A New Microscopic Theory of Superfluidity at all Temperatures

S. ADAMS[1] and J.-B. BRU[2]

Institut für Mathematik, Fakultät II, Sek. Ma 7-4, Technische Universität Berlin, Straße des 17. Juni 136, D-10623 Berlin, Germany
and
School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Rd., Dublin 4, Ireland.

Abstract

Following the program suggested in [1], we get a new microscopic theory of superfluidity for all temperatures and densities. In particular, the corresponding phase diagram of this theory exhibits:

(i) a thermodynamic behavior corresponding to the Mean-Field Gas for small densities or high temperatures,
(ii) the "Landau-type" excitation spectrum in the presence of non-conventional Bose condensation for high densities or small temperatures,
(iii) a coexistence of particles inside and outside the condensate with the formation of "Cooper pairs", even at zero-temperature (experimentally, an estimate of the fraction of condensate in liquid ^{4}He at T=0 K is 9 %, see [2, 3]).

In contrast to Bogoliubov's last approach and with the caveat that the full interacting Hamiltonian is truncated, the analysis performed here is rigorous by involving for the first time a complete thermodynamic analysis of a non-trivial continuous gas in the canonical ensemble.

Keywords : Bogoliubov, helium, superfluidity, Landau, excitation, spectrum, Cooper, Bose condensation, equivalence of ensembles.

[1]TU-Berlin, email: adams@math.tu-berlin.de
[2]School of Theoretical Physics, email: jbbru@stp.dias.ie

1. Introduction

The first microscopic theory of superfluidity was originally proposed in 1947 by Bogoliubov in [4–8]. A recent analysis of the Bogoliubov theory has already been performed in the review [9], itself containing a summary of [10–15]. The critical analysis performed in [1] leads us to use the Bogoliubov truncation of the full Hamiltonian within the framework of the canonical ensemble. In the grandcanonical ensemble, it corresponds to a weaker truncation than the Bogoliubov one by implying the superstable Bogoliubov Hamiltonian [16] defined in Section 2. This non-diagonal Bose gas is rigorously solved at the thermodynamic level: in the grandcanonical ensemble (Section 3) and in the canonical ensemble (Section 4).
In the case of *homogeneous* systems, these analyses provide a new (canonical) theory of superfluidity with a gapless spectrum at all particle densities and temperatures, leading us to a deeper understanding of the Bose condensation phenomenon in liquid helium explained in Section 5.

Actually, at any temperatures $\mathrm{T} \geq 0$ below a critical temperature T_c, the corresponding Bose gas is a mixture of particles inside and outside the Bose condensate. Even at zero-temperature, two Bose systems coexist: the Bose condensate and a second one, which is denoted here as the *Bogoliubov condensate*. This comes from a *non-diagonal* interaction, which implies an effective attraction between bosons in the zero kinetic energy state, i.e. in the Bose condensate [9, 12]. In contrast with the (conventional) Bose-Einstein condensation, these bosons pair up via the Bogoliubov condensate to form "Cooper pairs" as in the case of a superconductor. This Bose condensation constituted by Cooper pairs is non-conventional [9, 11, 12, 14, 17, 18], i.e. turned on by the Bose statistics but completely transformed by interaction phenomena.

The coherency due to the presence of the Bose condensation is not enough to make the Perfect Bose Gas superfluid, see discussions in [4–6]. The spectrum of elementary excitations has to be collective. In this theory, the particles outside the Bose condensate (*the Bogoliubov condensate*) follow a *new* statistics, different from the Bose statistics, which we call the *Bogoliubov statistics.* The Bogoliubov condensate is a system of "quasi-particles" with the Landau-type excitation spectrum. Therefore, following Landau's criterion of superfluidity [19, 20] it is a *superfluid* gas. The corresponding "quasi-particles" are created from two particles respectively of momenta p and $-p$ ($p \neq 0$) through the Bose condensate ($p = 0$) combined with phenomena of interaction.

The theoretical critical temperature where the Landau-type excitation spectrum holds equals $\mathrm{T}_c \approx 3.14$ K. For the liquid ^{4}He, the superfluid liquid already disappears at $\mathrm{T}_\lambda = 2.17$ K, but the Henshaw-Woods spectrum does *not* change drastically when the temperature crosses T_λ: it is still a Landau-type excitation spectrum for $\mathrm{T}_\lambda < \mathrm{T} < \widetilde{\mathrm{T}}_\lambda$. For a complete description of this theory in relation with liquid ^{4}He, see Sections 5.3 to 5.4.

The phenomenon of Cooper pairs between two fermions corresponds to the phenomenological explanation given for the existence of superfluidity and Bose condensation in ^{3}He [21–23]. Therefore, at the end (Section 5.5), we explain how this theory may also be a starting point for a microscopic theory of superfluidity for ^{3}He within the framework of Fermi systems.

Before finishing this short introduction, we recall again that this analysis is based on the Bogoliubov truncation in the canonical ensemble [1]. This *unique* truncation hypothesis is still not proven in this paper, but we will show that the theory is, at least, self-consistent. Note that it implies the exact solution of a *non-diagonal continuous model* far from the Perfect Bose Gas in the *canonical ensemble* at all temperatures and densities. This is the first time for such a rigorous thermodynamic analysis to be performed on a non-trivial continuous gas.

Remark 1.1. *This analysis is technically based on three papers. First we use the proof of the exactness of the Bogoliubov approximation in the grandcanonical ensemble for a superstable gas [24], as done by Ginibre [25]. Then, we use the "superstabilization" method [26, 27].*

2. Setup of the problem

Let an interacting *homogeneous* gas of n spinless bosons with mass m be enclosed in a cubic box $\Lambda = \underset{\alpha=1}{\overset{3}{\times}} L \subset \mathbb{R}^3$. We denote by $\varphi(x) = \varphi(\|x\|)$ a (real) *two-body* interaction potential and we assume that:

(A) $\varphi(x) \in L^1(\mathbb{R}^3)$.

(B) Its (real) Fourier transformation

$$\lambda_k = \int_{\mathbb{R}^3} d^3x \varphi(x) e^{-ikx}, \;\; k \in \mathbb{R}^3,$$

satisfies: $\lambda_0 > 0$ and $0 \le \lambda_k = \lambda_{-k} \le \lim_{\|k\| \to 0^+} \lambda_k$ for $k \in \mathbb{R}^3$.

(C) The interaction potential $\varphi(x)$ satisfies:

$$(\text{C1}) : \frac{\lambda_0}{2} + g_{00} \ge 0, \;\; \text{or } (\text{C2}) : \frac{\lambda_0}{2} + g_{00} < 0,$$

where the (*effective coupling)* constant g_{00} equals

$$g_{00} \equiv -\frac{1}{4(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{\lambda_k^2}{\varepsilon_k} < 0, \tag{2.1}$$

with the one-particle energy spectrum defined by $\varepsilon_k \equiv \hbar^2 k^2 / 2m$.

The last conditions (C1)-(C2) will be important only at the end of this paper and first appeared in the study of the Weakly Imperfect Bose Gas [9, 11, 12, 16]. In particular, we explain later in Section 5.2 the quantum interpretation given by [12] of the constant g_{00}.

Following [1], the full interacting Bose gas should be truncated within the framework of the canonical ensemble. Formally, the Mean-Field interaction does not change the "physical properties" of a Bose system (cf. [26, 27]). The "physical" effect of the interaction potential should express itself by the other terms of interaction. Actually, considering the existence of a Bose condensation in the zero-kinetic energy state, one should partially truncate the full interaction by keeping complete the Mean-Field interaction since it is a constant in the canonical ensemble. Within the framework of the grandcanonical ensemble, this procedure [1] implies the (*non-diagonal)* superstable Bogoliubov Hamiltonian [16]:

$$H_{\Lambda,\lambda_0}^{SB} \equiv H_{\Lambda,0}^{B} + U_{\Lambda}^{MF}, \tag{2.2}$$

where

$$U_{\Lambda}^{MF} \equiv \frac{\lambda_0}{2V} \sum_{k_1,k_2\in\Lambda^*} a_{k_1}^* a_{k_2}^* a_{k_2} a_{k_1} = \frac{\lambda_0}{2V} \left(N_\Lambda^2 - N_\Lambda\right), \tag{2.3}$$

$$H_{\Lambda,\lambda_0}^{B} \equiv T_\Lambda + U_\Lambda^D + U_\Lambda^{ND} + U_\Lambda^{BMF}, \tag{2.4}$$

and

$$N_\Lambda \equiv \sum_{k\in\Lambda^*} a_k^* a_k$$

$$T_\Lambda \equiv \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k,$$

$$U_\Lambda^D \equiv \frac{1}{2V} \sum_{k\in\Lambda^*\backslash\{0\}} \lambda_k a_0^* a_0 \left(a_k^* a_k + a_{-k}^* a_{-k}\right), \tag{2.5}$$

$$U_\Lambda^{ND} \equiv \frac{1}{2V} \sum_{k\in\Lambda^*\backslash\{0\}} \lambda_k \left(a_k^* a_{-k}^* a_0^2 + {a_0^*}^2 a_k a_{-k}\right), \tag{2.6}$$

$$U_\Lambda^{BMF} \equiv \frac{\lambda_0}{2V} {a_0^*}^2 a_0^2 + \frac{\lambda_0}{V} a_0^* a_0 \sum_{k\in\Lambda^*\backslash\{0\}} a_k^* a_k. \tag{2.7}$$

In the canonical ensemble, the important Hamiltonian to analyze is $H_{\Lambda,0}^B$. However, because of the instability of this model in the grandcanonical ensemble [1], we need first the grand-canonical thermodynamic properties of the superstable Bogoliubov Hamiltonian $H_{\Lambda,\lambda_0}^{SB}$ for any $\lambda_0 > 0$ (Section 3), in order to get the canonical thermodynamics of $H_{\Lambda,0}^B$ (Section 4). In particular, $H_{\Lambda,0}^B$ does not depend on $\lambda_0 > 0$, originally defined as the Fourier transformation of $\varphi(x)$ for $k = 0$. Then, we explain in Section 4 that λ_0 can be chosen freely as an arbitrary parameter and has to be taken sufficiently large to satisfy (C1).

Remark 2.1. *For $\lambda_0 > 0$, the Hamiltonian $H_{\Lambda,\lambda_0>0}^B$ represents in fact the original Bogoliubov Hamiltonian [4–8]. Since the truncation does not involve here the Mean-Field interaction U_Λ^{MF}, we get the superstabilized [26, 27] version of the Bogoliubov Hamiltonian for $\lambda_0 = 0$ (the Bogoliubov Hamiltonian without the term U_Λ^{BMF} coming from U_Λ^{MF} by using the Bogoliubov truncation).*

The Hamiltonian $H_{\Lambda,\lambda_0}^{SB}$ acts on the boson Fock space

$$\mathcal{F}_\Lambda^B \equiv \bigoplus_{n=0}^{+\infty} \mathcal{H}_B^{(n)},$$

with $\mathcal{H}_B^{(n)}$ defined as the symmetrized n-particle Hilbert spaces

$$\mathcal{H}_B^{(n)} \equiv \left(L^2\left(\Lambda^n\right)\right)_{\text{symm}},\ \mathcal{H}_B^{(0)} \equiv \mathbb{C},$$

see [24, 28]. Using periodic boundary conditions, let

$$\Lambda^* \equiv \left\{k \in \mathbb{R}^3 :\ k_\alpha = \frac{2\pi n_\alpha}{L},\ n_\alpha = 0, \pm1, \pm2, ..., \ \alpha = 1, 2, 3\right\}$$

be the set of wave vectors. Also, note that $a_k^{\#} = \{a_k^* \text{ or } a_k\}$ are the usual boson creation/annihilation operators in the one-particle state $\psi_k(x) = V^{-\frac{1}{2}} e^{ikx}, k \in \Lambda^*$, $x \in \Lambda$, acting on the boson Fock space $\mathcal{F}_\Lambda^B$. Under assumptions (A) and (B) on the interaction potential $\varphi(x)$ the Hamiltonian $H_{\Lambda,\lambda_0}^{SB}$ is superstable [24].

