Large deviations for the local particle densities

J.-B. BRU[1]

Center for Mathematical Sciences Research, Rutgers,
The State University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019
USA
and
School of Theoretical Physics, Dublin Institute for Advanced Studies, 10
Burlington Rd., Dublin 4, Ireland.

Abstract

We analyze the relations between the large deviation principle of the "local" particle densities of the $x-$ and $k-$spaces respectively. Here the $k-$space means the space of momentums (the Fourier transform counterpart of the $x-$ space). This study gives new insights on the results of papers [2], where the authors have found the corresponding large deviation principle of the local particle density in the $x-$ space. In particular, for a very large class of stable Hamiltonians we show that the "local" particle densities ($x-$ and $k-$spaces) are equal to each other from the point of view of the large deviation principle. In other words, the "local" particle densities in the $x-$ and $k-$spaces are in this case exponentially equivalent [1]. Applying this result to the specific case of the Perfect Bose Gas, we found an alternative proof to the one done in [2].

Keywords : Large deviation principle, exponentially equivalent, Bose systems.

[1]School of Theoretical Physics, email: jbbru@stp.dias.ie

1. Set up of the problem

We define by $\Lambda \subset \mathbb{R}^{d\geq 1}$ and $\widetilde{\Lambda} \subset \Lambda \subset \mathbb{R}^d$, two cubic boxes respectively of volume $V \equiv |\Lambda| = L^d$ and $\widetilde{V} \equiv \left|\widetilde{\Lambda}\right| = \widetilde{L}^d < V$.

1.1. Boson Fock spaces

1.1.1. The infinite volume boson Fock space

The *infinite volume* boson Fock space $\mathcal{F}_B^\infty$ is constructed over $L^2\left(\mathbb{R}^d\right)$ (see for example [3]):

$$\mathcal{F}_B^\infty \equiv \bigoplus_{p=0}^{+\infty} \mathcal{H}^{(p)} \equiv \bigoplus_{p=0}^{+\infty} L^2_{\text{sym.}}\left(\mathbb{R}^{dp}\right), \tag{1.1}$$

with $\mathcal{H}^{(p\neq 0)}$ defined as the symmetrized p-particle Hilbert space appropriate for bosons, whereas $\mathcal{H}^{(p=0)} \equiv \mathbb{C}$. One can also define a scalar product and so a norm in $\mathcal{F}_B^\infty$ deduced from

$$\forall f, h \in L^2\left(\mathbb{R}^d\right), \ \ (f,h)_{L^2\left(\mathbb{R}^d\right)} \equiv \int_{\mathbb{R}^d} \overline{f(x)} h(x)\, dx < +\infty.$$

For any $f \in L^2\left(\mathbb{R}^d\right)$, the action of the annihilation and creation operators is given for $\psi \in \mathcal{D}(f)$, a dense subset of $\mathcal{F}_B^\infty$, by

$$\begin{aligned} \left(a(f)\psi\right)^{(p)}(x_1,...,x_p) &\equiv (p+1)^{1/2} \int_{\mathbb{R}^d} dx \overline{f(x)} \psi^{(p+1)}(x, x_1, ..., x_p), \\ \left(a^*(f)\psi\right)^{(p)}(x_1,...,x_p) &\equiv p^{-1/2} \sum_{i=1}^{p} f(x_i)\, \psi^{(p-1)}(x_1..., \widehat{x}_i, ..., x_p), \end{aligned} \tag{1.2}$$

where $\widehat{x}_i$ means that the argument x_i is omitted, and

$$\forall f, h \in L^2\left(\mathbb{R}^d\right), \ \ [a(f), a^*(h)] = (f,h)_{L^2\left(\mathbb{R}^d\right)} \mathbb{I}_{\mathcal{F}_B^\infty},$$

with $\mathbb{I}_{\mathcal{F}_B^\infty}$ defined as the identity operator on $\mathcal{F}_B^\infty$.
The operator-valued distributions, i.e., the fields $a(x)$ and $a^*(x)$, are defined in the infinite system such that formally one has

$$\begin{aligned} a(f) &= \int_{\mathbb{R}^d} dx \overline{f(x)} a(x), \\ a^*(f) &= \int_{\mathbb{R}^d} dx f(x)\, a^*(x). \end{aligned} \tag{1.3}$$

Their corresponding action is formally given by

$$\begin{aligned} \left(a(x)\psi\right)^{(p)}(x_1,...,x_p) &= (p+1)^{1/2} \psi^{(p+1)}(x, x_1, ..., x_p), \\ \left(a^*(x)\psi\right)^{(p)}(x_1,...,x_p) &= p^{-1/2} \sum_{i=1}^{p} \delta(x - x_i)\, \psi^{(p-1)}(x_1..., \widehat{x}_i, ..., x_p), \end{aligned} \tag{1.4}$$

for some functions $\psi \in \mathcal{F}_B^\infty$. In fact the operator $a(x)$ is a well-defined operator acting on a dense subset $\mathcal{D}(\delta)$ of $\mathcal{F}_B^\infty$, whereas its adjoint $a^*(x)$ is not. However $a^*(x)$ is a well-defined quadratic form on $\mathcal{D}(\delta) \times \mathcal{D}(\delta)$. Then the two equalities in (1.3), or in (1.4), should be understood in the sense of quadratic forms or using, instead of standard functions f, the set of distributions: $a(x) = a(\delta(x))$, $a^*(x) = a^*(\delta(x))$.
We can also define the two (infinite volume) quadratic forms given by

$$\begin{aligned}(a_k\psi)^{(p)}(x_1,...,x_p) &= (p+1)^{1/2} \int_{\mathbb{R}^d} dx e^{-ikx}\psi^{(p+1)}(x,x_1,...,x_p), \\ (a_k^*\psi)^{(p)}(x_1,...,x_p) &= p^{-1/2}\sum_{i=1}^{p} e^{ikx_i}\psi^{(p-1)}(x_1...,\widehat{x}_i,...,x_p),\end{aligned} \tag{1.5}$$

for $k \in \mathbb{R}^d$ and $\psi \in \mathcal{D}\left(e^{ikx}\right)$, a dense subset of $\mathcal{F}_B^\infty$. The quadratic form a_k could be interpreted as a densely defined operator acting on $\mathcal{D}\left(e^{ikx}\right) \subset \mathcal{F}_B^\infty$ whereas its adjoint a_k^* is at least a well-defined quadratic form on $\mathcal{D}\left(e^{ikx}\right) \times \mathcal{D}\left(e^{ikx}\right)$.

1.1.2. The finite volume boson Fock space

For any cubic box $\Lambda_0 \subset \mathbb{R}^d$ of volume $V_0 \equiv |\Lambda_0| = L_0^d$, i.e., for $\Lambda_0 = \Lambda$ or $\Lambda_0 = \widetilde{\Lambda}$, the corresponding *finite volume* boson Fock space $\mathcal{F}_B^{(\Lambda_0)}$ is constructed over $L^2(\Lambda_0)$:

$$\mathcal{F}_B^{(\Lambda_0)} \equiv \bigoplus_{p=0}^{+\infty} \mathcal{H}^{(\Lambda_0,p)} \equiv \bigoplus_{p=0}^{+\infty} L^2_{\text{sym.}}\left(\Lambda_0^p\right), \tag{1.6}$$

where $\mathcal{H}^{(\Lambda_0,p\neq0)}$ is the symmetrized Hilbert space for p bosons enclosed in Λ_0, and $\mathcal{H}^{(\Lambda_0,p=0)} \equiv \mathcal{H}^{(p=0)} \equiv \mathbb{C}$.
Notice that $L^2\left(\Lambda_0^p\right) \subset L^2\left(\mathbb{R}^{dp}\right)$, so by (1.1) and (1.6) we have $\mathcal{F}_B^{(\Lambda_0)} \subset \mathcal{F}_B^\infty$. Then the projection operator P_{Λ_0} from $\mathcal{F}_B^\infty$ to $\mathcal{F}_B^{(\Lambda_0)}$ is defined for any $p > 0$ by

$$\begin{aligned}&\mathcal{H}^{(p\neq0)} \longrightarrow \mathcal{H}^{(\Lambda_0,p\neq0)} \\ &\varphi^{(p)}(X) \longmapsto \psi^{(p)}(X) = \left(P_{\Lambda_0}^{(p)}\varphi^{(p)}\right)(X) = \chi_{\Lambda_0^p}(X)\,\varphi^{(p)}(X)\end{aligned} \tag{1.7}$$

with $X = (x_1,...,x_p) \in \mathbb{R}^{dp}$. For any set $A \subset \mathbb{R}^d$, $\chi_{A^p}(X)$ is the corresponding characteristic function for $p > 0$ particles defined by

$$\chi_{A^p}(X) \equiv \prod_{i=1}^{p} \chi_A(x_i), \ \ \chi_A(x) \equiv \begin{cases} 1 \text{ for } x \in A. \\ 0 \text{ for } x \in \mathbb{R}^d \backslash A. \end{cases} \tag{1.8}$$

Notice that for $p = 0$, P_{Λ_0} is the identity of $\mathbb{C}$.
We denote by $a_{k,\Lambda_0}^{\#} = \left\{a_{k,\Lambda_0}^* \text{ or } a_{k,\Lambda_0}\right\}$, the standard operators defined by

$$a_{k,\Lambda_0}^{\#} \equiv a^{\#}\left(\frac{\chi_{\Lambda_0}(x)}{\sqrt{V_0}}e^{ikx}\right), \text{ for } k \in \Lambda_0^*, \tag{1.9}$$

on $\mathcal{F}_B^\infty$, see (1.2), where

$$\Lambda_0^* = \left\{k \in \mathbb{R}^d : \alpha = 1,...,d;\, k_\alpha = \frac{2\pi n_\alpha}{L_0} \text{ and } n_\alpha = 0, \pm1, \pm2, ...\right\} \tag{1.10}$$

is the "Fourier transform" of the box Λ_0 corresponding to *periodic boundary conditions* on Λ_0.

Remark 1.1. *If the cubic box* Λ *is such that* $\widetilde{\Lambda} \subset \Lambda \subset \mathbb{R}^d$ *with* $L = n\widetilde{L}$, $n \in \mathbb{N}\backslash\{0\}$, *notice that one has* $\widetilde{\Lambda}^* \subset \Lambda^*$, *see (1.10).*

Remark 1.2. *By (1.5) combining with* P_{Λ_0} *(1.7)-(1.8), the operator* a_{k,Λ_0} *(1.9), defined on* $\mathcal{F}_B^\infty$, *is equal to*

$$a_{k,\Lambda_0} = a_k \frac{P_{\Lambda_0}}{\sqrt{V_0}}, \text{ for } k \in \mathbb{R}^d.$$

1.2. Particle number operators

1.2.1. The finite volume particle number operators

Using *periodic boundary conditions* on Λ, the finite volume particle number operator is equal to

$$N_{\Lambda,\text{p.b.c}} \equiv \sum_{k\in\Lambda^*} a_{k,\Lambda}^* a_{k,\Lambda} = P_\Lambda \left[\frac{1}{V} \sum_{k\in\Lambda^*} a_k^* a_k \right] P_\Lambda, \tag{1.11}$$

with Λ^* defined by (1.10) for $\Lambda_0 = \Lambda$. The second part of (1.11) comes from Remark 1.2. The operator $N_{\Lambda,\text{p.b.c}}$ is well-defined on a dense subset

$$\mathcal{N}_B^{(\Lambda)} \equiv \left\{ \psi \in \mathcal{F}_B^\infty : \sum_{p=0}^{+\infty} \left\| p P_\Lambda^{(p)} \psi^{(p)} \right\|_{\mathcal{H}^{(p)}}^2 < +\infty \right\} \tag{1.12}$$

of the boson Fock space $\mathcal{F}_B^\infty$ (1.1). Here $P_\Lambda^{(p)}$ is defined by (1.7) for $\Lambda_0 = \Lambda$.

Remark 1.3. *The particle operator number in an infinite volume is formally equal to*

$$N_\infty \equiv \int_{\mathbb{R}^d} a^*(x)\, a(x)\, dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} a_k^* a_k dk, \tag{1.13}$$

see (1.4) and (1.5). Notice that N_∞ *is well-defined on a dense subset*

$$\mathcal{N}_B^\infty \equiv \left\{ \psi \in \mathcal{F}_B^\infty : \sum_{p=0}^{+\infty} \left\| p \psi^{(p)}(X) \right\|_{\mathcal{H}^{(p)}}^2 < +\infty \right\} \tag{1.14}$$

of the boson Fock space $\mathcal{F}_B^\infty$.

1.2.2. The local particle number operators

Let $\widetilde{\Lambda} \subset \Lambda \subset \mathbb{R}^d$ such that $\widetilde{\Lambda}^* \subset \Lambda^* \subset \mathbb{R}^d$, cf. (1.10) and Remark 1.1. Notice that the corresponding boson Fock spaces verify $\mathcal{F}_B^{(\widetilde{\Lambda})} \subset \mathcal{F}_B^{(\Lambda)} \subset \mathcal{F}_B^\infty$, see (1.1) and (1.6). Then, using the projection operator $P_{\widetilde{\Lambda}}$ from $\mathcal{F}_B^\infty$ to $\mathcal{F}_B^{(\widetilde{\Lambda})}$ (1.7)-(1.8), we define the following *local* sequences of operators:

- in the "$x-$space",

$$N_{\widetilde{\Lambda}} \equiv N_{\Lambda,\text{p.b.c}} P_{\widetilde{\Lambda}} = \int\limits_{\widetilde{\Lambda}} a^*(x)\, a(x)\, dx, \;\; \rho_{\widetilde{\Lambda}} \equiv \frac{N_{\widetilde{\Lambda}}}{\widetilde{V}}, \tag{1.15}$$

where $N_{\widetilde{\Lambda}}$ is called the ($x-$space) *local* particle number operator, whereas $\rho_{\widetilde{\Lambda}}$ is the *local* ($x-$space) particle density operator,

- in the "$k-$space" or momentum space,

$$N_{\widetilde{\Lambda}^*} \equiv \sum_{k \in \widetilde{\Lambda}^*} a^*_{k,\Lambda} a_{k,\Lambda}, \;\; \rho_{\widetilde{\Lambda}^*} \equiv \frac{N_{\widetilde{\Lambda}^*}}{\widetilde{V}}, \tag{1.16}$$

respectively denoted as the *local* ($k-$space or momentum space) particle number operator and the *local* ($k-$space) particle density operator. $\widetilde{\Lambda}^* \subset \Lambda^*$ is defined by (1.10) for $\Lambda_0 = \widetilde{\Lambda}$.

The two operators $N_{\widetilde{\Lambda}}$ (1.15) and $N_{\widetilde{\Lambda}^*}$ (1.16) are respectively well-defined on

$$\begin{aligned} \mathcal{N}_B^{(\widetilde{\Lambda})} &\equiv \left\{ \psi \in \mathcal{F}_B^\infty : \; \sum_{p=0}^{+\infty} \left\| p P_{\widetilde{\Lambda}}^{(p)} \psi^{(p)} \right\|^2_{\mathcal{H}^{(p)}} < +\infty \right\} \supseteq \mathcal{N}_B^\infty, \\ \mathcal{N}_B^{(\widetilde{\Lambda}^*)} &\equiv \left\{ \psi \in \mathcal{F}_B^\infty : \; \sum_{p=0}^{+\infty} \left\| p P_{\widetilde{\Lambda}^*}^{(p)} \widehat{\psi}^{(p)} \right\|^2_{\mathcal{H}^{(p)}} < +\infty \right\} \supseteq \mathcal{N}_B^\infty, \end{aligned} \tag{1.17}$$

where $\widehat{\psi}^{(p)}(K) = \widehat{\psi}^{(p)}(k_1, ..., k_p)$ is the fourier transform of $\psi^{(p)}(X) = \psi^{(p)}(x_1, ..., x_p)$. Here $P_{\widetilde{\Lambda}^*}^{(p)}$ is defined by (1.7) for $\Lambda_0 = \widetilde{\Lambda}^*$, with $K = (k_1, ..., k_p) \in \mathbb{R}^{dp}$ instead of $X = (x_1, ..., x_p) \in \mathbb{R}^{dp}$.

1.3. Grand-canonical Boson Gibbs States

1.3.1. Finite volume Hamiltonian

We consider a system X of bosons of mass m enclosed in the cubic box $\Lambda \subset \mathbb{R}^d$ of volume $V \equiv |\Lambda| = L^d$, defined by some Hamiltonian acting on a dense subset of $\mathcal{F}_B^\infty$ (1.1):

$$H^X_{\Lambda,\text{p.b.c}} \equiv \sum_{k \in \Lambda^*} \varepsilon_k a^*_{k,\Lambda} a_{k,\Lambda} + \frac{1}{2V} \sum_{k_1,k_2,q \in \Lambda^*} v(q)\, a^*_{k_1-q,\Lambda} a^*_{k_2+q,\Lambda} a_{k_2,\Lambda} a_{k_1,\Lambda} \equiv T_{\Lambda,\text{p.b.c}} + U^X_{\Lambda,\text{p.b.c}}, \tag{1.18}$$

Here $\varepsilon_k = \hbar^2 k^2 / 2m \geq 0$ is the one-particle energy spectrum of free bosons whereas the function $v(q)$ is interpreted as the fourier transform of an integrable two-body interaction potential $\varphi \in L^1(\mathbb{R}^d)$. All sums run over the set Λ^*, i.e., we use *periodic boundary conditions* on Λ.

Remark 1.4. *There is conservation of the particle number in the box Λ for the Bose gas X (1.18), i.e.,*

$$\left[H^X_{\Lambda,p.b.c}, N_{\Lambda,p.b.c} \right] = \left[U^X_{\Lambda,p.b.c}, N_{\Lambda,p.b.c} \right] = 0. \tag{1.19}$$

Remark 1.5. *For $\widetilde{\Lambda} \subset \Lambda$ such that $\widetilde{\Lambda}^* \subset \Lambda^*$, cf. (1.10) and Remark 1.1, by (1.18) combining with Remark 1.2 we have*

$$H^X_{\widetilde{\Lambda},p.b.c} = P_{\widetilde{\Lambda},p.b.c}\left(\frac{V}{\widetilde{\widetilde{V}}}T_{\Lambda,p.b.c} + \left(\frac{V}{\widetilde{\widetilde{V}}}\right)^2 U^X_{\Lambda,p.b.c}\right)P_{\widetilde{\Lambda},p.b.c},$$

where $P_{\widetilde{\Lambda},p.b.c}$ is the "projection" operator from $\mathcal{F}^\infty_B$ to the boson Fock space $\mathcal{F}^{(\widetilde{\Lambda})}_{B,p.b.c} \subset \mathcal{F}^{(\widetilde{\Lambda})}_B$ constructed on the Hilbert space $\left(L^2\left(\widetilde{\Lambda}\right)\right)_{p.b.c}$ of periodic functions on $\widetilde{\Lambda}$.

We assume that $v(q)$ is such that $U^X_{\Lambda,\mathrm{p.b.c}}$ is a stable interaction [5], i.e., there is $B \geq 0$ such that

$$U^X_{\Lambda,\mathrm{p.b.c}} \geq -BN_{\Lambda,\mathrm{p.b.c}}, \tag{1.20}$$

with $N_{\Lambda,\mathrm{p.b.c}}$ defined by (1.11). An example is given by $v(q) = 0$ for $q \in \mathbb{R}^d$, i.e., by the Perfect Bose Gas (Perfect Bose Gas) $T_{\Lambda,\mathrm{p.b.c}}$.

