$

where $V_e(n) \equiv V(2n)$ and $V_o(n) \equiv V(2n-1)$. After transformation:

$$\hat{A}(n) = R^{-1} A(n) R \equiv \hat{A}_e(n) + \hat{A}_o(n) \tag{3.44}$$

where:

$$\hat{A}_e(n) = \begin{pmatrix} \frac{1}{1-\kappa_-}\frac{1}{N_+} F^{-1} V_e(n) W_o^- F \frac{1}{1+\kappa_-} N_+ & \frac{1}{1-\kappa_-}\frac{1}{N_+} F^{-1} V_e(n) W_o^- F \frac{1}{1+\kappa_+} N_- \\ \frac{1}{1-\kappa_+}\frac{1}{N_-} F^{-1} V_e(n) W_o^- F \frac{1}{1+\kappa_-} N_+ & \frac{1}{1-\kappa_+}\frac{1}{N_-} F^{-1} V_e(n) W_o^- F \frac{1}{1+\kappa_+} N_- \end{pmatrix} \tag{3.45}$$

$$\hat{A}_o(n) = \begin{pmatrix} \frac{1}{1-\kappa_-^2}\frac{1}{N_+} F^{-1} W_e^- V_o(n) F N_+ & \frac{1}{1-\kappa_-^2}\frac{1}{N_+} F^{-1} W_e^- V_o(n) F N_- \\ \frac{1}{1-\kappa_+^2}\frac{1}{N_-} F^{-1} W_e^- V_o(n) F N_+ & \frac{1}{1-\kappa_+^2}\frac{1}{N_-} F^{-1} W_e^- V_o(n) F N_- \end{pmatrix} \tag{3.46}$$

3.5.2 Real symplectic form

For some practical purposes it is convenient to have a basis where T_0 is diagonal, but we will also need to write T_0 in a basis where its matrix elements stay real, and where it assumes a nice (quasi-block diagonal) symplectic form. Let us denote by : $\mathcal{V} = (v_1^+, \ldots, v_{2N}^+, v_1^-, \ldots, v_{2N}^-)$ the basis where T_0 assumes its diagonal form $\hat{T}_0$. Here v_l^σ denotes the vector that has a non-vanishing entry equal to one only in the $(l + \frac{1-\sigma}{N})$-th component. We want to find a basis where T_0 can be written in terms of real, channel preserving, rotations. As is well known, the eigenvectors of any two by two rotation matrix are given by $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$. This motivates the choice of the new basis $\mathcal{E} = (e_1^+, \ldots, e_{2N}^+, e_1^-, \ldots, e_{2N}^-)$ defined by the relations :

$$e_1^\pm = v_1^\pm, \quad e_2^\pm = v_2^\pm \tag{3.47}$$

$$e^{\sigma}_{2l+1} = \frac{1}{\sqrt{2}}(v^{\sigma}_{2l+1} + v^{\sigma}_{2l+2}), \quad e^{\sigma}_{2l+2} = \frac{\sigma i}{\sqrt{2}}(v^{\sigma}_{2l+1} - v^{\sigma}_{2l+2}) \tag{3.48}$$

for $l = 1, \dots N-2$, and :

$$e^{+}_{l} = \frac{1}{\sqrt{2}}(v^{+}_{l} + v^{-}_{l}), \quad e^{-}_{l} = \frac{i}{\sqrt{2}}(v^{+}_{l} - v^{-}_{l}) \tag{3.49}$$

for $l \in \{2N-1, 2N\}$.

Conversely :

$$v^{\sigma}_{2l+1} = \frac{1}{\sqrt{2}}(e^{\sigma}_{2l+1} - i\sigma e^{\sigma}_{2l+2}), \quad v^{\sigma}_{2l+2} = \frac{1}{\sqrt{2}}(e^{\sigma}_{2l+1} + i\sigma e^{\sigma}_{2l+2}) \tag{3.50}$$

for $l = 1, \dots N-2$, and :

$$v^{\sigma}_{l} = \frac{1}{\sqrt{2}}(e^{+}_{l} - i\sigma e^{-}_{l}) \tag{3.51}$$

for $l \in \{2N-1, 2N\}$.

The corresponding base change matrix is defined as follows. Let :

$$C_+ \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \qquad C_- \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{3.52}$$

and let C be the $4N \times 4N$ matrix defined by :

$$C \equiv \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \tag{3.53}$$

where $C_1, \dots, C_4$ are the $2N \times 2N$ matrices given by :

$$C_1 \equiv \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & C_+ & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & C_+ & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & C_+ & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \end{pmatrix} \quad C_2 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \end{pmatrix} \tag{3.54}$$

$$C_3 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{-\mathbf{i}}{\sqrt{\mathbf{2}}} \end{pmatrix} \quad C_4 \equiv \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & C_- & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & C_- & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & C_- & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{\mathbf{i}}{\sqrt{\mathbf{2}}} \end{pmatrix} \tag{3.55}$$

Then C (or more precisely $C^\dagger$) is a unitary matrix that takes T_0 from its diagonal form to the desired real symplectic form $\tilde{T}_0$. Namely :

$$CC^\dagger = C^\dagger C = \mathbb{I}_{4N} \;\; ; \;\; \tilde{T}_0 \equiv C\hat{T}_0 C^\dagger = \begin{pmatrix} \tilde{T}_0^1 & \tilde{T}_0^2 \\ \tilde{T}_0^3 & \tilde{T}_0^4 \end{pmatrix} \tag{3.56}$$

Here :

$$\tilde{T}_0^1 \equiv \begin{pmatrix} \mathrm{e}^{\eta_{\frac{N}{2}}} R_{\beta_{\frac{N}{2}}} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathrm{e}^{\eta_{\frac{N}{2}-1}} R_{\beta_{\frac{N}{2}-1}} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\eta_{\frac{N}{2}-1}} R_{\beta_{\frac{N}{2}-1}} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \mathrm{e}^{\eta_1} R_{\beta_1} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \cos\beta \end{pmatrix} \tag{3.57}$$

$$\tilde{T}_0^4 \equiv \begin{pmatrix} \mathrm{e}^{-\eta_{\frac{N}{2}}} R_{\beta_{\frac{N}{2}}} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathrm{e}^{-\eta_{\frac{N}{2}-1}} R_{\beta_{\frac{N}{2}-1}} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-\eta_{\frac{N}{2}-1}} R_{\beta_{\frac{N}{2}-1}} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \mathrm{e}^{-\eta_1} R_{\beta_1} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \cos\beta \end{pmatrix} \tag{3.58}$$

$$\tilde{T}_0^2 = -\tilde{T}_0^3 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -\sin\beta \end{pmatrix} \tag{3.59}$$

where :

$$\beta \equiv \begin{pmatrix} \beta_0^+ & 0 \\ 0 & \beta_0^- \end{pmatrix} \tag{3.60}$$

and :

$$R_\alpha \equiv \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{3.61}$$

is the 2×2 rotation matrix by some angle α.

4 Lyapunov exponents

Let

$$U_\lambda(L, E) \equiv T_\lambda(L) T_\lambda(L-1) \dots T_\lambda(2) T_\lambda(1). \tag{4.1}$$

be the (two)L-step transfer matrix. We introduce the notation $\mathcal{H}^{\otimes p}$ to denote the p-fold tensor product of copies of the same Hilbert space $\mathcal{H}$, and denote by $\mathcal{F}_p(\mathcal{H})$ the anti-symmetrization of this space (p-fermion space). Similarly, given an operator M on $\mathcal{H}$, we

denote by $M^{\otimes p}$ its p$^{\text{th}}$ tensor-power acting on $\mathcal{H}^{\otimes p}$, and by $\Lambda^p M$ its restriction to $\mathcal{F}_p(\mathcal{H})$.

The first $2N$ non-negative Lyapunov exponents $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_{2N}$ are the family of numbers defined by :

$$\sum_{i=1}^{p} \gamma_i(E) = \lim_{L\to\infty} \frac{1}{2L} \mathbb{E}\left(\ln \|\Lambda^p U_\lambda(L,E)\|\right), \ p \in \{1,\ldots,2N\} \tag{4.2}$$

where the operator norm is defined as usual, and the expectation is taken over all random variables. It is easy to see that if R is an arbitrary base change matrix in $\mathcal{H}_N$, one can replace $U_\lambda(L,E)$ with its expression $R^{-1}U_\lambda(L,E)R$ in the new basis without changing the result. Moreover, it is proved in [22], that one can also write :

$$\sum_{i=1}^{p} \gamma_i(E) = \lim_{L\to\infty} \frac{1}{2L} \mathbb{E}\left(\ln \|\Lambda^p U_\lambda(L,E) u_1 \wedge u_2 \wedge \ldots \wedge u_p\|\right), \ p \in \{1,\ldots,2N\} \tag{4.3}$$

where $u_1 \wedge u_2 \wedge \ldots \wedge u_p$ is an arbitrary nonzero element of $\mathcal{F}_p(\mathcal{H}_N)$. We choose to write the transfer matrix in the basis $\mathcal{E}$ where the free transfer matrix is given by $\tilde{T}_0$. The perturbed transfer matrix $\tilde{T}_\lambda(n)$ is a real and symplectic matrix in this basis. Concerning the "initial condition", we choose it, following [20] as a symplectic frame, i.e an orthonormal family of $2N$ vectors $\{u_1,\ldots,u_{2N}\}$ satisfying the relations :

$$\langle u_i, J u_j\rangle = 0 \quad i,j = 1,\ldots 2N \tag{4.4}$$

We also recursively define a (random) evolution of this symplectic frame by the set of $2N$ equations :

$$u_1(n) \wedge \ldots \wedge u_p(n) \equiv \frac{\Lambda^p \tilde{T}_\lambda(n)(u_1(n-1) \wedge \ldots \wedge u_p(n))}{\|\Lambda^p \tilde{T}_\lambda(n-1)(u_1(n-1) \wedge \ldots \wedge u_p(n-1))\|}, \qquad n \geq 1 \tag{4.5}$$

It is easy to show that the entire family of these equations for p ranging from 1 to $2N$ defines a unique symplectic frame $\{u_1(n),\ldots,u_{2N}(n)\}$, provided that $\{u_1(n-1),\ldots,u_{2N}(n-1)\}$ is itself a symplectic frame.

Let us here introduce some definitions that will be needed later on concerning the channels. As previously mentioned, there are $\frac{N}{2}+2$ channels that we number from 0 to $\frac{N}{2}+1$. We assign these numbers in such a way that increasing channel numbers correspond to decreasing exponents η. More precisely, for $k \in \{0,\ldots,\frac{N}{2}+1\}$ we define $\hat{\eta}_k = \eta_{\frac{N}{2}-k}$ and $\hat{\beta}_k = \beta_{\frac{N}{2}-k}$, where one has set $\eta_{-1} \equiv \eta_0$, $\beta_0 \equiv \beta_0^+$ and $\beta_{-1} \equiv \beta_0^-$. Moreover if $i \in \{0,\ldots,2N\}$ is some frame vector index, we denote by $\hat{\imath}$ the corresponding channel index, i.e. we let $\hat{\imath}$ be the entire part of $\frac{i+1}{4}$ if $i \leq 2N-1$, and we let $\hat{\imath} = \frac{N}{2}+1$ if $i = 2N$.

Now

$$\|\Lambda^p \tilde{U}_\lambda(L,E) u_1 \wedge \ldots \wedge u_p\|^2 = \|\Lambda^p \tilde{T}_\lambda(L) \Lambda^p U_\lambda(L-1,E) u_1 \wedge \ldots \wedge u_p\|^2$$

$$= \frac{\|\Lambda^p \tilde{T}_\lambda(L) \Lambda^p \tilde{U}_\lambda(L-1,E) u_1 \wedge \ldots \wedge u_p\|^2}{\|\Lambda^p \tilde{U}_\lambda(L-1,E) u_1 \wedge \ldots \wedge u_p\|^2} \|\Lambda^p \tilde{U}_\lambda(L-1,E) u_1 \wedge \ldots \wedge u_p\|^2$$

$$= \|\Lambda^p \tilde{T}_\lambda(L) u_1(L-1) \wedge \ldots \wedge u_p(L-1)\|^2 \|\Lambda^p \tilde{U}_\lambda(L-1, E) u_1 \wedge \ldots \wedge u_p\|^2 \tag{4.6}$$

iterating this procedure, we obtain :

$$\|\Lambda^p \tilde{U}_\lambda(L, E) u_1 \wedge \ldots \wedge u_p\|^2 = \prod_{n=0}^{L-1} \|\Lambda^p \tilde{T}_\lambda(n+1) u_1(n) \wedge \ldots \wedge u_p(n)\|^2 \tag{4.7}$$

and it follows that

$$\sum_{i=1}^{p} \gamma_i(E) = \lim_{L\to\infty} \frac{1}{4L} \mathbb{E} \sum_{n=0}^{L-1} \left(\ln \|\Lambda^p \tilde{T}_\lambda(n+1) u_1(n) \wedge \ldots \wedge u_p(n)\|^2 \right) \tag{4.8}$$

Introducing weighted frame vectors $\hat{u}_i = \mathrm{e}^{-\hat{\eta}_\hbar} u_i$, this can also be written :

$$\sum_{i=1}^{p} \left(\gamma_i(E) - \frac{\hat{\eta}_\hbar}{2} \right) = \lim_{L\to\infty} \frac{1}{4L} \mathbb{E} \sum_{n=0}^{L-1} \left(\ln \|\Lambda^p \tilde{T}_\lambda(n+1) \hat{u}_1(n) \wedge \ldots \wedge \hat{u}_p(n)\|^2 \right) \tag{4.9}$$

$$= \lim_{L\to\infty} \frac{1}{4L} \mathbb{E} \sum_{n=0}^{L-1} \left(\ln \det_p \left(\langle \hat{u}_i(n), \tilde{T}_\lambda^\dagger(n+1) \tilde{T}_\lambda(n+1) \hat{u}_j(n) \rangle \right)_{1\le i,j\le p} \right) \tag{4.10}$$

Now, $\ln \det_p = \mathrm{Tr}_p \ln$, so that :

$$\sum_{i=1}^{p} \left(\gamma_i(E) - \frac{\hat{\eta}_\hbar}{2} \right) = \lim_{L\to\infty} \frac{1}{4L} \mathbb{E} \sum_{n=0}^{L-1} \left(\mathrm{Tr}_p \left(\ln \left(\langle \hat{u}_i(n), \tilde{T}_\lambda^\dagger(n+1) \tilde{T}_\lambda(n+1) \hat{u}_j(n) \rangle \right)_{1\le i,j\le p} \right) \right) \tag{4.11}$$

Let us define the three $p \times p$ matrices $\mathcal{T}_i^p(n), i \in \{0, 1, 2\}$:

$$(\mathcal{T}_0^p(n))_{ij} = \langle \hat{u}_i(n), \tilde{T}_0^\dagger(n+1) \tilde{T}_0(n+1) \hat{u}_j(n) \rangle \tag{4.12}$$

$$(\mathcal{T}_1^p(n))_{ij} = \langle \hat{u}_i(n), \tilde{A}_\lambda^\dagger(n+1) \tilde{T}_0(n+1) + \tilde{T}_0^\dagger(n+1) \tilde{A}_\lambda(n+1) \hat{u}_j(n) \rangle \tag{4.13}$$

$$(\mathcal{T}_2^p(n))_{ij} = \langle \hat{u}_i(n), \tilde{A}_\lambda^\dagger(n+1) \tilde{A}_\lambda(n+1) \hat{u}_j(n) \rangle \tag{4.14}$$

where each time $1 \le i, j \le p$. Then :

$$\sum_{i=1}^{p} \left(\gamma_i(E) - \frac{\hat{\eta}_\hbar}{2} \right) = \lim_{L\to\infty} \frac{1}{4L} \mathbb{E} \sum_{n=0}^{L-1} \left(\mathrm{Tr}_p \left(\ln \left(\mathcal{T}_0^p(n) + \lambda \mathcal{T}_1^p(n) + \lambda^2 \mathcal{T}_2^p(n) \right) \right) \right) \tag{4.15}$$

