Large deviations provide good approximation to queueing system with dynamic routing

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Abstract

We consider a system with two infinite-buffer FCFS servers (of speed one). The arrivals processes are three independent Poisson flows Ξ_i, of rates λ_i, $i = 0, 1, 2$, each with IID task service times. The tasks from Ξ_i are directed to server i, $i = 1, 2$ (dedicated traffic). The tasks from Ξ_0 are directed to the server that has the shorter workload in the buffer at the time of arrival (opportunistic traffic). We compare the analytical data for the large deviation (LD) probabilities for the virtual waiting time in flow Ξ_0 and empercial delay freqencies from simulations.

1 Introduction

We consider queueing systems with dynamic routing. In particular large deviation (LD) probabilities of long delay in the stationary regime. Various problems of heavy load in systems with dynamic routing have already been investigated. For example, see [12, 13, 14, 8] and references therein. In particular, in [12, 13] a several-server system (with different speeds) and a single discretionary flow, where all tasks are directed to the least busy server, was studied (a $GI/GI/s/\infty$ load-balanced queue). In [14, 8] a system with two servers was considered, with different speeds and three Poisson flows, two dedicated and one discretionary

Here the delay probabilities are analyzed in a system with non-exponential service times. We consider a model with two servers and three Poisson flows, each with a general distribution of service times (a $M/GI/2/\infty$ load-balanced queue). We only discuss systems with equal server speed and do not consider LD probabilities for queue lengths.

We present a novel method to derive LD probabilities: we introduce an auxiliary system where the discretionary flow is partitioned between servers in some proportion, regardless of workloads. We then determine a 'genuine' proportion that occurs in the original system in the stationary LD regime.

We believe this procedure can be followed for more complicated queueing networks, where other approaches would prove too intricate. Even if the auxiliary flow approach is not rigorously justifiable for these more complicated networks, belief in its predictions can be gained through simulations such as those described in this paper (where the LD limits can be rigorously justified for Poisson processes).

The problem of the accuracy of probability estimates obtained by LD theory was discussed by R.L. Dobrushin in one of his first works on large deviations in queuing systems [3]. He wrote that in many real systems the system designer sets a significance level, p; he wishes that the waiting times ω experienced in the system exceed a given bound with probability smaller than p. Thus it is of interest to study the function $T(p)$ defined by

$$\mathbf{P}(\omega > T(p)) = p.$$

If the rate of decrease in the tail of the probability distribution describing the input flow is exponential or faster, it is naturally to expect

$$\lim_{p \to 0} \frac{T(p)}{\ln(1/p)} = \alpha,$$

for some constant α. Thus $T(p) \approx \alpha \ln(1/p)$ for small p, and it is of interest to find the constant α.

Is this approximation sufficient for applications? In the majority of cases only rough estimates for the desired p are specified. It is usual to use a scale of the type

$$p = 10^{-6}, 10^{-7}, 10^{-8}, \dots$$

Thus for LD approximations to be useful, they must be accurate with respect to this scale. This work presents examples where the LP approximation is useful.

We compare our LD predictions with simulation data. The comparison shows that for our system the LD probabilities provide good approximation, even for relatively small delays.

In Section 2 we describe the system under investigation, its auxiliary system and the main LD theorem is presented. In Section 3 we present simulation results for Poisson flows for the initial system and compare them with the LD predictions.

2 The initial and auxiliary systems

2.1 The initial model

We focus on a system with two infinite-buffer FCFS servers of speed one. The arrivals to this system are formed by three independent Poisson flows Ξ_0, Ξ_1 and Ξ_2, of rates λ_i, $i = 0, 1, 2$, each with IID service times. Flows Ξ_1 and Ξ_2 are dedicated: flow Ξ_1 is directed to server 1; flow Ξ_2 to server 2. Flow Ξ_0 is discretionary: its tasks join the queue with the smaller workload. We denote by $S^{(k)}$ the random service time in flow Ξ_k and by φ_k the Laplace transform of $S^{(k)}$,

$$\varphi_k(\theta) = \mathbf{E}e^{\theta S^{(k)}}. \tag{1}$$

We assume that the functions φ_k, $k = 0, 1, 2$, are defined on positive intervals and on these intervals take all values in $[1, \infty)$. Formally,

$$\Xi_0 = (\tau_j^{(0)}, \xi_j^{(0)})_{j=-\infty}^{\infty}, \Xi_1 = (\tau_j^{(1)}, \xi_j^{(1)})_{j=-\infty}^{\infty}, \Xi_2 = (\tau_j^{(2)}, \xi_j^{(2)})_{j=-\infty}^{\infty}, \tag{2}$$

where for the j^{th} message of the k^{th} flow, τ_j^k is its arrival time and ξ_j^k is its service time.