The grand-canonical pressure

$$p^X_\Lambda(\beta,\mu) \equiv \frac{1}{\beta V}\ln Tr_{\mathcal{F}^\infty_B}\left(P_\Lambda e^{-\beta\left(H^X_{\Lambda,\mathrm{p.b.c}}-\mu N_{\Lambda,\mathrm{p.b.c}}\right)}P_\Lambda\right) = \frac{1}{\beta V}\ln Tr_{\mathcal{F}^{(\Lambda)}_B}\left(e^{-\beta\left(H^X_{\Lambda,\mathrm{p.b.c}}-\mu N_{\Lambda,\mathrm{p.b.c}}\right)}\right) \tag{1.21}$$

exists, even in the thermodynamic limit, for $(\beta,\mu) \in Q^X$ with

$$(a)\ \ Q^X \equiv \{\beta > 0\} \times \left\{\mu < \mu_{\sup} < +\infty\right\} \ \text{ or } \ (b)\ \ Q^X \equiv \{\beta > 0\} \times \{\mu < +\infty\}, \tag{1.22}$$

see (1.20). Here $\beta > 0$ is the fixed inverse temperature whereas μ is the chemical potential.

1.3.2. The (finite volume) grand-canonical Gibbs state for a fixed particle density

In the grand-canonical ensemble (β,μ), the Hamiltonian $H^X_{\Lambda,\mathrm{p.b.c}}$ (1.18) defines a finite volume Gibbs state

$$\omega^X_{\Lambda,\mu}(-) \equiv \frac{Tr_{\mathcal{F}^\infty_B}\left((-)\,P_\Lambda e^{-\beta\left(H^X_{\Lambda,\mathrm{p.b.c}}-\mu N_{\Lambda,\mathrm{p.b.c}}\right)}P_\Lambda\right)}{Tr_{\mathcal{F}^\infty_B}\left(P_\Lambda e^{-\beta\left(H^X_{\Lambda,\mathrm{p.b.c}}-\mu N_{\Lambda,\mathrm{p.b.c}}\right)}P_\Lambda\right)} = \frac{Tr_{\mathcal{F}^{(\Lambda)}_B}\left((-)\,e^{-\beta\left(H^X_{\Lambda,\mathrm{p.b.c}}-\mu N_{\Lambda,\mathrm{p.b.c}}\right)}\right)}{Tr_{\mathcal{F}^{(\Lambda)}_B}\left(e^{-\beta\left(H^X_{\Lambda,\mathrm{p.b.c}}-\mu N_{\Lambda,\mathrm{p.b.c}}\right)}\right)} \tag{1.23}$$

well-defined on the C^*-algebra $\mathcal{A}^\infty_B$ of bounded operators acting on the boson Fock space $\mathcal{F}^\infty_B$ (1.6). P_Λ (1.7)-(1.8) is the projection operator from $\mathcal{F}^\infty_B$ to the boson Fock space $\mathcal{F}^{(\Lambda)}_B \subset \mathcal{F}^\infty_B$. Then we assume also that the corresponding particle density

$$\rho^X_\Lambda(\beta,\mu) \equiv \omega^X_{\Lambda,\mu}\left(\frac{N_{\Lambda,\mathrm{p.b.c}}}{V}\right) = \partial_\mu p^X_\Lambda(\beta,\mu) < +\infty,\ \ (\beta,\mu) \in Q^X, \tag{1.24}$$

as a function of μ is strictly increasing (i.e., $p^X_\Lambda(\beta,\mu)$ is strictly convex) and verifies

$$(a)\ \lim_{\mu\to\mu_{\sup}} \rho^X_\Lambda(\beta,\mu) = +\infty \text{ or } (b)\ \lim_{\mu\to+\infty} \rho^X_\Lambda(\beta,\mu) = +\infty. \tag{1.25}$$

In fact, the assumption (1.25), combining with the strict monotonicity of $\rho_\Lambda^X(\beta,\mu)$, allows to define a new Gibbs state $\omega_{\Lambda,\rho}^X(-)$ corresponding to a fixed particle density ρ in the *grand-canonical* ensemble:

$$\omega_{\Lambda,\rho}^X(A) \equiv \omega_{\Lambda,\mu_\Lambda^X(\rho)}^X(A), \; A \in \mathcal{A}_B^\infty, \tag{1.26}$$

with $\mu_\Lambda^X(\rho)$ defined as the unique solution of equation

$$\omega_{\Lambda,\mu_\Lambda^X(\rho)}^X\left(\frac{N_{\Lambda,\mathrm{p.b.c}}}{V}\right) = \rho_\Lambda^X\left(\beta,\mu_\Lambda^X(\rho)\right) = \rho. \tag{1.27}$$

Remark 1.6. *The norm on* $\mathcal{A}_B^\infty$ *is defined by*

$$\forall A \in \mathcal{A}_B^\infty, \;\; \|A\|_{\mathcal{A}_B^\infty}^2 = \sup_{\varphi\in\mathcal{F}_B^\infty}\left\{\frac{\|A\varphi\|_{\mathcal{F}_B^\infty}^2}{\|\varphi\|_{\mathcal{F}_B^\infty}^2}\right\} < +\infty.$$

Remark 1.7. *One always has*

$$\omega_{\Lambda,\rho}^X\left(\mathbb{I}_{\mathcal{F}_B^\infty}\right) = 1.$$

Remark 1.8. *Even if* $N_{\tilde\Lambda} \notin \mathcal{A}_B^\infty$ *(or* $N_{\Lambda,p.b.c} \notin \mathcal{A}_B^\infty$*), cf. (1.15), one has*

$$\omega_{\Lambda,\rho}^X\left(N_{\tilde\Lambda}\right) \leq \omega_{\Lambda,\rho}^X\left(N_{\Lambda,p.b.c}\right) = \rho V < +\infty,$$

see (1.26)-(1.27). Note also that the unbounded operator $N_{\tilde\Lambda}$ *could be interpreted as the limit of a sequence* $\left\{N_{\tilde\Lambda}^{(n)}\right\}_{n\in\mathbb{N}}$ *of bounded operators, i.e.,* $N_{\tilde\Lambda}^{(n)} \in \mathcal{A}_B^\infty$ *for* $n \in \mathbb{N}$*, and* $\omega_{\Lambda,\rho}^X\left(N_{\tilde\Lambda}\right)$ *could be seen as:*

$$\omega_{\Lambda,\rho}^X\left(N_{\tilde\Lambda}\right) = \lim_{n\to+\infty}\omega_{\Lambda,\rho}^X\left(N_{\tilde\Lambda}^{(n)}\right) < +\infty.$$

1.3.3. The (infinite volume) grand-canonical Gibbs state

The equality (1.26) defines a sequence of Gibbs states $\left\{\omega_{\Lambda,\rho}^X(-)\right\}_\Lambda$ over the C^*-algebra $\mathcal{A}_B^\infty$ but we don't know, à priori, if this sequence of linear functions $\mathcal{A}_B^\infty \to \mathbb{C}$ converges or does not. First, note that

$$\lim_\Lambda \omega_{\Lambda,\rho}^X\left(N_\infty\right) = \lim_\Lambda \omega_{\Lambda,\rho}^X\left(N_{\Lambda,\mathrm{p.b.c}}\right) = +\infty,$$

see (1.11), (1.13), and (1.23)-(1.27), whereas

$$\lim_\Lambda \omega_{\Lambda,\rho}^X\left(\frac{N_\infty}{V}\right) = \lim_\Lambda \omega_{\Lambda,\rho}^X\left(\frac{N_{\Lambda,\mathrm{p.b.c}}}{V}\right) = \rho. \tag{1.28}$$

In fact the problem in the thermodynamic limit does not comes from the fact that the operator N_∞ is unbounded but from the nonlocality of the operator N_∞.
For example, considering the Perfect Bose Gas, i.e., for $q \in \mathbb{R}^d$ $v(q) = 0$ in (1.18), one gets the same problem if we want to analyze the following limit

$$\lim_\Lambda \omega_{\Lambda,\rho}^{PBG}\left(a_k^* a_k\right) = \lim_\Lambda V\omega_{\Lambda,\rho}^{PBG}\left(a_{k,\Lambda}^* a_{k,\Lambda}\right) = +\infty,$$

see (1.5), (1.9) and Remark 1.2, whereas for any $k \in \mathbb{R}^d \setminus \{0\}$ one has

$$\lim_{\Lambda} \omega_{\Lambda,\rho}^{PBG} \left(\frac{a_k^* a_k}{V} \right) = \lim_{\Lambda} \omega_{\Lambda,\rho}^{PBG} \left(a_{k,\Lambda}^* a_{k,\Lambda} \right) = \frac{1}{e^{\beta \left(\varepsilon_k - \lim_{\Lambda} \mu_{\Lambda}^{PBG}(\rho) \right)} - 1} < +\infty.$$

In this case, the operators $\{N_k \equiv a_k^* a_k\}_{k \in \mathbb{R}^d}$ are nonlocal.

Then we add two following assumptions on the convergence of the sequence $\left\{\omega_{\Lambda,\rho}^X\right\}_\Lambda$ over the C^*-algebra $\mathcal{A}_B^\infty$:

- In the thermodynamic limit we assume that the Gibbs state $\omega_{\Lambda,\rho}^X$ (cf. (1.23), (1.26)-(1.27)) defines an *infinite volume* Gibbs state ω_ρ^X acting on $\mathcal{A}_B^\infty$ for a fixed particle density $\rho > 0$ and inverse temperature $\beta > 0$:

$$\omega_\rho^X (A) = \lim_{\Lambda} \omega_{\Lambda,\rho}^X (A), \text{ for any } A \in \mathcal{A}_B^\infty, \tag{1.29}$$

  i.e., the sequence $\left\{\omega_{\Lambda,\rho}^X\right\}_\Lambda$ converges weakly to ω_ρ^X.

- We assume that the Gibbs state ω_ρ^X is *invariant by translation.*

- Moreover for any finite box $\widetilde{\Lambda} \subset \Lambda$ of volume $\widetilde{V} < V$, we have

$$\lim_{\Lambda} \omega_{\Lambda,\rho}^X \left(N_{\widetilde{\Lambda}}\right) = \lim_{\Lambda} \omega_{\Lambda,\rho}^X \left(N_\infty P_{\widetilde{\Lambda}}\right) = \omega_\rho^X \left(N_\infty P_{\widetilde{\Lambda}}\right) = \omega_\rho^X \left(N_{\widetilde{\Lambda}}\right) < +\infty, \tag{1.30}$$

  see (1.15), i.e., if $\widetilde{V}$ remains finite, the Bose system X (1.18) has no collapse in some finite area of $\mathbb{R}^d$.

Remark 1.9. *The limit (1.30) could be seen as*

$$\lim_{\Lambda} \omega_{\Lambda,\rho}^X \left(N_{\widetilde{\Lambda}}\right) = \lim_{\Lambda} \lim_{n \to +\infty} \omega_{\Lambda,\rho}^X \left(N_{\widetilde{\Lambda}}^{(n)}\right) = \omega_\rho^X \left(N_{\widetilde{\Lambda}}\right) < +\infty, \tag{1.31}$$

where the sequence $\left\{N_{\widetilde{\Lambda}}^{(n)}\right\}_{n \in \mathbb{N}}$ *of bounded operators converges to the unbounded operator* $N_{\widetilde{\Lambda}} \notin \mathcal{A}_B^\infty$*, see Remark 1.8. Nevertheless, we don't know, à priori, if we can exchange the two limits in (1.31), i.e., if there is uniform convergence.*

Remark 1.10. *The equalities (1.19)-(1.22) and the condition (1.25) are also verified by the Bogoliubov Weakly Imperfect Bose Gas (cf. eq. (3.81) in [6]) with* $\mu_{\text{sup}} = 0$ *in the thermodynamic limit, see [7–9]. In fact, assuming the three last conditions of this subsection, cf. (1.29)-(1.30), all the proofs done below remain true for the Bogoliubov Weakly Imperfect Bose Gas.*

2. Large deviation principle of the local particle densities: $x-$ versus $k-$spaces.

Considering the two cubic boxes $\Lambda \subset \mathbb{R}^d$ and $\widetilde{\Lambda} \subset \Lambda \subset \mathbb{R}^d$, we assume that $n\left(\widetilde{V}\right) \equiv V/\widetilde{V} \to +\infty$. For example we define L as a function of $\widetilde{L}$ by

$$L = \gamma\left(\widetilde{L}\right) \widetilde{L}, \tag{2.1}$$

where

$$\lim_{x\rightarrow+\infty}\gamma\left(x\right)=+\infty;\ \forall x>0,\ \gamma\left(x\right)\in\mathbb{N}\backslash\left\{0\right\}\Rightarrow n\left(\widetilde{V}\right)\equiv V/\widetilde{V}\in\mathbb{N}\backslash\left\{0\right\}. \tag{2.2}$$

By (2.1)-(2.2) note that $\widetilde{\Lambda}^{*}\subset\Lambda^{*}$, see (1.10) and Remark 1.1. Then, similar to the analyze done in [2] for the Perfect Bose Gas (cf. (1.18) with $U_{\Lambda,\text{p.b.c}}=0$), the aim is to find a new method to evaluate the limit

$$\lim_{\widetilde{\Lambda}}\frac{1}{\beta\widetilde{V}}\ln\mathbb{P}\left\{\rho_{\widetilde{\Lambda}}\in I\right\}=\lim_{\widetilde{\Lambda}}\frac{1}{\beta\widetilde{V}}\ln\omega_{\Lambda,\rho}^{X}\left(\chi_{I}\left(\rho_{\widetilde{\Lambda}}\right)\right), \tag{2.3}$$

for any interval $I=[a,b]$. We recall that $\rho_{\widetilde{\Lambda}}$ (1.15) is the particle density operator in the box $\widetilde{\Lambda}$ and χ_{A} is the characteristic function of a set $A\subset\mathbb{R}$. In fact, for $\widetilde{\Lambda}\rightarrow\mathbb{R}^{d}$, our purpose is to show the very close relations between the behavior of the *local* ($x-$space) particle density $\rho_{\widetilde{\Lambda}}$ as a "random variable" and the one of the *local* ($k-$space) particle density operator $\rho_{\widetilde{\Lambda}^{*}}$ (1.16). Intuitively, our arguments are the following :

1° The operators $N_{\widetilde{\Lambda}}$ and $\rho_{\widetilde{\Lambda}}$ are defined by (1.15) where, à priori, there is *no* specific boundary condition on $\widetilde{\Lambda}$. However, using for any $k\in\widetilde{\Lambda}^{*}$ (cf. (1.10) for $\Lambda_{0}=\widetilde{\Lambda}$) the creation/annihilation operators

$$a_{k,\widetilde{\Lambda}}^{\#}=a^{\#}\left(\frac{\chi_{\widetilde{\Lambda}}\left(x\right)}{\sqrt{\widetilde{V}}}e^{ikx}\right), \tag{2.4}$$

(cf. (1.9)), we could also define the following operators

$$N_{\widetilde{\Lambda},\text{p.b.c}}\equiv\sum_{k\in\widetilde{\Lambda}^{*}}a_{k,\widetilde{\Lambda}}^{*}a_{k,\widetilde{\Lambda}},\ \rho_{\widetilde{\Lambda},\text{p.b.c}}\equiv\frac{N_{\widetilde{\Lambda},\text{p.b.c}}}{\widetilde{V}}, \tag{2.5}$$

see (1.11) with $\widetilde{\Lambda}$ instead of Λ. In fact, $N_{\widetilde{\Lambda},\text{p.b.c}}$ and $\rho_{\widetilde{\Lambda},\text{p.b.c}}$ are the local ($x-$space), respectively particle number and particle density, operators considering *periodic boundary conditions* on $\widetilde{\Lambda}$.

2° Notice that the set

$$\left\{\frac{\chi_{\widetilde{\Lambda}}\left(x\right)}{\sqrt{\widetilde{V}}}e^{ikx}\right\}_{k\in\widetilde{\Lambda}^{*}} \tag{2.6}$$

is an orthonormal basis of the Hilbert space $\left(L^{2}\left(\widetilde{\Lambda}\right)\right)_{\text{p.b.c}}$ of squared integrable functions in $\widetilde{\Lambda}$ with *periodic boundary conditions* on the box $\widetilde{\Lambda}$. Since $\left(L^{2}\left(\widetilde{\Lambda}\right)\right)_{\text{p.b.c}}$ is a dense subset of $L^{2}\left(\widetilde{\Lambda}\right)$ in which there is, à priori, *no periodic boundary conditions*, intuitively one should formally have

$$\rho_{\widetilde{\Lambda}}\equiv\frac{N_{\widetilde{\Lambda}}}{\widetilde{V}}\approx\rho_{\widetilde{\Lambda},\text{p.b.c}}\equiv\frac{N_{\widetilde{\Lambda},\text{p.b.c}}}{\widetilde{V}}. \tag{2.7}$$

In fact, since $\omega^X_{\Lambda,\rho}\left(\rho_{\widetilde{\Lambda}}\right) < +\infty$ and $\omega^X_{\Lambda,\rho}\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right) < +\infty$ (Remark 1.8), the Theorem B.1 in Appendix B implies for $\omega = \omega^X_{\Lambda,\rho}$ (cf. (1.23), (1.26)-(1.27)) that:

$$\begin{aligned} &\omega^X_{\Lambda,\rho}\left(\rho_{\widetilde{\Lambda}}\right) = \omega^X_{\Lambda,\rho}\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right), \\ &\omega^X_{\Lambda,\rho}\left(\chi_I\left(\rho_{\widetilde{\Lambda}}\right)\right) = \omega^X_{\Lambda,\rho}\left(\chi_I\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right)\right), \end{aligned} \tag{2.8}$$

for any interval $I = [a, b] \subset \mathbb{R}$, which by (2.3) implies

$$\lim_{\widetilde{\Lambda}} \frac{1}{\beta \widetilde{V}} \ln \mathbb{P}\left\{\rho_{\widetilde{\Lambda}} \in I\right\} = \lim_{\widetilde{\Lambda}} \frac{1}{\beta \widetilde{V}} \ln \mathbb{P}\left\{\rho_{\widetilde{\Lambda},\text{p.b.c}} \in I\right\}. \tag{2.9}$$

3° Then the study of the fluctuations of the $x-$space *local* density seems to be *equivalent* to analyze the fluctuations of the $k-$space local density, except that the corresponding plane waves e^{ikx} are, for a finite $\widetilde{\Lambda}$, truncated by the function $\chi_{\widetilde{\Lambda}}(x)$ (one should not worry about the term $\sqrt{\widetilde{V}}$ which is just a renormalization term). Therefore, since the *invariance by translation* of the Gibbs state ω^X_ρ, the fluctuations should remain the same in the limit $\widetilde{\Lambda} \to \mathbb{R}^d$ if you use, instead of the *truncated* plane wave (2.6), the *full* plane wave

$$\left\{\frac{\chi_\Lambda(x)}{\sqrt{V}} e^{ikx}\right\}_{k \in \widetilde{\Lambda}^*}$$

defined on the full box Λ and so, instead of the operator $a^{\#}_{k,\widetilde{\Lambda}}$ (2.4) the standard operators $a^{\#}_{k,\Lambda}$ (1.9), i.e., instead of $\rho_{\widetilde{\Lambda},\text{p.b.c}}$ (2.5) we analyze the *local* ($k-$space) particle density operator (1.16). Consequently by (2.9) we should have:

$$\lim_{\widetilde{\Lambda}} \frac{1}{\beta \widetilde{V}} \ln \mathbb{P}\left\{\rho_{\widetilde{\Lambda}} \in I\right\} = \lim_{\widetilde{\Lambda}} \frac{1}{\beta \widetilde{V}} \ln \mathbb{P}\left\{\rho_{\widetilde{\Lambda},\text{p.b.c}} \in I\right\} = \lim_{\widetilde{\Lambda}} \frac{1}{\beta \widetilde{V}} \ln \mathbb{P}\left\{\rho_{\widetilde{\Lambda}^*} \in I\right\},$$

for any interval $I = [a, b] \subset \mathbb{R}$. In other words, the "local" particle densities $\left\{\rho_{\widetilde{\Lambda}}\right\}_{\widetilde{\Lambda}}$, $\left\{\rho_{\widetilde{\Lambda},\text{p.b.c}}\right\}_{\widetilde{\Lambda}}$ and $\left\{\rho_{\widetilde{\Lambda}^*}\right\}_{\widetilde{\Lambda}}$ are exponentially equivalent [1].