Now, as in [20] :

$$\mathcal{T}_0^p(n) = \mathbb{I}_p + \tilde{\mathcal{T}}_0^p(n) \tag{4.16}$$

where $\tilde{\mathcal{T}}_0^p(n) = \mathcal{O}(\lambda)$. Let :

$$\mathcal{T}_\lambda^p(n) = \tilde{\mathcal{T}}_0^p(n) + \lambda \mathcal{T}_1^p(n) + \lambda^2 \mathcal{T}_2^p(n) \tag{4.17}$$

Then we get, expanding the logarithm :

$$\ln(\mathbb{I}_p + \mathcal{T}_\lambda^p(n)) = \mathcal{T}_\lambda^p(n) - \frac{1}{2} \mathcal{T}_\lambda^p(n) \mathcal{T}_\lambda^p(n) + \mathcal{O}(\lambda^3) \tag{4.18}$$

Taking the expectation value and using that $\mathbb{E}(\tilde{\mathcal{T}}_0^p(n) \mathcal{T}_1^p(n)) = \mathbb{E}(\mathcal{T}_1^p(n)) = 0$ we then get, neglecting the terms of order λ^3, the expression :

$$\mathbb{E}(\ln(\mathbb{I}_p + \mathcal{T}_\lambda^p(n))) = \mathbb{E}(\tilde{\mathcal{T}}_0^p(n)) + \lambda^2 \mathbb{E}(\tilde{\mathcal{T}}_2^p(n)) - \frac{1}{2} \{ \mathbb{E}(\tilde{\mathcal{T}}_0^p(n) \tilde{\mathcal{T}}_0^p(n)) + \lambda^2 \mathbb{E}(\tilde{\mathcal{T}}_1^p(n) \tilde{\mathcal{T}}_1^p(n)) \} + \mathcal{O}(\lambda^3) \tag{4.19}$$

where $\tilde{\mathcal{T}}_i^p(n), i \in \{1,2\}$ are obtained from (4.13) and (4.14) by replacing $\tilde{A}_\lambda(n)$ with $\tilde{A}(n)$, the remaining part giving rise to terms of order λ^3 or higher. Finally, we get the following expression for the sum of the p first Lyapunov exponents :

$$\sum_{i=1}^{p}\left(\gamma_i(E)-\frac{\hat{\eta}_\hbar}{2}\right)=\lim_{L\to\infty}\frac{1}{4L}\sum_{n=0}^{L-1}\Big(\mathrm{Tr}_p\Big(\mathbb{E}(\tilde{\mathcal{T}}_0^p(n))+\lambda^2\mathbb{E}(\tilde{\mathcal{T}}_2^p(n))-$$

$$\frac{1}{2}\{\mathbb{E}(\tilde{\mathcal{T}}_0^p(n)\tilde{\mathcal{T}}_0^p(n))+\lambda^2\mathbb{E}(\tilde{\mathcal{T}}_1^p(n)\tilde{\mathcal{T}}_1^p(n))\}\Big)\Big)+\mathcal{O}(\lambda^3) \tag{4.20}$$

The sum of the two lowest exponents can now be obtained by subtraction. Let Π be the $2N\times 2N$ matrix corresponding to the projection onto the last two indices :

$$(\Pi)_{ij}=(\delta_{i,2N-1}+\delta_{i,2N})\delta_{ij}\ , \tag{4.21}$$

where δ_{ij} is the Kronecker delta. Then, taking into account that $\eta_0=0$, we get :

$$\sum_{i=2N-1}^{2N}\gamma_i(E)=\lim_{L\to\infty}\frac{1}{4L}\sum_{n=0}^{L-1}\Big(\mathrm{Tr}_{2N}\Big(\mathbb{E}(\Pi\tilde{\mathcal{T}}_0^{2N}(n)\Pi)+\lambda^2\mathbb{E}(\Pi\tilde{\mathcal{T}}_2^{2N}(n)\Pi)-$$

$$\frac{1}{2}\Big\{2\mathbb{E}(\Pi\tilde{\mathcal{T}}_0^{2N}(n)\tilde{\mathcal{T}}_0^{2N}(n)\Pi)-\mathbb{E}(\Pi\tilde{\mathcal{T}}_0^{2N}(n)\Pi\tilde{\mathcal{T}}_0^{2N}(n)\Pi)+$$

$$\lambda^2\Big(2\mathbb{E}(\Pi\tilde{\mathcal{T}}_1^{2N}(n)\tilde{\mathcal{T}}_1^{2N}(n)\Pi)-\mathbb{E}(\Pi\tilde{\mathcal{T}}_1^{2N}(n)\Pi\tilde{\mathcal{T}}_1^{2N}(n)\Pi)\Big)\Big\}\Big)\Big)+\mathcal{O}(\lambda^3) \tag{4.22}$$

We are now ready to state the main result of this paper :

Theorem 4.1 *Let us suppose that the energy is in the band center :* $0<E<\sin\frac{\pi}{N}$ *and such that the two following conditions are satisfied for signs* $\sigma_1,\dots,\sigma_4\in\{\pm1\}$*, and elliptic channel indexes* $m_1,m_2\in\{\frac{N}{2},\frac{N}{2}+1\}$ *:*

$\mathrm{e}^{i(\sigma_1\hat{\beta}_{m_1}-\sigma_2\hat{\beta}_{m_2})}=1$ *holds if and only if* $\sigma_1=\sigma_2$ *and* $m_1=m_2$*.*

$\mathrm{e}^{i((\sigma_1+\sigma_2)\hat{\beta}_{m_1}-(\sigma_3+\sigma_4)\hat{\beta}_{m_2})}=1$ *holds if and only if* $\sigma_1+\sigma_2=\sigma_3+\sigma_4$ *and* $m_1=m_2$*, or if* $\sigma_1+\sigma_2=\sigma_3+\sigma_4=0$*.*

Then, the sum of the two lowest Lyapunov exponents reads :

$$\sum_{i=2N-1}^{2N}\gamma_i=\frac{\lambda^2}{4}\Bigg\{\sum_{i=2N-1}^{2N}\sum_{k=\frac{N}{2}}^{\frac{N}{2}+1}(d-d_1-d_2(k))\left\langle\rho_{i,k}\right\rangle-$$

$$\frac{\sigma^2}{2N}\sum_{i,j=2N-1}^{2N}\Bigg(\sum_\sigma\sum_k\frac{1}{\sin^2\frac{\hat{\beta}_k}{2}}\left\langle\rho_{i,k}^\sigma\rho_{j,k}^{-\sigma}\right\rangle+$$

$$\sum_{\{\sigma\}}\sum_{\{k\}}(1-\delta_{k_1k_2})\left(\frac{1}{2\sin\frac{\sigma_1\hat{\beta}_{k_1}}{2}}+\frac{1}{2\sin\frac{\sigma_2\hat{\beta}_{k_2}}{2}}\right)^2\Big\{\left\langle\rho_{j,k_1}^{\sigma_1}\rho_{i,k_2}^{\sigma_2}\right\rangle+$$

$$\left\langle \sqrt{\rho^{\sigma_2}_{i,k_2}\rho^{\sigma_1}_{j,k_1}\rho^{-\sigma_2}_{j,k_2}\rho^{-\sigma_1}_{i,k_1}}\mathrm{e}^{i\{(\theta^{\sigma_1}_{j,k_1}+\theta^{-\sigma_1}_{i,k_1})-(\theta^{-\sigma_2}_{j,k_2}+\theta^{\sigma_2}_{i,k_2})\}}\right\rangle\Big\}\Big)\Big\} + \mathcal{O}(\lambda^3) \tag{4.23}$$

where d, d_1 and $d_2(k)$ are the constants given by (4.52),(4.85) and (4.89) respectively, and the $\theta^{\sigma}_{j,k}$ are defined in (5.146).

Remark. We believe that the $\mathcal{O}(\lambda^3)$ term is bounded in N but did not check this in detail. This would imply that the above asymptotics hold for λ small compared to N^{-1}.

Proof :

To keep the main line of the proof clear, some calculations have been deferred to appendices. Before we start, let us introduce some useful definitions and properties.

By definition of the basis $\mathcal{V}$, one has :

If $j \in \{1, \ldots, 2N-2\}$, (i.e. j is not elliptic) :

$$T_0 v^{\sigma}_j = \mathrm{e}^{\sigma(\eta_{\hat{\jmath}} - ip(j)\hat{\beta}_{\hat{\jmath}})} v^{\sigma}_j \tag{4.24}$$

If $j \in \{2N-1, 2N\}$, (i.e. if j is elliptic) :

$$T_0 v^{\sigma}_j = \mathrm{e}^{i\sigma\hat{\beta}_{\hat{\jmath}}} v^{\sigma}_j \tag{4.25}$$

If $u_i(n)$ is a symplectic frame vector, we denote by :

$$\psi_i(n) = C^{\dagger} u_i(n) \tag{4.26}$$

its expression in the basis $\mathcal{V}$. Let $k \in \{0, \ldots, \frac{N}{2}+1\}$ be some channel index and $\sigma \in \{\pm 1\}$ some sign. We introduce the projections :

$$\pi^{\sigma}_k = \sum_{j:\hat{\jmath}=k} |v^{\sigma}_j\rangle\langle v^{\sigma}_j| \tag{4.27}$$

$$\pi_k = \pi^{+}_k + \pi^{-}_k \tag{4.28}$$

Since $\mathcal{V}$ is an orthonormal basis of $\mathbb{C}^{4N}$, we have :

$$\sum_k \pi_k = \mathbb{I}_{4N} \tag{4.29}$$

We also introduce the weight of the ith frame vector in the kth channel :

$$\rho^{\sigma}_{i,k}(n) \equiv \langle u_i(n), \pi^{\sigma}_k u_i(n)\rangle \ , \quad \rho_{i,k}(n) = \rho^{+}_{i,k}(n) + \rho^{-}_{i,k}(n) \tag{4.30}$$

Since the frame vectors are normalized, one has :

$$\sum_k \rho_{i,k}(n) = 1 \tag{4.31}$$

Moreover, for an elliptic channel one has :

$$\pi^{+}_k = (\pi^{-}_k)^* \tag{4.32}$$

It follows that, for such a k :

$$\rho_{i,k}(n) = 2\rho_{i,k}^{\pm} \tag{4.33}$$

We also introduce a notation for the average of some random quantity $f(n)$:

$$\langle f \rangle = \lim_{L\to+\infty} \frac{1}{L} \sum_{n=0}^{L-1} \mathbb{E}(f(n)) \tag{4.34}$$

whenever the limit exists.

The following facts have already been proved in [20] for large enough n :

If j is an elliptic frame vector index and k a hyperbolic channel index,

$$\rho_{j,k}^{\sigma}(n) = \mathcal{O}(\lambda^2) \tag{4.35}$$

If k is an elliptic channel index,

$$\pi_k^{\sigma}\psi_j(n) = \mathrm{e}^{i\sigma\hat{\beta}_k}\pi_k^{\sigma}\psi_j(n-1) + \mathcal{O}(\lambda) \tag{4.36}$$

If k is not elliptic, then

$$\sum_{j:\hat{j}=k} |u_j(n)\rangle\langle u_j(n)| = \pi_k^{+} + \mathcal{O}(\lambda) \tag{4.37}$$

and

$$\rho_{j,k}^{\sigma}(n) = \mathcal{O}(\lambda^2) \tag{4.38}$$

unless $\sigma = +$ and $\hat{\jmath} = k$.

If i is an elliptic frame vector index, then :

$$\left\langle \rho_{i,k}^{+} \right\rangle = \left\langle \rho_{i,k}^{-} \right\rangle + \mathcal{O}(\lambda^3) \tag{4.39}$$

We now compute the individual terms of (4.22):

First term

$$S_1 \equiv \lim_{L\to\infty} \frac{1}{4L} \sum_{n=0}^{L-1} \left(\mathrm{Tr}_{2N} \left(\mathbb{E}(\Pi \tilde{\mathcal{I}}_0^{2N}(n)\Pi) \right) \right) =$$

$$\lim_{L\to\infty} \frac{1}{4L} \sum_{i=2N-1}^{2N} \sum_{n=0}^{L-1} \left(\mathbb{E}(\langle \hat{u}_i(n), \left(\tilde{T}_0^{\dagger}(n+1)\tilde{T}_0(n+1) - 1 \right) \hat{u}_i(n) \rangle) \right). \tag{4.40}$$

But it follows from (4.24) and (4.25) that

$$T_0^{\dagger}T_0 = \sum_{\sigma=\pm1} \sum_{k=0}^{\frac{N}{2}+1} \mathrm{e}^{2\sigma\hat{\eta}_k}\pi_k^{\sigma} \tag{4.41}$$

and hence

$$S_1 = \lim_{L\to\infty} \frac{1}{4L} \sum_{n=0}^{L-1} \sum_{i=2N-1}^{2N} \sum_{\sigma=\pm1} \sum_{k=0}^{\frac{N}{2}+1} (\mathrm{e}^{2\sigma\hat{\eta}_k} - 1)\mathbb{E}(\langle u_i(n), \pi_k^{\sigma} u_i(n) \rangle)$$

$$= \frac{1}{4} \sum_{i=2N-1}^{2N} \sum_{\sigma=\pm 1} \sum_{k=0}^{\frac{N}{2}+1} (\mathrm{e}^{2\sigma\hat{\eta}_k} - 1) \left\langle \rho_{i,k}^{\sigma} \right\rangle_N \tag{4.42}$$

Using (4.39) we now obtain

$$S_1 = \frac{1}{4} \sum_{i=2N-1}^{2N} \sum_{k=0}^{\frac{N}{2}+1} (\cosh(2\hat{\eta}_k) - 1) \left\langle \rho_{i,k} \right\rangle_N + \mathcal{O}(\lambda^3) \tag{4.43}$$

$$= \frac{1}{2} \sum_{i=2N-1}^{2N} \sum_{k=0}^{\frac{N}{2}+1} \sinh^2(\hat{\eta}_k) \left\langle \rho_{i,k} \right\rangle_N + \mathcal{O}(\lambda^3) \tag{4.44}$$

Second term:

It is more convenient to write this term in the basis $\mathcal{V}$.

$$S_2 \equiv \lim_{L\to\infty} \frac{\lambda^2}{4L} \sum_{n=0}^{L-1} \left(\mathrm{Tr}_{2N} \left(\mathbb{E}(\Pi \tilde{\mathcal{I}}_0^{2N}(n) \Pi) \right) \right) =$$

$$\lim_{L\to\infty} \frac{\lambda^2}{4L} \sum_{i=2N-1}^{2N} \sum_{n=0}^{L-1} \left(\mathbb{E}(\langle \hat{\psi}_i(n), \hat{A}^\dagger(n+1)\hat{A}(n+1)\hat{\psi}_i(n) \rangle) \right). \tag{4.45}$$

Inserting the equality (4.29) twice, we get :

$$S_2 = \lim_{L\to\infty} \frac{\lambda^2}{4L} \sum_{i=2N-1}^{2N} \sum_{n=0}^{L-1} \sum_{\sigma_1,\sigma_2=\pm} \sum_{m_1,m_2=0}^{\frac{N}{2}+1} \left(\mathbb{E}(\langle \psi_i(n), \pi_{m_1}^{\sigma_1} \hat{A}^\dagger(n+1)\hat{A}(n+1)\pi_{m_2}^{\sigma_2} \psi_i(n) \rangle) \right). \tag{4.46}$$

But, because of (4.35), if $m \notin \{\frac{N}{2}, \frac{N}{2}+1\}$ and $k \in \{2N-1, 2N\}$:

$$\|\pi_m^\sigma \psi_k(n)\|^2 = \rho_{k,m}^\sigma = \mathcal{O}(\lambda^2) \tag{4.47}$$

and hence

$$= \lim_{L\to\infty} \frac{\lambda^2}{4L} \sum_{i=2N-1}^{2N} \sum_{n=0}^{L-1} \sum_{\sigma_1,\sigma_2=\pm} \sum_{m_1,m_2=\frac{N}{2}}^{\frac{N}{2}+1} \left(\mathbb{E}(\langle \psi_i(n), \pi_{m_1}^{\sigma_1} \hat{A}^\dagger(n+1)\hat{A}(n+1)\pi_{m_2}^{\sigma_2} \psi_i(n) \rangle) \right) + \mathcal{O}(\lambda^3) \tag{4.48}$$