Remark 2.2. *Here $\beta > 0$ is the inverse temperature, μ the chemical potential, $\rho > 0$ the fixed full particle density, whereas $n = [\rho V]$ defined as the integer of ρV, is the number of particles in the canonical ensemble. Also $\mathrm{T} = (k_B \beta)^{-1} \geq 0$ is the temperature where k_B is the Boltzmann constant.*

3. Thermodynamics in the grandcanonical ensemble

In this section we analyze the thermodynamic behavior of the superstable Bogoliubov Hamiltonian $H_{\Lambda,\lambda_0}^{SB}$ in the grandcanonical ensemble. Note that in the canonical ensemble for a given density ρ, i.e. on the Hilbert space $\mathcal{H}_B^{(n=[\rho V])}$, the Hamiltonians $H_{\Lambda,\lambda_0}^{SB}$ and $H_{\Lambda,0}^{B}$ differ only by a constant. Therefore one may be tempted to first analyze the easier Hamiltonian $H_{\Lambda,0}^{B}$. Indeed $H_{\Lambda,\lambda_0}^{SB}$ represents the superstabilized version of $H_{\Lambda,0}^{B}$ (remark 2.1) and from [26, 27] we could have found all the thermodynamic behavior of the superstable Bose system $H_{\Lambda,\lambda_0}^{SB}$, as it is done for the Mean-Field Bose Gas using the Perfect Bose Gas, cf. [29–35]. The thermodynamic behavior of $H_{\Lambda,0}^{B}$ is known [1], but, unfortunately, this Bose system is drastically instable at high densities in the grandcanonical ensemble. The terms of repulsion in the Hamiltonian $H_{\Lambda,0}^{B}$ are not strong enough to prevent the system from collapse. So the Bose gas $H_{\Lambda,0}^{B}$ gives only very partial results on its superstabilized version $H_{\Lambda,\lambda_0}^{SB}$, and in fact only at sufficiently low chemical potentials. Consequently, the thermodynamic behavior of $H_{\Lambda,\lambda_0}^{SB}$ has to be found directly.

Before entering this study recall the definitions of the grandcanonical pressure $p_\Lambda^{SB}(\beta,\mu)$ and particle density $\rho_\Lambda^{SB}(\beta,\mu)$ associated with $H_{\Lambda,\lambda_0}^{SB}$:

$$\begin{aligned} p_\Lambda^{SB}(\beta,\mu) &\equiv \frac{1}{\beta V} \ln Tr_{\mathcal{F}_\Lambda^B}\left(e^{-\beta\left(H_{\Lambda,\lambda_0}^{SB} - \mu N_\Lambda\right)}\right), \\ \rho_\Lambda^{SB}(\beta,\mu) &\equiv \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_{\Lambda,\lambda_0}^{SB}} (\beta,\mu) = \partial_\mu p_\Lambda^{SB}(\beta,\mu). \end{aligned}$$

Here $\langle - \rangle_{H_\Lambda^X}(\beta,\mu)$ represents the (finite volume) *grandcanonical* Gibbs state for some Hamiltonian H_Λ^X:

$$\langle - \rangle_{H_\Lambda^X}(\beta,\mu) \equiv \frac{Tr_{\mathcal{F}_\Lambda^B}\left((-)\, e^{-\beta\left(H_\Lambda^X - \mu N_\Lambda\right)}\right)}{Tr_{\mathcal{F}_\Lambda^B}\left(e^{-\beta\left(H_\Lambda^X - \mu N_\Lambda\right)}\right)}.$$

From the superstability of the Hamiltonian $H_{\Lambda,\lambda_0}^{SB}$ it follows that $p_\Lambda^{SB}(\beta,\mu)$ is defined for every pair

$$(\beta,\mu) \in Q^S \equiv \{\beta > 0\} \times \{\mu \in \mathbb{R}\},$$

even in the thermodynamic limit [24].

3.1. The grandcanonical pressure

The first step is to use the Bogoliubov approximation, i.e.

$$a_0/\sqrt{V} \to c \in \mathbb{C},\ a_0^*/\sqrt{V} \to \overline{c} \in \mathbb{C},$$

for the Hamiltonian $H_{\Lambda,\lambda_0}^{SB}(\mu) \equiv H_{\Lambda,\lambda_0}^{SB} - \mu N_\Lambda$. Since the model $H_{\Lambda,\lambda_0}^{SB}$ is superstable [24], Ginibre [25] proves the exactness of the Bogoliubov approximation in the sense that

$$p^{SB}(\beta,\mu) = \lim_{\Lambda} p_\Lambda^{SB}(\beta,\mu) = \sup_{c\in\mathbb{C}} p^{SB}(\beta,\mu,c) \equiv \lim_{\Lambda}\left\{\sup_{c\in\mathbb{C}} p_\Lambda^{SB}(\beta,\mu,c)\right\}, \tag{3.1}$$

with

$$p_\Lambda^{SB}(\beta,\mu,c) \equiv \frac{1}{\beta V}\ln Tr_{\mathcal{F}_B'} e^{-\beta H_{\Lambda,\lambda_0}^{SB}(\mu,c)},\ p^{SB}(\beta,\mu,c) \equiv \lim_{\Lambda} p_\Lambda^{SB}(\beta,\mu,c). \tag{3.2}$$

Here

$$\mathcal{F}_B' \equiv \bigoplus_{n=0}^{+\infty} \mathcal{H}_{B,k\neq 0}^{(n)}$$

is the boson Fock space of the symmetrized n-particle Hilbert spaces $\mathcal{H}_{B,k\neq 0}^{(n)}$ for non-zero momentum bosons.

Note that

$$H_{\Lambda,\lambda_0}^{SB}(\mu,c) = H_{\Lambda,\lambda_0}^{B}(\mu,c) + \frac{\lambda_0}{2V}\left(N_{\Lambda,k\neq 0}^2 - N_{\Lambda,k\neq 0}\right) \text{ with } \lambda_0 > 0, \tag{3.3}$$

where $H_{\Lambda,\lambda_0}^{B}(\mu,c)$ is defined by (A.1) in Appendix A and $N_{\Lambda,k\neq 0}$ is the operator of the number of particles outside the zero-mode. Then, the second step to find the thermodynamic limit $p^{SB}(\beta,\mu)$ of the pressure uses [26] and gives the following result:

Theorem 3.1. *Let $p_0^B(\beta,\alpha,x)$ (A.9) be the thermodynamic limit of the pressure of the non-superstable Hamiltonian $H_{\Lambda,0}^{B}(\alpha,c)$, then for any $(\beta,\mu)\in Q^S$, one has*

$$p^{SB}(\beta,\mu) = \sup_{c\in\mathbb{C}} p^{SB}(\beta,\mu,c) = \sup_{x\geq 0}\left\{\inf_{\alpha\leq 0}\left\{p_0^B(\beta,\alpha,x) + \frac{(\mu-\alpha)^2}{2\lambda_0}\right\}\right\}. \tag{3.4}$$

Proof. By (3.1) the proof consists of getting the thermodynamic limit $p^{SB}(\beta,\mu,c)$ of the pressure $p_\Lambda^{SB}(\beta,\mu,c)$ associated with the Hamiltonian $H_{\Lambda,\lambda_0}^{SB}(\mu,c)$. We calculate the pressure $p_\Lambda^{SB}(\beta,\mu,c)$ via the weak-equivalence of another related Hamiltonian defined in part (i). In (ii) we consider the specific free energy densities in the canonical ensemble, whereas in (iii) we show convexity, such that weak-equivalence of ensembles implies the theorem in (iv).

(i) Let

$$\widetilde{H}_{\Lambda,\lambda_0}^{B}(\gamma,c) \equiv H_{\Lambda,\lambda_0}^{B}(\gamma,c) + \gamma\,|c|^2\,V - \frac{\lambda_0}{2}\left(|c|^4\,V - |c|^2\right), \tag{3.5}$$

see (A.1). The Hamiltonians $\widetilde{H}_{\Lambda,\lambda_0}^{B}(\gamma,c)$ and $H_{\Lambda,\lambda_0}^{SB}(\mu,c)$ are well-defined on the boson Fock space $\mathcal{F}_B'$ for any fixed $c\in\mathbb{C}$. Here we use two chemical potentials γ and μ respectively for the models $\widetilde{H}_{\Lambda,\lambda_0}^{B}(\gamma,c)$ and $H_{\Lambda,\lambda_0}^{SB}(\mu,c)$. From Appendix A, the Hamiltonian $\widetilde{H}_{\Lambda,\lambda_0}^{B}(\gamma,c)$

is diagonalizable by the Bogoliubov canonical u-v transformation, see (A.7), and one gets a perfect Bose gas with a spectrum E^{B}_{k,λ_0} (A.5). We then have

$$\widetilde{p}^{B}_{\Lambda,\lambda_0}(\beta,\gamma,c) = p^{B}_{\Lambda,\lambda_0}(\beta,\gamma,c) - \gamma\left|c\right|^2 + \frac{\lambda_0}{2}\left(\left|c\right|^4 - \frac{\left|c\right|^2}{V}\right) \tag{3.6}$$

for

$$\gamma \leq \left|c\right|^2 \lambda_0 + \min_{k\in\Lambda^*\backslash\{0\}} \varepsilon_k,$$

see (A.6) and (A.8) in Appendix A. The thermodynamic limit follows as

$$\widetilde{p}^{B}_{\lambda_0}\left(\beta,\gamma,x=\left|c\right|^2\right) \equiv \lim_{\Lambda} \widetilde{p}^{B}_{\Lambda,\lambda_0}(\beta,\gamma,c) = p^{B}_{\lambda_0}(\beta,\gamma,x) - \gamma x + \frac{\lambda_0}{2}x^2, \tag{3.7}$$

cf. (A.9) for

$$\gamma \leq x\lambda_0 = \lim_{\Lambda}\left(\left|c\right|^2\lambda_0 + \min_{k\in\Lambda^*\backslash\{0\}}\varepsilon_k\right) \text{ and } \lambda_0 > 0.$$

(ii) Note that

$$\left[\widetilde{H}^{B}_{\Lambda,\lambda_0}(\gamma,c), N_{\Lambda,k\neq 0}\right] \neq 0, \quad \left[H^{SB}_{\Lambda,\lambda_0}(\mu,c), N_{\Lambda,k\neq 0}\right] \neq 0.$$

However, for a fixed particle density $\rho_1 > 0$, let $\widetilde{f}^{B}_{\Lambda,\lambda_0}(\beta,\rho_1,c)$ and $f^{SB}_{\Lambda}(\beta,\rho_1,c)$ be the free-energy densities:

$$\begin{aligned} \widetilde{f}^{B}_{\Lambda,\lambda_0}(\beta,\rho_1,c) &\equiv -\frac{1}{\beta V}\ln Tr_{\mathcal{H}^{(n)}_{B,k\neq 0}}\left(\left\{e^{-\beta \widetilde{H}^{B}_{\Lambda,\lambda_0}(0,c)}\right\}^{(n,k\neq 0)}\right), \\ f^{SB}_{\Lambda}(\beta,\rho_1,c) &\equiv -\frac{1}{\beta V}\ln Tr_{\mathcal{H}^{(n)}_{B,k\neq 0}}\left(\left\{e^{-\beta H^{SB}_{\Lambda,\lambda_0}(0,c)}\right\}^{(n,k\neq 0)}\right), \end{aligned} \tag{3.8}$$

where

$$A^{(n,k\neq 0)} \equiv A\lceil \mathcal{H}^{(n)}_{B,k\neq 0}$$

is the *restriction* of any operator A acting on the boson Fock space $\mathcal{F}'_B$ ($n = [\rho_1 V]$). Note that

$$\widetilde{p}^{B}_{\Lambda,\lambda_0}(\beta,\gamma,c) = \frac{1}{\beta V}\ln\sum_{n=0}^{+\infty} e^{\beta V\left\{\gamma\frac{n}{V} - \widetilde{f}^{B}_{\Lambda,\lambda_0}\left(\beta,\frac{n}{V},c\right)\right\}}. \tag{3.9}$$

The free-energy density $\widetilde{f}^{B}_{\Lambda,\lambda_0}(\beta,\rho_1,c)$ is in fact well-defined for any $\rho_1 > 0$ and $\beta > 0$ in the thermodynamic limit, i.e.

$$\widetilde{f}^{B}_{\lambda_0}\left(\beta,\rho_1,x=\left|c\right|^2\right) \equiv \lim_{\Lambda}\widetilde{f}^{B}_{\Lambda,\lambda_0}(\beta,\rho_1,c) < +\infty.$$

From (3.3) and (3.5) we then have

$$f^{SB}_{\Lambda}(\beta,\rho_1,c) = \widetilde{f}^{B}_{\Lambda,\lambda_0}(\beta,\rho_1,c) + \frac{\lambda_0}{2}\left(\rho_1^2 - \frac{\rho_1}{V}\right) - \mu\left|c\right|^2 + \frac{\lambda_0}{2}\left(\left|c\right|^4 - \frac{\left|c\right|^2}{V}\right),$$

which gives

$$f^{SB}\left(\beta,\rho_1,x=|c|^2\right) \equiv \lim_{\Lambda} f_{\Lambda}^{SB}\left(\beta,\rho_1,c\right) = \widetilde{f}_{\lambda_0}^{B}\left(\beta,\rho_1,x\right) + \frac{\lambda_0}{2}\rho_1^2 - \mu x + \frac{\lambda_0}{2}x^2.$$