This heuristic is in fact shown to be true, see Theorem 3.7 and Remarks 3.8-3.9 in the next section. For a direct application of all these results on the Perfect Bose Gas, see Section 3.4.

3. Rigorous results

Now, we present below the rigorous arguments and results.

3.1. Expectation value of the local particles densities

As a first stage, we analyse the expectation value of $\rho_{\widetilde{\Lambda}}$ (1.15) and $\rho_{\widetilde{\Lambda}^*}$ (1.16).

Theorem 3.1. *There is $\gamma_0(x)$ such that*

$$\lim_{\widetilde{\Lambda}} \omega^X_{\Lambda,\rho}\left(\rho_{\widetilde{\Lambda}}\right) = \lim_{\widetilde{\Lambda}} \omega^X_{\Lambda,\rho}\left(\rho_{\widetilde{\Lambda}^*}\right), \tag{3.1}$$

for any function $\gamma(x)$ in (2.1)-(2.2) verifying $\gamma(x) \geq \gamma_0(x)$ for $x > 0$.

Proof. From Remark 1.2 one has

$$a_{k,\Lambda} = a_k \frac{P_\Lambda}{\sqrt{V}},\ a_{k,\widetilde{\Lambda}} = a_k \frac{P_{\widetilde{\Lambda}}}{\sqrt{\widetilde{V}}}, \tag{3.2}$$

for any $k \in \mathbb{R}^d$. Using an *equivolume* partition $\left\{\widetilde{\Lambda}_i\right\}_{i=1}^{n(\widetilde{V})}$ of the full box Λ (cf. (2.1)-(2.2)):

$$\begin{aligned}
&n\left(\widetilde{V}\right) \equiv V/\widetilde{V} \in \mathbb{N}\backslash\{0\}, \text{ see (2.1)},\\
&\forall i \in \left\{1,...,n\left(\widetilde{V}\right)\right\},\ \left|\widetilde{\Lambda}_i\right| = \widetilde{V}_i = \widetilde{V},\\
&\forall i,j \in \left\{1,...,n\left(\widetilde{V}\right)\right\},\ i \neq j,\ \widetilde{\Lambda}_i \cap \widetilde{\Lambda}_j = \{\varnothing\},\\
&\bigcup_{i=1}^{n(\widetilde{V})} \widetilde{\Lambda}_i = \Lambda \text{ and notice that } \sum_{i=1}^{n(\widetilde{V})} \widetilde{V}_i = V,
\end{aligned} \tag{3.3}$$

one has

$$P_\Lambda = \sum_{i=1}^{n(\widetilde{V})} P_{\widetilde{\Lambda}_i}, \tag{3.4}$$

which by (3.2) implies

$$a_{k,\Lambda} = \sum_{i=1}^{n(\widetilde{V})} \frac{a_k}{\sqrt{V}} P_{\widetilde{\Lambda}_i} = \sum_{i=1}^{n(\widetilde{V})} \frac{\sqrt{\widetilde{V}}}{\sqrt{V}} a_{k,\widetilde{\Lambda}_i},\ \text{ for } k \in \widetilde{\Lambda}^*. \tag{3.5}$$

Therefore, since by (1.16) and (2.5) we have

$$\begin{aligned}
\left(\varphi, \rho_{\widetilde{\Lambda}^*}\varphi\right)_{\mathcal{F}_B^\infty} &= \frac{1}{\widetilde{\widetilde{V}}} \sum_{k \in \widetilde{\Lambda}^*} \left\|a_{k,\Lambda}\varphi\right\|^2_{\mathcal{F}_B^\infty},\\
\left(\varphi, \rho_{\widetilde{\Lambda},\mathrm{p.b.c}}\varphi\right)_{\mathcal{F}_B^\infty} &= \frac{1}{\widetilde{\widetilde{V}}} \sum_{k \in \widetilde{\Lambda}^*} \left\|a_{k,\widetilde{\Lambda}}\varphi\right\|^2_{\mathcal{F}_B^\infty},
\end{aligned}$$

for any $\varphi \in \mathcal{N}_B^{(\widetilde{\Lambda})} \cap \mathcal{N}_B^{(\widetilde{\Lambda}^*)}$ (1.17), by (3.5) we get

$$\begin{aligned}
\left(\varphi, \rho_{\widetilde{\Lambda}^*}\varphi\right)_{\mathcal{F}_B^\infty} &= \frac{1}{\widetilde{\widetilde{V}}} \sum_{k \in \widetilde{\Lambda}^*} \frac{\widetilde{V}}{V} \left\| \sum_{i=1}^{n(\widetilde{V})} a_{k,\widetilde{\Lambda}_i}\varphi \right\|^2_{\mathcal{F}_B^\infty} = \frac{1}{\widetilde{\widetilde{V}}} \sum_{k \in \widetilde{\Lambda}^*} \sum_{i=1}^{n(\widetilde{V})} \frac{\widetilde{V}}{V} \left\|a_{k,\widetilde{\Lambda}_i}\varphi\right\|^2_{\mathcal{F}_B^\infty}\\
&= \frac{\widetilde{V}}{V} \sum_{i=1}^{n(\widetilde{V})} \left\{ \frac{1}{\widetilde{\widetilde{V}}} \sum_{k \in \widetilde{\Lambda}^*} \left\|a_{k,\widetilde{\Lambda}_i}\varphi\right\|^2_{\mathcal{F}_B^\infty} \right\} = \frac{\widetilde{V}}{V} \sum_{i=1}^{n(\widetilde{V})} \left(\varphi, \rho_{\widetilde{\Lambda}_i,\mathrm{p.b.c}}\varphi\right)_{\mathcal{F}_B^\infty}.
\end{aligned} \tag{3.6}$$

Since the density matrix

$$d^X_{\Lambda,\rho} \equiv \frac{P_\Lambda e^{-\beta\left(H^X_{\Lambda,\mathrm{p.b.c}} - \mu^X_\Lambda(\rho) N_{\Lambda,\mathrm{p.b.c}}\right)} P_\Lambda}{Tr_{\mathcal{F}_B^\infty}\left(e^{-\beta\left(H^X_{\Lambda,\mathrm{p.b.c}} - \mu^X_\Lambda(\rho) N_{\Lambda,\mathrm{p.b.c}}\right)}\right)} \tag{3.7}$$

is a positive self-adjoint operator defined on $\mathcal{F}_B^\infty$, there is an operator D defined on $\mathcal{F}_B^\infty$ such that

$$d_{\Lambda,\rho}^X = D^* D. \tag{3.8}$$

Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis for $\mathcal{F}_B^\infty$, then by (3.8) one gets

$$\begin{aligned} Tr_{\mathcal{F}_B^\infty}\left(d_{\Lambda,\rho}^X A\right) &= Tr_{\mathcal{F}_B^\infty}\left(D^* D A\right) = Tr_{\mathcal{F}_B^\infty}\left(D A D^*\right) \\ &= \sum_{n=1}^{+\infty} \left(\varphi_n, D A D^* \varphi_n\right)_{\mathcal{F}_B^\infty} = \sum_{n=1}^{+\infty} \left(\left[D^* \varphi_n\right], A \left[D^* \varphi_n\right]\right)_{\mathcal{F}_B^\infty}, \end{aligned} \tag{3.9}$$

for any $A \in \mathcal{A}_B^\infty$. Therefore, using the Gibbs state $\omega_{\Lambda,\rho}^X$ (1.26) (see also (1.23)), from (3.9) extended for the unbounded operators $\rho_{\widetilde{\Lambda}^*}$ and $\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}$ the equality (3.6) implies that

$$\omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}^*}\right) = \frac{\widetilde{V}}{V} \sum_{i=1}^{n(\widetilde{V})} \omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right) = \frac{1}{n\left(\widetilde{V}\right)} \sum_{i=1}^{n(\widetilde{V})} \omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right). \tag{3.10}$$

Notice that the infinite volume Gibbs state ω_ρ^X (1.29) is *invariant by translation.* So $\forall i,j \in \left\{1,...,n\left(\widetilde{V}\right)\right\}$ one has

$$\omega_\rho^X\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right) = \lim_\Lambda \omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right) = \lim_\Lambda \omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}_j,\text{p.b.c}}\right) = \omega_\rho^X\left(\rho_{\widetilde{\Lambda}_j,\text{p.b.c}}\right),$$

and if V and $\widetilde{V}$ are here two *independent* parameters, one gets

$$\lim_{\widetilde{\Lambda}} \omega_\rho^X\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right) = \lim_{\widetilde{\Lambda}}\lim_\Lambda \omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right) = \lim_{\widetilde{\Lambda}}\lim_\Lambda \omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}_j,\text{p.b.c}}\right) = \lim_{\widetilde{\Lambda}} \omega_\rho^X\left(\rho_{\widetilde{\Lambda}_j,\text{p.b.c}}\right). \tag{3.11}$$

Let us consider the function $f(x,y)$, defined for $x>0$, $y>0$ by

$$f\left(\frac{1}{\widetilde{V}}, \frac{1}{V}\right) = \omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right) - \omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}_j,\text{p.b.c}}\right). \tag{3.12}$$

Then, considering V and $\widetilde{V}$ as two *independent* parameters, by (3.11) the function $f(x,y)$ (3.12) verifies:

$$\lim_{x\to 0^+}\lim_{y\to 0^+} f(x,y) = \lim_{\widetilde{\Lambda}}\lim_\Lambda \left[\omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right) - \omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}_j,\text{p.b.c}}\right)\right] = 0. \tag{3.13}$$

Using Lemma C.1 (Appendix C), there exists a function $y_0(x)>0$, i.e., $\Gamma\left(\widetilde{V}\right) \equiv \left\{y_0\left(1/\widetilde{V}\right)\right\}^{-1}$, such that

$$\begin{aligned} &\lim_{x\to 0^+} y_0(x) = 0, \\ &\lim_{x\to 0^+} f(x, y(x)) = \lim_{x\to 0^+}\lim_{y\to 0^+} f(x,y) = 0, \end{aligned}$$

for any $|y(x)| \le |y_0(x)|$ in a neighborhood of $(0,0)$, i.e.,

$$\begin{aligned} &\lim_{\widetilde{\Lambda}} \Gamma\left(\widetilde{V}\right) = +\infty, \\ &\lim_{\widetilde{\Lambda}} \left[\omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right) - \omega_{\Lambda,\rho}^X\left(\rho_{\widetilde{\Lambda}_j,\text{p.b.c}}\right)\right] = \lim_{\widetilde{\Lambda}} \left[\omega_\rho^X\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right) - \omega_\rho^X\left(\rho_{\widetilde{\Lambda}_j,\text{p.b.c}}\right)\right] = 0, \end{aligned} \tag{3.14}$$

for $V \geq \Gamma\left(\widetilde{V}\right)$ and $i,j \in \left\{1,...,n\left(\widetilde{V}\right)\right\}$. In fact, using the invariance by translation of the infinite volume Gibbs ω_{ρ}^{X} (1.29), $\forall i \in \left\{1,...,n\left(\widetilde{V}\right)\right\}$, by (3.14) one has

$$\lim_{\widetilde{\Lambda}} \omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right) = \lim_{\widetilde{\Lambda}} \omega_{\rho}^{X}\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right) = \lim_{\widetilde{\Lambda}} \omega_{\rho}^{X}\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right) = \lim_{\widetilde{\Lambda}} \omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right), \tag{3.15}$$

for $V \geq \Gamma\left(\widetilde{V}\right)$ or $L \geq \sqrt[3]{\Gamma\left(\widetilde{L}^3\right)}$, then for $\gamma(x) \geq \gamma_0(x) \equiv x^{-1}\sqrt[3]{\Gamma(x^3)}$ in (2.1)-(2.2). By (3.10), notice that $\omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}^*}\right)$ is similar to a sum of $n\left(\widetilde{V}\right) \equiv V/\widetilde{V}$ (3.3) expectation values $X_i\left(\widetilde{\Lambda}\right)$ of "random variables" $\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}$, i.e.,

$$\omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}^*}\right) = \frac{1}{n\left(\widetilde{V}\right)} \sum_{i=1}^{n(\widetilde{V})} \omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right) = \frac{1}{n\left(\widetilde{V}\right)} \sum_{i=1}^{n(\widetilde{V})} X_i\left(\widetilde{\Lambda}\right). \tag{3.16}$$

Since by (3.15) the expectation values $\left\{X_i\left(\widetilde{\Lambda}\right)\right\}_{i=1}^{n(\widetilde{V})}$ are all equal to $X = \omega_{\rho}^{X}\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right)$ in the limit $\Lambda \to \mathbb{R}^d$ then by (3.15)-(3.16) we obtain

$$\lim_{\widetilde{\Lambda}} \omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}^*}\right) = \lim_{\widetilde{\Lambda}} \omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right), \tag{3.17}$$

for $\gamma(x) \geq \gamma_0(x)$ which by (2.8) implies (3.1). ■

Remark 3.2. *For the Perfect Bose Gas, i.e., $U_{\Lambda,p.b.c}^{X} = 0$ (1.18), the corresponding finite volume Gibbs state $\omega_{\Lambda,\rho}^{PBG}$ is already translation invariant inside the box Λ, i.e., $\forall i \in \left\{1,...,n\left(\widetilde{V}\right)\right\}$ (3.3) we have*

$$\omega_{\Lambda,\rho}^{PBG}\left(\rho_{\widetilde{\Lambda}_i,p.b.c}\right) = \omega_{\Lambda,\rho}^{PBG}\left(\rho_{\widetilde{\Lambda},p.b.c}\right).$$

Then, from (2.8) and (3.10), the equality (3.1) is already verified in finite volume:

$$\omega_{\Lambda,\rho}^{PBG}\left(\rho_{\widetilde{\Lambda}^*}\right) = \omega_{\Lambda,\rho}^{PBG}\left(\rho_{\widetilde{\Lambda},p.b.c}\right) = \omega_{\Lambda,\rho}^{PBG}\left(\rho_{\widetilde{\Lambda}}\right).$$

3.2. The logarithmic moment generating functions

Let us consider $g_{\widetilde{\Lambda}}^{X}(\lambda)$ and $g_{\widetilde{\Lambda}^*}^{X}(\lambda)$ respectively defined by

$$\begin{aligned} g_{\widetilde{\Lambda}}^{X}(\lambda) &\equiv \frac{1}{\beta\widetilde{V}} \ln \omega_{\Lambda,\rho}^{X}\left(e^{\beta\lambda N_{\widetilde{\Lambda}}}\right), \\ g_{\widetilde{\Lambda}^*}^{X}(\lambda) &\equiv \frac{1}{\beta\widetilde{V}} \ln \omega_{\Lambda,\rho}^{X}\left(e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right). \end{aligned} \tag{3.18}$$

Lemma 3.3. *By (1.19)-(1.22) and (1.24)-(1.25), if the case (a) is verified, then for $\mu_{\Lambda}^{X}(\rho) < \mu_{\sup}$ one has:*

$$\begin{aligned} &\forall\lambda < \lambda_{\sup},\ \omega_{\Lambda,\rho}^{X}\left(e^{\beta\lambda N_{\widetilde{\Lambda}}}\right) < +\infty \text{ and } \omega_{\Lambda,\rho}^{X}\left(e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right) < +\infty, \\ &\lim_{\lambda\to\lambda_{\sup}^-} \omega_{\Lambda,\rho}^{X}\left(e^{\beta\lambda N_{\widetilde{\Lambda}}}\right) = +\infty \text{ and } \lim_{\lambda\to\lambda_{\sup}^-} \omega_{\Lambda,\rho}^{X}\left(e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right) = +\infty, \end{aligned} \tag{3.19}$$

with $\lambda_{\sup} \equiv \mu_{\sup} - \mu^X_\Lambda(\rho)$, *whereas if the Bose system satisfies* (b) *then* $\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\tilde\Lambda}}\right)$ *and* $\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\tilde\Lambda^*}}\right)$ *exist for any* $\lambda \in \mathbb{R}$ *and*

$$\lim_{\lambda\to+\infty} \omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\tilde\Lambda}}\right) = +\infty, \quad \lim_{\lambda\to+\infty} \omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\tilde\Lambda^*}}\right) = +\infty. \tag{3.20}$$

Proof. Let us consider the case (a), cf. (1.22) and (1.24)-(1.25), i.e., there is $\mu_{\sup} < +\infty$ such that for $\beta > 0$ one has

$$\begin{gathered} Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - \mu N_{\Lambda,\text{p.b.c}}\right)}\right) < +\infty, \text{ for } \mu < \mu_{\sup}, \\ \lim_{\mu\to\mu^-_{\sup}} Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - \mu N_{\Lambda,\text{p.b.c}}\right)}\right) = +\infty. \end{gathered} \tag{3.21}$$

(i) If $\lambda \le 0$ and $\mu < \mu_{\sup}$, i.e., $e^{\beta\lambda N_{\tilde\Lambda}} \le \mathbb{I}_{\mathcal{F}_B^\infty}$ and $e^{\beta\lambda N_{\tilde\Lambda^*}} \le \mathbb{I}_{\mathcal{F}_B^\infty}$, then by (1.19), we find

$$\begin{gathered} Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{\beta\lambda N_{\tilde\Lambda}} e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - \mu N_{\Lambda,\text{p.b.c}}\right)}\right) \le Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - \mu N_{\Lambda,\text{p.b.c}}\right)}\right), \\ Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{\beta\lambda N_{\tilde\Lambda^*}} e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - \mu N_{\Lambda,\text{p.b.c}}\right)}\right) \le Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - \mu N_{\Lambda,\text{p.b.c}}\right)}\right), \end{gathered} \tag{3.22}$$

and by (1.23) and (3.21) one gets

$$\omega^X_{\Lambda,\mu}\left(e^{\beta\lambda N_{\tilde\Lambda}}\right) < +\infty, \ \omega^X_{\Lambda,\mu}\left(e^{\beta\lambda N_{\tilde\Lambda^*}}\right) < +\infty, \text{ if } \lambda \le 0 \text{ and } \mu < \mu_{\sup}. \tag{3.23}$$