We make the following assumptions:

1. For $k = 0, 1, 2$, $\{\xi_j^{(k)}\}$ forms an i.i.d. sequence taking values in $\mathbf{R}_+$. There exists $\theta_+^{(k)} > 0,\ k = 0, 1, 2 > 0$, such that

$$\varphi_k(\theta) = \mathbf{E}\exp\{\theta\xi^{(k)}\} < \infty \quad \text{for} \quad \theta < \theta_+^{(k)} \tag{3}$$

   and $\lim\limits_{\theta \uparrow \theta_+^{(k)}} \varphi_k(\theta) = \infty$.

2. The sequences $\{\tau_j^{(k)}\}$ and $\{\xi_j^{(k)}\}$ are independent.

3. Each sequence of random variables $\{\tau_j^{(k)}\}$ forms a stationary Poisson point process on $\mathbf{R}$, where $\tau_j^{(k)} < \tau_{j+1}^{(k)}$ and $\tau_{-1}^{(k)} < 0 \le t_0$. The rate of the kth process is denoted by λ_k.

4. There exists a unique stationary regime; this is guaranteed by the stability condition

$$\lambda_i\varphi_i'(0) < 1, i = 1, 2, \sum_{i=0,1,2} \lambda_i\varphi_i'(0) < 2. \tag{4}$$

We consider a Markov process $(w^1(t), w^2(t))$ on $\mathbf{R}_+^{\mathbf{2}}$ defined by the generator

$$
\begin{aligned}
Lf(w^1, w^2) = & \\
& -\frac{\partial f}{\partial w^1}\mathbf{I}_{(w^1>0)} - \frac{\partial f}{\partial w^2}\mathbf{I}_{(w^2>0)} + \\
& \lambda_1 \left[\mathbf{E}f(w^1+\xi^1, w^2) - f(w^1, w^2)\right] + \\
& \lambda_2 \left[\mathbf{E}f(w^1, w^2+\xi^2) - f(w^1, w^2)\right] + \\
& \lambda_0 \left[\mathbf{E}f(w^1+\xi^0, w^2) - f(w^1, w^2)\right] \mathbf{I}_{(w^1<w^2)}(w_1, w_2) + \\
& \lambda_0 \left[\mathbf{E}f(w^1, w^2+\xi^0) - f(w^1, w^2)\right] \mathbf{I}_{(w^1>w^2)}(w_1, w_2).
\end{aligned} \tag{5}
$$

The vector $(w^1(t), w^2(t))$ describes the servers' workload at time t. On the other side $(w^1(t), w^2(t))$ is the vector of the virtual waiting times at time t.

We analyze the LD probabilities for the delay of a virtual task (of zero length) embedded into flow Ξ_0 at a fixed time. Let $w^1(t), w^2(t)$ be the workloads at the first and second servers at time t, say $t = 0$, and let

$$w^0(t) = \min\{w^1(t), w^2(t)\}, \tag{6}$$

the minimum workload. We consider the large deviation problem of existence and identification of the limit

$$I_0(d) = \lim_{n\to\infty} \frac{-1}{n} \log\ \mathbf{P}(w^0 \geq nd), \tag{7}$$

for $d > 0$.

In our problem the input flows to each server are neither independent nor Poisson, even though the source flows are. The process (w^1, w^2) takes values in the quarter-plane $\mathbf{R}_+^2 = \{(w^1, w^2) :\ w^1, w^2 \geq 0\}$ and has transition probabilities that are homogeneous in each half of the quarter-plane: $\{(w^1, w^2) :\ 0 < w^1 < w^2\}$ and $\{(w^1, w^2) :\ 0 < w^2 < w^1\}$. However, on the diagonal $\{(w^1, w^2) :\ 0 < w^1 = w^2\}$ the transition probabilities are discontinuous. The large deviation principal in the case of a two-dimensional Markov process with discontinuous transition probabilities is considered in [5, 6], [7], [2], [1] (thus we can call $I_0(d)$ the rate function of the initial problem). We do not follow these works. We introduce a new, different strategy to estimate the probability of large delay.