(iii) Notice that we do not know if the specific free energy $\widetilde{f}_{\lambda_0}^{B}\left(\beta,\rho_1,x\right)$ is convex as a function of ρ_1, which is crucial in order to use [26] for our proof. It is the next step of the proof.
By (3.6), there is a unique solution of $\gamma_\Lambda\left(\rho_1\right)$ with

$$\partial_\gamma \widetilde{p}_{\Lambda,\lambda_0}^{B}\left(\beta,\gamma_\Lambda\left(\rho_1\right),c\right) = \rho_1 \tag{3.10}$$

at all densities $\rho_1 > 0$. By direct computations of (3.10) done via (3.7), the corresponding thermodynamic limit

$$\gamma\left(\rho_1\right) \equiv \lim_{\Lambda} \gamma_\Lambda\left(\rho_1\right) = \begin{cases} < x\lambda_0 \text{ for } \rho_1 < \partial_\gamma \widetilde{p}_{\lambda_0}^{B}\left(\beta,\lambda_0 x,x\right), \\ \;\; x\lambda_0 \text{ for } \rho_1 \geq \partial_\gamma \widetilde{p}_{\lambda_0}^{B}\left(\beta,\lambda_0 x,x\right), \end{cases}$$

is an increasing continuous function of $\rho_1 > 0$. By (3.9) we also have

$$\begin{aligned} \widetilde{p}_{\lambda_0}^{B}\left(\beta,\gamma\left(\rho_1\right),x\right) &\equiv \lim_{\Lambda} \widetilde{p}_{\Lambda,\lambda_0}^{B}\left(\beta,\gamma_\Lambda\left(\rho_1\right),c\right) = \gamma\left(\rho_1\right)\left(\rho_1\right) - \widetilde{f}_{\lambda_0}^{B}\left(\beta,\rho_1,x\right) \\ &= \sup_{t>0}\left\{\gamma\left(\rho_1\right)t - \widetilde{f}_{\lambda_0}^{B}\left(\beta,t,x\right)\right\}. \end{aligned} \tag{3.11}$$

Therefore for any $\rho_1 > 0$,

$$\partial_{\rho_1} \widetilde{f}_{\lambda_0}^{B}\left(\beta,\rho_1,x\right) = \gamma\left(\rho_1\right)$$

is an increasing function of $\rho_1 > 0$, i.e. $\widetilde{f}_{\lambda_0}^{B}\left(\beta,\rho_1,x\right)$ is a convex function of $\rho_1 > 0$.

(iv) The weak equivalence of ensembles is then verified by the model $\widetilde{H}_{\Lambda,\lambda_0}^{B}\left(\gamma,c\right)$ for each fixed $x = |c|^2 \geq 0$, and using [26] combined with (3.3) and (3.5) we directly find

$$p^{SB}\left(\beta,\mu,c\right) = \left\{\inf_{\gamma\leq x\lambda_0}\left\{\widetilde{p}_{\lambda_0}^{B}\left(\beta,\gamma,x\right) + \frac{\left(\mu-\gamma\right)^2}{2\lambda_0}\right\} + \mu x - \frac{\lambda_0}{2}x^2\right\}\Bigg|_{x=|c|^2}. \tag{3.12}$$

Therefore the theorem follows by (3.1) and (3.7) and the last equality, if we take $\alpha = \gamma - x\lambda_0 \leq 0$ in the expression for the infimum and finally switch from $\widetilde{p}_{\lambda_0}^{B}\left(\beta,\gamma,x\right)$ to $p_0^B\left(\beta,\alpha,x\right)$. ■

Let us consider the Mean-Field Hamiltonian

$$H_\Lambda^{MF} = T_\Lambda + U_\Lambda^{MF} \equiv T_\Lambda + \frac{\lambda_0}{2V}\left(N_\Lambda^2 - N_\Lambda\right), \tag{3.13}$$

or the Imperfect Bose Gas

$$H_\Lambda^{IBG} = T_\Lambda + \frac{\lambda_0}{2V}N_\Lambda^2 \tag{3.14}$$

see [29–35]. Then, by theorem 3.1 and (A.9), we get the following lower bound for the pressure:

$$\begin{aligned} p^{SB}\left(\beta,\mu\right) &\geq \inf_{\alpha\leq 0}\left\{p_0^B\left(\beta,\alpha,0\right) + \frac{\left(\mu-\alpha\right)^2}{2\lambda_0}\right\} \\ &= \inf_{\alpha\leq 0}\left\{p^{PBG}\left(\beta,\alpha\right) + \frac{\left(\mu-\alpha\right)^2}{2\lambda_0}\right\} = p^{MF}\left(\beta,\mu\right) = p^{IBG}\left(\beta,\mu\right), \end{aligned} \tag{3.15}$$

where $p^{PBG}(\beta,\alpha)$, $p^{MF}(\beta,\mu)$ and $p^{IBG}(\beta,\mu)$ are the (infinite volume) pressures respectively for the Perfect Bose Gas, the Mean-Field Bose Gas and the Imperfect Bose Gas, see [26,29–35]. Let $\alpha(x) \equiv \alpha(\beta,\mu,x)$ be the solution of

$$\inf_{\alpha\leq 0}\left\{p_0^B(\beta,\alpha,x)+\frac{(\mu-\alpha)^2}{2\lambda_0}\right\}=\left\{p_0^B(\beta,\alpha,x)+\frac{(\mu-\alpha)^2}{2\lambda_0}\right\}\Big|_{\alpha=\alpha(x)} \tag{3.16}$$

for any fixed $x \geq 0$. Thus we have

$$\partial_\alpha\left\{p_0^B(\beta,\alpha,x)+\frac{(\mu-\alpha)^2}{2\lambda_0}\right\}\Big|_{\alpha=\alpha(x)\leq 0}=\left\{\rho_0^B(\beta,\alpha,x)-\frac{(\mu-\alpha)}{\lambda_0}\right\}\Big|_{\alpha=\alpha(x)\leq 0}=0 \tag{3.17}$$

for chemical potentials

$$\mu \leq \mu_c(\beta,x) \equiv \lambda_0\rho_0^B(\beta,0,x), \tag{3.18}$$

whereas for $\mu \geq \mu_c(\beta,x)$ and $\alpha \leq 0$ the corresponding derivative is negative:

$$\rho_0^B(\beta,\alpha,x)-\frac{(\mu-\alpha)}{\lambda_0}\leq 0 \text{ which implies } \alpha(x)=0. \tag{3.19}$$

Here,

$$\begin{aligned}\rho_0^B(\beta,\alpha,x) &\equiv \partial_\alpha p_0^B(\beta,\alpha,x)=x+\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{f_{k,0}}{E_{k,0}^B\left[e^{\beta E_{k,0}^B}-1\right]}d^3k\\ &+\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{x^2\lambda_k^2}{2E_{k,0}^B\left[f_{k,0}+E_{k,0}^B\right]}d^3k.\end{aligned} \tag{3.20}$$

Note that $\rho_0^B(\beta,\alpha,0)=\rho^{PBG}(\beta,\alpha\leq 0)$ is the critical density of the Perfect Bose Gas. Since all functions depend only on $x=|c|^2$, in the following we denote by $\widehat{x}=\widehat{x}(\beta,\mu)$ the solution of the variational problem of theorem 3.1:

$$p^{SB}(\beta,\mu)=\inf_{\alpha\leq 0}\left\{p_0^B(\beta,\alpha,\widehat{x})+\frac{(\mu-\alpha)^2}{2\lambda_0}\right\}, \tag{3.21}$$

and we solve it via the following theorem.

Theorem 3.2. *For any $\beta>0$, there exists a unique $\mu_c(\beta)$ such that*

$$p^{SB}(\beta,\mu)=\begin{cases} p_0^B(\beta,\alpha(0),0)+\dfrac{(\mu-\alpha(0))^2}{2\lambda_0}=p^{MF}(\beta,\mu)=p^{IBG}(\beta,\mu), & \text{for } \mu\leq\mu_c(\beta).\\ \left\{p_0^B(\beta,\alpha(x),x)+\dfrac{(\mu-\alpha(x))^2}{2\lambda_0}\right\}\Big|_{x=\widehat{x}>0}, & \text{for } \mu>\mu_c(\beta).\end{cases}$$

The function $\mu_c(\beta)$ is bijective from $[a,+\infty)$ to $\mathbb{R}_+$ and we denote by $\beta_c(\mu)\geq 0$ the inverse function of $\mu_c(\beta)$, see figure 3.1. Here $a=0$ if (C1) holds whereas if (C2) is satisfied $a=\mu_0<0$. The pressure $p^{SB}(\beta,\mu)$ is continuous for $\mu=\mu_c(\beta)$.

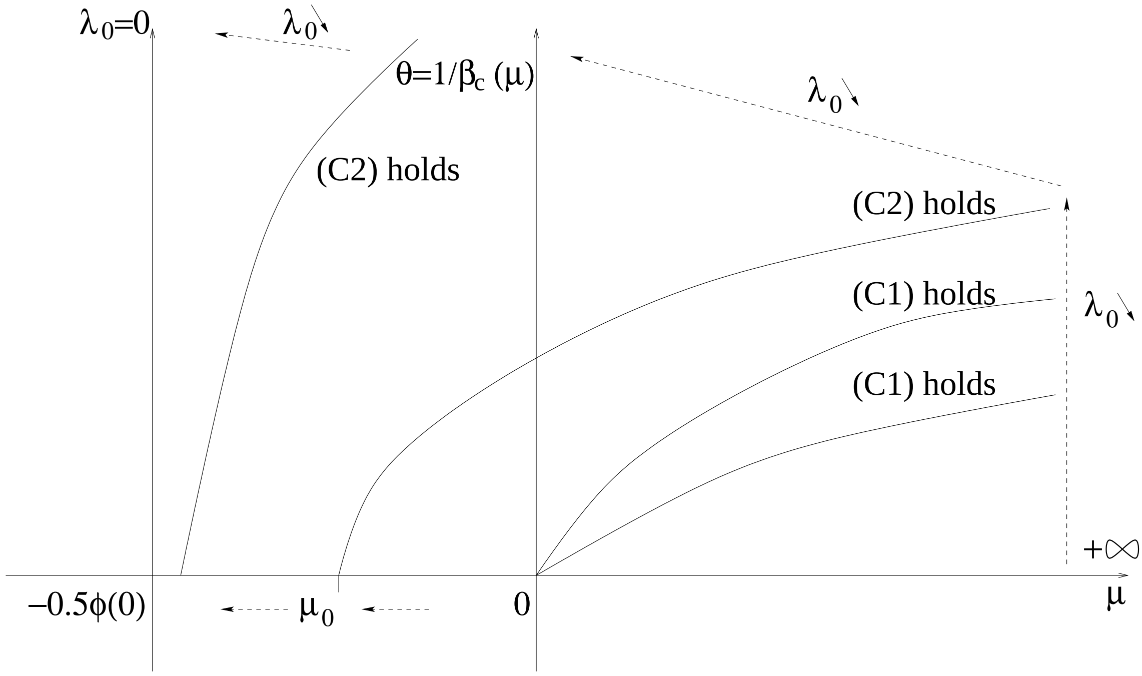

Abbildung 3.1: *Illustration of the critical temperature* $\theta_c = 1/\beta_c(\mu)$ *from* $\lambda_0 = +\infty$ *to* 0^+ *(model* $H^B_{\Lambda,0}$*, see [1]).*

Proof. From theorem 3.1 we get

$$p^{SB}(\beta,\mu) = \sup_{x\geq 0}\{F_\beta(\alpha(x),x)\} = \{F_\beta(\alpha(x),x)\}\Big|_{x=\widehat{x}},$$

where the function $F_\beta(\alpha,x)$ is given by

$$\begin{cases} F_\beta(\alpha,x) \equiv p_0^B(\beta,\alpha,x) + \dfrac{(\mu-\alpha)^2}{2\lambda_0} = \xi_0(\beta,\alpha,x) + \eta_0(\alpha,\beta) + \dfrac{(\mu-\alpha)^2}{2\lambda_0}. \\ F_\infty(\alpha,x) \equiv \lim\limits_{\beta\to\infty} F_\beta(\alpha,x) = \eta_0(\alpha,x) + \dfrac{(\mu-\alpha)^2}{2\lambda_0}. \end{cases} \tag{3.22}$$

We recall that $p_0^B(\beta,\alpha,x) = \xi_0(\beta,\alpha,x) + \eta_0(\alpha,x)$ is defined by (A.9) in Appendix A. So, we have to evaluate the sign of

$$\partial_x\{F_\beta(\alpha(x),x)\} = \{\partial_x F_\beta(\alpha,x)\}\Big|_{\alpha=\alpha(x)} + \{\partial_x\alpha(x)\,\partial_\alpha F_\beta(\alpha,x)\}\Big|_{\alpha=\alpha(x)}, \tag{3.23}$$

to obtain $x = \widehat{x}$ maximizing the function $F_\beta(\alpha(x),x)$.
The proof is then divided in four parts. First we get in **1**. an easier expression of the derivative of the functional $F_\beta(\alpha(x),x)$: the second term of (3.23) is in fact zero. In the second step **2**. we study the solution $\alpha(x)$ and the corresponding critical chemical potential $\mu_c(\beta,x)$ (3.18). In **3**. and **4**. we get the first results for $\beta\to\infty$ and then for arbitrary finite β.