(ii) If $\lambda \ge 0$ and $\lambda + \mu < \mu_{\sup}$, i.e.,

$$\mathbb{I}_{\mathcal{F}_B^{(\Lambda)}} \le e^{\beta\lambda N_{\tilde\Lambda}} \le e^{\beta\lambda N_{\Lambda,\text{p.b.c}}}, \ \mathbb{I}_{\mathcal{F}_B^\infty} \le e^{\beta\lambda N_{\tilde\Lambda^*}} \le e^{\beta\lambda N_{\Lambda,\text{p.b.c}}},$$

then using (1.19) one has

$$\begin{gathered} Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{\beta\lambda N_{\tilde\Lambda}} e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - \mu N_{\Lambda,\text{p.b.c}}\right)}\right) \le Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - (\mu+\lambda) N_{\Lambda,\text{p.b.c}}\right)}\right), \\ Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{\beta\lambda N_{\tilde\Lambda^*}} e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - \mu N_{\Lambda,\text{p.b.c}}\right)}\right) \le Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - (\mu+\lambda) N_{\Lambda,\text{p.b.c}}\right)}\right), \end{gathered}$$

and by (1.23) and (3.21) one gets

$$\omega^X_{\Lambda,\mu}\left(e^{\beta\lambda N_{\tilde\Lambda}}\right) < +\infty, \ \omega^X_{\Lambda,\mu}\left(e^{\beta\lambda N_{\tilde\Lambda^*}}\right) < +\infty, \text{ if } \lambda \ge 0 \text{ and } \lambda + \mu < \mu_{\sup}. \tag{3.24}$$

(iii) By Remark 1.5 combining with (2.5), $\mathcal{F}_B^{(\tilde\Lambda)} \subset \mathcal{F}_B^{(\Lambda)} \subset \mathcal{F}_B^\infty$ implies:

$$Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{\beta\lambda N_{\tilde\Lambda^*}} e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - \mu N_{\Lambda,\text{p.b.c}}\right)}\right) \ \ge \ Tr_{\mathcal{F}_B^{(\tilde\Lambda)}}\left(e^{-\beta\left[\frac{\tilde V}{V}\right]\left(T_{\tilde\Lambda,\text{p.b.c}} + \frac{\tilde V}{V} U^X_{\tilde\Lambda,\text{p.b.c}} - (\mu+\lambda) N_{\tilde\Lambda,\text{p.b.c}}\right)}\right), \tag{3.25}$$

and for $\lambda > 0$, $n\left(\widetilde V\right) = V/\widetilde V > 1$,

$$Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{\beta\lambda N_{\tilde\Lambda}} e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - \mu N_{\Lambda,\text{p.b.c}}\right)}\right) \ \ge \ Tr_{\mathcal{F}_B^{(\tilde\Lambda)}}\left(e^{-\left[\beta\frac{\tilde V}{V}\right]\left(T_{\tilde\Lambda,\text{p.b.c}} + \frac{\tilde V}{V} U^X_{\tilde\Lambda,\text{p.b.c}} - (\mu+\lambda) N_{\tilde\Lambda,\text{p.b.c}}\right)}\right). \tag{3.26}$$

(iv) Let us consider the Bogoliubov (convexity) inequality [5,10] applied to the two Hamiltonians $H^X_{\widetilde{\Lambda}}$ (1.18) and

$$\widetilde{H}^X_{\widetilde{\Lambda},\text{p.b.c}} = T_{\widetilde{\Lambda},\text{p.b.c}} + \frac{\widetilde{V}}{V} U^X_{\widetilde{\Lambda},\text{p.b.c}},$$

then one has

$$\ln Tr_{\mathcal{F}_B^{(\widetilde{\Lambda})}} \left(e^{-\beta_{\widetilde{\Lambda}}\left(\widetilde{H}^X_{\widetilde{\Lambda},\text{p.b.c}} - (\mu+\lambda) N_{\widetilde{\Lambda},\text{p.b.c}}\right)} \right) - \ln Tr_{\mathcal{F}_B^{(\widetilde{\Lambda})}} \left(e^{-\beta_{\widetilde{\Lambda}}\left(H^X_{\widetilde{\Lambda}} - (\mu+\lambda) N_{\widetilde{\Lambda},\text{p.b.c}}\right)} \right) \geq \beta_{\widetilde{\Lambda}} \left(1 - \frac{\widetilde{V}}{V}\right) \times \\ \times \omega^X_{\widetilde{\Lambda},\mu+\lambda}\left(U^X_{\widetilde{\Lambda},\text{p.b.c}}\right), \tag{3.27}$$

with $\beta_{\widetilde{\Lambda}} = \beta\widetilde{V}/V \geq 0$. Here $\omega^X_{\widetilde{\Lambda},\mu+\lambda}(-)$ (1.23) represents the grand-canonical Gibbs state for the Hamiltonian $H^X_{\widetilde{\Lambda}}$ with chemical potential $(\mu+\lambda)$, an inverse temperature $\beta_{\widetilde{\Lambda}}$ and using periodic boundary conditions on $\widetilde{\Lambda}$. Then by (1.20) and (3.27)

$$\ln Tr_{\mathcal{F}_B^{(\widetilde{\Lambda})}} \left(e^{-\beta_{\widetilde{\Lambda}}\left(\widetilde{H}^X_{\widetilde{\Lambda},\text{p.b.c}} - (\mu+\lambda) N_{\widetilde{\Lambda},\text{p.b.c}}\right)} \right) \geq \widetilde{V}\beta_{\widetilde{\Lambda}} \left[p^X_{\widetilde{\Lambda}}\left(\beta_{\widetilde{\Lambda}}, (\mu+\lambda)\right) - B\left(1 - \frac{\widetilde{V}}{V}\right) \rho^X_{\widetilde{\Lambda}}\left(\beta_{\widetilde{\Lambda}}, (\mu+\lambda)\right) \right], \tag{3.28}$$

where the pressure $p^X_{\widetilde{\Lambda}}\left(\beta_{\widetilde{\Lambda}}, \alpha\right)$ and the particle density $\rho^X_{\widetilde{\Lambda}}\left(\beta_{\widetilde{\Lambda}}, \alpha\right)$ are respectively defined for a box $\widetilde{\Lambda}$ by (1.21) and (1.24). Since

$$\rho^X_{\widetilde{\Lambda}}\left(\beta_{\widetilde{\Lambda}}, \alpha\right) = \partial_\alpha p^X_{\widetilde{\Lambda}}\left(\beta_{\widetilde{\Lambda}}, \alpha\right),$$

by (1.22) one gets

$$\lim_{\lambda \to [\mu_{\sup} - \mu]} \left[p^X_{\widetilde{\Lambda}}\left(\beta_{\widetilde{\Lambda}}, (\mu+\lambda)\right) - B\left(1 - \frac{\widetilde{V}}{V}\right) \rho^X_{\widetilde{\Lambda}}\left(\beta_{\widetilde{\Lambda}}, (\mu+\lambda)\right) \right] = +\infty, \tag{3.29}$$

which by (3.25) and (3.28) implies

$$\lim_{\lambda \to [\mu_{\sup} - \mu]} Tr_{\mathcal{F}_B^{(\Lambda)}} \left(e^{\beta\lambda N_{\widetilde{\Lambda}^*}} e^{-\beta\left(H^X_{\Lambda,\text{p.b.c}} - \mu N_{\Lambda,\text{p.b.c}}\right)} \right) = +\infty,$$

i.e., by (1.23)-(1.27) one has

$$\lim_{\lambda \to [\mu_{\sup} - \mu^X_\Lambda(\rho)]} \omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right) = +\infty. \tag{3.30}$$

Moreover by (3.26) combining with (3.28) and (3.29) we find also

$$\lim_{\lambda \to [\mu_{\sup} - \mu^X_\Lambda(\rho)]} \omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}}}\right) = +\infty. \tag{3.31}$$

(v) If one has the case (b) then $\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}}}\right)$ and $\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right)$ exist for any $\lambda \in \mathbb{R}$ and verify (3.20), see arguments in (i)-(iv) with $\mu_{\sup} \to +\infty$. ∎

Remark 3.4. *If the case* (a)*, cf. (1.22) and (1.24)-(1.25), is verified, the functions* $g_{\widetilde{\Lambda}}^{X}(\lambda)$ *and* $g_{\widetilde{\Lambda}^*}^{X}(\lambda)$ *(3.18) are strictly convex for* $\lambda < \lambda_{\sup} \equiv \mu_{\sup} - \mu_{\Lambda}^{X}(\rho)$ *and verifies*

$$\lim_{\lambda \to \lambda_{\sup}} g_{\widetilde{\Lambda}}^{X}(\lambda) = \lim_{\lambda \to \lambda_{\sup}} g_{\widetilde{\Lambda}^*}^{X}(\lambda) = +\infty, \tag{3.32}$$

see Lemma 3.3. If one has the case (b)*, then* $g_{\widetilde{\Lambda}}^{X}(\lambda)$ *and* $g_{\widetilde{\Lambda}^*}^{X}(\lambda)$ *(3.18) are strictly convex for* $\lambda \in \mathbb{R}$ *and*

$$\lim_{\lambda \to +\infty} g_{\widetilde{\Lambda}}^{X}(\lambda) = \lim_{\lambda \to +\infty} g_{\widetilde{\Lambda}^*}^{X}(\lambda) = +\infty. \tag{3.33}$$

Theorem 3.5. *In the domain of existence of* $g_{\widetilde{\Lambda}}^{X}(\lambda)$ *(3.18) (or of* $g_{\widetilde{\Lambda}^*}^{X}(\lambda)$*), there exists* $\widetilde{\gamma}_0(x)$ *such that*

$$g^{X}(\lambda) = \lim_{\widetilde{\Lambda}} g_{\widetilde{\Lambda}}^{X}(\lambda) = \lim_{\widetilde{\Lambda}} g_{\widetilde{\Lambda}^*}^{X}(\lambda), \tag{3.34}$$

for $\gamma(x) \geq \widetilde{\gamma}_0(x)$ *(2.1)-(2.2).*

Proof. (i) If the case (a) is verified then for $\mu_{\Lambda}^{X}(\rho) < \mu_{\sup}$, $\lambda \geq 0$ and $\lambda < \lambda_{\sup} \equiv \mu_{\sup} - \mu_{\Lambda}^{X}(\rho)$, by (1.19) and (1.26)-(1.27) one gets

$$\begin{aligned}
\omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}} e^{\beta\lambda N_{\widetilde{\Lambda}}}\right) &\leq \frac{V}{\widetilde{V}}\omega_{\Lambda,\rho}^{X}\left(\rho_{\Lambda,\mathrm{p.b.c}} e^{\beta\lambda N_{\Lambda,\mathrm{p.b.c}}}\right) = \frac{V}{\widetilde{V}}\rho_{\Lambda}^{X}\left(\beta, \mu_{\Lambda}^{X}(\rho)+\lambda\right) < +\infty, \\
\omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}} e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right) &\leq \frac{V}{\widetilde{V}}\rho_{\Lambda}^{X}\left(\beta, \mu_{\Lambda}^{X}(\rho)+\lambda\right) < +\infty, \\
\omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}^*} e^{\beta\lambda N_{\widetilde{\Lambda}}}\right) &\leq \frac{V}{\widetilde{V}}\rho_{\Lambda}^{X}\left(\beta, \mu_{\Lambda}^{X}(\rho)+\lambda\right) < +\infty, \\
\omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}^*} e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right) &\leq \frac{V}{\widetilde{V}}\rho_{\Lambda}^{X}\left(\beta, \mu_{\Lambda}^{X}(\rho)+\lambda\right) < +\infty.
\end{aligned} \tag{3.35}$$

with $\rho_{\widetilde{\Lambda}} = N_{\widetilde{\Lambda}}/\widetilde{V}$ (1.15), $\rho_{\widetilde{\Lambda}^*} = N_{\widetilde{\Lambda}^*}/\widetilde{V}$ (1.16) and $\rho_{\Lambda,\mathrm{p.b.c}} = N_{\Lambda,\mathrm{p.b.c}}/V$ (1.11). If $\lambda \leq 0$, by (1.26)-(1.27) one has

$$\begin{aligned}
\omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}} e^{\beta\lambda N_{\widetilde{\Lambda}}}\right) &\leq \frac{V}{\widetilde{V}}\omega_{\Lambda,\rho}^{X}\left(\rho_{\Lambda,\mathrm{p.b.c}}\right) = \frac{V}{\widetilde{V}}\rho < +\infty, \\
\omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}} e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right) &\leq \frac{V}{\widetilde{V}}\rho < +\infty, \\
\omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}^*} e^{\beta\lambda N_{\widetilde{\Lambda}}}\right) &\leq \frac{V}{\widetilde{V}}\rho < +\infty, \\
\omega_{\Lambda,\rho}^{X}\left(\rho_{\widetilde{\Lambda}^*} e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right) &\leq \frac{V}{\widetilde{V}}\rho < +\infty.
\end{aligned} \tag{3.36}$$

Therefore using (3.18), (3.35), (3.36) and Remark 3.4, one gets

$$\left|\frac{\omega_{\Lambda,\rho}^{X}\left(\left[\rho_{\widetilde{\Lambda}} - \rho_{\widetilde{\Lambda}^*}\right] e^{\beta\lambda N_{\widetilde{\Lambda}}}\right)}{\omega_{\Lambda,\rho}^{X}\left(e^{\beta\lambda N_{\widetilde{\Lambda}}}\right)}\right| < +\infty, \quad \left|\frac{\omega_{\Lambda,\rho}^{X}\left(\left[\rho_{\widetilde{\Lambda}} - \rho_{\widetilde{\Lambda}^*}\right] e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right)}{\omega_{\Lambda,\rho}^{X}\left(e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right)}\right| < +\infty, \tag{3.37}$$

for $\lambda < \lambda_{\sup} \equiv \mu_{\sup} - \mu_{\Lambda}^{X}(\rho)$.
If the case (b) is verified then, using the same arguments with $\mu_{\sup} \to +\infty$, (3.37) is also satisfied for $\lambda \in \mathbb{R}$.

(ii) Using the Bogoliubov inequality (A.17) (Appendix A), one gets

$$g^X_{\widetilde{\Lambda}^*}(\lambda) - g^X_{\widetilde{\Lambda}}(\lambda) \leq \lambda \zeta^X_{\lambda,\widetilde{\Lambda}^*}\left(\rho_{\widetilde{\Lambda}} - \rho_{\widetilde{\Lambda}^*}\right). \tag{3.38}$$

Here we define the new state $\zeta^X_{\lambda,\widetilde{\Lambda}^*}$ for $A \in \mathcal{A}^\infty_B$ by

$$\zeta^X_{\lambda,\widetilde{\Lambda}^*}(A) = Tr_{\mathcal{F}^\infty_B}\left(\sigma^X_{\lambda,\widetilde{\Lambda}^*}A\right),\ \sigma^X_{\lambda,\widetilde{\Lambda}^*} = \left[Z^X_{\lambda,\widetilde{\Lambda}^*}\right]^{-1} d^X_{\Lambda,\rho} e^{\beta\lambda N_{\widetilde{\Lambda}^*}}, \tag{3.39}$$

with

$$Z^X_{\lambda,\widetilde{\Lambda}^*} = \omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right) = Tr_{\mathcal{F}^\infty_B}\left(d^X_{\Lambda,\rho} e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right) < +\infty, \tag{3.40}$$

and the density matrix $d^X_{\Lambda,\rho}$ defined by (3.7).
(iii) Since $\sigma^X_{\lambda,\widetilde{\Lambda}^*}$ (3.39) is a density matrix, notice that the state $\zeta^X_{\lambda,\widetilde{\Lambda}^*}$ (3.39) is a normal state which converges to a quasi-normal state ζ^X_λ on $\mathcal{A}^\infty_B$ (see discussion p. 26 in [11]). In fact we conjecture that

$$\zeta^X_\lambda(A) = \lim_{\widetilde{\Lambda}} \omega^X_{\Lambda,\lambda+\lim_\Lambda \mu^X_\Lambda(\rho)}(A),\ \text{for any } A \in \mathcal{A}^\infty_B, \tag{3.41}$$

see (1.23) and (1.26). *Note that this last conjecture (3.41) is just a remark and is not necessary for the rest of this proof.*
Following the arguments using for the proof of Theorem 3.1 for the state $\zeta^X_{\lambda,\widetilde{\Lambda}^*}$ instead of $\omega^X_{\Lambda,\rho}$ one has

$$\zeta^X_{\lambda,\widetilde{\Lambda}^*}\left(\rho_{\widetilde{\Lambda}^*}\right) = \frac{1}{n\left(\widetilde{V}\right)} \sum_{i=1}^{n(\widetilde{V})} \zeta^X_{\lambda,\widetilde{\Lambda}^*}\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right),\ n\left(\widetilde{V}\right) = V/\widetilde{V} \in \mathbb{N}\backslash\{0\}\ \text{(2.1)-(2.2)}, \tag{3.42}$$

where we use an equivolume partition $\left\{\widetilde{\Lambda}_i\right\}_{i=1}^{n(\widetilde{V})}$ (3.3) of the full box Λ. Then, using the invariance by translation of $\zeta^X_{\lambda,\widetilde{\Lambda}^*}$ (3.39) in the limit $\Lambda \to \mathbb{R}^d$, there is $\gamma_1(x)$ such that

$$\lim_{\widetilde{\Lambda}} \zeta^X_{\lambda,\widetilde{\Lambda}^*}\left(\rho_{\widetilde{\Lambda}^*}\right) = \lim_{\widetilde{\Lambda}} \zeta^X_{\lambda,\widetilde{\Lambda}^*}\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right), \tag{3.43}$$

for $\gamma(x) \geq \gamma_1(x)$ (2.1)-(2.2) which, by Theorem B.1 in Appendix B for $\omega = \zeta^X_{\lambda,\widetilde{\Lambda}^*}$ and (3.38), implies

$$\lim_{\widetilde{\Lambda}} \left[g^X_{\widetilde{\Lambda}^*}(\lambda) - g^X_{\widetilde{\Lambda}}(\lambda)\right] \leq 0. \tag{3.44}$$