2.2 The auxiliary system

To use a general large deviation theory adapted to the situation where the transition probability are continuous we consider the auxiliary systems where there are two servers and two independent Poisson flows to these servers. More precisely, we introduce an auxiliary model where the discretionary flow Ξ_0 is divided into two independent Poisson sub-flows each directed to a particular server with no per-message routing. Using the large deviation principal for the Poisson processes (see [4]) of the auxiliary system , we find the optimal LD trajectories for achieving large workloads in both servers and determine a suitable proportion that divides flow Ξ^0 in a way that the probability of large workloads in the initial and auxiliary systems coincide. The idea of the auxiliary system was proposed in [9, 10]

We first describe the auxiliary system informally. Consider a queueing system with two servers and two Poisson flows Ξ_1^A and Ξ_2^A. The flows are directed to the first and the second server respectively. The flows' rates are

$$\lambda_1^A = \lambda_1 + \alpha\lambda_0, \quad \text{and} \quad \lambda_2^A = \lambda_2 + (1-\alpha)\lambda_0, \ \ 0 \le \alpha \le 1.$$

And the Laplace transforms of the service times are

$$\begin{aligned} \varphi_1^A(\theta) &= \frac{1}{\lambda_1 + \alpha\lambda_0}(\lambda_1\varphi_1(\theta) + \alpha\lambda_0\varphi_0(\theta)), \\ \varphi_2^A(\theta) &= \frac{1}{\lambda_2 + (1-\alpha)\lambda_0}(\lambda_2\varphi_2(\theta) + (1-\alpha)\lambda_0\varphi_0(\theta)). \end{aligned}$$

Where all φ_i and $\lambda_i, \ i = 0, 1, 2,$ are the same as in the initial problem. Later we shall determine the value of α that makes our initial and auxiliary systems equivalent with respect to LD probabilities.

In the auxiliary system arrivals of the discretionary flow Ξ_0 are directed to server 1 with probability α and to server 2 with probability $1-\alpha$, independently. It is convenient to represent flow Ξ_0 as a superposition of independent Poisson flows $\Xi_1^A = \{(\tau_{Aj}^1, \xi_{Aj}^1)\}_{j=-\infty}^{\infty}$ and $\Xi_2^A = \{(\tau_{Aj}^2, \xi_{Aj}^2)\}_{j=-\infty}^{\infty}$ of rates $\alpha\lambda_0$ and $(1-\alpha)\lambda_0$ with the same service-time distribution as in original flow Ξ^0.

Consider the workloads w_1^A and w_2^A at two servers of the auxiliary system:

$$w_i^A = \sup_{t\ge 0}\{a_i^A(t) - t\}, \ i = 1, 2, \tag{8}$$

where

$$a_i^A(t) = \begin{cases} \sum\limits_{j:\, 0<\tau_j^i\le t} \xi_j^{(i)} + \sum\limits_{j:\, 0<\tau_{Aj}^{(i)}\le t} \xi_{Aj}^{(i)}, & \text{if } t > 0, \\ -\sum\limits_{j:\, \tau<t_j^{(i)}\le 0} \xi_j^{(i)} - \sum\limits_{j:\, \tau<t_{Aj}^{(i)}\le 0} \xi_{Aj}^{(i)}, & \text{if } t \le 0. \end{cases} \tag{9}$$

We study the following event

$$\mathcal{A} = \left(\min_{i=1,2}\{w_i^A\} > nd\right), \tag{10}$$

and determine the function

$$I_0^A(d) = \lim_{n\to\infty} \frac{-1}{n} \log \ \mathbf{P}(\min_{i=1,2}\{w_i^A\} > nd), \ d > 0. \tag{11}$$