1. Through (3.22) one has

$$\partial_\alpha F_\beta(\alpha, x) = \rho_0^B(\beta, \alpha, x) - \frac{(\mu - \alpha)}{\lambda_0}. \tag{3.24}$$

Then, by (3.17)-(3.19), one has

$$\begin{cases} \{\partial_\alpha F_\beta(\alpha, x)\}\Big|_{\alpha=\alpha(x)<0} = 0 \text{ for } \mu < \mu_c(\beta, x). \\ \{\partial_\alpha F_\beta(\alpha, x)\}\Big|_{\alpha=\alpha(x)=0} = 0 \text{ for } \mu = \mu_c(\beta, x). \\ \{\partial_\alpha F_\beta(\alpha, x)\}\Big|_{\alpha=\alpha(x)=0} < 0 \text{ for } \mu > \mu_c(\beta, x). \end{cases} \tag{3.25}$$

Therefore

$$\{\partial_x \alpha(x)\, \partial_\alpha F_\beta(\alpha, x)\}\Big|_{\alpha=\alpha(x)} = 0$$

for any fixed μ and so (3.23) can be written as

$$\partial_x \{F_\beta(\alpha(x), x)\} = \{\partial_x F_\beta(\alpha, x)\}\Big|_{\alpha=\alpha(x)}. \tag{3.26}$$

Notice that $\alpha(x) = \alpha(\beta, \mu, x)$ is also a function of the inverse temperature and chemical potential and, in the same way, we get

$$\begin{cases} \{\partial_\beta \alpha(x)\, \partial_\alpha F_\beta(\alpha, x)\}\Big|_{\alpha=\alpha(x)} = 0. \\ \{\partial_\mu \alpha(x)\, \partial_\alpha F_\beta(\alpha, x)\}\Big|_{\alpha=\alpha(x)} = 0. \end{cases} \tag{3.27}$$

2. By (3.18) and (3.20) note that

$$\lim_{x\to+\infty} \mu_c(\beta, x) = +\infty.$$

Moreover we have

$$\begin{aligned} \partial_x \rho_0^B(\beta, \alpha, x) &= 1 + \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{x\lambda_k^2}{\sqrt{\varepsilon_k - \alpha}\,(\varepsilon_k - \alpha + 2x\lambda_k)^{3/2}} \left(1 + \frac{2}{e^{\beta E_{k,0}^B} - 1}\right) d^3k \\ &\quad - \frac{1}{4(2\pi)^3} \int_{\mathbb{R}^3} \lambda_k \left(\frac{\varepsilon_k - \alpha + x\lambda_k}{\varepsilon_k - \alpha + 2x\lambda_k}\right) \frac{\beta}{\sinh^2\left(\beta E_{k,0}^B/2\right)} d^3k. \end{aligned} \tag{3.28}$$

Since the last in term in (3.28) vanishes when $\beta \to \infty$ for all $x \geq 0$ and all $\alpha \leq 0$, $\partial_x \rho_0^B(\beta, \alpha, x) > 0$ for sufficiently large β. Thus

$$\inf_{x\geq 0} \mu_c(\beta, x) = \lambda_0 \inf_{x\geq 0} \rho_0^B(\beta, 0, x) = \mu_c(\beta, 0) = \lambda_0 \rho^{PBG}(\beta, 0) > 0 \tag{3.29}$$

for sufficiently large $\beta > 0$ and the critical chemical potential $\mu_c(\beta, x)$ is an increasing function of $x \geq 0$. Consequently there is for $\mu > \mu_c(\beta, 0)$ a solution $x_\mu > 0$ of

$$\mu_c(\beta, x_\mu) = \mu, \tag{3.30}$$

such that

$$\alpha(x) = \begin{cases} 0, \text{ for } 0 \leq x \leq x_\mu, \\ < 0, \text{ for } x > x_\mu > 0, \end{cases} \tag{3.31}$$

and for all $x_2 > x_1 > x_\mu$,

$$\alpha(x_2) < \alpha(x_1) \text{ and } \lim_{x \to +\infty} \alpha(x) = -\infty. \tag{3.32}$$

To summarize the behavior of $\alpha(x) = \alpha(\beta, \mu, x)$:

$$\begin{aligned} \lim_{x \to +\infty} \alpha(\beta, \mu, x) &= -\infty \text{ for } \beta, \mu \text{ fixed}, \\ \lim_{\mu \to -\infty} \alpha(\beta, \mu, x) &= -\infty \text{ for } \beta, x \text{ fixed}, \\ \lim_{\beta \to 0^+} \alpha(\beta, \mu, x) &= -\infty \text{ for } \mu, x \text{ fixed}. \end{aligned} \tag{3.33}$$

3. We consider now the limit $\beta \to \infty$. To analyze the derivative of the functional $F_\infty(\alpha(x), x)$ we only have to consider the partial derivative with respect to x, because we get the same results for $F_\infty(\alpha(x), x)$ as in (3.25) and (3.26) for the functional $F_\beta(\alpha(x), x)$. Thus by (3.22) we have for any $\alpha \geq 0$

$$\partial_x \lim_{\beta \to +\infty} F_\beta(\alpha, x) = \partial_x F_\infty(\alpha, x) = \alpha + \Omega(\alpha, x), \tag{3.34}$$

where

$$\Omega(\alpha, x) \equiv \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \lambda_k \left\{ 1 - \frac{\sqrt{\varepsilon_k - \alpha}}{\sqrt{\varepsilon_k - \alpha + 2x\lambda_k}} \right\} d^3k \geq 0. \tag{3.35}$$

By direct computations of the partial derivatives with respect to α and x, we find that $\Omega(\alpha, x)$ is a strictly increasing concave function of $x \geq 0$ for any fixed $\alpha \leq 0$ with

$$\Omega(\alpha, 0) = 0 \text{ and } \lim_{x \to +\infty} \Omega(\alpha, x) = \frac{\varphi(0)}{2}, \tag{3.36}$$

whereas for any fixed $x > 0$, $\Omega(\alpha, x)$ is a strictly increasing convex function of $\alpha \leq 0$ with

$$\lim_{\alpha \to -\infty} \Omega(\alpha, x) = 0 \leq \Omega(0, x) = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \lambda_k \left\{ 1 - \frac{\sqrt{\varepsilon_k}}{\sqrt{\varepsilon_k + 2x\lambda_k}} \right\} d^3k. \tag{3.37}$$

Via (3.29) one has

$$\lim_{\beta \to +\infty} \left\{ \inf_{x \geq 0} \mu_c(\beta, x) \right\} = 0,$$

i.e. we have to consider the cases $\mu > 0$ and $\mu \leq 0$.

3.1. Let us first discuss the case $\mu > 0$. By (3.30)-(3.32), there is $x_\mu > 0$ such that

$$\alpha(x) = \begin{cases} 0 \text{ , for } 0 \le x \le x_\mu . \\ < 0 \text{ , for } x > x_\mu > 0. \end{cases} \tag{3.38}$$

Combining (3.36)-(3.37) with the previous relation, we get

$$\Omega(0, x_\mu) \ge \Omega(0, x) > 0$$

for $0 < x \le x_\mu$ and $\mu > 0$ and the lower bound

$$\sup_{x \ge 0} \{F_\infty(\alpha(x), x)\} = \{F_\infty(\alpha(x), x)\}\Big|_{x=\widehat{x}} > \sup_{0 \le x \le x_\mu} \{F_\infty(\alpha(x), x)\} = F_\infty(0, x_\mu) \tag{3.39}$$

which implies $\widehat{x} > x_\mu > 0$ and $\alpha(\widehat{x}) < 0$ for $\mu > 0$. This first result proves the theorem for $\mu > 0$ and $\beta \to \infty$.

3.2. If $\mu \le 0$ the condition (3.17) is always satisfied and gives an expression for $\alpha = \alpha(x)$, i.e. $\alpha(x) = \mu - \lambda_0 \rho_0^B(\beta, \alpha(x), x)$. Hence, since the second term in (3.20) vanishes in the limit $\beta \to \infty$ we can rewrite (3.34):

$$\begin{aligned} \partial_x \{F_\infty(\alpha(x), x)\} &= \{\partial_x F_\infty(\alpha, x)\}\Big|_{\alpha=\alpha(x)} = \mu - \lambda_0 \rho_0^B(\beta, \alpha(x), x) + \Omega(\alpha(x), x) \\ &= \left\{ \mu + \Omega(\alpha, x) - \lambda_0 x - \frac{\lambda_0 x^2}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{\lambda_k^2}{E_{k,0}^B \left[f_{k,0} + E_{k,0}^B\right]} d^3k \right\}\Bigg|_{\alpha=\alpha(x)} . \end{aligned} \tag{3.40}$$

Moreover, notice that

$$\partial_x^2 F_\infty(\alpha, x) = -\lambda_0 + \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{\lambda_k^2 (\varepsilon_k - \alpha - x\lambda_0)}{\sqrt{\varepsilon_k - \alpha}\,(\varepsilon_k - \alpha + 2x\lambda_k)^{3/2}} d^3k < \partial_x^2 F_\infty(\alpha, x)\Big|_{x=0}, \tag{3.41}$$

for any $\alpha \le 0$ and $x > 0$, see (3.28) with $\beta \to +\infty$ for the derivative of the density $\rho_0^B(\beta, \alpha, x)$. We also have

$$\partial_x^2 F_\infty(\alpha, x)\Big|_{x=0} = -\lambda_0 + \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{\lambda_k^2}{(\varepsilon_k - \alpha)} d^3k \le \partial_x^2 F_\infty(\alpha, x)\Big|_{x=0, \alpha=0} = -2\left(\frac{\lambda_0}{2} + g_{00}\right),$$

see (2.1). Therefore, by fixing the sign of $\partial_x^2 F_\infty(\alpha, x)\Big|_{x=0,\alpha=0}$ the assumptions (C1)-(C2) imply two different behaviors for the solution $\widehat{x}$ of the variational problem.

(C1) If condition (C1) is satisfied, we find via the two previous expressions that

$$\partial_x^2 F_\infty(\alpha, x) < 0$$

for all $\alpha \le 0$ and $x > 0$, which via (3.40) implies

$$\begin{aligned} &\mu + \Omega(\alpha(x), x) - \lambda_0 x - \frac{\lambda_0 x^2}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{\lambda_k^2}{E_{k,0}^B \left[f_{k,0} + E_{k,0}^B\right]} d^3k \\ &< \mu + \Omega(\alpha(x), 0) = \mu + \Omega(\alpha(0), 0) = \mu \le 0, \end{aligned} \tag{3.42}$$

for $x>0$, i.e., $\partial_x \left\{F_\infty\left(\alpha\left(x\right),x\right)\right\}\Big|_{x>0} < 0$ for all $\mu \le 0$. Therefore we get $\widehat{x}=0$. Actually, by combining the results for $\mu>0$ and $\mu\le 0$, notice that

$$\begin{cases} \widehat{x}>0 \text{ for } \mu>0, \\ \lim\limits_{\mu\to 0^-}\widehat{x}=\lim\limits_{\mu\to 0^+}\widehat{x}=0, \\ \widehat{x}=0 \text{ for } \mu\le 0, \end{cases} \tag{3.43}$$

at infinite inverse temperature ($\beta\to\infty$).