(iv) Using again an equivolume partition $\left\{\widetilde{\Lambda}_i\right\}_{i=1}^{n(\widetilde{V})}$ (3.3) of the full box Λ and the operator $N_{\widetilde{\Lambda},\text{p.b.c}}$ (2.5), by the Bogoliubov inequality (A.17) (Appendix A) for $i \in \left\{1,...,n\left(\widetilde{V}\right)\right\}$ one has

$$\frac{1}{\beta V}\ln\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right) - \frac{1}{\beta V}\ln\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}_i,\text{p.b.c}}}\right) \geq \frac{\lambda}{V}\zeta^X_{\lambda,\widetilde{\Lambda}_i,\text{p.b.c}}\left(N_{\widetilde{\Lambda}_i,\text{p.b.c}} - N_{\widetilde{\Lambda}^*}\right), \tag{3.45}$$

where $\forall i \in \left\{1,...,n\left(\widetilde{V}\right)\right\}$ we define the new states $\zeta^X_{\lambda,\widetilde{\Lambda}_i,\text{p.b.c}}$ for $A \in \mathcal{A}^\infty_B$ by

$$\zeta^X_{\lambda,\widetilde{\Lambda}_i,\text{p.b.c}}(A) = Tr_{\mathcal{F}^\infty_B}\left(\sigma^X_{\lambda,\widetilde{\Lambda}_i,\text{p.b.c}}A\right),\ \sigma^X_{\lambda,\widetilde{\Lambda}_i,\text{p.b.c}} = \left[Z^X_{\lambda,\widetilde{\Lambda}_i,\text{p.b.c}}\right]^{-1} d^X_{\Lambda,\rho} e^{\beta\lambda N_{\widetilde{\Lambda}_i,\text{p.b.c}}}, \tag{3.46}$$

with

$$Z^X_{\lambda,\widetilde{\Lambda}_i,\mathrm{p.b.c}} = \omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}_i,\mathrm{p.b.c}}}\right) = Tr_{\mathcal{F}^\infty_B}\left(d^X_{\Lambda,\rho}e^{\beta\lambda N_{\widetilde{\Lambda}_i,\mathrm{p.b.c}}}\right) < +\infty, \tag{3.47}$$

see also (3.7) for the definition of $d^X_{\Lambda,\rho}$. Therefore by (3.45) we obtain

$$\sum_{i=1}^{n(\widetilde{V})}\left\{\frac{1}{\beta V}\ln\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right) - \frac{1}{\beta V}\ln\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}_i,\mathrm{p.b.c}}}\right)\right\} \geq \frac{\lambda}{V}\sum_{i=1}^{n(\widetilde{V})}\zeta^X_{\lambda,\widetilde{\Lambda}_i,\mathrm{p.b.c}}\left(N_{\widetilde{\Lambda}_i,\mathrm{p.b.c}} - N_{\widetilde{\Lambda}^*}\right),$$

which implies

$$\frac{1}{\beta\widetilde{V}}\ln\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right) - \frac{1}{n\left(\widetilde{V}\right)}\sum_{i=1}^{n(\widetilde{V})}\frac{1}{\beta\widetilde{V}}\ln\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}_i,\mathrm{p.b.c}}}\right) \geq \frac{\lambda}{n\left(\widetilde{V}\right)}\sum_{i=1}^{n(\widetilde{V})}\zeta^X_{\lambda,\widetilde{\Lambda}_i,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}_i,\mathrm{p.b.c}} - \rho_{\widetilde{\Lambda}^*}\right). \tag{3.48}$$

Following the arguments done from (3.2) to (3.10) for the states $\zeta^X_{\lambda,\widetilde{\Lambda}_j,\mathrm{p.b.c}}$ (3.46) instead of $\omega^X_{\Lambda,\rho}$ one has

$$\forall j \in \left\{1,...,n\left(\widetilde{V}\right)\right\},\ \zeta^X_{\lambda,\widetilde{\Lambda}_j,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}^*}\right) = \frac{1}{n\left(\widetilde{V}\right)}\sum_{i=1}^{n(\widetilde{V})}\zeta^X_{\lambda,\widetilde{\Lambda}_j,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}_i,\mathrm{p.b.c}}\right). \tag{3.49}$$

If $\lambda \geq 0$, then

$$\forall i,j \in \left\{1,...,n\left(\widetilde{V}\right)\right\},\ i\neq j,\ \zeta^X_{\lambda,\widetilde{\Lambda}_j,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}_i,\mathrm{p.b.c}}\right) \leq \zeta^X_{\lambda,\widetilde{\Lambda}_j,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}_j,\mathrm{p.b.c}}\right),$$

and by (3.49) we obtain

$$\lambda\left\{\frac{1}{n\left(\widetilde{V}\right)}\sum_{i=1}^{n(\widetilde{V})}\zeta^X_{\lambda,\widetilde{\Lambda}_i,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}_i,\mathrm{p.b.c}}\right) - \zeta^X_{\lambda,\widetilde{\Lambda}_j,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}^*}\right)\right\} \geq 0,\ \text{for } \lambda \geq 0. \tag{3.50}$$

If $\lambda \leq 0$, then

$$\forall i,j \in \left\{1,...,n\left(\widetilde{V}\right)\right\},\ i\neq j,\ \zeta^X_{\lambda,\widetilde{\Lambda}_j,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}_i,\mathrm{p.b.c}}\right) \geq \zeta^X_{\lambda,\widetilde{\Lambda}_j,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}_j,\mathrm{p.b.c}}\right),$$

and by (3.49) we obtain

$$\lambda\left\{\frac{1}{n\left(\widetilde{V}\right)}\sum_{i=1}^{n(\widetilde{V})}\zeta^X_{\lambda,\widetilde{\Lambda}_i,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}_i,\mathrm{p.b.c}}\right) - \zeta^X_{\lambda,\widetilde{\Lambda}_j,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}^*}\right)\right\} \geq 0,\ \text{for } \lambda \leq 0. \tag{3.51}$$

Since

$$\begin{aligned}
\frac{1}{n\left(\widetilde{V}\right)}\sum_{i=1}^{n(\widetilde{V})}\zeta^X_{\lambda,\widetilde{\Lambda}_i,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}_i,\mathrm{p.b.c}} - \rho_{\widetilde{\Lambda}^*}\right) &\leq \frac{1}{n\left(\widetilde{V}\right)}\sum_{i=1}^{n(\widetilde{V})}\zeta^X_{\lambda,\widetilde{\Lambda}_i,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}_i,\mathrm{p.b.c}}\right) - \zeta^X_{\lambda,\widetilde{\Lambda}_{i_{\min}},\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}^*}\right),\\
\frac{1}{n\left(\widetilde{V}\right)}\sum_{i=1}^{n(\widetilde{V})}\zeta^X_{\lambda,\widetilde{\Lambda}_i,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}_i,\mathrm{p.b.c}} - \rho_{\widetilde{\Lambda}^*}\right) &\geq \frac{1}{n\left(\widetilde{V}\right)}\sum_{i=1}^{n(\widetilde{V})}\zeta^X_{\lambda,\widetilde{\Lambda}_i,\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}_i,\mathrm{p.b.c}}\right) - \zeta^X_{\lambda,\widetilde{\Lambda}_{i_{\max}},\mathrm{p.b.c}}\left(\rho_{\widetilde{\Lambda}^*}\right),
\end{aligned}$$

with $i_{\min}$, $i_{\max} \in \left\{1, ..., n\left(\widetilde{V}\right)\right\}$ such that

$$\forall i \in \left\{1, ..., n\left(\widetilde{V}\right)\right\},\ \zeta^X_{\lambda,\widetilde{\Lambda}_{i_{\min}},\text{p.b.c}}\left(\rho_{\widetilde{\Lambda}^*}\right) \leq \zeta^X_{\lambda,\widetilde{\Lambda}_i,\text{p.b.c}}\left(\rho_{\widetilde{\Lambda}^*}\right) \leq \zeta^X_{\lambda,\widetilde{\Lambda}_{i_{\max}},\text{p.b.c}}\left(\rho_{\widetilde{\Lambda}^*}\right),$$

then from (3.50) and (3.51)

$$\frac{\lambda}{n\left(\widetilde{V}\right)}\sum_{i=1}^{n\left(\widetilde{V}\right)}\zeta^X_{\lambda,\widetilde{\Lambda}_i,\text{p.b.c}}\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}-\rho_{\widetilde{\Lambda}^*}\right) \geq \lambda\left\{\frac{1}{n\left(\widetilde{V}\right)}\sum_{i=1}^{n\left(\widetilde{V}\right)}\zeta^X_{\lambda,\widetilde{\Lambda}_i,\text{p.b.c}}\left(\rho_{\widetilde{\Lambda}_i,\text{p.b.c}}\right)-\zeta^X_{\lambda,\widetilde{\Lambda}_l,\text{p.b.c}}\left(\rho_{\widetilde{\Lambda}^*}\right)\right\} \geq 0, \tag{3.52}$$

with

$$l = \begin{cases} i_{\max} \text{ if } \lambda \geq 0. \\ i_{\min} \text{ if } \lambda \leq 0. \end{cases}$$

Moreover using the same arguments than for the proof of the Theorem 3.1 (invariance by translation of ω^X_ρ (1.29)), there is $\gamma_2(x)$ such that

$$\lim_{\widetilde{\Lambda}}\frac{1}{n\left(\widetilde{V}\right)}\sum_{i=1}^{n\left(\widetilde{V}\right)}\frac{1}{\beta\widetilde{V}}\ln\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}_i,\text{p.b.c}}}\right)=\lim_{\widetilde{\Lambda}}\frac{1}{\beta\widetilde{V}}\ln\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda},\text{p.b.c}}}\right), \tag{3.53}$$

for $\gamma(x) \geq \gamma_2(x)$ (2.1)-(2.2), and so, by (3.18) and (3.48) combining with (3.52), we get

$$\lim_{\widetilde{\Lambda}} g^X_{\widetilde{\Lambda}^*}(\lambda) \geq \lim_{\widetilde{\Lambda}}\frac{1}{\beta\widetilde{V}}\ln\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda},\text{p.b.c}}}\right). \tag{3.54}$$

(v) Again by the Bogoliubov inequality (A.17) (Appendix A) one has

$$\lambda\zeta^X_{\lambda,\widetilde{\Lambda}}\left(\rho_{\widetilde{\Lambda}}-\rho_{\widetilde{\Lambda},\text{p.b.c}}\right) \leq \frac{1}{\beta\widetilde{V}}\ln\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda},\text{p.b.c}}}\right)-g^X_{\widetilde{\Lambda}}(\lambda) \leq \lambda\zeta^X_{\lambda,\widetilde{\Lambda},\text{p.b.c}}\left(\rho_{\widetilde{\Lambda}}-\rho_{\widetilde{\Lambda},\text{p.b.c}}\right). \tag{3.55}$$

where we define the new state $\zeta^X_{\lambda,\widetilde{\Lambda}}$ for $A \in \mathcal{A}^\infty_B$ by

$$\zeta^X_{\lambda,\widetilde{\Lambda}}(A) = Tr_{\mathcal{F}^\infty_B}\left(\sigma^X_{\lambda,\widetilde{\Lambda}}A\right),\ \sigma^X_{\lambda,\widetilde{\Lambda}} = \left[Z^X_{\lambda,\widetilde{\Lambda}}\right]^{-1} d^X_{\Lambda,\rho}e^{\beta\lambda N_{\widetilde{\Lambda}}}, \tag{3.56}$$

with

$$Z^X_{\lambda,\widetilde{\Lambda}} = \omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}}}\right) = Tr_{\mathcal{F}^\infty_B}\left(d^X_{\Lambda,\rho}e^{\beta\lambda N_{\widetilde{\Lambda}}}\right) < +\infty. \tag{3.57}$$

Using the Theorem B.1 in Appendix B for $\omega = \zeta^X_{\lambda,\widetilde{\Lambda},\text{p.b.c}}$ (3.46) and $\omega = \zeta^X_{\lambda,\widetilde{\Lambda}}$ (3.56) one gets

$$\zeta^X_{\lambda,\widetilde{\Lambda},\text{p.b.c}}\left(\rho_{\widetilde{\Lambda}}-\rho_{\widetilde{\Lambda},\text{p.b.c}}\right) = \zeta^X_{\lambda,\widetilde{\Lambda}}\left(\rho_{\widetilde{\Lambda}}-\rho_{\widetilde{\Lambda},\text{p.b.c}}\right) = 0,$$

which by (3.55) implies

$$g^X_{\widetilde{\Lambda}}(\lambda) = \frac{1}{\beta\widetilde{V}}\ln\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda},\text{p.b.c}}}\right). \tag{3.58}$$

Consequently, combining (3.44) with (3.54), by (3.58) we find (3.34) for $\gamma(x) \geq \widetilde{\gamma}_0(x) \equiv \sup\{\gamma_1(x), \gamma_2(x)\}$ (2.1)-(2.2). ■

Remark 3.6. *For the Perfect Bose Gas, i.e.,* $U^{PBG}_{\Lambda,p.b.c}=0$ *(1.18), the corresponding finite volume Gibbs state* $\omega^{PBG}_{\Lambda,\rho}$ *is already translation invariant inside the box* Λ*. Then we have*

$$\left\{\begin{array}{l}\zeta^{PBG}_{\lambda,\widetilde{\Lambda}^*}\left(\rho_{\widetilde{\Lambda}^*}\right)=\zeta^{PBG}_{\lambda,\widetilde{\Lambda}^*}\left(\rho_{\widetilde{\Lambda},p.b.c}\right)=\zeta^{PBG}_{\lambda,\widetilde{\Lambda}^*}\left(\rho_{\widetilde{\Lambda}}\right)\ \text{cf. (3.42)-(3.43) and Theorem B.1 (Appendix B)},\\ \dfrac{1}{n\left(\widetilde{V}\right)}\displaystyle\sum_{i=1}^{n(\widetilde{V})}\frac{1}{\beta\widetilde{V}}\ln\omega^{PBG}_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}_i,p.b.c}}\right)=\frac{1}{\beta\widetilde{V}}\ln\omega^{PBG}_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda},p.b.c}}\right),\ \text{cf. (3.53)},\end{array}\right\}$$

which, by (3.38) and by (3.48) combining with (3.52) and (3.58), implies

$$g^{PBG}_{\widetilde{\Lambda}^*}(\lambda)=g^{PBG}_{\widetilde{\Lambda}}(\lambda),$$

in the domain of existence of $g^{PBG}_{\widetilde{\Lambda}}(\lambda)$ *or* $g^{PBG}_{\widetilde{\Lambda}^*}(\lambda)$*, i.e. for* $\lambda<\mu^{PBG}_{\Lambda}(\rho)$ *(*$\mu_{\sup}=0$ *for the Perfect Bose Gas).*

3.3. Main results

Notice that we restrict our analysis on a domain of (β,μ) in which

$$\lim_{\widetilde{\Lambda}}\omega^X_{\Lambda,\rho}\left(\rho_{\widetilde{\Lambda}}\right)<+\infty,\ \ \lim_{\widetilde{\Lambda}}\omega^X_{\Lambda,\rho}\left(\rho_{\widetilde{\Lambda}^*}\right)<+\infty.$$

We choose also $\gamma(x)$ in (2.1)-(2.2) which diverges to ∞ sufficiently quickly (cf. (2.2)) such that Theorems 3.1 and 3.5 are verified. Note that

$$\left\{g^X_{\widetilde{\Lambda}}(\lambda)=\frac{1}{\beta\widetilde{V}}\ln\omega^X_{\Lambda}\left(e^{\beta\lambda N_{\widetilde{\Lambda}}}\right)\right\}_{\widetilde{\Lambda}},\ \ \left\{g^X_{\widetilde{\Lambda}^*}(\lambda)=\frac{1}{\beta\widetilde{V}}\ln\omega^X_{\Lambda}\left(e^{\beta\lambda N_{\widetilde{\Lambda}^*}}\right)\right\}_{\widetilde{\Lambda}}$$

are two sequences of convex functions (cf. Remark 3.4) which verify (3.32) or (3.33) and also

$$\partial_\lambda g^X_{\widetilde{\Lambda}}(\lambda)=\zeta^X_{\lambda,\widetilde{\Lambda}}\left(\frac{N_{\widetilde{\Lambda}}}{\widetilde{V}}\right),\ \partial_\lambda g^X_{\widetilde{\Lambda}^*}(\lambda)=\zeta^X_{\lambda,\widetilde{\Lambda}^*}\left(\frac{N_{\widetilde{\Lambda}^*}}{\widetilde{V}}\right),$$

with $\zeta^X_{\lambda,\widetilde{\Lambda}^*}$ and $\zeta^X_{\lambda,\widetilde{\Lambda}}$ respectively defined by (3.39)-(3.40) and (3.56)-(3.57). So, for any $a>0$ there are two sequences $\left\{\lambda_{\widetilde{\Lambda},a}\right\}_{\widetilde{\Lambda}}$ and $\left\{\lambda_{\widetilde{\Lambda}^*,a}\right\}_{\widetilde{\Lambda}}$ such that

$$\partial_\lambda g^X_{\widetilde{\Lambda}}\left(\lambda_{\widetilde{\Lambda},a}\right)=a,\ \partial_\lambda g^X_{\widetilde{\Lambda}^*}\left(\lambda_{\widetilde{\Lambda}^*,a}\right)=a. \tag{3.59}$$

From Theorem 3.5, note that

$$\lambda_a=\lim_{\widetilde{\Lambda}}\lambda_{\widetilde{\Lambda}^*,a}=\lim_{\widetilde{\Lambda}}\lambda_{\widetilde{\Lambda},a}. \tag{3.60}$$

Then we can express the main statement of this section:

Theorem 3.7. *If for any* $a>0$,

$$\lim_{\widetilde{\Lambda}}\zeta^X_{\lambda_{\widetilde{\Lambda}^*,a},\widetilde{\Lambda}^*}\left(\chi_{[a,a+\frac{1}{V}]}\left(\rho_{\widetilde{\Lambda}^*}\right)\right)=\lim_{\widetilde{\Lambda}}\zeta^X_{\lambda_{\widetilde{\Lambda},a},\widetilde{\Lambda}}\left(\chi_{[a,a+\frac{1}{V}]}\left(\rho_{\widetilde{\Lambda}}\right)\right)=0, \tag{3.61}$$

then for any interval $I=[a,b]$ *one has*

$$\lim_{\widetilde{\Lambda}}\frac{1}{\beta\widetilde{V}}\ln\mathbb{P}\left\{\rho_{\widetilde{\Lambda}}\in I\right\}=\lim_{\widetilde{\Lambda}}\frac{1}{\beta\widetilde{V}}\ln\mathbb{P}\left\{\rho_{\widetilde{\Lambda}^*}\in I\right\}. \tag{3.62}$$

The states $\zeta^X_{\lambda,\widetilde{\Lambda}^*}$ *and* $\zeta^X_{\lambda,\widetilde{\Lambda}}$ *are defined by (3.39)-(3.40) and (3.56)-(3.57) respectively.*