To use LD principle of [4] we re-scale our processes to take the LD limit. For time t and space x the rescaling is

$$t \to t/n = t^{(n)}, \quad \text{and } x \to x/n = x^{(n)}$$

thus the arrival processes are rescaled $\Xi_i \to \Xi_i^{(n)}$ in the following way:

$$\tau_j^{(n)} = \tau_j/n, \quad \xi_j^{(n)} = \xi_j/n, \quad -\infty < j < \infty,$$

$$w_i^{(n)}(t^{(n)}) = w_i(t)/n, \quad i = 0, 1, 2.$$

In the new notation, we have

$$\mathcal{A} = \left(\min_{i=1,2} \{w_1^{(n)}, w_2^{(n)}\} > d \right).$$

The rate function is defined as follows. Let the trajectory $\mathbf{x}(t) = (x_1(t), x_2(t)$ be absolutely continuous on $[0, \infty]$ then the value $I(\mathbf{x})$ of the rate function on $\mathbf{x}$ is

$$I(\mathbf{x}) = \int_0^\infty \sup_{\theta_1, \theta_2} \left\{ \theta_1 \dot{x}_1(t) + \theta_2 \dot{x}_2(t) - \lambda_1^A \left[\varphi_1^A(\theta_1) - 1 \right] - \lambda_2^A \left[\varphi_2^A(\theta_2) - 1 \right] \right\} \mathrm{d}t. \tag{12}$$

We have to find $\inf_{\mathbf{x} \in \mathcal{A}} I(\mathbf{x})$. Belonging $\mathbf{x} \in \mathcal{A}$ means that

$$\min_{i=1,2} \left\{ \sup_{t/geq0} \{\mathbf{x}_1(t) - t\}, \sup_{t/geq0} \{\mathbf{x}_2(t) - t\} \right\} > d.$$

At fixed α the problem is reduced to minimization of

$$T_1 \left(\sup_{\theta_1} \{v_1\theta_1 - \lambda[\varphi_1(\theta_1) - 1]\} \right) + T_2 \left(\sup_{\theta_2} \{v_2\theta_2 - \lambda[\varphi_2(\theta_2) - 1]\} \right) \tag{13}$$

over T_1, T_2, v_1 and v_2, where $v_i = \frac{T_i - d}{T_i}, \ i = 1, 2$. However, making the auxiliary system approximating the original one, requires to chose α in a way such that $T_1 = T_2$ and thus $v_1 = v_2$.

We give an explicit identification for the mininum of expression (13) (this minimum is $I_0^A(d)$ defined in equation (11)). It is determined in terms of solution ϑ to (14) and the solutions $\theta_i, \theta_{0,j}, \ i = 1, 2, \ j = 3 - i$, to (15),(16),

$$\vartheta = \frac{1}{2} \left(\sum_{i=0,1,2} \lambda_i(\varphi_i(\vartheta) - 1) \right), \tag{14}$$

and

$$\theta_i + \theta_{0,j} = \lambda_i(\varphi_i(\theta_i) - 1) + \lambda_j(\varphi_j(\theta_{0,j}) - 1) + \lambda_0(\varphi_0(\theta_{0,j}) - 1), \tag{15}$$

$$\lambda_i \varphi_i'(\theta_i) = \lambda_j \varphi_j'(\theta_{0,j}) + \lambda_0 \varphi_0'(\theta_{0,j}), \tag{16}$$

where

$$i = 1, 2, \quad j = 3 - i.$$

Theorem 1.

Let ϑ be a solution to (14)

A. *In the case*

$$\lambda_0 \varphi_0'(\vartheta) \geq \left| \lambda_1 \varphi_1'(\vartheta) - \lambda_2 \varphi_2'(\vartheta) \right|, \tag{17}$$

$I_0^A(d)$ *has the form*

$$I_0^A(d) = 2d\vartheta, \tag{18}$$

This case is called a balanced case. Here

$$\alpha = \frac{1}{2} + \frac{\lambda_2 \varphi_2'(\vartheta) - \lambda_1 \varphi_1'(\vartheta)}{\lambda_0 \varphi_0'(\vartheta)}.$$

B. *In the case*

$$\lambda_2 \varphi_2'(\vartheta) > \lambda_1 \varphi_1'(\vartheta) + \lambda_0 \varphi_0'(\vartheta), \tag{19}$$

$I_0^A(d)$ *has the form*

$$I_0^A(d) = d(\theta_2 + \theta_{0,1}). \tag{20}$$

where θ_2, θ_{01} are the solution to (15), (16) *with $i = 2,\ j = 1$.*

This case is called an unbalanced case. Here $\alpha = 1$.