Now we are going to use an oscillatory sum argument as in [20] : Use the equality (4.36) to obtain up to order λ^3 :

$$\begin{aligned} S_2 \;=\; & \lim_{L\to\infty} \frac{\lambda^2}{4L} \sum_{i=2N-1}^{2N} \sum_{n=0}^{L-1} \sum_{\sigma_1,\sigma_2=\pm} \sum_{m_1,m_2=\frac{N}{2}}^{\frac{N}{2}+1} \Big(\mathbb{E}\big(\mathrm{e}^{i(\sigma_2\hat{\beta}_{m_2} - \sigma_1\hat{\beta}_{m_1})} \\ & \times \langle \psi_i(n), \pi_{m_1}^{\sigma_1} \hat{A}^\dagger(n+1)\hat{A}(n+1)\pi_{m_2}^{\sigma_2} \psi_i(n) \rangle \big) \Big) \end{aligned} \tag{4.49}$$

Comparing with the previous equation, one sees that this is only possible if $\mathrm{e}^{i(\sigma_2\hat{\beta}_{m_2}-\sigma_1\hat{\beta}_{m_1})}=1$. By the hypothesis of the theorem this, in turn, is only possible if $\sigma_1=\sigma_2$ and $m_1=m_2$. One obtains :

$$S_2=\lim_{L\to\infty}\frac{\lambda^2}{4L}\sum_{i=2N-1}^{2N}\sum_{n=0}^{L-1}\sum_{\sigma=\pm}\sum_{m=\frac{N}{2}}^{\frac{N}{2}+1}\Big(\mathbb{E}(\langle\psi_i(n),\pi_m^\sigma\hat{A}^\dagger(n+1)\hat{A}(n+1)\pi_m^\sigma\psi_i(n)\rangle)\Big)+\mathcal{O}(\lambda^3). \tag{4.50}$$

We show in the Appendix (Section 5.2) that

$$\mathbb{E}(\pi_m^\sigma\hat{A}^\dagger(n+1)\hat{A}(n+1)\pi_m^\sigma)=d\pi_m^\sigma \tag{4.51}$$

where

$$d=\frac{\sigma^2}{4N}\Big\{\ \mathrm{Tr}_w(D_e+\tilde{D}_o)\Big\} \tag{4.52}$$

and

$$D_e=\frac{1}{|1-\kappa_-|^2}\frac{1}{N_+^2}+\frac{1}{|1-\kappa_+|^2}\frac{1}{N_-^2}, \tag{4.53}$$

$$\tilde{D}_o=KD_o \tag{4.54}$$

with :

$$D_o=\frac{1}{|1-\kappa_-^2|^2}\frac{1}{N_+^2}+\frac{1}{|1-\kappa_+^2|^2}\frac{1}{N_-^2} \tag{4.55}$$

and

$$K=|\mu|+2E^2\Pi_{[0]} \tag{4.56}$$

with

$$(\Pi_{[0]})_{ij}=(\delta_{i,1}+\delta_{i,2})\delta_{ij} \tag{4.57}$$

Moreover, the weighted trace Tr_w is defined by

$$\mathrm{Tr}_w(B)=\sum_{l=0}^{2N}\big\{\chi_{mix}([l])\mathrm{atanh}^2(x_{(l)})+\delta([l]=0)+\delta([l]=N-1)\big\}\,B_{ll}. \tag{4.58}$$

Here, for $k\in\{1,\dots,2N\}$ we have defined $[k]$ to be the entire part of $\frac{k-1}{2}$: $[k]=\frac{k-1}{2}$ for odd k, and $[k]=\frac{k-2}{2}$ for even k, so that $[k]\in\{0,\dots,N-1\}$. We have also introduced the number $(k)=\frac{N}{2}-\tilde{k}$, where $\tilde{k}$ is the entire part of $\frac{k+1}{4}$ (hence $(k)\in\{0,\dots,\frac{N}{2}\}$), and we have defined the function

$$\chi_{mix}([l])\equiv 1-(\delta([l]=0)+\delta([l]=N-1)) \tag{4.59}$$

Thus :

$$S_2=\frac{d\lambda^2}{4}\sum_{i=2N-1}^{2N}\sum_{m=\frac{N}{2}}^{\frac{N}{2}+1}\langle\rho_{i,m}\rangle_N+\mathcal{O}(\lambda^3). \tag{4.60}$$

But, by (4.35),

$$\sum_{m=\frac{N}{2}}^{\frac{N}{2}+1}\langle\rho_{i,m}\rangle_N=1+\mathcal{O}(\lambda^2), \tag{4.61}$$

so that finally :

$$S_2=\frac{d\lambda^2}{2}+\mathcal{O}(\lambda^3) \tag{4.62}$$

Third term:

$$S_3 = \lim_{L\to\infty} \frac{1}{4L} \sum_{n=0}^{L-1} \left(\mathrm{Tr}_{2N} \left(\mathbb{E}(\Pi \tilde{\mathcal{T}}_0^{2N}(n) \tilde{\mathcal{T}}_0^{2N}(n) \Pi) \right) \right) \tag{4.63}$$

$$= \lim_{L\to\infty} \frac{1}{4L} \sum_{n=0}^{L-1} \sum_{i=2N-1}^{2N} \sum_{j=1}^{2N} \left\{ \sum_{\sigma_1=\pm} \sum_{m_1=0}^{\frac{N}{2}+1} (\mathrm{e}^{2\sigma_1 \eta_{m_1} - \eta_{\hat{\jmath}}} - 1) \langle u_i(n), \pi_{m_1}^{\sigma_1} u_j(n) \rangle \right\} \times$$

$$\left\{ \sum_{\sigma_2=\pm} \sum_{m_2=0}^{\frac{N}{2}+1} (\mathrm{e}^{2\sigma_2 \eta_{m_2} - \eta_{\hat{\jmath}}} - 1) \langle u_j(n), \pi_{m_2}^{\sigma_2} u_i(n) \rangle \right\} \tag{4.64}$$

if $\hat{\jmath} < \frac{N}{2}$ and $m_1 = \hat{\jmath}$, $\sigma_1 = +$ or $m_1 \in \{\frac{N}{2}, \frac{N}{2}+1\}$:

$$\langle u_i(n), \pi_{m_1}^{\sigma_1} u_j(n) \rangle = \mathcal{O}(\lambda) \tag{4.65}$$

The remaining terms are either of order λ^2 or have a vanishing prefactor. Hence (4.64) becomes

$$\lim_{L\to\infty} \frac{1}{4L} \sum_{n=0}^{L-1} \sum_{i=2N-1}^{2N} \sum_{j=1}^{2N-2} \left\{ (\mathrm{e}^{\hat{\eta}_{\hat{\jmath}}} - 1) \langle u_i(n), \pi_{\hat{\jmath}}^{+} u_j(n) \rangle + \sum_{\sigma_1=\pm} \sum_{m_1=\frac{N}{2}}^{\frac{N}{2}+1} (\mathrm{e}^{-\hat{\eta}_{\hat{\jmath}}} - 1) \langle u_i(n), \pi_{m_1}^{\sigma_1} u_j(n) \rangle \right\}$$

$$\times \left\{ (\mathrm{e}^{\hat{\eta}_{\hat{\jmath}}} - 1) \langle u_j(n), \pi_{\hat{\jmath}}^{+} u_i(n) \rangle + \sum_{\sigma_2=\pm} \sum_{m_2=\frac{N}{2}}^{\frac{N}{2}+1} (\mathrm{e}^{-\hat{\eta}_{\hat{\jmath}}} - 1) \langle u_j(n), \pi_{m_2}^{\sigma_2} u_i(n) \rangle \right\} + \mathcal{O}(\lambda^3) \tag{4.66}$$

Now,

$$\sum_{\sigma=\pm} \sum_{m=\frac{N}{2}-1}^{\frac{N}{2}} \pi_m^{\sigma} = \mathbb{I}_{4N} - \sum_{\sigma=\pm} \sum_{m=0}^{\frac{N}{2}-2} \pi_m^{\sigma}, \tag{4.67}$$

and since for $j \neq i$ one has $\langle u_i(n), u_j(n) \rangle = 0$, it follows by (4.35) and (4.38) that

$$\langle u_j(n), \sum_{\sigma=\pm} \sum_{m=\frac{N}{2}-1}^{\frac{N}{2}} \pi_m^{\sigma} u_i(n) \rangle = -\langle u_j(n), \pi_{\hat{\jmath}}^{+} u_i(n) \rangle + \mathcal{O}(\lambda^2). \tag{4.68}$$

Therefore,

$$S_3 = \lim_{L\to\infty} \frac{1}{4L} \sum_{n=0}^{L-1} \sum_{i=2N-1}^{2N} \sum_{j=1}^{2N-2} \left\{ (\mathrm{e}^{\hat{\eta}_{\hat{\jmath}}} - \mathrm{e}^{\hat{\eta}_{\hat{\jmath}}})^2 \langle u_i(n), \pi_{\hat{\jmath}}^{+} u_j(n) \rangle \langle u_j(n), \pi_{\hat{\jmath}}^{+} u_i(n) \rangle \right\} + \mathcal{O}(\lambda^3) \tag{4.69}$$

For $j < 2N-1 \leq i$ one has by (4.35), (4.37) and (4.39) that

$$\sum_{j:\hat{\jmath}=k} \langle u_i(n), \pi_{\hat{\jmath}}^{+} u_j(n) \rangle \langle u_j(n), \pi_{\hat{\jmath}}^{+} u_i(n) \rangle = \rho_{i,k}^{+}(n) + \mathcal{O}(\lambda^3) = \frac{1}{2} \rho_{i,k}(n) + \mathcal{O}(\lambda^3), \tag{4.70}$$

so that finally :

$$S_3 = \frac{1}{2} \sum_{i=2N-1}^{2N} \sum_{k=0}^{\frac{N}{2}+1} \sinh^2(\hat{\eta}_k) \left\langle \rho_{i,k} \right\rangle_N + \mathcal{O}(\lambda^3) \tag{4.71}$$

It follows that the third term cancels the first one to highest order.

Fourth term:

This term is easily seen to be of order λ^4 since for $i,j \in \{2N-1, 2N\}$:

$$\sum_{k=\frac{N}{2}}^{\frac{N}{2}+1} \sum_{\sigma=\pm} \langle u_i(n), \pi_k^\sigma u_j(n)\rangle - \delta_{ij} = \mathcal{O}(\lambda^2) \tag{4.72}$$

Fifth term:

For this term it is again more convenient to use the basis $\mathcal{V}$. If A and B are two arbitrary $2N \times 2N$ matrices, let $A \cdot B \equiv AB + (AB)^\dagger$. We introduce the matrix $P(n) \equiv \hat{T}_0^{-1}\hat{A}(n)$, so that $\hat{A}^\dagger \cdot \hat{T}_0 = P^\dagger \cdot |\hat{T}_0|^2$. Then :

$$S_5 = \lim_{L\to\infty} \frac{\lambda^2}{4L} \sum_{n=0}^{L-1} \mathrm{Tr}_{2N}\left(\mathbb{E}(\Pi \tilde{\mathcal{T}}_1^{2N}(n) \tilde{\mathcal{T}}_1^{2N}(n)\Pi)\right) \tag{4.73}$$

$$= \lim_{L\to\infty} \frac{\lambda^2}{4L} \sum_{n=0}^{L-1} \sum_{i=2N-1}^{2N} \sum_{j=1}^{2N} \mathrm{e}^{-2\hat{\eta}_{\mathfrak{j}}} \mathbb{E}\Big(\langle \psi_i(n), P^\dagger \cdot |\hat{T}_0|^2 (n+1)\psi_j(n)\rangle$$

$$\times \langle \psi_j(n), P^\dagger \cdot |\hat{T}_0|^2 (n+1)\psi_i(n)\rangle \Big) \tag{4.74}$$

now by (4.24) and (4.38), if $j < 2N-1$:

$$|\hat{T}_0|^2 \psi_j(n) = \mathrm{e}^{2\hat{\eta}_{\mathfrak{j}}} \pi_{\mathfrak{j}}^+ \psi_j(n) + \mathcal{O}(\lambda) \tag{4.75}$$

and by (4.24) and (4.35), if $j \geq 2N-1$:

$$|\hat{T}_0|^2 \psi_j(n) = \sum_{\sigma=\pm} \sum_{k=\frac{N}{2}}^{\frac{N}{2}+1} \pi_k^\sigma \psi_j(n) + \mathcal{O}(\lambda) \tag{4.76}$$

it follows :

$$S_5 = \lim_{L\to\infty} \frac{\lambda^2}{4L} \sum_{n=0}^{L-1} \sum_{i=2N-1}^{2N} \Bigg\{ \sum_{j=2N-1}^{2N} \sum_{\sigma_1,\sigma_2=\pm} \sum_{k_1,k_2=\frac{N}{2}}^{\frac{N}{2}+1} \mathbb{E}\Big(\langle \psi_i(n), P^\dagger \cdot \pi_{k_1}^{\sigma_1}(n+1)\psi_j(n)\rangle$$

$$\times \langle \psi_j(n), P^\dagger \cdot \pi_{k_2}^{\sigma_2}(n+1)\psi_i(n)\rangle \Big) + \sum_{j=1}^{2N-2} \sum_{\sigma_1,\sigma_2=\pm} \sum_{k_1,k_2=\frac{N}{2}}^{\frac{N}{2}+1} \mathbb{E}\Big(\langle \psi_i(n), \pi_{k_1}^{\sigma_1}(P\mathrm{e}^{-\hat{\eta}_{\mathfrak{j}}} + \mathrm{e}^{\hat{\eta}_{\mathfrak{j}}} P^\dagger \pi_{\mathfrak{j}}^+)\psi_j(n)\rangle$$

$$\times \langle \psi_j(n), (\pi_{\mathfrak{j}}^+ P \mathrm{e}^{\hat{\eta}_{\mathfrak{j}}} + \mathrm{e}^{-\hat{\eta}_{\mathfrak{j}}} P^\dagger)\pi_{k_2}^{\sigma_2}\psi_i(n)\rangle \Big) \Bigg\} + \mathcal{O}(\lambda^3) \tag{4.77}$$

We are now going to treat the first (elliptic) and second (hyperbolic) term inside the brackets separately. For the hyperbolic part, we use (4.37) to obtain :

$$\lim_{L\to\infty} \quad \frac{\lambda^2}{4L}\sum_{n=0}^{L-1}\sum_{i=2N-1}^{2N}\sum_{\hat{\jmath}=0}^{\frac{N}{2}-1}\sum_{\sigma_1,\sigma_2=\pm}\sum_{k_1,k_2=\frac{N}{2}}^{\frac{N}{2}+1}$$
$$\mathbb{E}\Big(\langle\psi_i(n),\pi_{k_1}^{\sigma_1}(P\mathrm{e}^{-\hat{\eta}_{\hat{\jmath}}}+\mathrm{e}^{\hat{\eta}_{\hat{\jmath}}}P^\dagger)\,\pi_{\hat{\jmath}}^{+}(P\mathrm{e}^{\hat{\eta}_{\hat{\jmath}}}+\mathrm{e}^{-\hat{\eta}_{\hat{\jmath}}}P^\dagger)\pi_{k_2}^{\sigma_2}\psi_i(n)\rangle\Big). \tag{4.78}$$