(C2) Assuming now condition (C2), there is a critical value $\alpha_0<0$ such that the upper bound of (3.41) becomes positive:

$$\partial_x^2 F_\infty\left(\alpha,x\right)\Big|_{x=0} > 0, \tag{3.44}$$

for any $\alpha_0<\alpha\le 0$. Since for $\mu=0$ one has $\alpha\left(0\right)=0$ from (3.17), by (3.41) and (3.44) we have

$$\begin{cases} \partial_x \left\{F_\infty\left(\alpha\left(x\right),x\right)\right\}\Big|_{\mu=0,x=0} = 0, \\ \partial_x^2 \left\{F_\infty\left(\alpha\left(x\right),x\right)\right\}\Big|_{\mu=0,x<\delta} > 0, \end{cases} \tag{3.45}$$

for sufficiently small $\delta>0$, because of continuity. Therefore from the definition of $p^{SB}(\beta,\mu)$ and (3.22) we have $\widehat{x}>0$ for $\mu=0$. Actually, there is a $\mu_0<0$ such that

$$\begin{cases} \widehat{x}>0 \text{ for } \mu\ge 0. \\ \lim\limits_{\mu\to 0^+}\widehat{x}=\lim\limits_{\mu\to 0^-}\widehat{x}. \\ \widehat{x}>0 \text{ for } \mu_0<\mu\le 0. \\ \lim\limits_{\mu\to \mu_0^+}\widehat{x}>\lim\limits_{\mu\to \mu_0^-}\widehat{x}=0. \\ \widehat{x}=0 \text{ for } \mu<\mu_0. \end{cases} \tag{3.46}$$

4. Now we consider the case of finite inverse temperatures $\beta<+\infty$. By (3.22), (3.26) and (3.27) one has

$$(i)\ \begin{cases} \partial_x \left\{F_\beta\left(\alpha\left(x\right),x\right)\right\} = \left\{\partial_x \xi_0\left(\beta,\alpha,x\right)+\partial_x F_\infty\left(\alpha,x\right)\right\}\Big|_{\alpha=\alpha(x)} < \left\{\partial_x F_\infty\left(\alpha,x\right)\right\}\Big|_{\alpha=\alpha(x)}, \\ \lim\limits_{x\to+\infty}\xi_0\left(\beta,\alpha\left(x\right),x\right)=0, \end{cases}$$

$$(ii)\ \begin{cases} \partial_\beta \left\{F_\beta\left(\alpha\left(x\right),x\right)\right\} = \left\{\partial_\beta \xi_0\left(\beta,\alpha,x\right)\right\}\Big|_{\alpha=\alpha(x)} < 0, \\ \lim\limits_{\beta\to+\infty}\xi_0\left(\beta,\alpha,x\right)=0, \end{cases}$$

for fixed $\mu\in\mathbb{R}$. By (i) for fixed μ, if $\widehat{x}=0$ for $\beta\to\infty$, then $\widehat{x}=0$ for any $\beta\ge 0$.
Let $\mu>0$. By definition of $F_\beta\left(\alpha,x\right)$ one has

$$F_\beta\left(\alpha,x\right) > F_\infty\left(\alpha,x\right),$$

for $\mu > 0$ and any fixed $\alpha \leq 0$. Since by (ii) the function $F_{\beta}(\alpha(x), x)$ is monotonically decreasing for $\beta \nearrow \infty$, we find that

$$F_{\beta}(\alpha(0), 0) < F_{\infty}(\alpha(\widehat{x}), \widehat{x} > 0) < \sup_{x\geq 0}\{F_{\beta}(\alpha(x), x)\},$$

for sufficiently large β and $\mu > 0$, i.e. $\widehat{x} > 0$. Since one has (3.33), $\partial_x \xi_0(\beta, \alpha, x) < 0$ (i) and

$$\partial_\beta \partial_x \xi_0(\beta, \alpha, x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k \frac{E_{k,0}^B e^{\beta E_{k,0}^B}}{\left(1 - e^{\beta E_{k,0}^B}\right)^2}\partial_x E_{k,0}^B > 0,$$

for $\mu > 0$, there is an inverse temperature $\beta_c(\mu) > 0$ such that $\widehat{x} > 0$ for $\beta > \beta_c(\mu > 0)$ and $\widehat{x} = 0$ for $\beta < \beta_c(\mu > 0)$. The function $\beta_c(\mu > 0)$ is bijective so we define by $\mu_c(\beta) > 0$ the inverse function of $\beta_c(\mu)$. Note that if $\mu > \mu_c(\beta, 0)$ (3.18) then the arguments done in **2**. (cf. (3.28)-(3.32)) and **3**.**1**. still work. So, $\widehat{x} > 0$ for $\mu > \mu_c(\beta, 0)$. Consequently

$$\mu_c(\beta) \leq \mu_c(\beta, 0) = \lambda_0 \rho_0^B(\beta, 0, 0), \tag{3.47}$$

and $\beta_c(\mu)$ is a strictly increasing function from $[0, +\infty)$ to $[0, +\infty)$.
If the condition (C2) is verified, the arguments done here in **4**. for $\mu > 0$ work also for $\mu > \mu_0$ and the function $\beta_c(\mu > \mu_0)$ is bijective. In particular the inverse function $\mu_c(\beta)$ of $\beta_c(\mu)$ verifies:

$$\lim_{\beta\to+\infty}\mu_c(\beta) = \mu_0 < 0,$$

and (3.46) holds for $\beta > 0$. An illustration of the critical temperature $\theta_c(\mu) = 1/\beta_c(\mu)$ as a function of μ is given by figure 3.1. ■

Remark 3.3. *The solution $\widehat{x} = \widehat{x}(\beta, \mu)$ of (3.21) always satisfies*

$$\left\{\begin{array}{l} \left\{\partial_{|c|^2} p^{SB}(\beta, \mu, c)\right\}\Big|_{|c|^2=\widehat{x}} = \left\{\partial_x p_0^B(\beta, \alpha, x)\right\}\Big|_{x=\widehat{x}, \alpha=\alpha(\widehat{x})} = 0, \text{ for } \mu > \mu_c(\beta), \\ \widehat{x} = \widehat{x}(\beta, \mu) = 0 \text{ for } \mu < \mu_c(\beta), \end{array}\right. \tag{3.48}$$

see (3.22) and (3.26). Moreover, from the proof of the previous theorem we can see that the solution $\alpha(\widehat{x}) = \alpha(\beta, \mu, \widehat{x})$ of (3.16) is always strictly negative for any $\mu \neq \mu_c(\beta)$ or $\beta \neq \beta_c(\mu)$. In particular, one always has (3.17) for $\alpha = \alpha(\widehat{x})$, $x = \widehat{x}$ (3.21), and $\mu \neq \mu_c(\beta)$ or $\beta \neq \beta_c(\mu)$.

Remark 3.4. *Actually, as an extension for finite β of (3.43) and (3.46) we get two behaviors for $\widehat{x}$ depending on conditions (C1) and (C2):*

$$\text{(C1)}: \left\{\begin{array}{c} \widehat{x} = 0 \text{ for } \mu \leq \mu_c(\beta). \\ \lim\limits_{\mu\to\mu_c^-(\beta)}\widehat{x} = \lim\limits_{\mu\to\mu_c^+(\beta)}\widehat{x} = 0. \\ \widehat{x} > 0 \text{ for } \mu > \mu_c(\beta). \end{array}\right\} \text{ or (C2)}: \left\{\begin{array}{c} \widehat{x} = 0 \text{ for } \mu < \mu_c(\beta). \\ 0 = \lim\limits_{\mu\to\mu_c^-(\beta)}\widehat{x} < \lim\limits_{\mu\to\mu_c^+(\beta)}\widehat{x}. \\ \widehat{x} > 0 \text{ for } \mu > \mu_c(\beta). \end{array}\right\}.$$

Remark 3.5. *If condition (C1) holds, using arguments from the proof of theorem 3.2 (**3**.**2**. and **4**.) we have*

$$\mu_c(\beta) \geq \inf_{x\geq 0}\mu_c(\beta, x).$$

Therefore, for sufficiently large β, i.e., for small temperatures (compare (3.29) and (3.47)), we have

$$\mu_c(\beta) = \mu_c(\beta, 0) = \lambda_0 \rho^{PBG}(\beta, 0).$$

We recall that $\mu_c(\beta, x) = \lambda_0 \rho_0^B(\beta, 0, x)$ is defined by (3.18) and $\rho^{PBG}(\beta, 0)$ is the critical density of the Perfect Bose Gas.

Remark 3.6. *Notice that the proof of theorem 3.2 does not depend on the fact that λ_0 is the Fourier transformation of the interaction potential for $k = 0$. Actually, one could have taken as an arbitrary (strictly positive) parameter satisfying either (C1) or (C2). In Section 4, we explain that λ_0 has no physical relevance for a fixed particle density and is then taken arbitrary large enough such that only (C1) holds with a strict inequality.*

3.2. Non-conventional Bose condensation and Bogoliubov statistics

By lemma B.1 (Appendix B) or by remark 3.3 for $\mu \neq \mu_c(\beta)$ or $\beta \neq \beta_c(\mu)$ the full particle density equals:

$$\begin{aligned}\rho^{SB}(\beta, \mu) &\equiv \lim_{\Lambda} \left\langle \frac{N_\Lambda}{V} \right\rangle_{H^{SB}_{\Lambda,\lambda_0}} (\beta, \mu) = \frac{\mu - \alpha(\widehat{x})}{\lambda_0} \\ &= \begin{cases} \rho^{PBG}(\beta, \alpha(0)) \text{ for } \mu < \mu_c(\beta) \text{ or } \beta < \beta_c(\mu), \\ \left\{\rho_0^B(\beta, \alpha(\widehat{x}), \widehat{x})\right\} \text{ for } \mu > \mu_c(\beta) \text{ or } \beta > \beta_c(\mu), \end{cases}\end{aligned} \tag{3.49}$$

where $\rho_0^B(\beta, \alpha(x), x)$ is defined by (3.20) and with $\widehat{x} = \widehat{x}(\beta, \mu)$ and $\alpha(x) = \alpha(\beta, \mu, x)$ the solutions of the variational problems. Now our main results concern the particle densities inside and outside the zero-mode.

Theorem 3.7. *Under the assumptions of the previous theorems it follows:*
(i) A non-conventional Bose condensation induced by the non-diagonal interaction U_Λ^{ND} [9,12] for high chemical potentials (high particles densities), or low temperatures:

$$\lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H^{SB}_{\Lambda,\lambda_0}} (\beta, \mu) = \widehat{x}(\beta, \mu) = \begin{cases} = 0 \text{ for } \mu < \mu_c(\beta) \text{ or } \beta < \beta_c(\mu). \\ > 0 \text{ for } \mu > \mu_c(\beta) \text{ or } \beta > \beta_c(\mu). \end{cases}$$

(ii) No Bose condensation (of any type I, II or III [38–40]) outside the zero-mode for any chemical potentials, particles densities or temperatures:

$$\left\{\begin{array}{l} \forall k \in \Lambda^* \backslash \{0\}, \ \lim_{\Lambda} \left\langle \frac{a_k^* a_k}{V} \right\rangle_{H^{SB}_{\Lambda,\lambda_0}} (\beta, \mu) = 0 \\ \lim_{\delta \to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, 0 < \|k\| \leq \delta\}} \langle a_k^* a_k \rangle_{H^{SB}_{\Lambda,\lambda_0}} (\beta, \mu) = 0 \end{array}\right\} \text{ for any } (\beta, \mu) \in Q^S.$$

(iii) A particle density outside the zero-mode equal to:

$$\lim_{\Lambda} \frac{1}{V} \sum_{k \in \Lambda^* \backslash \{0\}} \langle a_k^* a_k \rangle_{H^{SB}_{\Lambda,\lambda_0}} (\beta, \mu) = \left\{ \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{f_{k,0}}{E^B_{k,0} \left[e^{\beta E^B_{k,0}} - 1 \right]} d^3k \right\} \Bigg|_{x = \widehat{x}, \alpha = \alpha(\widehat{x})}$$

$$+\left\{\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\frac{x^2\lambda_k^2}{2E_{k,0}^B\left[f_{k,0}+E_{k,0}^B\right]}d^3k\right\}\Bigg|_{x=\widehat{x},\alpha=\alpha(\widehat{x})}.$$

Note that the last limit equals the full particle density $\rho^{PBG}(\beta,\alpha(0)\leq 0)$ of the Perfect Bose Gas for $\mu<\mu_c(\beta)$ or $\beta<\beta_c(\mu)$.

(iv) There is no discontinuity of the particle densities (full density (3.49), density in the zero-mode (i) or outside the zero-mode (iii)) only if condition (C1) is satisfied. Assuming condition (C2), a discontinuity of the three densities appears with a strictly positive jump.