C. *In the case*

$$\lambda_1 \varphi_1'(\vartheta) > \lambda_2 \varphi_2'(\vartheta) + \lambda_0 \varphi_0'(\vartheta). \tag{21}$$

$I^A(d)$ *has the form*

$$I_0^A(d) = d(\theta_1 + \theta_{0,2}), \tag{22}$$

where $\theta_1,\ \theta_{0,2}$ are the solution to (15), (16) *with $i = 1,\ j = 2$*

This case is also an unbalanced case. Here $\alpha = 0$.

Further, $I_0(d) = I_0^A(d)$.

Remark 1 *Observe that the expression for I_0^A in the balanced case is identical to the system where all flows are fed into a single-server queue working at speed 2. (See, e.g. [11], [4] and the references there.)*

Remark 2 *In the unbalanced case, alpha is* 0 *or* 1*, corresponding to the whole of the discretionary flow joining a specific server.*

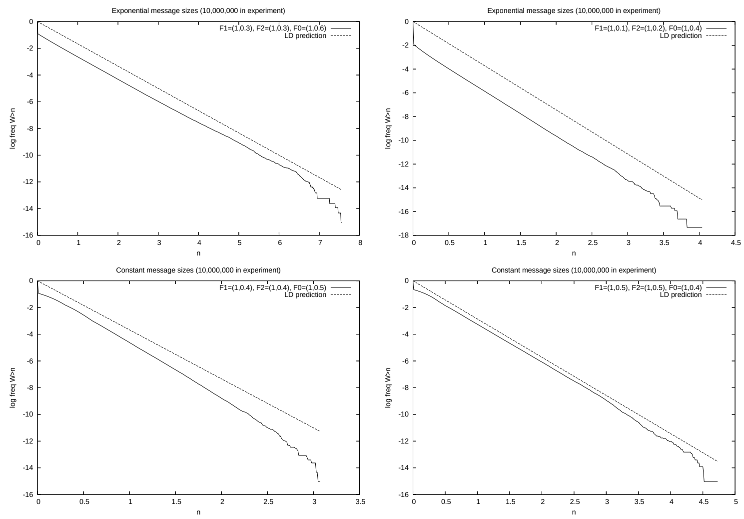

Figure 1: Balanced case.

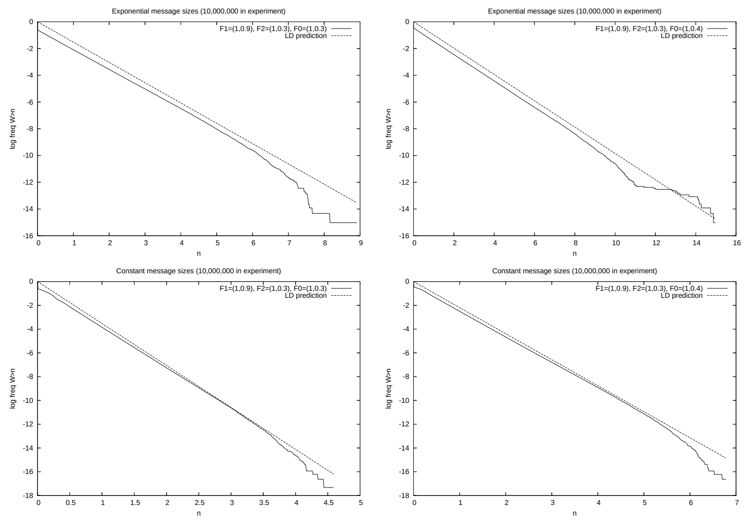

Figure 2: Unbalanced case.

3 Simulation results

A series of simulations were performed to calculate $\log \text{freq}(w^0 \geq d)$ empirically and to compare these data with $I_0^A(d)$. According to the LD theory as $n \to \infty$

$$- \log \ \mathbf{P}(w^0 \geq nd) = nI_0(d) + o\Big(n\Big). \tag{23}$$

Fundamental to determining the merit of approximating $\mathbf{P}(w^0 \geq nd)$ by $\exp(-nI(d))$ is understanding the character of the error $o\Big(n\Big)$.