Using an oscillatory sum argument as before, this can be simplified to give :

$$\lim_{L\to\infty}\frac{\lambda^2}{4L}\sum_{n=0}^{L-1}\sum_{i=2N-1}^{2N}\sum_{\hat{\jmath}=0}^{\frac{N}{2}-1}\sum_{\sigma=\pm}\sum_{k=\frac{N}{2}}^{\frac{N}{2}+1}\mathbb{E}\left(\langle\psi_i(n),\pi_k^{\sigma}(P\mathrm{e}^{-\hat{\eta}_{\hat{\jmath}}}+\mathrm{e}^{\hat{\eta}_{\hat{\jmath}}}P^\dagger)\pi_{\hat{\jmath}}^{+}(P\mathrm{e}^{\hat{\eta}_{\hat{\jmath}}}+\mathrm{e}^{-\hat{\eta}_{\hat{\jmath}}}P^\dagger)\pi_k^{\sigma}\psi_i(n)\rangle\right). \tag{4.79}$$

But, for $k\in\{\frac{N}{2},\frac{N}{2}+1\}$,

$$\pi_k^{\sigma}\psi_i(n)=\psi_{i,k}^{\sigma}(n)v_k^{\sigma} \tag{4.80}$$

where $\psi_{i,k}^{\sigma}(n)$ is a complex number with modulus $\sqrt{\rho_{i,k}^{\sigma}(n)}$, so that (4.79) becomes :

$$\frac{\lambda^2}{4}\sum_{i=2N-1}^{2N}\sum_{\hat{\jmath}=0}^{\frac{N}{2}-1}\sum_{\sigma=\pm}\sum_{k=\frac{N}{2}}^{\frac{N}{2}+1}\left\langle\rho_{i,k}^{\sigma}\right\rangle_N\mathbb{E}\left(\langle v_k^{\sigma},(P\mathrm{e}^{-\hat{\eta}_{\hat{\jmath}}}+\mathrm{e}^{\hat{\eta}_{\hat{\jmath}}}P^\dagger)\pi_{\hat{\jmath}}^{+}(P\mathrm{e}^{\hat{\eta}_{\hat{\jmath}}}+\mathrm{e}^{-\hat{\eta}_{\hat{\jmath}}}P^\dagger)v_k^{\sigma}\rangle\right). \tag{4.81}$$

Now, by definition of P,

$$\mathbb{E}(\langle v_k^{\sigma},P\pi_{\hat{\jmath}}^{+}Pv_k^{\sigma}\rangle)=\mathbb{E}(\langle v_k^{\sigma},P^\dagger\pi_{\hat{\jmath}}^{+}P^\dagger v_k^{\sigma}\rangle)^* \tag{4.82}$$

and moreover,

$$\mathcal{R}e(\mathbb{E}(\langle v_k^{\sigma},P\pi_{\hat{\jmath}}^{+}Pv_k^{\sigma}\rangle))=-\mathcal{R}e(\mathbb{E}(\langle v_k^{-\sigma},P\pi_{\hat{\jmath}}^{+}Pv_k^{-\sigma}\rangle)) \tag{4.83}$$

so that, by the relation

$$\left\langle\rho_{i,k}^{\sigma}\right\rangle_N=\left\langle\rho_{i,k}^{-\sigma}\right\rangle_N+\mathcal{O}(\lambda^2), \tag{4.84}$$

these two terms give no contribution to highest order. For the remaining two terms, one gets (conf. the Appendix, Section 5.4.1):

$$d_1\equiv\sum_{\hat{\jmath}=0}^{\frac{N}{2}-1}\mathrm{e}^{2\hat{\eta}_{\hat{\jmath}}}\mathbb{E}(\langle v_k^{\sigma},P^\dagger\pi_{\hat{\jmath}}^{+}Pv_k^{\sigma}\rangle)=\frac{\sigma^2}{4N}\left\{\mathrm{Tr}_w(Q_e+\tilde{Q}_o)\right\} \tag{4.85}$$

where

$$Q_e=\frac{1}{N_+^2}\frac{1}{|1-\kappa_-|^2}(\mathbb{I}_{2N}-\Pi) \tag{4.86}$$

and

$$\tilde{Q}_o=KQ_o \tag{4.87}$$

with

$$Q_o=\frac{1}{N_+^2}\frac{1}{|1-\kappa_-^2|^2}(\mathbb{I}_{2N}-\Pi). \tag{4.88}$$

Moreover,

$$d_2(k) \equiv \sum_{\hat{\jmath}=0}^{\frac{N}{2}-1} \mathrm{e}^{-2\hat{\eta}_j} \mathbb{E}(\langle v_k^\sigma, P\pi_{\hat{\jmath}}^+ P^\dagger v_k^\sigma\rangle) = \frac{\sigma^2}{4N\sin^2\frac{\hat{\beta}_{\hat{k}}}{2}} \left\{(1+E^2)\mathrm{Tr}(G_e) + 2EF(G_e) + \mathrm{Tr}(G_o)\right\}, \tag{4.89}$$

where

$$G_e = \frac{1}{|1+\kappa_+|^2} N_+^2(\mathbb{I}_{2N} - \Pi), \tag{4.90}$$

$$G_o = |\kappa_-|^2 N_+^2(\mathbb{I}_{2N} - \Pi) \tag{4.91}$$

and

$$F(B) \equiv \sum_{l=0}^{2N} \left\{ \chi_{mix}([l]) \frac{\sin 3\alpha_{(l)}}{\cosh x_{(l)}} - \delta([l] = N-1)p(l) \right\} B_{ll}, \tag{4.92}$$

so that the hyperbolic term gives a contribution

$$\frac{\lambda^2}{4} \sum_{i=2N-1}^{2N} \sum_{k=\frac{N}{2}}^{\frac{N}{2}+1} (c_1 + c_2(k)) \left\langle \rho_{i,k} \right\rangle_N + \mathcal{O}(\lambda^3) \tag{4.93}$$

We now turn to the elliptic term :

$$\lim_{L\to\infty} \frac{\lambda^2}{4L} \sum_{n=0}^{L-1} \sum_{i,j=2N-1}^{2N} \sum_{\sigma_1,\sigma_2=\pm} \sum_{k_1,k_2=\frac{N}{2}}^{\frac{N}{2}+1} \mathbb{E}\Big(\langle \psi_i(n), P^\dagger \cdot \pi_{k_1}^{\sigma_1}(n+1)\psi_j(n)\rangle \times \langle \psi_j(n), P^\dagger \cdot \pi_{k_2}^{\sigma_2}(n+1)\psi_i(n)\rangle \Big). \tag{4.94}$$

$\psi_i(n)$ and $\psi_j(n)$ are elliptic frame vectors, so that up to an error of order λ,

$$\mathbb{E}\left(\langle \psi_i(n), P^\dagger \cdot \pi_{k_1}^{\sigma_1}(n+1)\psi_j(n)\rangle \langle \psi_j(n), P^\dagger \cdot \pi_{k_2}^{\sigma_2}(n+1)\psi_i(n)\rangle\right) =$$

$$\sum_{\sigma_3,\sigma_4=\pm} \sum_{k_3,k_4=\frac{N}{2}}^{\frac{N}{2}+1} \mathbb{E}\left(\langle \psi_i(n), (\pi_{k_3}^{\sigma_3} P^\dagger) \cdot \pi_{k_1}^{\sigma_1}(n+1)\psi_j(n)\rangle \langle \psi_j(n), (\pi_{k_4}^{\sigma_4} P^\dagger) \cdot \pi_{k_2}^{\sigma_2}(n+1)\psi_i(n)\rangle\right) \tag{4.95}$$

This yields (conf. the Appendix, Section 5.4.2) a contribution

$$\frac{\lambda^2\sigma^2}{4N} \sum_{i,j=2N-1}^{2N} \left\{ \sum_\sigma \sum_k \frac{1}{\sin^2\frac{\hat{\beta}_k}{2}} \left\langle \rho_{i,k}^\sigma \rho_{j,k}^{-\sigma} \right\rangle + \sum_{\{\sigma\}} \sum_{\{k\}} (1-\delta_{k_1 k_2}) \left(\frac{1}{2\sin\frac{\sigma_1\hat{\beta}_{k_1}}{2}} + \frac{1}{2\sin\frac{\sigma_2\hat{\beta}_{k_2}}{2}} \right)^2 \left\{ \left\langle \rho_{j,k_1}^{\sigma_1} \rho_{i,k_2}^{\sigma_2} \right\rangle + \left\langle \sqrt{\rho_{i,k_2}^{\sigma_2} \rho_{j,k_1}^{\sigma_1} \rho_{j,k_2}^{-\sigma_2} \rho_{i,k_1}^{-\sigma_1}}\, \mathrm{e}^{i\{(\theta_{j,k_1}^{\sigma_1} + \theta_{i,k_1}^{-\sigma_1}) - (\theta_{j,k_2}^{-\sigma_2} + \theta_{i,k_2}^{\sigma_2})\}} \right\rangle \right\} \right\} \tag{4.96}$$

Sixth term:

This term equals half the elliptic part of the previous term.

Final expression for the sum of the two lowest exponents:

Finally, putting all the previous results together yields the announced expression for the sum of the two lowest Lyapunov exponents :

$$\sum_{i=2N-1}^{2N}\gamma_i=\frac{\lambda^2}{4}\left\{\sum_{i=2N-1}^{2N}\sum_{k=\frac{N}{2}}^{\frac{N}{2}+1}(d-d_1-d_2(k))\left\langle\rho_{i,k}\right\rangle-\right.$$

$$\frac{\sigma^2}{2N}\sum_{i,j=2N-1}^{2N}\left(\sum_{\sigma}\sum_{k}\frac{1}{\sin^2\frac{\hat{\beta}_k}{2}}\left\langle\rho_{i,k}^{\sigma}\rho_{j,k}^{-\sigma}\right\rangle+\right.$$

$$\sum_{\{\sigma\}}\sum_{\{k\}}(1-\delta_{k_1k_2})\left(\frac{1}{2\sin\frac{\sigma_1\hat{\beta}_{k_1}}{2}}+\frac{1}{2\sin\frac{\sigma_2\hat{\beta}_{k_2}}{2}}\right)^2\left\{\left\langle\rho_{j,k_1}^{\sigma_1}\rho_{i,k_2}^{\sigma_2}\right\rangle+\right.$$

$$\left.\left.\left.\left\langle\sqrt{\rho_{i,k_2}^{\sigma_2}\rho_{j,k_1}^{\sigma_1}\rho_{j,k_2}^{-\sigma_2}\rho_{i,k_1}^{-\sigma_1}}\mathrm{e}^{i\{(\theta_{j,k_1}^{\sigma_1}+\theta_{i,k_1}^{-\sigma_1})-(\theta_{j,k_2}^{-\sigma_2}+\theta_{i,k_2}^{\sigma_2})\}}\right\rangle\right\}\right)\right\} \tag{4.97}$$

□

5 Appendix : calculations relative to Theorem 4.1

5.1 Some notations and an expression for the matrix F

For $k\in\{1,\dots,2N\}$ we let $[k]$ be the entire part of $\frac{k-1}{2}$: $[k]=\frac{k-1}{2}$ for odd k, and $[k]=\frac{k-2}{2}$ for even k, so that $[k]\in\{0,\dots,N-1\}$. We also introduce the number $(k)=\frac{N}{2}-\hat{k}$, where $\hat{k}$ is the entire part of $\frac{k+1}{4}$. Hence $(k)\in\{0,\dots,\frac{N}{2}\}$. We moreover introduce the function $\delta^o(l)$ (resp. $\delta^e(l)$) which is equal to 1 if l is odd (resp. even) and equal to zero otherwise. We also write $p(l)=(-1)^l$.

Then we can write

$$F_{kl}=\frac{\chi_{mix}([l])}{\sqrt{2N\cosh(x_{(l)})}}\left\{\delta^o([l])z_{kl}(x_{(l)})\mathrm{e}^{i(\alpha_{(l)}-\frac{\pi}{2})\delta^o(k)}+\delta^e([l])\overline{z}_{kl}(-x_{(l)})\mathrm{e}^{i(\alpha_{(l)}-\frac{\pi}{2})\delta^e(k)}\right\}$$

$$+\delta([l]=0)\frac{\mathrm{e}^{i\pi[k]}}{\sqrt{N}}\delta^e(k+l)+\delta([l]=N-1)\frac{(-p(l))^{\delta^e(k)}}{\sqrt{2N}} \tag{5.1}$$

where

$$z_{kl}(x_{(l)})=\mathrm{e}^{i2[k]\alpha_{(l)}}\mathrm{e}^{\frac{x_{(l)}}{2}p(k)p(l)} \tag{5.2}$$

$$\chi_{mix}([l])\equiv 1-(\delta([l]=0)+\delta([l]=N-1)) \tag{5.3}$$

Similarly,

$$F_{kl}^{-1}=\chi_{mix}([k])\frac{\mathrm{argth}^2(x_{(k)})}{\sqrt{2N\cosh(x_{(k)})}}\left\{\delta^o([k])\left(\overline{z}_{lk}(x_{(k)})-\overline{z}_{lk}(-x_{(k)})\cosh^{-1}(x_{(k)})\right)\mathrm{e}^{-i(\alpha_{(k)}-\frac{\pi}{2})\delta^o(l)}\right.$$

$$+\delta^e([k])\left(z_{lk}(-x_{(k)})-z_{lk}(x_{(k)})\cosh^{-1}(x_{(k)})\right)\mathrm{e}^{-i(\alpha_{(k)}-\frac{\pi}{2})\delta^e(l)}\Big\}+\delta([k]=0)\frac{\mathrm{e}^{i\pi[l]}}{\sqrt{N}}\delta^e(k+l)$$

$$+\delta([k]=N-1)\frac{(-p(k))^{\delta^e(l)}}{\sqrt{2N}} \tag{5.4}$$

$$(W_e)_{kl}=-\{\delta^e(k)\delta^o(l)\delta(k=l+1)+\delta^e(l)\delta^o(k)\delta(l=k+1)\} \tag{5.5}$$

$$(W_o)_{kl}=-\{\delta^o(k)\delta^e(l)\delta(k=(l+1)_{2N})+\delta^o(l)\delta^e(k)\delta(l=(k+1)_{2N})\} \tag{5.6}$$

$$\mathbb{E}(V_eMV_e)=\mathbb{E}(V_oMV_o)=\sigma^2\mathrm{diag}(M) \tag{5.7}$$

where

$$\mathrm{diag}(M)_{ij}\equiv M_{ii}\delta_{ij} \tag{5.8}$$

5.2 Appendix: second term

5.2.1 Even part

Let

$$\hat{A}_e\equiv\begin{pmatrix}A_1 & A_2\\ A_3 & A_4\end{pmatrix} \tag{5.9}$$

$$\pi_m^+\equiv\begin{pmatrix}P_m & 0\\ 0 & 0\end{pmatrix}\qquad\pi_m^-\equiv\begin{pmatrix}0 & 0\\ 0 & P_m\end{pmatrix} \tag{5.10}$$

$$\pi_m^+\hat{A}_e^\dagger\hat{A}_e\pi_m^+=\begin{pmatrix}P_m(A_1^\dagger A_1+A_3^\dagger A_3)P_m & 0\\ 0 & 0\end{pmatrix} \tag{5.11}$$

$$\pi_m^-\hat{A}_e^\dagger\hat{A}_e\pi_m^-=\begin{pmatrix}0 & 0\\ 0 & P_m(A_2^\dagger A_2+A_4^\dagger A_4)P_m\end{pmatrix} \tag{5.12}$$

One reads off the definition of $\hat{A}_e$ that :

$$F_m(\hat{\eta},\hat{\beta})\equiv(\mathbb{E}(P_mA_1^\dagger A_1P_m)=\mathbb{E}(P_mA_2^\dagger A_2P_m) \tag{5.13}$$

and

$$\mathbb{E}(P_mA_3^\dagger A_3P_m)=\mathbb{E}(P_mA_4^\dagger A_4P_m)=F_m(-\hat{\eta},-\hat{\beta}) \tag{5.14}$$

We will show that

$$F_m(\hat{\eta},\hat{\beta})=c_m(\hat{\eta},\hat{\beta})P_m, \tag{5.15}$$

where $c_m(\hat{\eta},\hat{\beta})$ is a constant. Hence

$$\mathbb{E}(\pi_m^\sigma\hat{A}_e^\dagger\hat{A}_e\pi_m^\sigma)=(c_m(\hat{\eta},\hat{\beta})+c_m(-\hat{\eta},-\hat{\beta}))\pi_m^\sigma. \tag{5.16}$$