(v) For $\mu<\mu_c(\beta)$ or $\beta<\beta_c(\mu)$ one has the Bose statistics for a corresponding chemical potential $\alpha(0)<0$:

$$\forall k\in\Lambda^*\backslash\{0\},\quad \lim_{\Lambda}\langle a_k^*a_k\rangle_{H_{\Lambda,\lambda_0}^{SB}}(\beta,\mu)=\frac{1}{e^{\beta(\varepsilon_k-\alpha(0))}-1}.$$

But for $\mu>\mu_c(\beta)$ or $\beta>\beta_c(\mu)$, i.e. in the presence of a Bose condensation, we get another one, which we call the Bogoliubov statistics, for a corresponding chemical potential $\alpha(\widehat{x})<0$:

$$\lim_{\Lambda}\langle a_k^*a_k\rangle_{H_{\Lambda,\lambda_0}^{SB}}(\beta,\mu)=\left\{\frac{f_{k,0}}{E_{k,0}^B\left(e^{\beta E_{k,0}^B}-1\right)}+\frac{x^2\lambda_k^2}{2E_{k,0}^B\left(f_{k,0}+E_{k,0}^B\right)}\right\}\Bigg|_{x=\widehat{x},\alpha=\alpha(\widehat{x})}$$

for any $k\in\Lambda^*\backslash\{0\}$.

Before going to the proof let us point out the following remark:

Remark 3.8. *(a) Assuming condition (C2), a discontinuity of the densities appears because the direct term of repulsion*

$$\frac{\lambda_0}{2V}a_0^{*^2}a_0^2=\frac{\lambda_0}{2V}\left(N_0^2-N_0\right),\ \textit{with}\ N_0\equiv a_0^*a_0, \tag{3.50}$$

in (2.2) becomes too weak to beat the attraction induced by U_Λ^{ND} (2.6). U_Λ^{ND} express itself via the effective coupling constant g_{00} (2.1) [9,12] (see also discussions in Section 5.2 and figure 5.3).

(b) As expected the Bogoliubov statistics corresponds for $\lambda_0=0$ also to the one found [9,14] at a chemical potential $\alpha=0$ (high density regimes) in the thermodynamic behavior of the Bogoliubov Hamiltonian $H_{\Lambda,\lambda_0>0}^B$ (2.4). Actually this results for $\alpha=0$ in [9,14] could have easily been extended to $\alpha<0$ as soon as the non-conventional Bose condensation exists.

Proof. (i) Using lemma B.1 for $\mathcal{E}_\Lambda=\{0\}$ combined with remark 3.4 for $\mu\neq\mu_c(\beta)$, one gets (i).

(ii) The first limit comes directly from lemma B.2 of Appendix B. Also lemma B.1 with $\mathcal{E}_\Lambda=\{k\in\Lambda^*,0<\|k\|\leq\delta\}$ implies:

$$\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*,0<\|k\|\leq\delta\}}\langle a_k^*a_k\rangle_{H_{\Lambda,\lambda_0}^{SB}}(\beta,\mu)=\frac{1}{(2\pi)^3}\int\limits_{\|k\|\leq\delta}\xi_{\beta,\mu}(k)\,d^3k \tag{3.51}$$

where $\xi_{\beta,\mu}(k)$ is a continuous function on $k \in \mathbb{R}^3$ defined by

$$\xi_{\beta,\mu}(k) \equiv \frac{1}{e^{\beta(\varepsilon_k - \alpha(0))} - 1}, \tag{3.52}$$

for $\mu < \mu_c(\beta)$ or $\beta < \beta_c(\mu)$ whereas for $\mu > \mu_c(\beta)$ or $\beta > \beta_c(\mu)$

$$\xi_{\beta,\mu}(k) \equiv \left[\frac{f_{k,0}}{E^B_{k,0}\left(e^{\beta E^B_{k,0}} - 1\right)} + \frac{x^2 \lambda_k^2}{2E^B_{k,0}\left(f_{k,0} + E^B_{k,0}\right)} \right]_{x = \widehat{x}, \alpha(\widehat{x})}. \tag{3.53}$$

Therefore by taking the limit $\delta \to 0^+$ in (3.51) we get the second limit of (ii).

(iii) Since

$$N_\Lambda = a_0^* a_0 + \sum_{k \in \Lambda^* \setminus \{0\}} a_k^* a_k,$$

the limit is deduced from (3.49) and (i).

(iv) is a direct consequence of remark 3.4 combined with (i) and (iii).

(v) Notice that the mean particle values $\langle a_k^* a_k \rangle_{H^{SB}_{\Lambda,\lambda_0}}$ are defined on the discrete set $\Lambda^* \subset \mathbb{R}^3$. Below we denote by

$$g_{\beta,\mu,\Lambda}(k) \equiv \langle a_k^* a_k \rangle_{H^{SB}_{\Lambda,\lambda_0}}(\beta,\mu) \tag{3.54}$$

a continuous interpolation of these values from the set Λ^* to $\mathbb{R}^3$ and we define by $g_{\beta,\mu}(k)$ the corresponding thermodynamic limit:

$$g_{\beta,\mu}(k) \equiv \lim_\Lambda g_{\beta,\mu,\Lambda}(k) \text{ for } k \in \mathbb{R}^3 \setminus \{0\}. \tag{3.55}$$

By lemma B.2 note that the thermodynamic limit (3.55) exists and it is uniformly bounded for all $k \in \mathbb{R}^3 \setminus \{0\}$. Moreover for any interval (a,b) with $0 < a < b$, we have the convergence of the Riemann sums to the integral:

$$\lim_\Lambda \frac{1}{V} \sum_{k \in \Lambda^*, \|k\| \in (a>0,b)} \langle a_k^* a_k \rangle_{H^{SB}_{\Lambda,\lambda_0}}(\beta,\mu) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} g_{\beta,\mu}(k) \chi_{(a,b)}(\|k\|)\, d^3k,$$

where $\chi_{(a,b)}(\|k\|)$ is the characteristic function of (a,b), $0 < a < b$. Then the lemma B.1 with $\mathcal{E}_\Lambda = (a,b) \cap \Lambda^*$ implies

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} g_{\beta,\mu}(k) \chi_{(a,b)}(\|k\|)\, d^3k = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \xi_{\beta,\mu}(k) \chi_{(a,b)}(\|k\|)\, d^3k \tag{3.56}$$

with the continuous function $\xi_{\beta,\mu}(k)$ defined by (3.52) and (3.53). Since the relation (3.56) is valid for any interval $(a,b) \subset \mathbb{R}$ with $0 < a < b$ one gets

$$g_{\beta,\mu}(k) = \xi_{\beta,\mu}(k), \; k \in \mathbb{R}^3 \setminus \{0\}.$$

By this and (3.54)-(3.56) combined with (3.52)-(3.53) we finally get the statements in (v) for $k \in \Lambda^* \setminus \{0\}$. ■

3.3. The particle density as parameter in the grandcanonical ensemble

Let us consider the fixed particle density ρ in the *grandcanonical* ensemble which defines a unique chemical potential $\mu_{\beta,\rho}$ satisfying

$$\rho^{SB}\left(\beta,\mu_{\beta,\rho}\right)=\rho. \tag{3.57}$$

Actually, at a fixed inverse temperature β the function $\mu_{\beta,\rho}$ is the inverse function of the mean particle density $\rho^{SB}(\beta,\mu)$ of the superstable Bogoliubov Hamiltonian.

(i) Let

$$\left\{\begin{array}{l}\rho_{c,\inf}(\beta)\equiv\lim\limits_{\mu\rightarrow\mu_c^-(\beta)}\rho^{SB}(\beta,\mu).\\ \rho_{c,\sup}(\beta)\equiv\lim\limits_{\mu\rightarrow\mu_c^+(\beta)}\rho^{SB}(\beta,\mu).\end{array}\right. \tag{3.58}$$

Recall that $\mu_c(\beta)$ and $\beta_c(\mu)$ are defined in theorem 3.2 (figure 3.1). Through (iv) of theorem 3.7 combined with (3.47) and (3.49), we deduce that $\rho_c(\beta)\equiv\rho_{c,\inf}(\beta)=\rho_{c,\sup}(\beta)\leq\rho^{PBG}(\beta,0)$ if condition (C1) is satisfied.
(ii) By remark 3.3 and (3.49) we have

$$\mu_{\beta,\rho}-\lambda_0\rho=\alpha(\widehat{x})<0 \text{ for } \rho\notin\left[\rho_{c,\inf}(\beta),\rho_{c,\sup}(\beta)\right] \text{ or } \beta\neq\beta_c\left(\mu_{\beta,\rho}\right). \tag{3.59}$$

(iii) Combining theorem 3.2 with (3.49) and (3.59) we get

$$p^{SB}\left(\beta,\mu_{\beta,\rho}\right)=\left\{p_0^B(\beta,\alpha(x),x)\right\}\Big|_{x=\widehat{x}}+\frac{\lambda_0}{2}\rho^2 \tag{3.60}$$

for any $\rho>0$ where $\alpha(\widehat{x})<0$ is the unique solution of the *Bogoliubov density equation*:

$$\rho=\rho_0^B(\beta,\alpha,\widehat{x}) \text{ for } \rho\notin\left[\rho_{c,\inf}(\beta),\rho_{c,\sup}(\beta)\right] \text{ or } \beta\neq\beta_c\left(\mu_{\beta,\rho}\right). \tag{3.61}$$

(iv) For $\rho<\rho_{c,\inf}(\beta)$, one has $\mu_{\beta,\rho}<\mu_c(\beta)$ whereas $\mu_{\beta,\rho}>\mu_c(\beta)$ for $\rho>\rho_{c,\sup}(\beta)$, and theorem 3.7 is still valid for any $\rho\notin\left[\rho_{c,\inf}(\beta),\rho_{c,\sup}(\beta)\right]$, i.e. for $\mu_{\beta,\rho}\neq\mu_c(\beta)$ or $\beta\neq\beta_c\left(\mu_{\beta,\rho}\right)$. In particular, for any $\rho>\rho_{c,\sup}(\beta)$ there is only one Bose condensation in the zero mode, whereas for $\rho<\rho_{c,\inf}(\beta)\leq\rho^{PBG}(\beta,0)$ the system behaves as the Mean-Field Bose Gas (3.13) with no Bose condensations. Actually, for any $\rho>0$, there are no Bose condensations outside the zero-mode ((ii) of theorem 3.7), but for $\rho\notin\left[\rho_{c,\inf}(\beta),\rho_{c,\sup}(\beta)\right]$ the question of Bose condensation in the zero-mode is still open in the grandcanonical ensemble. We explain below in Section 4 that this question is not relevant in the canonical ensemble.

4. Thermodynamics in the canonical ensemble

The aim of this section is to examine the superstable Bogoliubov Hamiltonian $H_{\Lambda,\lambda_0}^{SB}$ in the canonical ensemble specified by (β,ρ). Actually, this gives the complete thermodynamic behavior of the model $H_{\Lambda,0}^{B}$, since *for any* $\lambda_0>0$ the Hamiltonians $H_{\Lambda,\lambda_0}^{SB}$ and $H_{\Lambda,0}^{B}$ differ only by a constant on the symmetrized n-particle Hilbert spaces $\mathcal{H}_B^{(n=[\rho V])}$. The model $H_{\Lambda,0}^{B}$ turns out to be not sufficient for a microscopic theory of superfluidity in the grandcanonical ensemble [1]. However, we explain here that the Bose gas $H_{\Lambda,0}^{B}$ can be solved in the canonical ensemble by taking λ_0 as a large enough parameter in order to use the analysis of $H_{\Lambda,\lambda_0}^{SB}$ in the grandcanonical one combined with the strong equivalence of ensembles. This is the basis of a new microscopic theory of superfluidity obtained from $H_{\Lambda,0}^{B}$ and explained in the next Section 5.