Proof. (The proof is just an application of the one done in [2]). We recall that

$$Q_{\widetilde{\Lambda}} = \mathbb{P}\left\{\rho_{\widetilde{\Lambda}} \in I\right\} = \omega^X_{\Lambda,\rho}\left(\chi_I\left(\rho_{\widetilde{\Lambda}}\right)\right), \tag{3.63}$$

see (2.3), and

$$Q_{\widetilde{\Lambda}^*} = \mathbb{P}\left\{\rho_{\widetilde{\Lambda}^*} \in I\right\} = \omega^X_{\Lambda,\rho}\left(\chi_I\left(\rho_{\widetilde{\Lambda}^*}\right)\right), \tag{3.64}$$

where χ_A is the characteristic function of $A \subseteq \mathbb{R}$.
Let us consider the case (a), cf. (1.19)-(1.22) and (1.24)-(1.25). Let $a > \rho$. Since $\rho^X_\Lambda(\beta,\mu)$ is strictly increasing in μ, we have $0 < \lambda_{\widetilde{\Lambda},a} < \lambda_{\sup} \equiv \mu_{\sup} - \mu^X_\Lambda(\rho)$, cf. Remark 3.4 and (3.59).
(i) Using the state $\zeta^X_{\lambda,\widetilde{\Lambda}}$ (3.56)-(3.57), (3.63) is equal to

$$Q_{\widetilde{\Lambda}} = Z^X_{\lambda,\widetilde{\Lambda}}\zeta^X_{\lambda,\widetilde{\Lambda}}\left(e^{-\beta\lambda N_{\widetilde{\Lambda}}}\chi_I\left(\rho_{\widetilde{\Lambda}}\right)\right), \tag{3.65}$$

with the partition function $Z^X_{\lambda,\widetilde{\Lambda}}$ defined by (3.57). Using the exponential Chebychev inequality we find a first upper bound for $Q_{\widetilde{\Lambda}}$:

$$Q_{\widetilde{\Lambda}} \leq \omega^X_{\lambda,\widetilde{\Lambda}}\left(e^{\beta\lambda\left(N_{\widetilde{\Lambda}}-a\widetilde{V}\right)}\right) = Z^X_{\lambda,\widetilde{\Lambda}}e^{-\beta\lambda a\widetilde{V}}, \tag{3.66}$$

for any $0 < \lambda < \lambda_{\sup}$.
Therefore using (3.61) for $c \in (a,b)$ one gets

$$\begin{aligned} Q_{\widetilde{\Lambda}} &\geq Z^X_{\lambda_{\widetilde{\Lambda},a},\widetilde{\Lambda}}\zeta^X_{\lambda_{\widetilde{\Lambda},a},\widetilde{\Lambda}}\left(e^{-\beta\lambda_{\widetilde{\Lambda},a}N_{\widetilde{\Lambda}}}\chi_{[a,c]}\left(\rho_{\widetilde{\Lambda}}\right)\right) \\ &\geq Z^X_{\lambda_{\widetilde{\Lambda},a},\widetilde{\Lambda}}e^{-\beta c\widetilde{V}\lambda_{\widetilde{\Lambda},a}}\zeta^X_{\lambda_{\widetilde{\Lambda},a},\widetilde{\Lambda}}\left(\chi_{[a,c]}\left(\rho_{\widetilde{\Lambda}}\right)\right) \\ &\geq \alpha Z^X_{\lambda_{\widetilde{\Lambda},a},\widetilde{\Lambda}}e^{-\beta\widetilde{V}c\lambda_{\widetilde{\Lambda},a}}, \alpha \in (0,1). \end{aligned} \tag{3.67}$$

Therefore combining (3.66) with (3.67) we find

$$g^X_{\widetilde{\Lambda}}\left(\lambda_{\widetilde{\Lambda},a}\right) - c\lambda_{\widetilde{\Lambda},a} + o(1) \leq \frac{1}{\beta\widetilde{V}}\ln Q_{\widetilde{\Lambda}} \leq g^X_{\widetilde{\Lambda}}\left(\lambda_{\widetilde{\Lambda},a}\right) - a\lambda_{\widetilde{\Lambda},a} + o(1),$$

for any $c \in (a,b)$, which implies

$$\lim_{\widetilde{\Lambda}} \frac{1}{\beta\widetilde{V}}\ln Q_{\widetilde{\Lambda}} = g^X(\lambda_a) - a\lambda_a, \tag{3.68}$$

with λ_a defined by (3.60) and $g^X(\lambda_a)$ defined by (3.34).
(ii) Using exactly the same argument for $Q_{\widetilde{\Lambda}^*}$ (3.64), by (3.61) we find

$$g^X_{\widetilde{\Lambda}^*}\left(\lambda_{\widetilde{\Lambda}^*,a}\right) - c\lambda_{\widetilde{\Lambda}^*,a} + o(1) \leq \frac{1}{\beta\widetilde{V}}\ln Q_{\widetilde{\Lambda}^*} \leq g^X_{\widetilde{\Lambda}^*}\left(\lambda_{\widetilde{\Lambda}^*,a}\right) - a\lambda_{\widetilde{\Lambda}^*,a} + o(1), \tag{3.69}$$

with $\lambda_{\widetilde{\Lambda}^*,a}$ solution of (3.59). By (3.60), (3.68) and (3.69) one gets

$$\lim_{\widetilde{\Lambda}} \frac{1}{\beta\widetilde{V}}\ln Q_{\widetilde{\Lambda}^*} = g^X(\lambda_a) - a\lambda_a = \lim_{\widetilde{\Lambda}} \frac{1}{\beta\widetilde{V}}\ln Q_{\widetilde{\Lambda}^*},$$

i.e., (3.62) (see (3.63) and (3.64)).
Using exactly the same kinds of argument for $a < \rho$, i.e., for $\lambda < 0$, one gets the same result. If the case (b) is verified then, following the same arguments with $\mu_{\sup} \to +\infty$, (3.62) is also satisfied. ■

Remark 3.8. *In fact, the condition (3.61) is necessary to prove Theorem 3.7 only if there is a conventional Bose-Einstein condensation for the fixed particle density ρ and/or a fixed local particle density a in the Bose gas X (1.18), see for example (III.2)-(III.14) in [2].*

Remark 3.9. *Moreover, it is not clear if the first two limits in (3.61) are correlated to each other and if this technical assumption can be deleted in the general setting. Actually, this condition (3.61) appears to be only sufficient in order to imply the existence of a large deviation principle for $\rho_{\widetilde{\Lambda}^*}$ and $\rho_{\widetilde{\Lambda}}$ respectively, with the correct constant $\left\{\left|\widetilde{\Lambda}\right| = \widetilde{V}\right\}$.*

3.4. A direct application to the Perfect Bose Gas

(**i**) Using periodic boundary conditions, the Perfect Bose Gas is of course defined by (1.18) with $U^X_{\Lambda,\mathrm{p.b.c}} = 0$, i.e. $H^X_{\Lambda,\mathrm{p.b.c}} = T_{\Lambda,\mathrm{p.b.c}}$. As we have already seen that the proofs are really *simpler* in this case, see for example Remarks 3.2 and 3.6.

(**ii**) Moreover, we can directly compute the logarithmic moment generating function $g^{PBG}_{\widetilde{\Lambda}^*}(\lambda)$ (3.18) associated with the local particles density $\rho_{\widetilde{\Lambda}^*}$ (1.16):

$$g^{PBG}_{\widetilde{\Lambda}^*}(\lambda) = p^{PBG}_{\widetilde{\Lambda}}\left(\beta, \mu^{PBG}_{\Lambda}(\rho) + \lambda\right) - p^{PBG}_{\widetilde{\Lambda}}\left(\beta, \mu^{PBG}_{\Lambda}(\rho)\right),$$

where $p^{PBG}_{\widetilde{\Lambda}}(\beta,\mu)$ is the Perfect Bose Gas pressure for a finite box $\widetilde{\Lambda}$. Therefore, by Theorem 3.5 (more precisely, see Remark 3.6), we directly get

$$g^{PBG}_{\widetilde{\Lambda},\mathrm{p.b.c}}(\lambda) = g^{PBG}_{\widetilde{\Lambda}}(\lambda) = g^{PBG}_{\widetilde{\Lambda}^*}(\lambda) = p^{PBG}_{\widetilde{\Lambda}}\left(\beta, \mu^{PBG}_{\Lambda}(\rho) + \lambda\right) - p^{PBG}_{\widetilde{\Lambda}}\left(\beta, \mu^{PBG}_{\Lambda}(\rho)\right). \quad (3.70)$$

The computation of the logarithmic moment generating function $g^{PBG}_{\widetilde{\Lambda}}(\lambda)$ was already found in [2] with completely different arguments, which use the high specificity of $T_{\Lambda,\mathrm{p.b.c}}$. In particular, these arguments [2] could not have been extended to a quartic Hamiltonian in term of creation/annihilation operators.

(**iii**) Let

$$\rho^{PBG}_c \equiv \lim_{\mu\to 0^-} \rho^{PBG}_c(\beta,\mu) \equiv \lim_{\mu\to 0^-}\lim_{\Lambda} \omega^{PBG}_{\Lambda,\mu}(\rho_\Lambda) \equiv \lim_{\mu\to 0^-}\lim_{\Lambda} \omega^{PBG}_{\Lambda,\mu}\left(\frac{N_{\Lambda,\mathrm{p.b.c}}}{V}\right)$$

be the critical density of the Perfect Bose Gas, see (1.23) for $H^X_{\Lambda,\mathrm{p.b.c}} = T_{\Lambda,\mathrm{p.b.c}}$ ($\mu_{\sup} = 0$). Then, if for any $a > 0$,

$$\liminf_{\widetilde{\Lambda}} \frac{1}{\widetilde{V}} \ln \zeta^{PBG}_{\lambda_{\widetilde{\Lambda}^*,a},\widetilde{\Lambda}^*}\left(\chi_{\left[a,a+\frac{1}{\widetilde{V}}\right]}\left(\rho_{\widetilde{\Lambda}^*}\right)\right) = \liminf_{\widetilde{\Lambda}} \frac{1}{\widetilde{V}} \ln \zeta^{PBG}_{\lambda_{\widetilde{\Lambda},a},\widetilde{\Lambda}}\left(\chi_{\left[a,a+\frac{1}{\widetilde{V}}\right]}\left(\rho_{\widetilde{\Lambda}}\right)\right) = 0, \quad (3.71)$$

(which is proven in [2] only for $\rho < \rho^{PBG}_c$ and which may be false for $\rho > \rho^{PBG}_c$) then for any interval $I = [a,b]$ one has

$$\lim_{\widetilde{\Lambda}} \frac{1}{\beta\widetilde{V}} \ln \mathbb{P}\left\{\rho_{\widetilde{\Lambda}} \in I\right\} = \lim_{\widetilde{\Lambda}} \frac{1}{\beta\widetilde{V}} \ln \mathbb{P}\left\{\rho_{\widetilde{\Lambda},\mathrm{p.b.c}} \in I\right\} = \lim_{\widetilde{\Lambda}} \frac{1}{\beta\widetilde{V}} \ln \mathbb{P}\left\{\rho_{\widetilde{\Lambda}^*} \in I\right\},$$

(as in [2] we assume that the infinite volume Gibbs state ω_{ρ}^{PBG} of the Perfect Bose Gas makes sense, see Section 1.3.3).

(**iv**) By (3.70) one directly finds

$$\lim_{\widetilde{\Lambda}} \frac{1}{\beta \widetilde{V}} \ln \mathbb{P}\left\{\rho_{\widetilde{\Lambda}} \in I\right\} = \sup_{x\in[a,b]} f^{PBG}_{\mu^{PBG}(\rho),\beta}(x),$$

as soon as (3.71) is verified. Here

$$\mu^{PBG}(\rho) = \lim_{\widetilde{\Lambda}} \mu^{PBG}_{\widetilde{\Lambda}}(\rho) \leq \mu_{\sup} = 0,$$

see (1.27) for $H^X_{\Lambda,\text{p.b.c}} = T_{\Lambda,\text{p.b.c}}$, and the *rate function* $f^{PBG}_{\mu,\beta}(x)$ is defined as the Fenchel-Legendre transform of $g^{PBG}_{\mu,\beta}(\lambda)$, i.e.

$$f^{PBG}_{\mu,\beta}(x) = \inf_{\lambda\in(-\infty,-\mu]} \left\{g^{PBG}_{\mu,\beta}(\lambda) - \lambda x\right\} = g^{PBG}_{\mu,\beta}(\lambda_x) - x\lambda_x. \tag{3.72}$$

The kth cumulant of $\xi_{\widetilde{\Lambda}} = \left(N_{\widetilde{\Lambda}} - \omega^{PBG}_{\Lambda,\rho}\left(N_{\widetilde{\Lambda}}\right)\right)/\sqrt{\widetilde{V}}$ is given by

$$C_{\Lambda}(k) = \frac{1}{\beta^k \widetilde{V}^{k/2}} \left[\frac{d^k}{d\lambda^k} \ln \omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_{\widetilde{\Lambda}}}\right)\right]_{\lambda=0}, \ k \geq 2,$$

Then, as in [2], by (3.70), one gets:

- for $\rho < \rho_c^{PBG}$, $\xi_{\widetilde{\Lambda}}$ converge, as $\widetilde{\Lambda} \nearrow \mathbb{R}^d$, to a gaussien variable with variance

$$\sigma = \lim_{\widetilde{\Lambda}} \beta^{-1}\left(\partial^2_{\mu} p^{PBG}_{\widetilde{\Lambda}}\left(\beta, \mu^{PBG}_{\widetilde{\Lambda}}(\rho)\right)\right) < +\infty;$$

- for $\rho > \rho_c^{PBG}$, the variable $\xi_{\widetilde{\Lambda}}$ does not converge, as $\widetilde{\Lambda} \nearrow \mathbb{R}^d$, to those of a gaussien since

$$\lim_{\widetilde{\Lambda}} \beta^{-1}\left(\partial^2_{\mu} p^{PBG}_{\widetilde{\Lambda}}\left(\beta, \mu^{PBG}_{\widetilde{\Lambda}}(\rho)\right)\right) = +\infty.$$

Appendix A. .

The aim of this appendix is to prove the Bogoliubov convexity inequality [5, 12–16] for the two logarithmic moment generating functions

$$g^X_1(\lambda) = \frac{1}{\beta\widetilde{V}} \ln \omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_1}\right), \ g^X_2(\lambda) = \frac{1}{\beta\widetilde{V}} \ln \omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_1}\right), \tag{A.1}$$

where the operators N_1 and N_2 are two local self-adjoint operators acting on $\mathcal{F}^{\infty}_B$ satisfying $N_1 = P_{\Lambda} N_1 P_{\Lambda}$, $N_2 = P_{\Lambda} N_2 P_{\Lambda}$, i.e., the restrictions of N_1 and N_2 on $\mathcal{F}^{(\Lambda)}_B$ are also self-adjoint and

$$\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_1}\right) < +\infty, \ \omega^X_{\Lambda,\rho}\left(e^{\beta\lambda N_2}\right) < +\infty,$$

for a domain of λ. Here P_Λ (1.7)-(1.8) is the projection operator from $\mathcal{F}_B^\infty$ to the boson Fock space $\mathcal{F}_B^{(\Lambda)} \subset \mathcal{F}_B^\infty$. First we recall that $\omega_{\Lambda,\rho}^X$ (cf. (1.23)-(1.27) and (1.29)) is normal [17] which means the existence of a density matrix $d_{\Lambda,\rho}^X$ (3.7), i.e., a positive trass-class operator $d_{\Lambda,\rho}^X$ on $\mathcal{F}_B^\infty$ with

$$Tr_{\mathcal{F}_B^\infty}\left(d_{\Lambda,\rho}^X\right) = 1, \tag{A.2}$$

such that

$$\omega_{\Lambda,\rho}^X(A) = Tr_{\mathcal{F}_B^\infty}\left(d_{\Lambda,\rho}^X A\right),\ A \in \mathcal{A}_B^\infty, \tag{A.3}$$

see (B.2), (B.3) and Theorem 2.4.21 in [17]. Let us define for any $A \in \mathcal{A}_B^\infty$ the two states $\zeta_{\lambda,1}^X$ and $\zeta_{\lambda,2}^X$:

$$\begin{aligned} \zeta_{\lambda,1}^X(A) &= Tr_{\mathcal{F}_B^\infty}\left(\sigma_{\lambda,1}^X A\right),\ \sigma_{\lambda,1}^X = \left[Z_{\lambda,1}^X\right]^{-1} d_{\Lambda,\rho}^X e^{\beta\lambda N_1}, \\ \zeta_{\lambda,2}^X(A) &= Tr_{\mathcal{F}_B^\infty}\left(\sigma_{\lambda,2}^X A\right),\ \sigma_{\lambda,2}^X = \left[Z_{\lambda,2}^X\right]^{-1} d_{\Lambda,\rho}^X e^{\beta\lambda N_2}, \end{aligned} \tag{A.4}$$

with

$$\begin{aligned} Z_{\lambda,1}^X &= \omega_{\Lambda,\rho}^X\left(e^{\beta\lambda N_1}\right) = Tr_{\mathcal{F}_B^\infty}\left(d_{\Lambda,\rho}^X e^{\beta\lambda N_1}\right) < +\infty, \\ Z_{\lambda,2}^X &= \omega_{\Lambda,\rho}^X\left(e^{\beta\lambda N_2}\right) = Tr_{\mathcal{F}_B^\infty}\left(d_{\Lambda,\rho}^X e^{\beta\lambda N_2}\right) < +\infty. \end{aligned} \tag{A.5}$$

Since the restriction of N_1 and N_2 are self-adjoint in $\mathcal{F}_B^{(\Lambda)}$, there are, in $\mathcal{F}_B^{(\Lambda)}$, two orthonormal basis $\left\{\varphi_{1,n}\right\}_{n=1}^{+\infty}$ and $\left\{\varphi_{2,n}\right\}_{n=1}^{+\infty}$, respectively two sets of eigenvectors for N_1 and N_2 with real eigenvalues $\{E_{1,n}\}_{n=1}^{+\infty}$ and $\{E_{2,n}\}_{n=1}^{+\infty}$, i.e.,

$$\begin{aligned} &N_1\varphi_{1,n} = E_{1,n}\varphi_{1,n},\ N_2\varphi_{2,n} = E_{2,n}\varphi_{2,n}, \\ &e^{\beta\lambda N_1}\varphi_{1,n} = e^{\beta\lambda E_{1,n}}\varphi_{1,n},\ e^{\beta\lambda N_2}\varphi_{2,n} = e^{\beta\lambda E_{2,n}}\varphi_{2,n}, \end{aligned}$$

with $\varphi_{1,n}, \varphi_{2,n} \in \mathcal{F}_B^{(\Lambda)}$ for $n \geq 1$. The family $\left\{e^{\beta\lambda N_1}\right\}_{\beta>0}$ (or $\left\{e^{\beta\lambda N_2}\right\}_{\beta>0}$) got the name Gibbs semigroup generated by N_1 (or N_2), see [14].