Let us compute :

$$\mathbb{E}(P_m\hat{A}_1^\dagger A_1P_m)=\frac{1}{8\cos^2(\frac{\hat{\beta}_m}{2})}\mathbb{E}(P_mF^\dagger W_o^-V_e(F^{-1})^\dagger DF^{-1}V_eW_o^-FP_m) \tag{5.17}$$

where

$$D(\hat{\eta},\hat{\beta})=\frac{1}{|1-\kappa_-|^2}\frac{1}{N_+^2} \tag{5.18}$$

D is a diagonal matrix satisfying

$$D_{kl} = d_{(l)}\delta_{kl}. \tag{5.19}$$

Using (5.7) we get :

$$\mathbb{E}(P_m \hat{A}_1^\dagger A_1 P_m) = \frac{\sigma^2}{8\cos^2(\frac{\hat{\beta}_m}{2})} P_m F^\dagger W_o^- \mathrm{diag}((F^{-1})^\dagger D F^{-1}) W_o^- F P_m$$

$$= \frac{\sigma^2}{8\cos^2(\frac{\hat{\beta}_m}{2})} (F^\dagger W_o^- \mathrm{diag}((F^{-1})^\dagger D F^{-1}) W_o^- F)_{mm} P_m \tag{5.20}$$

Let :

$$M = (F^{-1})^\dagger D F^{-1} \tag{5.21}$$

$$M_{jj} = ((F^{-1})^\dagger D F^{-1})_{jj} = \sum_{l=1}^{2N} d_l |F_{lj}^{-1}|^2 \tag{5.22}$$

$$(F^\dagger W_o^- \mathrm{diag}(M) W_o^- F)_{mm} = \sum_{kjn} F_{km}^* (W_o^-)_{kj} (W_o^-)_{jn} F_{nm} M_{jj}$$

$$= \frac{1}{2N} \sum_{kjn} (-p(m))^{\delta^e(k)+\delta^e(n)} (W_o^-)_{kj} (W_o^-)_{jn} M_{jj} \tag{5.23}$$

Now,

$$\sum_{kjn} (-p(m))^{\delta^e(k)+\delta^e(n)} (W_o)_{kj} (W_o)_{jn} M_{jj} = \mathrm{Tr}(M) \tag{5.24}$$

$$-E \sum_{kjn} (-p(m))^{\delta^e(k)+\delta^e(n)} (W_o)_{kj} \delta_{jn} M_{jj} = \tag{5.25}$$

$$-E \sum_{kjn} (-p(m))^{\delta^e(k)+\delta^e(n)} (W_o)_{jn} \delta_{kj} M_{jj} = -Ep(m)\mathrm{Tr}(M) \tag{5.26}$$

$$E^2 \sum_{kjn} (-p(m))^{\delta^e(k)+\delta^e(n)} (\delta)_{jn} \delta_{kj} M_{jj} = E^2 \mathrm{Tr}(M) \tag{5.27}$$

so that

$$(F^\dagger W_o^- \mathrm{diag}(M) W_o^- F)_{mm} = \frac{1}{2N}(1 - p(m)E)^2 \mathrm{Tr}(M) = \frac{4\cos^2 \frac{\hat{\beta}_m}{2}}{2N} \mathrm{Tr}(M). \tag{5.28}$$

But

$$\sum_j |F_{lj}^{-1}|^2 = \chi_{mix}([l])\mathrm{argth}^2 x_{(l)} + \delta([l] = 0) + \delta([l] = N-1) \tag{5.29}$$

so that

$$c_m(\hat{\eta}, \hat{\beta}) = \frac{\sigma^2}{4N} \mathrm{Tr}_w(D), \tag{5.30}$$

where

$$\mathrm{Tr}_w(D) = \sum_{l=0}^{2N} \{\chi_{mix}([l])\mathrm{argth}^2 x_{(l)} + \delta([l] = 0) + \delta([l] = N-1)\} D_{ll}. \tag{5.31}$$

Since the weighted trace Tr_w is linear, and

$$D(-\hat{\eta}, -\hat{\beta}) = \frac{1}{|1-\kappa_+|^2} \frac{1}{N_-^2}, \tag{5.32}$$

we obtain finally,

$$\mathbb{E}(\pi_m^\sigma \hat{A}_e^\dagger \hat{A}_e \pi_m^\sigma) = \frac{\sigma^2}{4N} \mathrm{Tr}_w(D_e)\pi_m^\sigma, \tag{5.33}$$

where

$$D_e = \frac{1}{|1-\kappa_-|^2}\frac{1}{N_+^2} + \frac{1}{|1-\kappa_+|^2}\frac{1}{N_-^2} \tag{5.34}$$

5.2.2 Odd part

Similarly to the even part,

$$\mathbb{E}(A_1^\dagger A_1)_{mm} = \frac{1}{2}\mathbb{E}(F^\dagger V_o W_e^- (F^{-1})^\dagger D F^{-1} W_e^- V_o F)_{mm} \tag{5.35}$$

where

$$D(\hat{\eta}, \hat{\beta}) = \frac{1}{|1-\kappa_-^2|^2}\frac{1}{N_+^2} \tag{5.36}$$

Let

$$M = W_e^- (F^{-1})^\dagger D F^{-1} W_e^-. \tag{5.37}$$

Then

$$(F^\dagger \mathrm{diag}(M) F)_{mm} = \sum_{j=0}^{2N} |F_{jm}|^2 M_{jj} = \frac{1}{2N}\mathrm{Tr}(M) \tag{5.38}$$

Let

$$Q = (F^{-1})^\dagger D F^{-1} \tag{5.39}$$

so that

$$\mathrm{Tr}(M) = \sum_{ijk} (W_e^-)_{ij}(W_e^-)_{ki} Q_{jk}. \tag{5.40}$$

Now :

$$\sum_{ijk} (W_e)_{ij}(W_e)_{ki} Q_{jk} = \mathrm{Tr}(Q) \tag{5.41}$$

$$E^2 \sum_{ijk} \delta_{ij}\delta_{ki} Q_{jk} = E^2 \mathrm{Tr}(Q) \tag{5.42}$$

and

$$-E\sum_{ijk}(W_e)_{ij}\delta_{ki}Q_{jk} = -E\sum_{ijk}\delta_{ij}(W_e)_{ki}Q_{jk} = E\sum_{j=0}^{2N}\{\delta^o(j)Q_{jj+1} + \delta^e(j)Q_{jj-1}\} \tag{5.43}$$

It follows from the definition of Q that :

$$\sum_{j=0}^{2N}\{\delta^o(j)Q_{jj+1} + \delta^e(j)Q_{jj-1}\} = 2\mathcal{R}e\left(\sum_{j=0}^{2N}\delta^o(j)Q_{jj+1}\right), \tag{5.44}$$

and, taking into account that $[j] = [j+1]$ if j is odd, a short computation yields :

$$\mathcal{R}e\left(\sum_{j=0}^{2N}\delta^o(j)(F_{lj}^{-1})^* F_{lj+1}^{-1}\right) = -\frac{1}{2}\left\{E\chi_{mix}([l])\mathrm{argth}^2 x_{(l)} + \delta([l]=N-1)p(l)\right\}. \tag{5.45}$$

Hence,

$$\mathcal{R}e\left(\sum_{j=0}^{2N}\delta^o(j)Q_{jj+1}\right) = -\frac{1}{2}\sum_{l=0}^{2N}\left\{E\chi_{mix}([l])\text{argth}^2 x_{(l)} + \delta([l]=N-1)p(l)\right\}D_{ll} \tag{5.46}$$

Remember, moreover, from the previous section that :

$$\text{Tr}(Q) = \text{Tr}_w(D) \tag{5.47}$$

Putting all the terms together yields

$$\begin{aligned}\text{Tr}(M) &= \sum_{l=0}^{2N}\left\{\chi_{mix}([l])(1-E^2)\text{argth}^2 x_{(l)} + \delta([l]=0)(1+E^2) + \right.\\ &\left. + \delta([l]=N-1)(1-p(l)E)^2\right\}D_{ll}\end{aligned} \tag{5.48}$$

Now let K be the diagonal $2N\times 2N$ matrix defined by :

$$K_{kl} = \left\{\chi_{mix}([l])(1-E^2) + \delta([l]=0)(1+E^2) + \delta([l]=N-1)(1-p(l)E)^2\right\}\delta_{kl} \tag{5.49}$$

Note that :

$$K = |\mu| + 2E^2\Pi_0 \tag{5.50}$$

then :

$$\text{Tr}(M) = \text{Tr}_w(KD) \tag{5.51}$$

and :

$$\mathbb{E}(\pi_m^\sigma \hat{A}_o^\dagger \hat{A}_o \pi_m^\sigma) = \frac{\sigma^2}{4N}\text{Tr}_w(KD_o)\pi_m^\sigma \tag{5.52}$$

where

$$D_o = \frac{1}{|1-\kappa_-^2|^2}\frac{1}{N_+^2} + \frac{1}{|1-\kappa_+^2|^2}\frac{1}{N_-^2} \tag{5.53}$$

together with (5.33) we finally obtain

$$\mathbb{E}(\pi_m^\sigma \hat{A}^\dagger \hat{A}\pi_m^\sigma) = \frac{\sigma^2}{4N}\left\{\text{Tr}_w(D_e) + \text{Tr}_w(KD_o)\right\}\pi_m^\sigma. \tag{5.54}$$

5.3 Appendix: fifth term, hyperbolic part

5.3.1 First term

$$P = T_0^{-1}\hat{A} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \tag{5.55}$$

$$\text{e}^{2\hat{\eta}_j}\mathbb{E}(\langle v_k^+, P^\dagger \pi_{\hat{\jmath}}^+ P v_k^+\rangle) = \text{e}^{2\hat{\eta}_j}\mathbb{E}(P_1^\dagger \Pi_{\hat{\jmath}} P_1)_{kk} \tag{5.56}$$

$$\text{e}^{2\hat{\eta}_j}\mathbb{E}(\langle v_k^-, P^\dagger \pi_{\hat{\jmath}}^+ P v_k^-\rangle) = \text{e}^{2\hat{\eta}_j}\mathbb{E}(P_2^\dagger \Pi_{\hat{\jmath}} P_2)_{kk} \tag{5.57}$$

Looking at the definitions of P_1 and P_2, it turns out that

$$\mathbb{E}(P_1^\dagger \Pi_{\hat{\jmath}} P_1)_{kk} = \mathbb{E}(P_2^\dagger \Pi_{\hat{\jmath}} P_2)_{kk} \tag{5.58}$$

Hence

$$\mathrm{e}^{2\hat{\eta}_j}\mathbb{E}(\langle v_k^\sigma, P^\dagger \pi_{\hat{\jmath}}^+ P v_k^\sigma\rangle) = \mathrm{e}^{2\hat{\eta}_j}\mathbb{E}(P_1^\dagger \Pi_{\hat{\jmath}} P_1)_{kk} \tag{5.59}$$

One reads off the definition that

$$\mathrm{e}^{2\hat{\eta}_j}\mathbb{E}(P_1^\dagger \Pi_{\hat{\jmath}} P_1)_{kk} = \mathbb{E}(\hat{A}^\dagger \Pi_{\hat{\jmath}} \hat{A})_{kk} \tag{5.60}$$

and the latter expression can be computed similarly to the previous section, yielding

$$\mathrm{e}^{2\hat{\eta}_j}\mathbb{E}(\langle v_k^\sigma, P^\dagger \pi_{\hat{\jmath}}^+ P v_k^\sigma\rangle) = \frac{\sigma^2}{4N}\left\{\mathrm{Tr}_w(G_e^j) + \mathrm{Tr}_w(KG_o^j)\right\}, \tag{5.61}$$

where K is given by (5.49) and :

$$G_e^{\hat{\jmath}} = \frac{1}{N_+^2}\frac{1}{|1-\kappa_-|^2}\Pi_{\hat{\jmath}} \qquad ; \qquad G_o^{\hat{\jmath}} = \frac{1}{N_+^2}\frac{1}{|1-\kappa_-^2|^2}\Pi_{\hat{\jmath}} \tag{5.62}$$

summing over $\hat{\jmath}$ finally gives :

$$\sum_{\hat{\jmath}=0}^{\frac{N}{2}-1} \mathrm{e}^{2\hat{\eta}_j}\mathbb{E}(\langle v_k^\sigma, P^\dagger \pi_{\hat{\jmath}}^+ P v_k^\sigma\rangle) = \frac{\sigma^2}{4N}\left\{\mathrm{Tr}_w(G_e) + \mathrm{Tr}_w(KG_o)\right\} \tag{5.63}$$

where :

$$G_e = \sum_{\hat{\jmath}=0}^{\frac{N}{2}-1} G_e^{\hat{\jmath}} = \frac{1}{N_+^2}\frac{1}{|1-\kappa_-|^2}(\mathbb{I}_{2N} - \Pi) \tag{5.64}$$

and :

$$G_o = \sum_{\hat{\jmath}=0}^{\frac{N}{2}-1} G_o^{\hat{\jmath}} = \frac{1}{N_+^2}\frac{1}{|1-\kappa_-^2|^2}(\mathbb{I}_{2N} - \Pi) \tag{5.65}$$

5.3.2 Second term

$$\mathrm{e}^{-2\hat{\eta}_j}\mathbb{E}(\langle v_k^+, P\pi_{\hat{\jmath}}^+ P^\dagger v_k^+\rangle) = \mathrm{e}^{-2\hat{\eta}_j}\mathbb{E}(P_1 \Pi_{\hat{\jmath}} P_1^\dagger)_{kk} = \mathrm{e}^{-2\hat{\eta}_j}\mathbb{E}(A_1 \Pi_{\hat{\jmath}} A_1^\dagger)_{kk} \tag{5.66}$$

$$\mathrm{e}^{-2\hat{\eta}_j}\mathbb{E}(\langle v_k^-, P\pi_{\hat{\jmath}}^+ P^\dagger v_k^-\rangle) = \mathrm{e}^{-2\hat{\eta}_j}\mathbb{E}(P_3 \Pi_{\hat{\jmath}} P_3^\dagger)_{kk} = \mathrm{e}^{-2\hat{\eta}_j}\mathbb{E}(A_3 \Pi_{\hat{\jmath}} A_3^\dagger)_{kk} \tag{5.67}$$

From the definition one has that

$$\mathbb{E}(A_1 \Pi_{\hat{\jmath}} A_1^\dagger)_{kk} = \mathbb{E}(A_3 \Pi_{\hat{\jmath}} A_3^\dagger)_{kk} \tag{5.68}$$

Hence, we only need to compute $\mathbb{E}(A_1 \Pi_{\hat{\jmath}} A_1^\dagger)_{kk}$. As before, we treat the even and odd part separately.