4.1. The weak equivalence of ensembles: free-energy density

Remark that $H_{\Lambda,\lambda_0}^{SB}$ commutes with the particle number operator

$$\left[H_{\Lambda,\lambda_0}^{SB}, N_\Lambda\right] = 0.$$

Let $f_\Lambda^{SB}(\beta,\rho)$ be the free-energy density for a fixed particle density $\rho > 0$:

$$f_\Lambda^{SB}(\beta,\rho) \equiv -\frac{1}{\beta V} \ln Tr_{\mathcal{H}_B^{(n)}} \left(\left\{ e^{-\beta H_{\Lambda,\lambda_0}^{SB}} \right\}^{(n=[\rho V])} \right).$$

Since the particle density (3.49) as the derivative of the pressure $p^{SB}(\beta, \mu \neq \mu_c(\beta))$ is continuous (as a function of $\mu \neq \mu_c(\beta)$), using a Tauberien theorem proven in [41], the existence of $p^{SB}(\beta,\mu)$ already implies the convexity of the thermodynamic limit $f^{SB}(\beta,\rho)$ for $\rho > 0$ of $f_\Lambda^{SB}(\beta,\rho)$ and the weak equivalence of the canonical and grandcanonical ensemble:

$$\begin{aligned} p^{SB}(\beta,\mu) = &\sup_{\rho>0} \left\{ \mu\rho - f^{SB}(\beta,\rho) \right\} = \mu\rho^{SB}(\beta,\mu) - f^{SB}\left(\beta, \rho^{SB}(\beta,\mu)\right), \ \mu \in \mathbb{R}, \\ f^{SB}(\beta,\rho) = &\sup_{\mu\in\mathbb{R}} \left\{ \mu\rho - p^{SB}(\beta,\mu) \right\} = \mu_{\beta,\rho}\rho - p^{SB}\left(\beta, \mu_{\beta,\rho}\right), \ \rho > 0, \end{aligned} \tag{4.1}$$

for $\rho \notin \left[\rho_{c,\inf}(\beta), \rho_{c,\sup}(\beta)\right]$. With (ii) and (iii) of Section 3.3 the Legendre transformation (4.1) implies an *explicit expression* of the corresponding free-energy density:

$$f^{SB}(\beta,\rho) = \left\{ \alpha(x)\rho - p_0^B(\beta,\alpha(x),x) \right\} \Big|_{x=\widehat{x}} + \frac{\lambda_0}{2}\rho^2, \tag{4.2}$$

with $p_0^B(\beta,\alpha,x)$ defined by (A.9). Now we give an interpretation of this last equality to show that $\widehat{x}$ and $\alpha(\widehat{x})$ are also solutions of variational problems in the canonical ensemble.

Since the Hamiltonian $H_{\Lambda,0}^B(0,c)$ (A.1) does not commute with the particle number operator, we have to use the new set of operators $\{\zeta_k\}_{k\in\Lambda^*\backslash\{0\}}$,

$$\zeta_k = a_0^*(N_0 + \mathrm{I})^{-1/2} a_k, \ \zeta_k^* = a_k^*(N_0 + \mathrm{I})^{-1/2} a_0, \ k \in \Lambda^*, \tag{4.3}$$

instead of $\{a_k\}_{k\in\Lambda^*\backslash\{0\}}$. So, we consider the Hamiltonian

$$\begin{aligned} \widehat{H}_{\Lambda,0}^B(c) = &\sum_{k\in\Lambda^*\backslash\{0\}} \varepsilon_k \zeta_k^*\zeta_k + \frac{1}{2}\sum_{k\in\Lambda^*\backslash\{0\}} \lambda_k |c|^2 \left[\zeta_k^*\zeta_k + \zeta_{-k}^*\zeta_{-k}\right] \\ &+\frac{1}{2}\sum_{k\in\Lambda^*\backslash\{0\}} \lambda_k \left[c^2\zeta_k^*\zeta_{-k}^* + \overline{c}^2\zeta_k\zeta_{-k}\right], \end{aligned} \tag{4.4}$$

instead, i.e. $H_{\Lambda,0}^B(0,c)$ with $\{a_k\}_{k\in\Lambda^*\backslash\{0\}} \to \{\zeta_k\}_{k\in\Lambda^*\backslash\{0\}}$. From (A.7) and without taking into account the constant terms, it follows that the Hamiltonian $\widehat{H}_{\Lambda,0}^B(c \neq 0)$ corresponds to a perfect Bose gas of quasi-particles with a spectrum of excitation given by $E_{k,0}^B$ (A.5). Then the thermodynamic limit of the pressure is

$$\begin{aligned} \widehat{p}_0^B(\beta,\alpha,x) &= \lim_\Lambda \frac{1}{\beta V} \ln Tr_{\mathcal{F}_\Lambda^B} e^{-\beta\left(\widehat{H}_{\Lambda,0}^B(c) - \alpha N_{\Lambda,k\neq 0}\right)} \\ &= p_0^B(\beta,\alpha,x) - \alpha(x)x. \end{aligned}$$

The free-energy density

$$\widehat{f}^B_{\Lambda,0}(\beta,\rho_1,c) \equiv -\frac{1}{\beta V}\ln Tr_{\mathcal{H}_B^{(n)}}\left(\left\{e^{-\beta \widehat{H}^B_{\Lambda,0}(c)}\right\}^{(n=[\rho_1 V])}\right)$$

for finite Λ and its thermodynamic limit are well-defined for any $\rho_1 > 0$ and $\beta > 0$. Moreover, for each fixed $x \geq 0$

$$\widehat{f}^B_0\left(\beta,\rho_1,x=|c|^2\right) \equiv \lim_{\Lambda} \widehat{f}^B_{\Lambda,0}(\beta,\rho_1,c) \tag{4.5}$$

is a convex function for $\rho_1 > 0$, i.e. for $\rho_1 \equiv \rho - x > 0$,

$$\widehat{f}^B_0(\beta,\rho_1,x) = \sup_{\alpha\leq 0}\left\{\alpha\rho_1 - \widehat{p}^B_0(\beta,\alpha,x)\right\} = \alpha(\rho_1,x)(\rho_1+x) - p^B_0(\beta,\alpha(\rho_1,x),x), \tag{4.6}$$

with $\alpha(\rho_1,x)$ defined as a solution of the Bogoliubov density equation $\rho = \rho^B_0(\beta,\alpha(\rho_1,x),x)$ (3.20) for $\rho_1 \equiv \rho - x > 0$.
To simplify we consider in the following $\rho \notin \left[\rho_{c,\inf}(\beta),\rho_{c,\sup}(\beta)\right]$, i.e. $\mu_{\beta,\rho} \neq \mu_c(\beta)$ or $\beta \neq \beta_c\left(\mu_{\beta,\rho}\right)$. For $x = \widehat{x}$ (3.21) the solution $\alpha(\widehat{x}) < 0$ is also the unique solution of the Bogoliubov density equation (3.61). Therefore

$$\left\{\alpha(\rho - x, x) = \alpha(x)\right\}\Big|_{x=\widehat{x}}, \tag{4.7}$$

which by (4.2) and (4.6) implies

$$f^{SB}(\beta,\rho) = \left\{\widehat{f}^B_0(\beta,\rho-x,x)\right\}\Big|_{x=\widehat{x}} + \frac{\lambda_0}{2}\rho^2. \tag{4.8}$$

The last equality is natural since

$$f^{SB}(\beta,\rho) = f^B_0(\beta,\rho) + \frac{\lambda_0}{2}\rho^2, \tag{4.9}$$

with

$$f^B_0(\beta,\rho) \equiv \lim_{\Lambda} -\frac{1}{\beta V}\ln Tr_{\mathcal{H}_B^{(n)}}\left(\left\{e^{-\beta H^B_{\Lambda,0}}\right\}^{(n=[\rho V])}\right). \tag{4.10}$$

The two models $H^B_{\Lambda,0}$ and $H^{SB}_{\Lambda,\lambda_0}$ are equivalent in the canonical ensemble, in the sense that their (infinite volume) free-energy densities at fixed densities differ only by a constant. Actually their Gibbs states are equal to each other for all (β,ρ).
The free-energy density $\widehat{f}^B_0(\beta,\rho-\widehat{x},\widehat{x})$ in (4.8) can be understood as the result of the Bogoliubov approximation done in the canonical ensemble and applied to the model $H^B_{\Lambda,0}$. The equality (4.8) finally means that the *non-diagonal* models $H^B_{\Lambda,0}$ and $H^{SB}_{\Lambda,\lambda_0}$ are thermodynamically equivalent to the model $\left\{\widehat{H}^B_{\Lambda,0}(c)\right\}_{|c|^2=\widehat{x}}$ as stated in [1].
From (4.8)-(4.10) combined with lemma B.3 we obtain

$$f^B_0(\beta,\rho) = \left\{\widehat{f}^B_0(\beta,\rho-x,x)\right\}\Big|_{x=\widehat{x}} = \inf_{x\geq 0}\left\{\widehat{f}^B_0(\beta,\rho-x,x)\right\}. \tag{4.11}$$

The solution $\widehat{x} = \widehat{x}\left(\beta, \mu_{\beta,\rho}\right) = \widehat{x}\left(\beta, \rho\right)$ of the variational problem (3.21) is also solution of (4.11) for a fixed density $\rho > 0$. Via (4.7), remark also that the solution $\alpha\left(\widehat{x}\right)$ of the variational problem (3.16) is solution in the canonical ensemble of (4.6) with $\rho_1 \equiv \rho - \widehat{x} > 0$. Now we add some remarks to highlight the important points in order to prepare the discussions of the next subsection.

Remark 4.1. *By (4.1) and (4.8)-(4.9) we obtain*

$$\mu_{\beta,\rho} = \partial_\rho f_0^B\left(\beta,\rho\right) + \lambda_0\rho = \partial_\rho\left\{\widehat{f}_0^B\left(\beta,\rho - x, x\right)\right\}\Big|_{x=\widehat{x}} + \lambda_0\rho.$$

Remark 4.2. *Direct computations (lemma B.3) imply*

$$\partial_{\lambda_0} f_0^B\left(\beta,\rho\right) = \partial_{\lambda_0}\widehat{x}\left\{\partial_x \widehat{f}_0^B\left(\beta,\rho_1,x\right) - \partial_{\rho_1}\widehat{f}_0^B\left(\beta,\rho_1,x\right)\right\}\Big|_{\rho_1=\rho-\widehat{x},x=\widehat{x}} = 0,$$

which can alternatively be found directly from (4.11) and the fact that $H_{\Lambda,0}^B$ does not depend on λ_0.

Remark 4.3. *Via (3.59) combined with remark 4.1 it immediately follows that*

$$\alpha\left(\widehat{x}\right) = \partial_\rho f_0^B\left(\beta,\rho\right)$$

and $\partial_{\lambda_0}\left\{\alpha\left(\widehat{x}\right)\right\} = 0$ for any $\rho \notin \left[\rho_{c,\inf}\left(\beta\right), \rho_{c,\sup}\left(\beta\right)\right]$.

Remark 4.4. *For a fixed density $\rho \notin \left[\rho_{c,\inf}\left(\beta\right), \rho_{c,\sup}\left(\beta\right)\right]$ we have via (4.11) $\partial_{\lambda_0}\widehat{x} = 0$. We can see this results using remark 3.3. Indeed for densities $\rho < \rho_{c,\inf}\left(\beta\right)$ one has $\partial_{\lambda_0}\widehat{x} = 0$ whereas for $\rho > \rho_{c,\sup}\left(\beta\right)$, $\widehat{x} > 0$ is solution of*

$$\left\{\partial_x p_0^B\left(\beta,\alpha,x\right)\right\}\Big|_{x=\widehat{x},\alpha=\partial_\rho f_0^B(\beta,\rho)} = 0$$

which implies $\partial_{\lambda_0}\widehat{x} = 0$. Other proofs may be to consider the full particle density (3.49) or to use direct computations, see lemma B.3. An illustration of the behavior of $\widehat{x}$ for a fixed density is performed in figure 4.1.

4.2. The strong equivalence of ensembles: particles densities

The two models $H_{\Lambda,0}^B$ and $H_{\Lambda,\lambda_0}^{SB}$ are equivalent in the canonical ensemble, see (4.9), i.e. their Gibbs states are equal for all (β,ρ). Since $H_{\Lambda,0}^B$ does not depend on λ_0, one has to check if the grandcanonical densities for $H_{\Lambda,\lambda_0}^{SB}$ depends on λ_0 for any fixed particle density.