Lemma A.1. *(Jensen inequality) Let ξ be a real random variable with expectation $\mathbb{E}(|\xi|) < \infty$. For any real convex function g on $\mathbb{R}^1$ one has*

$$\mathbb{E}(g(\xi)) \geq g(\mathbb{E}(\xi)). \tag{A.6}$$

Proof. By convexity of g there are two numbers $x_0 \in \mathbb{R}^1$ and $\lambda(x_0)$ such that

$$g(x) \geq g(x_0) + \lambda(x_0)(x - x_0). \tag{A.7}$$

Let $x = \xi$ and $x_0 = \mathbb{E}(\xi)$. Then by (A.7)

$$\mathbb{E}(g(\xi)) \geq g(\mathbb{E}(\xi)),$$

which proves (A.6). ■

Lemma A.2. *(Peierls-Bogoliubov inequality) Let $\left\{e^{\beta\lambda H}\right\}_{\beta>0}$ be a Gibbs semigroup generated by the self-adjoint operator H. Then for any orthonormal basis $\{\eta_n\}_{n=1}^{+\infty}$ in $\mathcal{F}_B^{(\Lambda)}$ one gets:*

$$Tr_{\mathcal{F}_B^\infty}\left(e^{\beta\lambda H} d_{\Lambda,\rho}^X\right) = Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{\beta\lambda H} d_{\Lambda,\rho}^X\right) \geq \sum_{n=1}^{+\infty}\left(\eta_n, d_{\Lambda,\rho}^X \eta_n\right)_{\mathcal{F}_B^\infty} \exp\left\{\beta\lambda\left(\widetilde{\eta}_n, d_{\Lambda,\rho}^X H\widetilde{\eta}_n\right)_{\mathcal{F}_B^\infty}\right\}, \tag{A.8}$$

with

$$\widetilde{\eta}_n = \frac{\eta_n}{\sqrt{\left(\eta_n, d_{\Lambda,\rho}^X \eta_n\right)_{\mathcal{F}_B^\infty}}},\ \text{for } n \geq 1.$$

Proof. Let $u \in \mathcal{F}_B^\infty$ be a vector such that $\left(u, d_{\Lambda,\rho}^X u\right)_{\mathcal{F}_B^\infty} = 1$. If $\{\varphi_n\}_{n=1}^{+\infty}$ is an orthonormal basis of eigenvectors of H associated with the eigenvalues $\{E_n\}_{n=1}^\infty$, then $u = \sum_{n=1}^{+\infty} u_n \varphi_n$ and

$$\left(u, d_{\Lambda,\rho}^X e^{\beta\lambda H} u\right)_{\mathcal{F}_B^\infty} = \sum_{n=1}^{+\infty} e^{\beta\lambda E_n} u_n \left(u, d_{\Lambda,\rho}^X \varphi_n\right)_{\mathcal{F}_B^\infty} . \tag{A.9}$$

Since $\left(u, d_{\Lambda,\rho}^X u\right)_{\mathcal{F}_B^\infty} = \sum_{n=1}^{+\infty} u_n \left(u, d_{\Lambda,\rho}^X \varphi_n\right)_{\mathcal{F}_B^\infty} = 1$, we could consider (A.9) as an expectation of the convex function of the random variable $\{E_n\}_{n=1}^{+\infty}$ with respect to the probability distribution defined by $\left\{u_n \left(u, d_{\Lambda,\rho}^X \varphi_n\right)_{\mathcal{F}_B^\infty}\right\}_{n=1}^{+\infty}$. Then by the Jensen inequality (A.6) one gets

$$\left(u, e^{\beta\lambda H} u\right)_{\mathcal{F}_B^\infty} \geq e^{\beta\lambda \sum_{n=1}^{+\infty} E_n u_n \left(u, d_{\Lambda,\rho}^X \varphi_n\right)_{\mathcal{F}_B^\infty}} = e^{\beta\lambda \left(u, d_{\Lambda,\rho}^X H u\right)_{\mathcal{F}_B^\infty}} . \tag{A.10}$$

By definition (3.7) of the density matrix $d_{\Lambda,\rho}^X$, note that

$$Tr_{\mathcal{F}_B^\infty}\left(e^{\beta\lambda H} d_{\Lambda,\rho}^X\right) = Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{\beta\lambda H} d_{\Lambda,\rho}^X\right) ,$$

and $d_{\Lambda,\rho}^X$ is a *strictly positive* operator in $\mathcal{F}_B^{(\Lambda)}$ whereas for any $\varphi \in \mathcal{F}_B^\infty \backslash \mathcal{F}_B^{(\Lambda)}$, $d_{\Lambda,\rho}^X \varphi = 0$. Then for any orthonormal basis $\{\eta_n\}_{n=1}^{+\infty}$ in $\mathcal{F}_B^{(\Lambda)}$ one has

$$Tr_{\mathcal{F}_B^\infty}\left(e^{\beta\lambda H} d_{\Lambda,\rho}^X\right) = Tr_{\mathcal{F}_B^\infty}\left(d_{\Lambda,\rho}^X e^{\beta\lambda H}\right) = Tr_{\mathcal{F}_B^{(\Lambda)}}\left(d_{\Lambda,\rho}^X e^{\beta\lambda H}\right) = \sum_{n=1}^{+\infty} \left(\eta_n, d_{\Lambda,\rho}^X e^{\beta\lambda H} \eta_n\right)_{\mathcal{F}_B^\infty} ,$$

Therefore, by the strictly positivity of $d_{\Lambda,\rho}^X$ (3.7) in $\mathcal{F}_B^{(\Lambda)}$, we find

$$Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{\beta\lambda H} d_{\Lambda,\rho}^X\right) = \sum_{n=1}^{+\infty} \left(\eta_n, d_{\Lambda,\rho}^X \eta_n\right)_{\mathcal{F}_B^\infty} \left(\widetilde{\eta}_n, d_{\Lambda,\rho}^X e^{\beta\lambda H} \widetilde{\eta}_n\right)_{\mathcal{F}_B^\infty} ,$$

with

$$\widetilde{\eta}_n = \frac{\eta_n}{\sqrt{\left(\eta_n, d_{\Lambda,\rho}^X \eta_n\right)_{\mathcal{F}_B^\infty}}}, \; n \geq 1.$$

Since $\left(\widetilde{\eta}_n, d_{\Lambda,\rho}^X \widetilde{\eta}_n\right)_{\mathcal{F}_B^\infty} = 1$, by (A.10) one gets

$$\begin{aligned} Tr_{\mathcal{F}_B^{(\Lambda)}}\left(e^{\beta\lambda H} d_{\Lambda,\rho}^X\right) &= \sum_{n=1}^{+\infty} \left(\eta_n, d_{\Lambda,\rho}^X \eta_n\right)_{\mathcal{F}_B^\infty} \left(\widetilde{\eta}_n, d_{\Lambda,\rho}^X e^{\beta\lambda H} \widetilde{\eta}_n\right)_{\mathcal{F}_B^\infty} \\ &\geq \sum_{n=1}^{+\infty} \left(\eta_n, d_{\Lambda,\rho}^X \eta_n\right)_{\mathcal{F}_B^\infty} \exp\left\{\beta\lambda \left(\widetilde{\eta}_n, d_{\Lambda,\rho}^X H \widetilde{\eta}_n\right)_{\mathcal{F}_B^\infty}\right\} , \end{aligned} \tag{A.11}$$

which proves (A.8). The rest is a consequence of invariance of the $Tr_{\mathcal{F}_B^{(\Lambda)}}(.)$ with respect to the choice of an orthonormal basis in $\mathcal{F}_B^{(\Lambda)}$. ■

Remark A.3. *The right side of the last inequality (A.8) could also be considered as an expectation of the convex function* exp *of the random variable* $\left\{\left(\widetilde{\eta}_n, d^X_{\Lambda,\rho}H\widetilde{\eta}_n\right)_{\mathcal{F}^\infty_B}\right\}_{n=1}^{+\infty}$ *with respect to the probability distribution defined by* $\left\{\left(\eta_n, d^X_{\Lambda,\rho}\eta_n\right)_{\mathcal{F}^\infty_B}\right\}_{n=1}^{+\infty}$ *using the fact that*

$$Tr_{\mathcal{F}^\infty_B}\left(d^X_{\Lambda,\rho}\right) = Tr_{\mathcal{F}^{(\Lambda)}_B}\left(d^X_{\Lambda,\rho}\right) = \sum_{n=1}^{+\infty}\left(\eta_n, d^X_{\Lambda,\rho}\eta_n\right)_{\mathcal{F}^\infty_B} = \sum_{n=1}^{+\infty}\left(\eta_n, d^X_{\Lambda,\rho}\eta_n\right)_{\mathcal{F}^\infty_B} = 1.$$

Then by the Jensen inequality (A.6) one gets

$$\begin{aligned}Tr_{\mathcal{F}^\infty_B}\left(e^{\beta\lambda H}d^X_{\Lambda,\rho}\right) = Tr_{\mathcal{F}^{(\Lambda)}_B}\left(e^{\beta\lambda H}d^X_{\Lambda,\rho}\right) &\geq \exp\left\{\sum_{n=1}^{+\infty}\left(\eta_n, d^X_{\Lambda,\rho}\eta_n\right)_{\mathcal{F}^\infty_B}\beta\lambda\left(\widetilde{\eta}_n, d^X_{\Lambda,\rho}H\widetilde{\eta}_n\right)_{\mathcal{F}^\infty_B}\right\} = \\ &= \exp\left\{\sum_{n=1}^{+\infty}\beta\lambda\left(\eta_n, d^X_{\Lambda,\rho}H\eta_n\right)_{\mathcal{F}^\infty_B}\right\} = \exp\left\{\beta\lambda Tr_{\mathcal{F}^{(\Lambda)}_B}\left(d^X_{\Lambda,\rho}H\right)\right\} = \\ &= \exp\left\{\beta\lambda Tr_{\mathcal{F}^\infty_B}\left(d^X_{\Lambda,\rho}H\right)\right\},\end{aligned}$$

i.e.,

$$\omega^X_{\Lambda,\rho}\left(e^{\beta\lambda H}\right) \geq \exp\left\{\beta\lambda\omega^X_{\Lambda,\rho}(H)\right\}.$$

Theorem A.4. *(Bogoliubov convexity inequality) Let N_1 and N_2 be two self-adjoint generators of Gibbs semigroups. Suppose that $Tr_{\mathcal{F}^\infty_B}\left[e^{\beta\lambda N_2}(N_1 - N_2)\right]$ and $Tr_{\mathcal{F}^\infty_B}\left[e^{\beta\lambda N_1}(N_1 - N_2)\right]$ are bounded with $N_1 = P_\Lambda N_1 P_\Lambda$, $N_2 = P_\Lambda N_2 P_\Lambda$, cf. (1.7)-(1.8). Then*

$$\beta\lambda\zeta^X_{\lambda,1}(N_1 - N_2) \leq \ln Tr_{\mathcal{F}^\infty_B}\left(d^X_{\Lambda,\rho}e^{\beta\lambda N_2}\right) - \ln Tr_{\mathcal{F}^\infty_B}\left(d^X_{\Lambda,\rho}e^{\beta\lambda N_1}\right) \leq \beta\lambda\zeta^X_{\lambda,2}(N_1 - N_2), \quad \text{(A.12)}$$

where $\zeta^X_{\lambda,1}$ and $\zeta^X_{\lambda,2}$ are defined by (A.4)-(A.5).

Proof. Since $N_1 = P_\Lambda N_1 P_\Lambda$, let $\left\{\varphi_{1,n}\right\}_{n=1}^{+\infty}$ be an orthonormal basis of eigenvectors of N_1 in $\mathcal{F}^{(\Lambda)}_B$. Now by (A.8) one gets

$$\begin{aligned}\frac{Tr_{\mathcal{F}^\infty_B}\left(d^X_{\Lambda,\rho}e^{\beta\lambda N_2}\right)}{Tr_{\mathcal{F}^\infty_B}\left(d^X_{\Lambda,\rho}e^{\beta\lambda N_1}\right)} &= \frac{Tr_{\mathcal{F}^{(\Lambda)}_B}\left(d^X_{\Lambda,\rho}e^{\beta\lambda N_2}\right)}{Tr_{\mathcal{F}^{(\Lambda)}_B}\left(d^X_{\Lambda,\rho}e^{\beta\lambda N_1}\right)} = \frac{Tr_{\mathcal{F}^{(\Lambda)}_B}\left(d^X_{\Lambda,\rho}e^{\beta\lambda(N_1+N_2-N_1)}\right)}{Tr_{\mathcal{F}^{(\Lambda)}_B}\left(d^X_{\Lambda,\rho}e^{\beta\lambda N_1}\right)} = \\ &= \frac{\sum_{n=1}^{+\infty}\left(\varphi_{1,n}, d^X_{\Lambda,\rho}e^{\beta\lambda(N_1+N_2-N_1)}\varphi_{1,n}\right)_{\mathcal{F}^\infty_B}}{Tr_{\mathcal{F}^{(\Lambda)}_B}\left(d^X_{\Lambda,\rho}e^{\beta\lambda N_1}\right)} \\ &\geq \frac{\sum_{n=1}^{+\infty}e^{\beta\lambda E_{1,n}}\left(\varphi_{1,n}, d^X_{\Lambda,\rho}\varphi_{1,n}\right)_{\mathcal{F}^\infty_B}e^{\left\{\beta\lambda\left(\widetilde{\varphi}_{1,n}, d^X_{\Lambda,\rho}(N_2-N_1)\widetilde{\varphi}_{1,n}\right)_{\mathcal{F}^\infty_B}\right\}}}{\sum_{n=1}^{+\infty}e^{\beta\lambda E_{1,n}}\left(\varphi_{1,n}, d^X_{\Lambda,\rho}\varphi_{1,n}\right)_{\mathcal{F}^\infty_B}},\end{aligned} \quad \text{(A.13)}$$

with

$$\widetilde{\varphi}_{1,n} = \frac{\varphi_{1,n}}{\sqrt{\left(\varphi_{1,n}, d^X_{\Lambda,\rho}\varphi_{1,n}\right)_{\mathcal{F}^\infty_B}}}, \; n \geq 1. \quad \text{(A.14)}$$

Therefore using now (A.6), from the inequality (A.13) one obtains

$$\frac{Tr_{\mathcal{F}_B^\infty}\left(d_{\Lambda,\rho}^X e^{\beta\lambda N_2}\right)}{Tr_{\mathcal{F}_B^\infty}\left(d_{\Lambda,\rho}^X e^{\beta\lambda N_1}\right)} \geq \exp\left\{\beta\lambda\left(\sum_{n=1}^{+\infty}\frac{e^{\beta\lambda E_{1,n}}\left(\varphi_{1,n}, d_{\Lambda,\rho}^X\varphi_{1,n}\right)_{\mathcal{F}_B^\infty}\left(\widetilde{\varphi}_{1,n}, d_{\Lambda,\rho}^X\left(N_2-N_1\right)\widetilde{\varphi}_{1,n}\right)_{\mathcal{F}_B^\infty}}{\sum\limits_{n=1}^{+\infty} e^{\beta\lambda E_{1,n}}\left(\varphi_{1,n}, d_{\Lambda,\rho}^X\varphi_{1,n}\right)_{\mathcal{F}_B^\infty}}\right)\right\},$$

which, by (A.4) and (A.14), implies

$$\begin{aligned}\frac{Tr_{\mathcal{F}_B^\infty}\left(d_{\Lambda,\rho}^X e^{\beta\lambda N_2}\right)}{Tr_{\mathcal{F}_B^\infty}\left(d_{\Lambda,\rho}^X e^{\beta\lambda N_1}\right)} &\geq \exp\left\{\beta\lambda\left(\sum_{n=1}^{+\infty}\frac{e^{\beta\lambda E_{1,n}}\left(\varphi_{1,n}, d_{\Lambda,\rho}^X\left(N_2-N_1\right)\varphi_{1,n}\right)_{\mathcal{F}_B^\infty}}{\sum\limits_{n=1}^{+\infty} e^{\beta\lambda E_{1,n}}\left(\varphi_{1,n}, d_{\Lambda,\rho}^X\varphi_{1,n}\right)_{\mathcal{F}_B^\infty}}\right)\right\} = \\ &= \exp\left\{\beta\lambda\zeta_{\lambda,1}^X\left(N_2-N_1\right)\right\}.\end{aligned} \tag{A.15}$$

If we rename N_2 and N_1, then (A.15) reads as

$$\frac{Tr_{\mathcal{F}_B^\infty}\left(d_{\Lambda,\rho}^X e^{\beta\lambda N_1}\right)}{Tr_{\mathcal{F}_B^\infty}\left(d_{\Lambda,\rho}^X e^{\beta\lambda N_2}\right)} \geq \exp\left\{\beta\lambda\zeta_{\lambda,2}^X\left(N_1-N_2\right)\right\}. \tag{A.16}$$

The inequalities (A.15) and (A.16) imply (A.12). ■

Corollary A.5. *For the functions $g_1^X(\lambda)$ and $g_2^X(\lambda)$ (cf. (A.1)), one gets the Bogoliubov inequality:*

$$\lambda\zeta_{\lambda,1}^X\left(\frac{N_1}{\widetilde{V}}-\frac{N_2}{\widetilde{V}}\right) \leq g_2^X(\lambda) - g_1^X(\lambda) \leq \lambda\zeta_{\lambda,2}^X\left(\frac{N_1}{\widetilde{V}}-\frac{N_2}{\widetilde{V}}\right). \tag{A.17}$$

Appendix B. .

First we recall the definition of $\rho_{\widetilde{\Lambda}} = N_{\widetilde{\Lambda}}/\widetilde{V}$ (1.15) and $\rho_{\widetilde{\Lambda},\text{p.b.c}} = N_{\widetilde{\Lambda},\text{p.b.c}}/\widetilde{V}$ (2.5):

$$\begin{aligned}\rho_{\widetilde{\Lambda}} &= \frac{N_{\widetilde{\Lambda}}}{\widetilde{V}} = \frac{1}{\widetilde{V}}\int\limits_{\widetilde{\Lambda}} a^*(x)\,a(x)\,dx, \\ \rho_{\widetilde{\Lambda},\text{p.b.c}} &= \frac{N_{\widetilde{\Lambda},\text{p.b.c}}}{\widetilde{V}} = \frac{1}{\widetilde{V}}\sum_{k\in\widetilde{\Lambda}^*} a_{k,\widetilde{\Lambda}}^* a_{k,\widetilde{\Lambda}},\end{aligned} \tag{B.1}$$

Now let us consider a normal state ω defined on $\mathcal{A}_B^\infty$ [17], i.e., a state defined by a density matrix ρ_ω (a positive trace-class operator ρ_ω on $\mathcal{F}_B^\infty$) with

$$Tr_{\mathcal{F}_B^\infty}(\rho_\omega) = 1, \tag{B.2}$$

such that

$$\forall A \in \mathcal{A}_B^\infty, \omega(A) = Tr_{\mathcal{F}_B^\infty}(\rho_\omega A). \tag{B.3}$$

Then we have the following result:

Theorem B.1. *If*

$$\omega\left(\rho_{\widetilde{\Lambda}}\right) < +\infty,\ \omega\left(\rho_{\widetilde{\Lambda},p.b.c}\right) < +\infty, \tag{B.4}$$

then one gets:
(i)

$$\omega\left(\rho_{\widetilde{\Lambda}}\right) = \omega\left(\rho_{\widetilde{\Lambda},p.b.c}\right); \tag{B.5}$$

(ii) for any continuous function $h(x)$ vanishing at ∞,

$$\omega\left(h\left(\rho_{\widetilde{\Lambda}}\right)\right) = \omega\left(h\left(\rho_{\widetilde{\Lambda},p.b.c}\right)\right); \tag{B.6}$$

(iii) for any interval $I = [a, b] \subset \mathbb{R}$,

$$\omega\left(\chi_I\left(\rho_{\widetilde{\Lambda}}\right)\right) = \omega\left(\chi_I\left(\rho_{\widetilde{\Lambda},p.b.c}\right)\right). \tag{B.7}$$

Here χ_A is the characteristic function of a set $A \subset \mathbb{R}$ whereas $\rho_{\widetilde{\Lambda}} = N_{\widetilde{\Lambda}}/\widetilde{V}$ and $\rho_{\widetilde{\Lambda},p.b.c} = N_{\widetilde{\Lambda},p.b.c}/\widetilde{V}$ are respectively defined by (1.15) and (2.5) (or see (B.1)).