Even part

$$\mathrm{e}^{-2\hat{\eta}_j}\mathbb{E}(A_1 \Pi_{\hat{\jmath}} A_1^\dagger)_{kk} = \frac{1}{2\cos^2\frac{\hat{\beta}_k}{2}}\mathbb{E}\left(F^{-1}V_e W_o^- F D F^\dagger W_o^- V_e (F^{-1})^\dagger\right)_{kk} \tag{5.69}$$

where :

$$D = \frac{1}{|1+\kappa_+|^2} N_+^2 \Pi_{\hat{\jmath}}. \tag{5.70}$$

Using (5.7) one obtains

$$\mathrm{e}^{-2\hat{\eta}_j} \mathbb{E}(A_1 \Pi_{\hat{\jmath}} A_1^\dagger)_{kk} = \frac{\sigma^2}{2\cos^2 \frac{\hat{\beta}_k}{2}} \left(F^{-1} \mathrm{diag}(W_o^- F D F^\dagger W_o^-)(F^{-1})^\dagger\right)_{kk} \tag{5.71}$$

Let

$$M = \mathrm{diag}(W_o^- F D F^\dagger W_o^-) \tag{5.72}$$

One has :

$$\left(F^{-1} M (F^{-1})^\dagger\right)_{kk} = \sum_{l=0}^{2N} |F_{kl}^{-1}|^2 M_{ll} \tag{5.73}$$

and since k is an elliptic index, we get

$$\left(F^{-1} M (F^{-1})^\dagger\right)_{kk} = \frac{1}{2N} \mathrm{Tr}(M) \tag{5.74}$$

Now define

$$G = F D F^\dagger. \tag{5.75}$$

Then

$$M_{mm} = \sum_{kl} (W_o^-)_{mk} G_{kl} (W_o^-)_{lm} \tag{5.76}$$

Now,

$$E^2 \sum_{kl} \delta_{mk} G_{kl} \delta_{lm} = E^2 G_{mm} \tag{5.77}$$

and

$$\sum_{kl} (W_o)_{mk} G_{kl} (W_o)_{lm} = \sum_k \{\delta^o(m)\delta(m = (k+1)_{2N}) + \delta^e(m)\delta(k = (m+1)_{2N})\} G_{kk} \tag{5.78}$$

$$-E \sum_{kl} (W_o)_{mk} G_{kl} \delta_{lm} = E \sum_k \{\delta^o(m)\delta(m = (k+1)_{2N}) + \delta^e(m)\delta(k = (m+1)_{2N})\} G_{km} \tag{5.79}$$

$$-E \sum_{kl} \delta_{mk} G_{kl} (W_o)_{lm} = E \sum_k \{\delta^o(m)\delta(m = (k+1)_{2N}) + \delta^e(m)\delta(k = (m+1)_{2N})\} G_{mk}. \tag{5.80}$$

It follows from the definition of G that G is self adjoint. Hence

$$-E \sum_{kl} \{(W_o)_{mk} G_{kl} \delta_{lm} + \delta_{mk} G_{kl} (W_o)_{lm}\}$$

$$= 2E \sum_k \{\delta^o(m)\delta(m = (k+1)_{2N}) + \delta^e(m)\delta(k = (m+1)_{2N})\} \mathcal{R}e(G_{km}) \tag{5.81}$$

so that :

$$\mathrm{Tr}(M) = (1+E^2)\mathrm{Tr}(G) + 2E \sum_{km} \{\delta^o(m)\delta(m = (k+1)_{2N}) + \delta^e(m)\delta(k = (m+1)_{2N})\} \mathcal{R}e(G_{km}) \tag{5.82}$$

We start by computing

$$\mathrm{Tr}(G) = \sum_{kl} |F_{kl}|^2 d_l. \tag{5.83}$$

It follows from its definition that

$$\sum_{k} |F_{kl}|^2 = 1 \tag{5.84}$$

Hence

$$\mathrm{Tr}(G) = \mathrm{Tr}(D) \tag{5.85}$$

Next, we have

$$\sum_{km} \{\delta^o(m)\delta(m = (k+1)_{2N}) + \delta^e(m)\delta(k = (m+1)_{2N})\}\mathcal{R}e(G_{km}) =$$

$$\sum_{klm} \{\delta^o(m)\delta(m = (k+1)_{2N}) + \delta^e(m)\delta(k = (m+1)_{2N})\}\mathcal{R}e(F_{kl}F^*_{ml})d_l =$$

$$2\sum_{kl} \delta^e(k)\mathcal{R}e(F_{kl}F^*_{(k+1)_{2N}l})d_l \tag{5.86}$$

Once again using the definition, it follows that

$$\sum_{kl} \delta^e(k)\mathcal{R}e(F_{kl}F^*_{(k+1)_{2N}l}) = \frac{1}{2}\left(\chi_{mix}([l])\frac{\sin 3\alpha_{(l)}}{\cosh(x_{(l)})} - \delta([l] = N-1)p(l)\right) \tag{5.87}$$

Thus

$$\mathrm{e}^{-2\hat{\eta}_j}\mathbb{E}(A_1\Pi_{\hat{\jmath}}A_1^\dagger)_{kk} = \frac{\sigma^2}{4N\cos^2\frac{\hat{\beta}_k}{2}}\left\{(1+E^2)\mathrm{Tr}(D) + 2E\tilde{\mathrm{Tr}}_w(D)\right\}, \tag{5.88}$$

where

$$\tilde{\mathrm{Tr}}_w(D) = \sum_{l=0}^{2N}\left(\chi_{mix}([l])\frac{\sin 3\alpha_{(l)}}{\cosh(x_{(l)})} - \delta([l] = N-1)p(l)\right)D_{ll} \tag{5.89}$$

Odd part

$$\mathrm{e}^{-2\hat{\eta}_j}\mathbb{E}(A_1\Pi_{\hat{\jmath}}A_1^\dagger)_{kk} = \frac{1}{2\sin^2\hat{\beta}_k}\mathbb{E}\left(F^{-1}W_e^-V_oFDF^\dagger V_oW_e^-F^{-1})^\dagger\right)_{kk} \tag{5.90}$$

where

$$D = N_+^2|\kappa_-|^2\Pi_{\hat{\jmath}} \tag{5.91}$$

Again using (5.7) we have

$$\mathrm{e}^{-2\hat{\eta}_j}\mathbb{E}(A_1\Pi_{\hat{\jmath}}A_1^\dagger)_{kk} = \frac{\sigma^2}{2\sin^2\hat{\beta}_k}\left(F^{-1}W_e^-\mathrm{diag}(FDF^\dagger)W_e^-(F^{-1})^\dagger\right)_{kk} \tag{5.92}$$

Let

$$M = W_e^-\mathrm{diag}(FDF^\dagger)W_e^- \tag{5.93}$$

then :

$$\left(F^{-1}M(F^{-1})^\dagger\right)_{kk} = \frac{1}{2N}\sum_{lm}(-p(k))^{\delta^e(m)+\delta^e(l)}M_{lm} \tag{5.94}$$

Let moreover :

$$G = \mathrm{diag}(FDF^\dagger) \tag{5.95}$$

Using the fact that D is a diagonal matrix that is constant within a given channel :

$$D_{kl} = d_{[k]}\delta_{kl} \tag{5.96}$$

together with the definition (5.1) of F, a short computation shows that :

$$G = \frac{1}{2N}\mathrm{Tr}(D)\mathbb{I}_{2N} \tag{5.97}$$

Hence :

$$M = \frac{1}{2N}\mathrm{Tr}(D)\left\{(1+E^2) - 2EW_e\right\} \tag{5.98}$$

but :

$$\frac{1}{2N}\sum_{lm}(-p(k))^{\delta^e(m)+\delta^e(l)}\delta_{lm} = 1 \tag{5.99}$$

and :

$$\frac{1}{2N}\sum_{lm}(-p(k))^{\delta^e(m)+\delta^e(l)}(W_e)_{lm} = p(k) \tag{5.100}$$

Hence :

$$\mathrm{e}^{-2\hat{\eta}_j}\mathbb{E}(A_1\Pi_{\hat{\jmath}}A_1^\dagger)_{kk} = \frac{\sigma^2}{4N\sin^2\hat{\beta}_k}(1-p(k)E)^2\mathrm{Tr}(D) = \frac{\sigma^2}{4N\sin^2\frac{\hat{\beta}_k}{2}}\mathrm{Tr}(D) \tag{5.101}$$

5.3.3 Final result

Finally, the entire second term of the hyperbolic part reads :

$$\mathrm{e}^{-2\hat{\eta}_j}\mathbb{E}(\langle v_k^\sigma, P\pi_{\hat{\jmath}}^+P^\dagger v_k^\sigma\rangle) = \frac{\sigma^2}{4N}\left\{\frac{1}{\cos^2\frac{\hat{\beta}_k}{2}}\left((1+E^2)\mathrm{Tr}(Q_e^{\hat{\jmath}}) + 2E\tilde{\mathrm{Tr}}_w(Q_e^{\hat{\jmath}})\right) + \frac{1}{\sin^2\frac{\hat{\beta}_k}{2}}\mathrm{Tr}(Q_o^{\hat{\jmath}})\right\} \tag{5.102}$$

where :

$$Q_o^{\hat{\jmath}} = N_+^2|\kappa_-|^2\Pi_{\hat{\jmath}} \quad ; \qquad Q_e^{\hat{\jmath}} = \frac{1}{|1+\kappa_+|^2}N_+^2\Pi_{\hat{\jmath}} \tag{5.103}$$

and :

$$\tilde{\mathrm{Tr}}_w(D) = \sum_{l=0}^{2N}\left(\chi_{mix}([l])\frac{\sin 3\alpha_{(l)}}{\cosh(x_{(l)})} - \delta([l]=N-1)p(l)\right)D_{ll} \tag{5.104}$$

5.4 Appendix: fifth term, elliptic part

5.4.1 Preliminaries

For k_1, k_2, k_3 and k_4 elliptic indexes :

$$C_e(k_1,k_2,k_3,k_4) \equiv \mathbb{E}\left((F^{-1}V_eW_o^-F)_{k_1k_2}(F^{-1}V_eW_o^-F)_{k_3k_4}\right) =$$

$$\frac{1}{(2N)^2}\mathbb{E}\left(\sum_{n_1,p_1}\sum_{n_2,p_2}(-p(k_1))^{\delta^e(n_1)}(-p(k_2))^{\delta^e(p_1)}(-p(k_3))^{\delta^e(n_2)}(-p(k_4))^{\delta^e(p_2)}\right.$$

$$\left.(V_e)_{n_1n_1}(V_e)_{n_2n_2}(W_o^-)_{n_1p_1}(W_o^-)_{n_2p_2}\right) =$$

$$\frac{\sigma^2}{4N^2}\sum_{np_1p_2}(p(k_1)p(k_3))^{\delta^e(n)}(-p(k_2))^{\delta^e(p_1)}(-p(k_4))^{\delta^e(p_2)}(W_o^-)_{np_1}(W_o^-)_{np_2} \tag{5.105}$$

Now

$$\frac{\sigma^2}{4N^2}\sum_{np_1p_2}(p(k_1)p(k_3))^{\delta^e(n)}(-p(k_2))^{\delta^e(p_1)}(-p(k_4))^{\delta^e(p_2)}(W_o)_{np_1}(W_o)_{np_2} =$$

$$\frac{\sigma^2}{4N^2}\sum_{np_1}(p(k_1)p(k_3))^{\delta^e(n)}(p(k_2)p(k_4))^{\delta^e(p_1)}(-(W_o)_{np_1}) = \frac{\sigma^2}{4N}\left(p(k_1)p(k_3)+p(k_2)p(k_4)\right) \tag{5.106}$$

Moreover,

$$-E\frac{\sigma^2}{4N^2}\sum_{np_1p_2}(p(k_1)p(k_3))^{\delta^e(n)}(-p(k_2))^{\delta^e(p_1)}(-p(k_4))^{\delta^e(p_2)}(W_o)_{np_1}\delta_{np_2} =$$

$$-E\frac{\sigma^2}{4N}\left(p(k_1)p(k_3)p(k_4)+p(k_2)\right) \tag{5.107}$$

and

$$-E\frac{\sigma^2}{4N^2}\sum_{np_1p_2}(p(k_1)p(k_3))^{\delta^e(n)}(-p(k_2))^{\delta^e(p_1)}(-p(k_4))^{\delta^e(p_2)}\delta_{np_1}(W_o)_{np_2} =$$

$$-E\frac{\sigma^2}{4N}\left(p(k_1)p(k_2)p(k_3)+p(k_4)\right) \tag{5.108}$$

Finally,

$$E^2\frac{\sigma^2}{4N^2}\sum_{np_1p_2}(p(k_1)p(k_3))^{\delta^e(n)}(-p(k_2))^{\delta^e(p_1)}(-p(k_4))^{\delta^e(p_2)}\delta_{np_1}\delta_{np_2} =$$

$$E^2\frac{\sigma^2}{4N}\left(1+p(k_1)p(k_2)p(k_3)p(k_4)\right)) \tag{5.109}$$

Hence

$$C_e(k_1,k_2,k_3,k_4) = \frac{\sigma^2}{4N}\left\{\left(p(k_1)p(k_3)+p(k_2)p(k_4)\right)\right.$$

$$\left.-E\left(p(k_2)+p(k_4)\right)\left(1+p(k_1)p(k_3)\right)+E^2\left(1+p(k_1)p(k_2)p(k_3)p(k_4)\right)\right\} =$$

$$\frac{\sigma^2}{2N}\left((1-p(k_2)E)^2\delta_{k_1k_3}\delta_{k_2k_4}+(E^2-1)(1-\delta_{k_1k_3})(1-\delta_{k_2k_4})\right) \tag{5.110}$$

We also need to compute For k_1,k_2,k_3 and k_4 elliptic indexes :

$$C_o(k_1,k_2,k_3,k_4) \equiv \mathbb{E}\left((F^{-1}W_e^-V_oF)_{k_1k_2}(F^{-1}W_e^-V_oF)_{k_3k_4}\right) =$$

$$\frac{\sigma^2}{4N^2}\sum_{n_1,p_1}\sum_{n_2,p_2}(-p(k_1))^{\delta^e(n_1)}(-p(k_2))^{\delta^e(p_1)}(-p(k_3))^{\delta^e(n_2)}(-p(k_4))^{\delta^e(p_2)}(W_e^-)_{n_1p_1}(W_e^-)_{n_2p_2}\delta_{p_1p_2} =$$

$$\frac{\sigma^2}{4N^2}\sum_{n_1,n_2,p}(-p(k_1))^{\delta^e(n_1)}(-p(k_3))^{\delta^e(n_2)}(p(k_2)p(k_4))^{\delta^e(p)}(W_e^-)_{n_1p}(W_e^-)_{n_2p} \tag{5.111}$$

Now :

$$\frac{\sigma^2}{4N^2}\sum_{n_1,n_2,p}(-p(k_1))^{\delta^e(n_1)}(-p(k_3))^{\delta^e(n_2)}(p(k_2)p(k_4))^{\delta^e(p)}(W_e)_{n_1p}(W_e)_{n_2p}=$$

$$\frac{\sigma^2}{4N^2}\sum_{n_1,p}(p(k_1)p(k_3))^{\delta^e(n_1)}(p(k_2)p(k_4))^{\delta^e(p)}(-W_e^-)_{n_1p}=\frac{\sigma^2}{4N}\left(p(k_1)p(k_3)+p(k_2)p(k_4)\right)$$

Moreover,

$$-E\frac{\sigma^2}{4N^2}\sum_{n_1,n_2,p}(-p(k_1))^{\delta^e(n_1)}(-p(k_3))^{\delta^e(n_2)}(p(k_2)p(k_4))^{\delta^e(p)}(W_e)_{n_1p}\delta_{n_2p}=$$

$$-E\frac{\sigma^2}{4N^2}\sum_{n_1,p}(-p(k_1))^{\delta^e(n_1)}(-p(k_2)p(k_3)p(k_4))^{\delta^e(p)}(W_e)_{n_1p}=-E\frac{\sigma^2}{4N}\left(p(k_1)+p(k_2)p(k_3)p(k_4)\right) \tag{5.112}$$

and

$$-E\frac{\sigma^2}{4N^2}\sum_{n_1,n_2,p}(-p(k_1))^{\delta^e(n_1)}(-p(k_3))^{\delta^e(n_2)}(p(k_2)p(k_4))^{\delta^e(p)}\delta_{n_1p}(W_e)_{n_2p}=$$

$$-E\frac{\sigma^2}{4N}\left(p(k_3)+p(k_1)p(k_2)p(k_4)\right) \tag{5.113}$$

and

$$E^2\frac{\sigma^2}{4N^2}\sum_{n_1,n_2,p}(-p(k_1))^{\delta^e(n_1)}(-p(k_3))^{\delta^e(n_2)}(p(k_2)p(k_4))^{\delta^e(p)}\delta_{n_1p}\delta_{n_2p}=$$

$$\frac{\sigma^2}{4N}\left(1+p(k_1)p(k_2)p(k_3)p(k_4)\right) \tag{5.114}$$

It follows that

$$C_o(k_1,k_2,k_3,k_4)=\frac{\sigma^2}{2N}\left((1-p(k_1)E)^2\delta_{k_1k_3}\delta_{k_2k_4}+(E^2-1)(1-\delta_{k_1k_3})(1-\delta_{k_2k_4})\right)$$