In simulation, the original system was modeled. Starting with an empty system, 10 million messages were generated between all three flows. Records were kept of the waiting times experienced by messages from flow 0. Figures 1 and 2 show the log empirical frequencies with which w^0 exceeds n vs. n. Also shown is the LD approximations to these probabilities, $-nI_0(1)$. The legends on the graphs correspond to the flow parameters in the experiment; $F_i(\lambda, c)$ represents the Poisson flow $\Xi_i, \ i = 0, 1, 2$, of rate λ_i with mean task length c_i. Constant and exponential message sizes were investigated.

All the empirical traces show similar behavior: a non-zero intercept; initial curvature; a straight line; and then noise. The noise is caused due to scarcity of data. If the experiments are run for longer, the place at which the noise occurs moves further to the right. The non-zero intercept and initial curvature are due to the details of the process. The LD probabilities should, in theory, match the slope of the straight line $-nI_0(1)$. The LD prediction is plotted from the origin, for convenience.

The graphs in Figure 1 show the balanced case and the graphs in Figure 2 show the unbalanced case, for exponential and constant message sizes. Good agreement of slope is seen in all cases. Indeed, the approximation works well for not only large values of n, but also quite small values. That the difference between prediction and the linear LD approximation is approximately constant suggests $\mathbf{P}(w^0 > n) \approx \exp(\mu - nI_0(1))$. That is the error in using the LD approximation, $o\big(n\big)$, is constant.

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References

[1] Murat Alanyali and Bruce Hajek, *On large deviations in load sharing networks*, Ann. Appl. Probab., **8**, 67–97, 1998.

[2] Blinovski V.M. and Dobrushin R.L. *Process level large deviations for a class of piecewise homogeneous random walks*, In The Dynkin Festschrift: Progr. Probab., **34**, 1–59, Birkhuser Boston, Boston, MA, 1994.

[3] Dobrushin R.L., Pechersky E.A., *Large deviations for tandem queueing systems* J. Appl. Math. Stochastic Anal. 7 (1994), no. 3, 301–330.

[4] Dobrushin, R.L. and Pechersky, E.A.*Large deviations for random processes with independent increments on infinite intervals*, Probl. Inform. Transm., **34**,4, pp 354-382, 1998.

[5] Dupuis, P., Ellis, R.S. and Weiss, A., *Large deviations for Markov processes with discontinuous statistics, I: general upper bound.* Ann. Probab., **19**, 1280-1297, 1991.

[6] Dupuis, P. and Ellis, R.S., *Large deviations for Markov processes with discontinuous statistics, II: random walks*, Probab. Theory Related Fields, **91** (1992), 153-194.

[7] Ignatyuk, I.A., Malyshev, V.A. and Shcherbakov, V.V., *The influence of boundaries in problems on large deviations*, Uspekhi Mat. Nauk **49**, (1994), 43-102.

[8] McDonald, D.R. and Turner, S.R.E., *Resource Pooling in Distributed Queueing Networks*, Fields Inst. Communications,**28**, 107-131, 2000.

[9] Pechersky E.A., Suhov Y.M. and Vvedenskaya N.D. *Large deviation in two-server system with dynamic routing*, Preprint, Isaac Newton Institute for Math. Sci., NI03075-IGS, 1 December 2003.

[10] Pechersky E.A., Suhov Y.M. and Vvedenskaya N.D. *Large deviations in a two-server system with dynamic routing', Proceedings* , 2004 IEEE Internat. Symposiom on Inform. Theory , Chicago, USA, p.114.

[11] Shwartz, A., and Weiss, A., *Large deviations for performance analysis: queues, communications and computing,* Chapman & Hall, 1995.

[12] Sadovski, J.S. and Szpankovski, W.,*The probability of large queue lengths and waiting times in a heterogeneous multiserver queue I: tight limits*, Adv. Appl. Prob., **27**, 532-566, 1995.

[13] Sadovski, J.S. *The probability of large queue lengths and waiting times in a heterogeneous multiserver queue* II:*positive recurrence and logarithmic limits*, Adv. Appl. Prob., **27**, 567-583, 1995.

[14] Turner, S.R.E., *Large deviations for Join the Shortest Queue* Fields Inst. Communications,**28**, 95-106, 2000.