Proof. (i) Since the set

$$\left\{\frac{\chi_{\widetilde{\Lambda}}(x)}{\sqrt{\widetilde{V}}} e^{ikx}\right\}_{k\in\widetilde{\Lambda}^*}$$

is an orthonormal basis of the Hilbert space $\left(L^2\left(\widetilde{\Lambda}\right)\right)_{\text{p.b.c}}$ of squared integrable functions in $\widetilde{\Lambda}$ with *periodic boundary conditions* on the box $\widetilde{\Lambda}$, one has

$$\left(\rho_{\widetilde{\Lambda}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1} \varphi_{\widetilde{\Lambda},\text{p.b.c.}} = \left(\rho_{\widetilde{\Lambda},\text{p.b.c}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1} \varphi_{\widetilde{\Lambda},\text{p.b.c.}}, \tag{B.8}$$

for any periodic function $\varphi_{\widetilde{\Lambda},\text{p.b.c.}} \in \mathcal{F}_B^{(\widetilde{\Lambda})}$. Now a function $\varphi \in \mathcal{F}_B^\infty$ could be written as

$$\varphi = P_{\widetilde{\Lambda}}\varphi + \left(1 - P_{\widetilde{\Lambda}}\right)\varphi = \varphi_{\widetilde{\Lambda}} + \varphi_{\mathbb{R}^d\setminus\widetilde{\Lambda}}, \tag{B.9}$$

with the projection operator $P_{\widetilde{\Lambda}}$ defined by (1.7)-(1.8) and

$$\begin{aligned} \left(\rho_{\widetilde{\Lambda}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1} \varphi &= \varphi_{\mathbb{R}^d\setminus\widetilde{\Lambda}} + \left(\rho_{\widetilde{\Lambda}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1} \varphi_{\widetilde{\Lambda}}, \\ \left(\rho_{\widetilde{\Lambda},\text{p.b.c}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1} \varphi &= \varphi_{\mathbb{R}^d\setminus\widetilde{\Lambda}} + \left(\rho_{\widetilde{\Lambda},\text{p.b.c}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1} \varphi_{\widetilde{\Lambda}}. \end{aligned} \tag{B.10}$$

Note that $\varphi_{\widetilde{\Lambda}} \in \mathcal{F}_B^{(\widetilde{\Lambda})}$ may be not periodic on $\widetilde{\Lambda}$. However from $\varphi_{\widetilde{\Lambda}}$ one can arbitrary change the value of $\varphi_{\widetilde{\Lambda}}(x)$ for $x \in \partial\widetilde{\Lambda}$ and define a periodic function $\varphi_{\widetilde{\Lambda},\text{p.b.c.}} \in \mathcal{F}_B^{(\widetilde{\Lambda})}$ with

$$\varphi_{\widetilde{\Lambda},\text{p.b.c.}}(x) = \varphi_{\widetilde{\Lambda}}(x) \text{ for } x \in \widetilde{\Lambda}\setminus\partial\widetilde{\Lambda}.$$

Therefore

$$\varphi_{\widetilde{\Lambda}} = \varphi_{\widetilde{\Lambda},\text{p.b.c.}} + \left(\varphi_{\widetilde{\Lambda}} - \varphi_{\widetilde{\Lambda},\text{p.b.c.}}\right) \equiv \varphi_{\widetilde{\Lambda},\text{p.b.c.}} + \varphi_{\partial\widetilde{\Lambda}}. \tag{B.11}$$

So, from (B.8)-(B.11) one gets

$$\begin{aligned}\left\|\left[\left(\rho_{\widetilde{\Lambda}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1} - \left(\rho_{\widetilde{\Lambda},\text{p.b.c}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}\right]\varphi\right\|_{\mathcal{F}_B^\infty}^2 &\leq \left\|\left(\rho_{\widetilde{\Lambda}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}\varphi_{\partial\widetilde{\Lambda}}\right\|_{\mathcal{F}_B^\infty}^2 + \\ &+ \left\|\left[\left(\rho_{\widetilde{\Lambda},\text{p.b.c}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}\right]\varphi_{\partial\widetilde{\Lambda}}\right\|_{\mathcal{F}_B^\infty}^2 \\ &\leq 2\left\|\varphi_{\partial\widetilde{\Lambda}}\right\|_{\mathcal{F}_B^\infty}^2 = 0,\end{aligned}$$

which implies

$$\left\|\left[\left(\rho_{\widetilde{\Lambda}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1} - \left(\rho_{\widetilde{\Lambda},\text{p.b.c}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}\right]\varphi\right\|_{\mathcal{F}_B^\infty}^2 = 0. \tag{B.12}$$

Since

$$\left(\rho_{\widetilde{\Lambda},\text{p.b.c}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1} - \left(\rho_{\widetilde{\Lambda}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1} = \left(\rho_{\widetilde{\Lambda},\text{p.b.c}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}\left(\rho_{\widetilde{\Lambda},\text{p.b.c}} - \rho_{\widetilde{\Lambda}}\right)\left(\rho_{\widetilde{\Lambda}} \pm i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1},$$

notice that the equality (B.12) implies also

$$\left\|\left(\rho_{\widetilde{\Lambda},\text{p.b.c}} - \rho_{\widetilde{\Lambda}}\right)\varphi\right\|_{\mathcal{F}_B^\infty}^2 = 0, \tag{B.13}$$

for any φ in the domain of definition of $\rho_{\widetilde{\Lambda}}$ and $\rho_{\widetilde{\Lambda},\text{p.b.c}}$.
Now, since ω is a normal state defined by (B.3), if (B.4) is satisfied, using an orthonormal basis $\{\varphi_n\}_{n=1}^{+\infty}$ of $\mathcal{F}_B^\infty$ in the definition domain of $\rho_{\widetilde{\Lambda}}$ and $\rho_{\widetilde{\Lambda},\text{p.b.c}}$, one gets

$$\omega\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right) = Tr_{\mathcal{F}_B^\infty}\left(\rho_\omega\rho_{\widetilde{\Lambda},\text{p.b.c}}\right) = \sum_{n=1}^{+\infty}\left(\rho_\omega^*\varphi_n, \rho_{\widetilde{\Lambda},\text{p.b.c}}\varphi_n\right)_{\mathcal{F}_B^\infty}. \tag{B.14}$$

The equality (B.13) implies

$$\left(\psi_1, \left(\rho_{\widetilde{\Lambda},\text{p.b.c}} - \rho_{\widetilde{\Lambda}}\right)\psi_2\right)_{\mathcal{F}_B^\infty} = 0, \tag{B.15}$$

for any ψ_1, ψ_2 in the domain of definition of $\rho_{\widetilde{\Lambda}}$ and $\rho_{\widetilde{\Lambda},\text{p.b.c}}$ then with $\psi_1 = \rho_\omega^*\varphi_n$ and $\psi_2 = \varphi_n$, by (B.14) we obtain

$$\begin{aligned}\omega\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right) &= \sum_{n=1}^{+\infty}\left(\rho_\omega^*\varphi_n, \rho_{\widetilde{\Lambda},\text{p.b.c}}\varphi_n\right)_{\mathcal{F}_B^\infty} = \sum_{n=1}^{+\infty}\left(\rho_\omega^*\varphi_n, \rho_{\widetilde{\Lambda}}\varphi_n\right)_{\mathcal{F}_B^\infty} \\ &= Tr_{\mathcal{F}_B^\infty}\left(\rho_\omega\rho_{\widetilde{\Lambda}}\right) = \omega\left(\rho_{\widetilde{\Lambda}}\right),\end{aligned} \tag{B.16}$$

i.e., (B.5).
(ii) Let us consider a continuous function $h(x)$ vanishing at ∞. Since the polynomials in $(x \pm i)^{-1}$ are dense (using the norm $\|-\|_\infty$) in the set of continuous functions vanishing at ∞, for a given $\varepsilon > 0$, there exists a polynomial $P(s,t)$ such that

$$\left\|h(x) - P\left(\frac{1}{x+i}, \frac{1}{x-i}\right)\right\|_\infty = \sup_{x\in\mathbb{R}}\left|h(x) - P\left(\frac{1}{x+i}, \frac{1}{x-i}\right)\right| < \frac{\varepsilon}{2}. \tag{B.17}$$

Therefore

$$\left\| h\left(\rho_{\widetilde{\Lambda}}\right) - P\left(\left(\rho_{\widetilde{\Lambda}} + i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}, \left(\rho_{\widetilde{\Lambda}} - i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}\right)\right\|_{\mathcal{A}_B^\infty}^2 < \frac{\varepsilon}{2}, \tag{B.18}$$

and

$$\left\| h\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right) - P\left(\left(\rho_{\widetilde{\Lambda},\text{p.b.c}} + i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}, \left(\rho_{\widetilde{\Lambda},\text{p.b.c}} - i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}\right)\right\|_{\mathcal{A}_B^\infty}^2 < \frac{\varepsilon}{2}. \tag{B.19}$$

Here we recall that the norm on $\mathcal{A}_B^\infty$ is defined in Remark 1.6. Moreover, by extending (B.12) we find

$$\left\| P\left(\left(\rho_{\widetilde{\Lambda}} + i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}, \left(\rho_{\widetilde{\Lambda}} - i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}\right) - P\left(\left(\rho_{\widetilde{\Lambda},\text{p.b.c}} + i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}, \left(\rho_{\widetilde{\Lambda},\text{p.b.c}} - i\mathbb{I}_{\mathcal{F}_B^\infty}\right)^{-1}\right)\right\|_{\mathcal{A}_B^\infty}^2 = 0. \tag{B.20}$$

Then for any $\varepsilon > 0$, by (B.18)-(B.20) we get

$$\left\| h\left(\rho_{\widetilde{\Lambda}}\right) - h\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right)\right\|_{\mathcal{A}_B^\infty}^2 < \varepsilon,$$

which implies that

$$\left\| h\left(\rho_{\widetilde{\Lambda}}\right) - h\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right)\right\|_{\mathcal{A}_B^\infty}^2 = 0. \tag{B.21}$$

Consequently for any $\varphi \in \mathcal{F}_B^\infty$ we have

$$\left\| \left[h\left(\rho_{\widetilde{\Lambda}}\right) - h\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right)\right]\varphi\right\|_{\mathcal{F}_B^\infty}^2 = 0. \tag{B.22}$$

Therefore using the same arguments than for (i), cf. (B.13)-(B.16) one gets (B.6).
(iii) Notice that the last result (B.21) is true only for a *continuous* function vanishing at ∞ but we are interested in evaluating the limit (B.21) for the characteristic function $\chi_I(x)$, i.e., for a continuous function only for $x \in \mathbb{R}\backslash\{a,b\}$ with compact support $I = [a,b]$. Therefore for a given $\varepsilon > 0$, and any compact $[c,d] \subset \mathbb{R}\backslash\{a-\delta, b+\delta\}$, $\delta > 0$, there exists a polynomial $P(s,t)$ such that

$$\sup_{x\in[c,d]}\left|\chi_{I_\delta}(x) - P\left(\frac{1}{x+i}, \frac{1}{x-i}\right)\right| < \frac{\varepsilon}{2}, \tag{B.23}$$

with $I_\delta = [a-\delta, b+\delta]$. Using $\mathcal{P}_{a,b}$ and $\widehat{\mathcal{P}}_{a,b}$ defined as the projections on the eigenspaces associated with all the eigenvalues $\rho \in \mathbb{R}\backslash\{(a-\delta-1,a)\cup(b,b+\delta+1)\}$ of respectively $\rho_{\widetilde{\Lambda}}$ and $\rho_{\widetilde{\Lambda},\text{p.b.c}}$ $(\delta > 0)$, (B.12) remains true even if we use, instead of $\rho_{\widetilde{\Lambda}}$ and $\rho_{\widetilde{\Lambda},\text{p.b.c}}$, the operators $\mathcal{P}_{a,b}\rho_{\widetilde{\Lambda}}\mathcal{P}_{a,b}$ and $\widehat{\mathcal{P}}_{a,b}\rho_{\widetilde{\Lambda},\text{p.b.c}}\widehat{\mathcal{P}}_{a,b}$. Then by (B.23) we get:

$$\lim_{\widetilde{\Lambda}}\left\|\chi_{I_\delta}\left(\mathcal{P}_{a,b}\rho_{\widetilde{\Lambda}}\mathcal{P}_{a,b}\right) - \chi_{I_\delta}\left(\widehat{\mathcal{P}}_{a,b}\rho_{\widetilde{\Lambda},\text{p.b.c}}\widehat{\mathcal{P}}_{a,b}\right)\right\|_{\mathcal{A}_B^\infty} = 0, \tag{B.24}$$

Consequently, since

$$\chi_I\left(\rho_{\widetilde{\Lambda}}\right) = \chi_{I_\delta}\left(\mathcal{P}_{a,b}\rho_{\widetilde{\Lambda}}\mathcal{P}_{a,b}\right),\ \chi_I\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right) = \chi_{I_\delta}\left(\widehat{\mathcal{P}}_{a,b}\rho_{\widetilde{\Lambda},\text{p.b.c}}\widehat{\mathcal{P}}_{a,b}\right),$$

by (B.24) one obtains

$$\lim_{\widetilde{\Lambda}}\left\|\chi_I\left(\rho_{\widetilde{\Lambda}}\right) - \chi_I\left(\rho_{\widetilde{\Lambda},\text{p.b.c}}\right)\right\|_{\mathcal{A}_B^\infty} = 0, \tag{B.25}$$

which implies (B.7). ∎

Corollary B.2. *If*

$$\omega\left(\rho_{\widetilde{\Lambda}}\right) < +\infty,\ \omega\left(\rho_{\widetilde{\Lambda},p.b.c}\right) < +\infty,$$

then one gets:

$$\omega\left(\chi_{\left[a,a+\frac{1}{V}\right]}\left(\rho_{\widetilde{\Lambda}}\right)\right) = \omega\left(\chi_{\left[a,a+\frac{1}{V}\right]}\left(\rho_{\widetilde{\Lambda},p.b.c}\right)\right),$$

with $\rho_{\widetilde{\Lambda}}$ *and* $\rho_{\widetilde{\Lambda},p.b.c}$ *respectively defined by (1.15) and (2.5) (or see (B.1)).*

Appendix C. .

Lemma C.1. *Let us consider a real function* $f(x,y)$ *defined in a neighborhood of* $(0,0)$*. If*

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0, \tag{C.1}$$

then there is a function $y_0(x)$ *such that*

$$\begin{aligned} &\lim_{x\to 0} y_0(x) = 0, \\ &\lim_{x\to 0} f(x,y(x)) = \lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0, \end{aligned} \tag{C.2}$$

for any $|y(x)| \leq |y_0(x)|$ *in a neighborhood of* $(0,0)$*.*

Proof. From (C.1), $\forall\varepsilon > 0, \exists\eta > 0$ such that $\forall x, |x| < \eta$, one has

$$\left|\lim_{y\to 0} f(x,y) - \lim_{x\to 0}\lim_{y\to 0} f(x,y)\right| < \frac{\varepsilon}{2}, \tag{C.3}$$

and

$$\exists\widetilde{\eta}(x) > 0 : \forall y, |y| < \widetilde{\eta}(x),\ \ \left|f(x,y) - \lim_{y\to 0} f(x,y)\right| < \frac{\varepsilon}{2}. \tag{C.4}$$

So, there is a function $y_0(x)$ such that

$$\forall\varepsilon > 0, \exists\eta > 0, \forall x, |x| < \eta,\ \ |y_0(x)| < \min\{\widetilde{\eta}(x), \varepsilon\} \leq \varepsilon, \tag{C.5}$$

i.e.,

$$\lim_{x\to 0} y_0(x) = 0.$$

Therefore, $\forall\varepsilon > 0, \exists\eta > 0$ such that $\forall x, |x| < \eta$, one has

$$\left|f(x,y(x)) - \lim_{x\to 0}\lim_{y\to 0} f(x,y)\right| < \left|f(x,y(x)) - \lim_{y\to 0} f(x,y)\right| + \left|\lim_{y\to 0} f(x,y) - \lim_{x\to 0}\lim_{y\to 0} f(x,y)\right|,$$

for any $|y(x)| \leq |y_0(x)|$ in a neighborhood of $(0,0)$, which by (C.3) and (C.4) combining with (C.5), implies

$$\left|f(x,y(x)) - \lim_{x\to 0}\lim_{y\to 0} f(x,y)\right| < \varepsilon,$$

i.e., (C.2). ■

References

[1] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications,* 2nd ed. Springer-Verlag, New-York (1998).

[2] Marco Lenci J.L. Lebowitz and Herbert Spohn, Large deviations for ideal quantum systems, J. Math. Phys. **41**:1224 (2000).

[3] J.T. Cannon, Infinite Volume Limits of the Canonical Free Bose Gas States on the Weyl Algebra, Commun. Math. Phys. **29**:89 (1973).

[4] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, *Vol. II: Functional Analysis*, Academic Press, New York-London, 1972.

[5] D. Ruelle, *Statistical Mechanics: Rigorous Results*, Benjamin, New-York, 1969.

[6] N.N. Bogoliubov, *Lectures on Quantum Statistics,* Vol. **I***,* Gordon and Breach Science Publishers, New York-London-Paris, 1970.

[7] J.-B. Bru and V.A. Zagrebnov, Exact solution of the Bogoliubov Hamiltonian for weakly imperfect Bose gas, *J. Phys. A: Math. Gen. A* **31**:9377 (1998).

[8] J.-B. Bru and V.A. Zagrebnov, On condensations in the Bogoliubov Weakly Imperfect Bose-Gas, *J. Stat. Phys.* **99**:1297 (2000).

[9] V.A. Zagrebnov and J.-B. Bru, The Bogoliubov Model of Weakly Imperfect Bose Gas, *Phys. Rep.* **350**:291 (2001).

[10] N.N. Bogoliubov (Jr), J.G. Brankov, V.A. Zagrebnov, A.M. Kurbatov and N.S. Tonchev, Some classes of exactly soluble models of problems in Quantum Statistical Mechanics : the method of the approximating Hamiltonian, *Russian Math. Surveys* **39**:1 (1984).

[11] O. Brattelli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Vol **II**, 2nd ed. Springer-Verlag, New York (1996).

[12] N.N. Bogoliubov (Jr), A method of studying model Hamiltonians (Pergamon, Oxford, 1977).

[13] N.N. Bogoliubov (Jr), J.G. Brankov, V.A. Zagrebnov, A.M. Kurbatov and N.S. Tonchev, *The Approximating Hamiltonian Method in Statistical Physics*, Publ. Bulgarian Akad. Sciences, Sofia, 1981.

[14] V.A. Zagrebnov, Perturbations of Gibbs Semigroups, *Commun. Math. Phys.* **120**:653 (1989).

[15] H.D. Maison, Analyticity of the partition function for finite quantum systems, *Commun. Math. Phys.* **22**:166 (1971).

[16] B. Simon, *The Statistical Mechanics of Lattice Gases,* Vol. **I**, Princeton Univ. Press, Princeton, 1993.

[17] O. Brattelli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Vol **I**, 2nd ed. Springer-Verlag, New York (1996).