Note that

$$C_e(k_1,k_2,k_3,k_4)=C_o(k_2,k_1,k_4,k_3) \tag{5.115}$$

It follows directly from the definition of P, that

$$\langle v_{k_1}^{\sigma_1},P_ev_{k_2}^{\sigma_2}\rangle=(P_e)_{k_1,k_2}^{\sigma_1,\sigma_2}(F^{-1}V_eW_o^-F)_{k_1k_2} \tag{5.116}$$

where

$$(P_e)_{k_1,k_2}^{\sigma_1,\sigma_2}=\frac{1}{\mathrm{e}^{i\sigma_1\hat{\beta}_{k_1}}-1}\frac{1}{\mathrm{e}^{-i\sigma_2\hat{\beta}_{k_2}}+1}=\frac{\mathrm{e}^{\frac{i}{2}(\sigma_2\hat{\beta}_{k_2}-\sigma_1\hat{\beta}_{k_1})}}{4i\sin\frac{\sigma_1\hat{\beta}_{k_1}}{2}\cos\frac{\sigma_2\hat{\beta}_{k_2}}{2}} \tag{5.117}$$

and

$$\langle v_{k_1}^{\sigma_1},P_ov_{k_2}^{\sigma_2}\rangle=(P_o)_{k_1,k_2}^{\sigma_1,\sigma_2}(F^{-1}W_e^-V_oF)_{k_1k_2} \tag{5.118}$$

where

$$(P_o)_{k_1,k_2}^{\sigma_1,\sigma_2}=\frac{1}{\mathrm{e}^{i\sigma_1\hat{\beta}_{k_1}}-\mathrm{e}^{-i\sigma_1\hat{\beta}_{k_1}}}=\frac{1}{2i\sin\sigma_1\hat{\beta}_{k_1}} \tag{5.119}$$

Moreover,

$$\langle v_{k_1}^{\sigma_1},P_e^\dagger v_{k_2}^{\sigma_2}\rangle=(P_e)_{k_2,k_1}^{-\sigma_2,-\sigma_1}(F^{-1}V_eW_o^-F)_{k_2k_1} \tag{5.120}$$

and

$$\langle v_{k_1}^{\sigma_1},P_o^\dagger v_{k_2}^{\sigma_2}\rangle=(P_o)_{k_2,k_1}^{-\sigma_2,-\sigma_1}(F^{-1}W_e^-V_oF)_{k_2k_1} \tag{5.121}$$

5.4.2 Computation of the elliptic term

Consider

$$\sum_{\sigma_1,\sigma_2=\pm}\sum_{k_1,k_2=\frac{N}{2}}^{\frac{N}{2}+1}\mathbb{E}\left(\langle\psi_i(n),P^\dagger\cdot\pi_{k_1}^{\sigma_1}(n+1)\psi_j(n)\rangle\langle\psi_j(n),P^\dagger\cdot\pi_{k_2}^{\sigma_2}(n+1)\psi_i(n)\rangle\right) \tag{5.122}$$

$\psi_i(n)$ and $\psi_j(n)$ are elliptic frame vectors, so that up to an error of order λ :

$$\mathbb{E}\left(\langle\psi_i(n),P^\dagger\cdot\pi_{k_1}^{\sigma_1}(n+1)\psi_j(n)\rangle\langle\psi_j(n),P^\dagger\cdot\pi_{k_2}^{\sigma_2}(n+1)\psi_i(n)\rangle\right)=$$

$$\sum_{\sigma_3,\sigma_4=\pm}\sum_{k_3,k_4=\frac{N}{2}}^{\frac{N}{2}+1}\mathbb{E}\left(\langle\psi_i(n),(\pi_{k_3}^{\sigma_3}P^\dagger)\cdot\pi_{k_1}^{\sigma_1}(n+1)\psi_j(n)\rangle\langle\psi_j(n),(\pi_{k_4}^{\sigma_4}P^\dagger)\cdot\pi_{k_2}^{\sigma_2}(n+1)\psi_i(n)\rangle\right). \tag{5.123}$$

thus

$$\sum_{\sigma_1,\sigma_2=\pm}\sum_{k_1,k_2=\frac{N}{2}}^{\frac{N}{2}+1}\mathbb{E}\left\langle\psi_i(n),P^\dagger\cdot\pi_{k_1}^{\sigma_1}(n+1)\psi_j(n)\rangle\langle\psi_j(n),P^\dagger\cdot\pi_{k_2}^{\sigma_2}(n+1)\psi_i(n)\rangle\right)=$$

$$\begin{aligned}\sum_{\{\sigma\}}\sum_{\{k\}}\quad&\Big\{\overline{\psi}_{i,k_3}^{\sigma_3}\psi_{j,k_1}^{\sigma_1}\overline{\psi}_{j,k_4}^{\sigma_4}\psi_{i,k_2}^{\sigma_2}\mathbb{E}(\langle v_{k_3}^{\sigma_3},P^\dagger v_{k_1}^{\sigma_1}\rangle\langle v_{k_4}^{\sigma_4},P^\dagger v_{k_2}^{\sigma_2}\rangle)+\\&\overline{\psi}_{i,k_1}^{\sigma_1}\psi_{j,k_3}^{\sigma_3}\overline{\psi}_{j,k_2}^{\sigma_2}\psi_{i,k_4}^{\sigma_4}\mathbb{E}(\langle v_{k_1}^{\sigma_1},Pv_{k_3}^{\sigma_3}\rangle\langle v_{k_2}^{\sigma_2},Pv_{k_4}^{\sigma_4}\rangle)+\\&\overline{\psi}_{i,k_3}^{\sigma_3}\psi_{j,k_1}^{\sigma_1}\overline{\psi}_{j,k_2}^{\sigma_2}\psi_{i,k_4}^{\sigma_4}\mathbb{E}(\langle v_{k_3}^{\sigma_3},P^\dagger v_{k_1}^{\sigma_1}\rangle\langle v_{k_2}^{\sigma_2},Pv_{k_4}^{\sigma_4}\rangle)+\\&\overline{\psi}_{i,k_1}^{\sigma_1}\psi_{j,k_3}^{\sigma_3}\overline{\psi}_{j,k_4}^{\sigma_4}\psi_{i,k_2}^{\sigma_2}\mathbb{E}(\langle v_{k_1}^{\sigma_1},Pv_{k_3}^{\sigma_3}\rangle\langle v_{k_4}^{\sigma_4},P^\dagger v_{k_2}^{\sigma_2}\rangle)\Big\}\end{aligned} \tag{5.124}$$

Let us introduce the shorthand notation :

$$\begin{aligned}C(k_1,k_2,k_3,k_4)\cdot(P_{k_1,k_2}^{\sigma_1,\sigma_2}P_{k_4,k_4}^{\sigma_3,\sigma_3})=C_e(k_1,k_2,k_3,k_4)(P_e)_{k_1,k_2}^{\sigma_1,\sigma_2}(P_e)_{k_4,k_4}^{\sigma_3,\sigma_3}+\\C_o(k_1,k_2,k_3,k_4)(P_o)_{k_1,k_2}^{\sigma_1,\sigma_2}(P_o)_{k_4,k_4}^{\sigma_3,\sigma_3}\end{aligned} \tag{5.125}$$

It follows that we need to compute:

$$\begin{aligned}\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\Big\{&\overline{\psi}_{i,k_3}^{\sigma_3}\psi_{j,k_1}^{\sigma_1}\overline{\psi}_{j,k_4}^{\sigma_4}\psi_{i,k_2}^{\sigma_2}C(k_1,k_3,k_2,k_4)\cdot P_{k_1,k_3}^{-\sigma_1,-\sigma_3}P_{k_2,k_4}^{-\sigma_2,-\sigma_4}+\\&\overline{\psi}_{i,k_1}^{\sigma_1}\psi_{j,k_3}^{\sigma_3}\overline{\psi}_{j,k_2}^{\sigma_2}\psi_{i,k_4}^{\sigma_4}C(k_1,k_3,k_2,k_4)\cdot P_{k_1,k_3}^{\sigma_1,\sigma_3}P_{k_2,k_4}^{\sigma_2,\sigma_4}+\\&\overline{\psi}_{i,k_3}^{\sigma_3}\psi_{j,k_1}^{\sigma_1}\overline{\psi}_{j,k_2}^{\sigma_2}\psi_{i,k_4}^{\sigma_4}C(k_1,k_3,k_2,k_4)\cdot P_{k_1,k_3}^{-\sigma_1,-\sigma_3}P_{k_2,k_4}^{\sigma_2,\sigma_4}+\\&\overline{\psi}_{i,k_1}^{\sigma_1}\psi_{j,k_3}^{\sigma_3}\overline{\psi}_{j,k_4}^{\sigma_4}\psi_{i,k_2}^{\sigma_2}C(k_1,k_3,k_2,k_4)\cdot P_{k_1,k_3}^{\sigma_1,\sigma_3}P_{k_2,k_4}^{-\sigma_2,-\sigma_4}\Big\}\end{aligned} \tag{5.126}$$

Let us note that for elliptic indexes $\{k_1,k_2,k_3,k_4\}$ one has :

$$(E^2-1)(1-\delta_{k_1k_2})(1-\delta_{k_3k_4})=-4\cos(\frac{\hat{\beta}_{k_1}}{2})\cos(\frac{\hat{\beta}_{k_2}}{2})(1-\delta_{k_1k_2})\left(\delta_{k_1k_3}\delta_{k_2k_4}+\delta_{k_1k_4}\delta_{k_2k_3}\right)$$

$$= -4\cos(\frac{\hat{\beta}_{k_3}}{2})\cos(\frac{\hat{\beta}_{k_4}}{2})(1-\delta_{k_1k_2})\left(\delta_{k_1k_3}\delta_{k_2k_4}+\delta_{k_1k_4}\delta_{k_2k_3}\right) \tag{5.127}$$

It follows that :

$$C_e(k_1,k_3,k_2,k_4)(P_e)^{\sigma_1,\sigma_3}_{k_1,k_3}(P_e)^{\sigma_2,\sigma_4}_{k_2,k_4} = -\left(\frac{\sigma_1\sigma_2 \mathrm{e}^{\frac{i}{2}((\sigma_3+\sigma_4)\hat{\beta}_{k_3}-(\sigma_1+\sigma_2)\hat{\beta}_{k_1})}}{4\sin^2\frac{\hat{\beta}_{k_1}}{2}}\right)\delta_{k_1k_2}\delta_{k_3k_4}$$

$$+\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{4\sin(\frac{\hat{\beta}_{k_1}}{2})\sin(\frac{\hat{\beta}_{k_2}}{2})}\left(\mathrm{e}^{\frac{i}{2}((\sigma_3-\sigma_1)\hat{\beta}_{k_1}+(\sigma_4-\sigma_2)\hat{\beta}_{k_2})}\delta_{k_1k_3}\delta_{k_2k_4}+\mathrm{e}^{\frac{i}{2}((\sigma_4-\sigma_1)\hat{\beta}_{k_1}+(\sigma_3-\sigma_2)\hat{\beta}_{k_2})}\delta_{k_1k_4}\delta_{k_2k_3}\right) \tag{5.128}$$

similarly :

$$C_o(k_1,k_3,k_2,k_4)(P_o)^{\sigma_1,\sigma_3}_{k_1,k_3}(P_o)^{\sigma_2,\sigma_4}_{k_2,k_4} = -\left(\frac{\sigma_1\sigma_2}{4\sin^2\frac{\hat{\beta}_{k_1}}{2}}\right)\delta_{k_1k_2}\delta_{k_3k_4}$$

$$+\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{4\sin(\frac{\hat{\beta}_{k_1}}{2})\sin(\frac{\hat{\beta}_{k_2}}{2})}\left(\delta_{k_1k_3}\delta_{k_2k_4}+\delta_{k_1k_4}\delta_{k_2k_3}\right)$$

As we will now see, for each term appearing in the sum (5.126), an oscillatory sum argument will allow us to discard all the terms for which the phase factors appearing in (5.128) are not one, so that the odd and the even part give the same contribution. To do this we will consider the first two terms and the last two terms in (5.126) separately. Let us start with the first half :

$$\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\left\{\overline{\psi}^{\sigma_3}_{i,k_3}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_4}_{j,k_4}\psi^{\sigma_2}_{i,k_2}C_e(k_1,k_3,k_2,k_4)(P_e)^{-\sigma_1,-\sigma_3}_{k_1,k_3}(P_e)^{-\sigma_2,-\sigma_4}_{k_2,k_4}+\right.$$

$$\left.\overline{\psi}^{\sigma_1}_{i,k_1}\psi^{\sigma_3}_{j,k_3}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{\sigma_4}_{i,k_4}C_e(k_1,k_3,k_2,k_4)(P_e)^{\sigma_1,\sigma_3}_{k_1,k_3}(P_e)^{\sigma_2,\sigma_4}_{k_2,k_4}\right\} \tag{5.129}$$

Each of the summands in the latter equation gives rise to three terms : The first one is preceded by a factor $\delta_{k_1k_2}\delta_{k_3k_4}$ the second by a factor $(1-\delta_{k_1k_2})\delta_{k_1k_3}\delta_{k_2k_4}$ and the third by $(1-\delta_{k_1k_2})\delta_{k_1k_4}\delta_{k_2k_3}$. The first contribution reads :

$$-\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\left(\frac{\sigma_1\sigma_2}{4\sin^2\frac{\hat{\beta}_{k_1}}{2}}\right)\delta_{k_1k_2}\delta_{k_3k_4}\left\{\overline{\psi}^{\sigma_3}_{i,k_3}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_4}_{j,k_3}\psi^{\sigma_2}_{i,k_1}\mathrm{e}^{\frac{i}{2}((\sigma_3+\sigma_4)\hat{\beta}_{k_3}-(\sigma_1+\sigma_2)\hat{\beta}_{k_1})}+\right.$$

$$\left.\mathrm{e}^{-\frac{i}{2}((\sigma_3+\sigma_4)\hat{\beta}_{k_3}-(\sigma_1+\sigma_2)\hat{\beta}_{k_1})}\overline{\psi}^{\sigma_1}_{i,k_1}\psi^{\sigma_3}_{j,k_3}\overline{\psi}^{\sigma_2}_{j,k_1}\psi^{\sigma_4}_{i,k_3}\right\} \tag{5.130}$$

An oscillatory sum argument now implies that only the terms for which the phase factors are equal to one survive, i.e. only the terms with $\sigma_1=-\sigma_2$ and $\sigma_3=-\sigma_4$ or with $k_3=k_1$ and $\sigma_1=\sigma_2=\sigma_3=\sigma_4$. The contribution coming from (5.130) thus reads :

$$\lim_{L\to\infty}\quad\frac{1}{L}\sum_{n=0}^{L-1}\left\{\sum_{\{\sigma\}}\sum_{\{k\}}\left(\frac{1}{4\sin^2\frac{\hat{\beta}_{k_1}}{2}}+\frac{1}{4\sin^2\frac{\hat{\beta}_{k_3}}{2}}\right)\overline{\psi}^{\sigma_3}_{i,k_3}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{-\sigma_3}_{j,k_3}\psi^{-\sigma_1}_{i,k_1}\right.$$

$$\left.-\sum_{\sigma}\sum_{k}\left(\frac{1}{2\sin^2\frac{\hat{\beta}_{k}}{2}}\right)\overline{\psi}^{\sigma}_{i,k}\psi^{\sigma}_{j,k}\overline{\psi}^{\sigma}_{j,k}\psi^{\sigma}_{i,k}\right\} \tag{5.131}$$

The second contribution to (5.129) reads :

$$\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{4\sin(\frac{\hat\beta_{k_1}}{2})\sin(\frac{\hat\beta_{k_2}}{2})}\delta_{k_1k_3}\delta_{k_2k_4}\Big\{\overline{\psi}^{\sigma_3}_{i,k_1}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_4}_{j,k_2}\psi^{\sigma_2}_{i,k_2}\mathrm{e}^{\frac{i}{2}((\sigma_3-\sigma_1)\hat\beta_{k_1}+(\sigma_4-\sigma_2)\hat\beta_{k_2})}+$$

$$\overline{\psi}^{\sigma_1}_{i,k_1}\psi^{\sigma_3}_{j,k_1}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{\sigma_4}_{i,k_2}\mathrm{e}^{-\frac{i}{2}((\sigma_3-\sigma_1)\hat\beta_{k_1}+(\sigma_4-\sigma_2)\hat\beta_{k_2})}\Big\} \tag{5.132}$$

Again, an oscillatory sum argument implies that only those terms with unit phase factor survive, i.e. only the terms with $\sigma_1=\sigma_3$ and $\sigma_2=\sigma_4$. The contribution coming from (5.132) thus reads :

$$\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{2\sin(\frac{\hat\beta_{k_1}}{2})\sin(\frac{\hat\beta_{k_2}}{2})}\overline{\psi}^{\sigma_1}_{i,k_1}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{\sigma_2}_{i,k_2} \tag{5.133}$$

Finally, the last contribution to (5.129) reads :

$$\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{4\sin(\frac{\hat\beta_{k_1}}{2})\sin(\frac{\hat\beta_{k_2}}{2})}\delta_{k_1k_4}\delta_{k_2k_3}\Big\{\overline{\psi}^{\sigma_3}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_4}_{j,k_1}\psi^{\sigma_2}_{i,k_2}\mathrm{e}^{\frac{i}{2}((\sigma_4-\sigma_1)\hat\beta_{k_1}+(\sigma_3-\sigma_2)\hat\beta_{k_2})}+$$

$$\overline{\psi}^{\sigma_1}_{i,k_1}\psi^{\sigma_3}_{j,k_2}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{\sigma_4}_{i,k_1}\mathrm{e}^{-\frac{i}{2}((\sigma_4-\sigma_1)\hat\beta_{k_1}+(\sigma_3-\sigma_2)\hat\beta_{k_2})}\Big\} \tag{5.134}$$

Again, an oscillatory sum argument allows us to keep only the terms with $\sigma_4=\sigma_1$ and $\sigma_2=\sigma_3$. The contribution coming from (5.134) thus reads :

$$\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{2\sin(\frac{\hat\beta_{k_1}}{2})\sin(\frac{\hat\beta_{k_2}}{2})}\overline{\psi}^{\sigma_2}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_1}_{j,k_1}\psi^{\sigma_2}_{i,k_2} \tag{5.135}$$

Hence, adding (5.131), (5.133) and (5.135) yields the total contribution

$$\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\Bigg\{\sum_{\{\sigma\}}\sum_{\{k\}}\left(\frac{1}{4\sin^2\frac{\hat\beta_{k_1}}{2}}+\frac{1}{4\sin^2\frac{\hat\beta_{k_2}}{2}}\right)\overline{\psi}^{\sigma_2}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{-\sigma_2}_{j,k_2}\psi^{-\sigma_1}_{i,k_1}$$

$$-\sum_{\sigma}\sum_{k}\left(\frac{1}{2\sin^2\frac{\hat\beta_k}{2}}\right)\overline{\psi}^{\sigma}_{i,k}\psi^{\sigma}_{j,k}\overline{\psi}^{\sigma}_{j,k}\psi^{\sigma}_{i,k}$$

$$+\sum_{\{\sigma\}}\sum_{\{k\}}\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{2\sin(\frac{\hat\beta_{k_1}}{2})\sin(\frac{\hat\beta_{k_2}}{2})}\Big\{\overline{\psi}^{\sigma_1}_{i,k_1}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{\sigma_2}_{i,k_2}+\overline{\psi}^{\sigma_2}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_1}_{j,k_1}\psi^{\sigma_2}_{i,k_2}\Big\}\Bigg\} \tag{5.136}$$

for the contribution coming from (5.129).

We now turn to the second half of (5.126) :

$$\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\Big\{\overline{\psi}^{\sigma_3}_{i,k_3}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{\sigma_4}_{i,k_4}C_e(k_1,k_3,k_2,k_4)(P_e)^{-\sigma_1,-\sigma_3}_{k_1,k_3}(P_e)^{\sigma_2,\sigma_4}_{k_2,k_4}+$$

$$\overline{\psi}^{\sigma_1}_{i,k_1}\psi^{\sigma_3}_{j,k_3}\overline{\psi}^{\sigma_4}_{j,k_4}\psi^{\sigma_2}_{i,k_2}C_e(k_1,k_3,k_2,k_4)(P_e)^{\sigma_1,\sigma_3}_{k_1,k_3}(P_e)^{-\sigma_2,-\sigma_4}_{k_2,k_4}\Big\} \tag{5.137}$$

Again, each term in the latter sum gives rise to three terms with prefactors $\delta_{k_1k_2}\delta_{k_3k_4}$, $(1-\delta_{k_1k_2})\delta_{k_1k_3}\delta_{k_2k_4}$ and $(1-\delta_{k_1k_2})\delta_{k_1k_4}\delta_{k_2k_3}$ respectively. The first contribution reads :

$$\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\left(\frac{\sigma_1\sigma_2}{4\sin^2\frac{\hat\beta_{k_1}}{2}}\right)\delta_{k_1k_2}\delta_{k_3k_4}\left\{\overline{\psi}^{\sigma_3}_{i,k_3}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_2}_{j,k_1}\psi^{\sigma_4}_{i,k_3}\mathrm{e}^{\frac{i}{2}((\sigma_4-\sigma_3)\hat\beta_{k_3}-(\sigma_2-\sigma_1)\hat\beta_{k_1})}+\right.$$
$$\left.\mathrm{e}^{-\frac{i}{2}((\sigma_4-\sigma_3)\hat\beta_{k_3}-(\sigma_2-\sigma_1)\hat\beta_{k_1})}\overline{\psi}^{\sigma_1}_{i,k_1}\psi^{\sigma_3}_{j,k_3}\overline{\psi}^{\sigma_4}_{j,k_3}\psi^{\sigma_2}_{i,k_1}\right\} \tag{5.138}$$

An oscillatory sum argument shows that only the terms with either $\sigma_3=\sigma_4$ and $\sigma_2=\sigma_1$ or the terms with $k_1=k_3$ and $\sigma_2=\sigma_4=-\sigma_3=-\sigma_1$ survive. One obtains

$$\lim_{L\to\infty}\quad\frac{1}{L}\sum_{n=0}^{L-1}\left\{\sum_{\{\sigma\}}\sum_{\{k\}}\left(\frac{1}{4\sin^2\frac{\hat\beta_{k_1}}{2}}+\frac{1}{4\sin^2\frac{\hat\beta_{k_3}}{2}}\right)\overline{\psi}^{\sigma_3}_{i,k_3}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_1}_{j,k_1}\psi^{\sigma_3}_{i,k_3}\right.$$
$$\left.-\sum_{\sigma}\sum_{k}\left(\frac{1}{2\sin^2\frac{\hat\beta_k}{2}}\right)\overline{\psi}^{\sigma}_{i,k}\psi^{\sigma}_{j,k}\overline{\psi}^{-\sigma}_{j,k}\psi^{-\sigma}_{i,k}\right\}. \tag{5.139}$$

The second contribution to (5.137) reads :

$$-\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{4\sin(\frac{\hat\beta_{k_1}}{2})\sin(\frac{\hat\beta_{k_2}}{2})}\delta_{k_1k_3}\delta_{k_2k_4}\left\{\overline{\psi}^{\sigma_3}_{i,k_1}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{\sigma_4}_{i,k_2}\mathrm{e}^{\frac{i}{2}((\sigma_1-\sigma_3)\hat\beta_{k_1}+(\sigma_4-\sigma_2)\hat\beta_{k_2})}+\right.$$
$$\left.\overline{\psi}^{\sigma_1}_{i,k_1}\psi^{\sigma_3}_{j,k_1}\overline{\psi}^{\sigma_4}_{j,k_2}\psi^{\sigma_2}_{i,k_2}\mathrm{e}^{-\frac{i}{2}((\sigma_1-\sigma_3)\hat\beta_{k_1}+(\sigma_4-\sigma_2)\hat\beta_{k_2})}\right\} \tag{5.140}$$

and an oscillatory sum argument shows that only the terms with $\sigma_1=\sigma_3$ and $\sigma_2=\sigma_4$ survive. One thus obtains the expression :

$$-\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{2\sin(\frac{\hat\beta_{k_1}}{2})\sin(\frac{\hat\beta_{k_2}}{2})}\overline{\psi}^{\sigma_1}_{i,k_1}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{\sigma_2}_{i,k_2} \tag{5.141}$$

for (5.140). The last contribution to (5.137) reads :

$$-\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{4\sin(\frac{\hat\beta_{k_1}}{2})\sin(\frac{\hat\beta_{k_2}}{2})}\delta_{k_1k_4}\delta_{k_2k_3}\left\{\overline{\psi}^{\sigma_3}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{\sigma_4}_{i,k_1}\mathrm{e}^{\frac{i}{2}((\sigma_4+\sigma_1)\hat\beta_{k_1}-(\sigma_3+\sigma_2)\hat\beta_{k_2})}+\right.$$
$$\left.\overline{\psi}^{\sigma_1}_{i,k_1}\psi^{\sigma_3}_{j,k_2}\overline{\psi}^{\sigma_4}_{j,k_1}\psi^{\sigma_2}_{i,k_2}\mathrm{e}^{-\frac{i}{2}((\sigma_4+\sigma_1)\hat\beta_{k_1}-(\sigma_3+\sigma_2)\hat\beta_{k_2})}\right\} \tag{5.142}$$

Again, an oscillatory sum argument allows to keep only the terms with $\sigma_1=-\sigma_4$ and $\sigma_3=-\sigma_2$. (5.142) thus reads :

$$-\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\sum_{\{\sigma\}}\sum_{\{k\}}\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{2\sin(\frac{\hat\beta_{k_1}}{2})\sin(\frac{\hat\beta_{k_2}}{2})}\overline{\psi}^{-\sigma_2}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{-\sigma_1}_{i,k_1} \tag{5.143}$$

Hence, adding (5.139), (5.141) and (5.143), we obtain

$$\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\Bigg\{\sum_{\{\sigma\}}\sum_{\{k\}}\left(\frac{1}{4\sin^2\frac{\hat\beta_{k_1}}{2}}+\frac{1}{4\sin^2\frac{\hat\beta_{k_2}}{2}}\right)\overline{\psi}^{\sigma_2}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_1}_{j,k_1}\psi^{\sigma_2}_{i,k_2}$$
$$-\sum_{\sigma}\sum_{k}\left(\frac{1}{2\sin^2\frac{\hat\beta_k}{2}}\right)\overline{\psi}^{\sigma}_{i,k}\psi^{\sigma}_{j,k}\overline{\psi}^{-\sigma}_{j,k}\psi^{-\sigma}_{i,k}$$
$$-\sum_{\{\sigma\}}\sum_{\{k\}}\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{2\sin(\frac{\hat\beta_{k_1}}{2})\sin(\frac{\hat\beta_{k_2}}{2})}\left(\overline{\psi}^{\sigma_1}_{i,k_1}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{\sigma_2}_{i,k_2}+\overline{\psi}^{-\sigma_2}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{-\sigma_1}_{i,k_1}\right)\Bigg\} \quad (5.144)$$

The total contribution from the elliptic part is obtained by addition of (5.136) and (5.144). As pointed out before, the odd part gives the same contribution, so that finally the elliptic term reads :

$$\frac{\sigma^2}{N}\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\Bigg\{\sum_{\{\sigma\}}\sum_{\{k\}}\left(\frac{1}{4\sin^2\frac{\hat\beta_{k_1}}{2}}+\frac{1}{4\sin^2\frac{\hat\beta_{k_2}}{2}}\right)\left\{\overline{\psi}^{\sigma_2}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{-\sigma_2}_{j,k_2}\psi^{-\sigma_1}_{i,k_1}+\overline{\psi}^{\sigma_2}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_1}_{j,k_1}\psi^{\sigma_2}_{i,k_2}\right\}$$
$$-\sum_{\sigma}\sum_{k}\left(\frac{1}{2\sin^2\frac{\hat\beta_k}{2}}\right)\left\{\overline{\psi}^{\sigma}_{i,k}\psi^{\sigma}_{j,k}\overline{\psi}^{\sigma}_{j,k}\psi^{\sigma}_{i,k}+\overline{\psi}^{\sigma}_{i,k}\psi^{\sigma}_{j,k}\overline{\psi}^{-\sigma}_{j,k}\psi^{-\sigma}_{i,k}\right\}+$$
$$\sum_{\{\sigma\}}\sum_{\{k\}}\frac{\sigma_1\sigma_2(1-\delta_{k_1k_2})}{2\sin(\frac{\hat\beta_{k_1}}{2})\sin(\frac{\hat\beta_{k_2}}{2})}\left\{\overline{\psi}^{\sigma_2}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_1}_{j,k_1}\psi^{\sigma_2}_{i,k_2}-\overline{\psi}^{-\sigma_2}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_2}_{j,k_2}\psi^{-\sigma_1}_{i,k_1}\right\}\Bigg\}$$
$$=\frac{\sigma^2}{N}\lim_{L\to\infty}\frac{1}{L}\sum_{n=0}^{L-1}\Bigg\{\sum_{\sigma}\sum_{k}\frac{1}{\sin^2\frac{\hat\beta_k}{2}}\overline{\psi}^{-\sigma}_{i,k}\psi^{\sigma}_{j,k}\overline{\psi}^{\sigma}_{j,k}\psi^{-\sigma}_{i,k}+$$
$$\sum_{\{\sigma\}}\sum_{\{k\}}(1-\delta_{k_1k_2})\left(\frac{1}{2\sin\frac{\sigma_1\hat\beta_{k_1}}{2}}+\frac{1}{2\sin\frac{\sigma_2\hat\beta_{k_2}}{2}}\right)^2\left\{\overline{\psi}^{\sigma_2}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{-\sigma_2}_{j,k_2}\psi^{-\sigma_1}_{i,k_1}+\overline{\psi}^{\sigma_2}_{i,k_2}\psi^{\sigma_1}_{j,k_1}\overline{\psi}^{\sigma_1}_{j,k_1}\psi^{\sigma_2}_{i,k_2}\right\}\Bigg\} \quad (5.145)$$

Now :

$$\pi^{\sigma}_k\psi_i(n)\equiv\psi^{\sigma}_{i,k}(n)v^{\sigma}_k=\sqrt{\rho^{\sigma}_{i,k}(n)}\mathrm{e}^{i\theta^{\sigma}_{i,k}(n)}v^{\sigma}_k \quad (5.146)$$

It follows that the elliptic term reads :

$$=\frac{\sigma^2}{N}\Bigg\{\sum_{\sigma}\sum_{k}\frac{1}{\sin^2\frac{\hat\beta_k}{2}}\left\langle\rho^{\sigma}_{i,k}\rho^{-\sigma}_{j,k}\right\rangle+$$
$$\sum_{\{\sigma\}}\sum_{\{k\}}(1-\delta_{k_1k_2})\left(\frac{1}{2\sin\frac{\sigma_1\hat\beta_{k_1}}{2}}+\frac{1}{2\sin\frac{\sigma_2\hat\beta_{k_2}}{2}}\right)^2\Bigg\{\left\langle\rho^{\sigma_1}_{j,k_1}\rho^{\sigma_2}_{i,k_2}\right\rangle+$$
$$\left\langle\sqrt{\rho^{\sigma_2}_{i,k_2}\rho^{\sigma_1}_{j,k_1}\rho^{-\sigma_2}_{j,k_2}\rho^{-\sigma_1}_{i,k_1}}\mathrm{e}^{i\{(\theta^{\sigma_1}_{j,k_1}+\theta^{-\sigma_1}_{i,k_1})-(\theta^{-\sigma_2}_{j,k_2}+\theta^{\sigma_2}_{i,k_2})\}}\right\rangle\Bigg\}\Bigg\} \quad (5.147)$$

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