Actually, the solutions $\alpha\left(\widehat{x}\right) = \alpha\left(\widehat{x},\lambda_0\right)$ and $\widehat{x}\left(\lambda_0\right)$ of the variational problems (3.16) and (3.21) are the key points of this first study. This is done via remarks 4.3 and 4.4 (figure 4.1): the solutions $\alpha\left(\widehat{x},\lambda_0\right)$ and $\widehat{x}\left(\lambda_0\right)$ are also solutions in the canonical ensemble of the variational problems (4.6) with $\rho_1 \equiv \rho - \widehat{x} > 0$, and (4.11) respectively and they do not depend on λ_0 for a fixed particle density $\rho \notin \left[\rho_{c,\inf}\left(\beta\right), \rho_{c,\sup}\left(\beta\right)\right]$. Consequently, as expected all densities in the

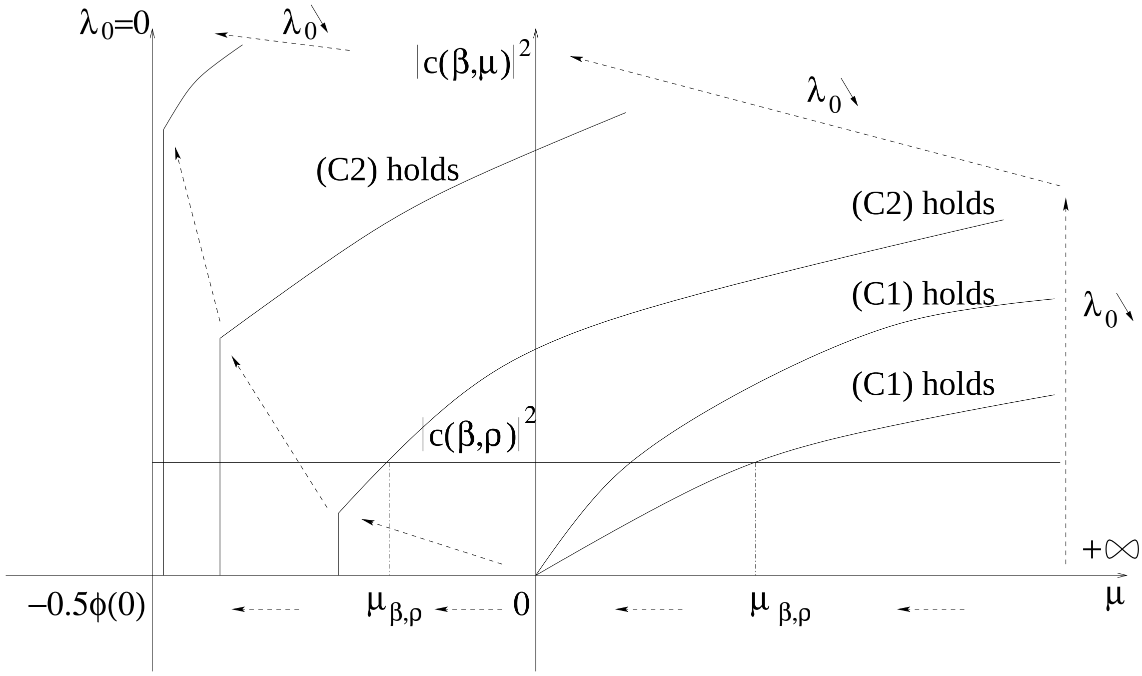

Abbildung 4.1: *Illustration of* $\widehat{x}(\beta,\mu)$ *and* $\widehat{x}\left(\beta,\mu_{\beta,\rho}\right)=\widehat{x}(\beta,\rho)$ *for a fixed particle density* $\rho>0$ *in the grandcanonical ensemble with* λ_0 *going from* $\lambda_0=+\infty$ *to* 0^+ *(model* $H^B_{\Lambda,0}$*, see [1]).*

grandcanonical ensemble do not depend on λ_0 at fixed particle densities ρ outside the phase transition.

The parameter λ_0 has no influence on the "physical" thermodynamic behavior of the system for a fixed particle density. Thus in the canonical ensemble the value of λ_0 can be chosen freely *as an arbitrary parameter* (see also remark 3.6).

If we *choose* λ_0 such that (C1) is *verified*, the weak equivalence (4.1) exists for any $\rho>0$ since

$$\rho_c(\beta)=\rho_{c,\inf}(\beta)=\rho_{c,\sup}(\beta)\,, \tag{4.12}$$

cf. (3.58). The Hamiltonian $H^{SB}_{\Lambda,\lambda_0}$ is the "superstabilization" [26] of the model $H^{SB}_{\Lambda,\widetilde{\lambda}_0}$ for a $\widetilde{\lambda}_0$ such that

$$H^{SB}_{\Lambda,\lambda_0}=H^{SB}_{\Lambda,\widetilde{\lambda}_0}+\frac{\lambda}{2V}\left(N_\Lambda^2-N_\Lambda\right)\ \text{with}\ \lambda=\lambda_0-\widetilde{\lambda}_0>0$$

and

$$\frac{\lambda_0}{2}+g_{00}>\frac{\widetilde{\lambda}_0}{2}+g_{00}\geq 0. \tag{4.13}$$

Because of (4.13) the model $H^{SB}_{\Lambda,\widetilde{\lambda}_0}$ satisfies the weak equivalence of ensembles for any density $\rho>0$ and therefore the Hamiltonian $H^{SB}_{\Lambda,\lambda_0}$ satisfies the strong equivalence of ensembles [27] for any $\rho\neq\rho_c(\beta)$ and $\lambda_0>0$ sufficiently large (see [42–44] for the notion of strong equivalence). We mean here the following:
Let us consider by A_Λ a (positive) quasi-local operator acting on

$$\mathcal{F}^B_\Lambda\subset\mathcal{F}^B_\infty\equiv\bigoplus_{n=0}^{+\infty}\left(L^2\left(\mathbb{R}^{nd}\right)\right)_{\text{symm}}$$

such that

$$\lim_{\Lambda} \langle A_\Lambda \rangle_{H^{SB}_{\Lambda,\lambda_0}} (\beta, \rho) < +\infty \text{ and } \lim_{\Lambda} \langle A_\Lambda \rangle_{H^{SB}_{\Lambda,\lambda_0}} (\beta, \mu) < +\infty, \tag{4.14}$$

for any $\beta > 0$ and $\rho > 0$. For $\beta > 0$, $\rho > 0$ ($\rho \neq \rho_c(\beta)$) and $\mu_{\Lambda,\beta,\rho}$ defined by

$$\left\langle \frac{N_\Lambda}{V} \right\rangle_{H^{SB}_{\Lambda,\lambda_0}} (\beta, \mu_{\Lambda,\beta,\rho}) = \rho,$$

it follows from [27] that

$$\lim_{\Lambda} \langle A_\Lambda \rangle_{H^{B}_{\Lambda,0}} (\beta, \rho) = \lim_{\Lambda} \langle A_\Lambda \rangle_{H^{SB}_{\Lambda,\lambda_0}} (\beta, \rho) = \lim_{\Lambda} \langle A_\Lambda \rangle_{H^{SB}_{\Lambda,\lambda_0}} \left(\beta, \mu_{\Lambda,\beta,\rho}\right), \tag{4.15}$$

i.e. the strong equivalence of ensemble is verified by the model $H^{SB}_{\Lambda,\lambda_0}$. Therefore the thermodynamic properties in the canonical ensemble of $H^{SB}_{\Lambda,\lambda_0}$ and $H^{B}_{\Lambda,0}$ correspond for a fixed particle density ρ to the one described in the last Section 3 *when (C1) holds* and with a chemical potential given by $\mu = \mu_{\beta,\rho} = \lim_{\Lambda} \mu_{\Lambda,\beta,\rho}$.

Remark 4.5. *For $\rho > \rho_c(\beta)$, there is a non-conventional Bose condensation whereas no Bose condensation (of any type I, II, or III [38–40]) appears outside the zero-mode at all densities $\rho > 0$ (theorem 3.7). The theory is self-consistent with the corresponding truncation of the full Hamiltonian in the canonical ensemble.*

At this point, the reader may be confused about the problem of the non-continuity of the grandcanonical particle density in the phase transition regime if (C2) holds (see (iv) of theorem 3.7). This in fact appears because a (direct) coupling constant $\lambda_0/2$ satisfying (C2) is *too small* to restore the problem of strict convexity of $f_0^B(\beta, \rho)$, see (4.9). This comes from the effective attraction g_{00} on the zero-mode arising from the non-diagonal interaction U_Λ^{ND} (2.6) (cf. [9,12], figures 5.2, 5.3 and 5.4). On the other hand, for λ_0 large enough, i.e. (4.13) is satisfied, the free-energy density $f^{SB}(\beta, \rho)$ (4.9) becomes strictly convex. Thus, in this case the grandcanonical density is continuous and the two ensembles are in fact strong equivalent, cf. (4.15).

Remark 4.6. *For low dimensions $d = 1, 2$, the effective coupling constant equals $g_{00} = -\infty$ and the hypothesis (C2) is verified for any interaction potential $\varphi(x)$ satisfying (A) and (B). Therefore we should have the existence of a non-conventional Bose condensation for $d = 1, 2$. However the method used here to find the canonical thermodynamic properties fails since λ_0 is never large enough to satisfy the condition (C1).*

5. A new microscopic theory of superfluidity

The aim of this section is to explain why the models $H^{B}_{\Lambda,0}$ can imply a new microscopic theory of superfluidity for Bose systems. It is essential here to note that in the canonical ensemble the conditions relating to the interaction potential $\varphi(x)$ may be relaxed as follows. The Fourier transformation of $\varphi(x)$ for $k = 0$ may be infinite since it has no physical impact in the canonical ensemble (see remark 3.6 and Section 4). However, the (effective coupling) constant g_{00} (2.1) and $\varphi(0)$ have to exist.
From Section 4, we recall that the canonical thermodynamic behavior of $H^{B}_{\Lambda,0}$ corresponds to the grandcanonical one of $H^{SB}_{\Lambda,\lambda_0}$ (Section 3) at fixed density $\rho > 0$ and inverse temperature $\beta > 0$, when (C1) holds with a strict inequality (cf. (4.13)).

5.1. Landau-type excitation spectrum in the presence of Bose condensation

In order to obtain a microscopic theory of superfluidity we have to get a Landau-type excitation spectrum [19, 20] as Bogoliubov did [4–8] for a *suitable choice* of $c-$numbers. So, we have to find the excitation spectrum of the *non-diagonal* model $H_{\Lambda,0}^{B}$ in order to check if we formally get a microscopic theory of ^{4}He. This analysis has to be done in the canonical ensemble.

1. Following the discussions of Section 4 and, as Landau's predictions [19,20], the Bose gas $H_{\Lambda,0}^{B}$ is thermodynamically equivalent to $\left\{\widehat{H}_{\Lambda,0}^{B}\left(c\right)\right\}_{\left|c\right|^{2}=\widehat{x}}$ in the canonical ensemble. This means that it is equivalent to a *"gas of collective elementary excitations"* or *"quasi-particles"* with a Bose condensate density

$$\lim_{\Lambda}\left\langle \frac{a_{0}^{*}a_{0}}{V}\right\rangle_{H_{\Lambda,0}^{B}}\left(\beta,\rho\right)=\widehat{x}\left(\beta,\rho\right),$$

cf. (i) of theorem 3.7. We recall that $\widehat{x}=\widehat{x}\left(\beta,\mu_{\beta,\rho}\right)=\widehat{x}\left(\beta,\rho\right)$ is the solution of the variational problems, (3.21) in the grandcanonical ensemble, or (4.11) in the canonical ensemble, and the critical density $\rho_{c}\left(\beta\right)$ is given by (4.12).
Consequently, the spectrum of excitations, which is macroscopically relevant, equals the Bogoliubov spectrum at inverse temperatures $\beta>0$ and particle densities $\rho>0$:

$$E_{k}^{B}\left(\beta,\rho\right)=\begin{cases}\varepsilon_{k}=\hbar^{2}k^{2}/2m & \text{for } \beta<\beta_{c}\left(\mu_{\beta,\rho}\right) \text{ or } \rho\leq\rho_{c}\left(\beta\right),\\ \sqrt{\varepsilon_{k}\left(\varepsilon_{k}+2\widehat{x}\lambda_{k}\right)} & \text{for } \beta>\beta_{c}\left(\mu_{\beta,\rho}\right) \text{ or } \rho>\rho_{c}\left(\beta\right),\end{cases}\tag{5.1}$$

see $E_{k,0}^{B}$ in (A.5) [4–8] with $x=\widehat{x}\left(\beta,\rho\right)$ and $\alpha=0$. The collective excitation spectrum $E_{k}^{B}\left(\beta,\rho\right)$ has *no gap for any densities or temperatures.*
Considering that the Bogoliubov approximation works also in the canonical ensemble, the excitation spectrum (5.1) was intuitively clear from the beginning [1]. The main difficulties are to find thermodynamic properties of the Hamiltonian $H_{\Lambda,0}^{B}$ in the canonical ensemble.

2. Now, to find the exact Landau-type excitation spectrum from (5.1), i.e. to get the "phonons" part and the "rotons" one, we can reason along the standard lines of Bogoliubov microscopic theory of superfluidity, see [4–9].
For this approach, we have to assume some specific conditions relating to the two-body interaction potential $\varphi\left(x\right)$. In particular, the two-body potential $\varphi\left(x\right)$ should verify (A)-(B), and we take again λ_{0} as the Fourier transformation of $\varphi\left(x\right)$ for $k=0$. Here λ_{k} is spherically-symmetric, i.e. $\lambda_{k}=\lambda_{\left\|k\right\|}$, and additionally, as Bogoliubov did, we assume the absolute integrability of $x^{2}\varphi\left(x\right)\in L^{1}\left(\mathbb{R}^{3}\right)$. Actually, we need here the last assumption and the Fourier transformation of $\varphi\left(x\right)$ for $k=0$ in order to have a Taylor expansion

$$\lambda_{k}=\lambda_{0}+\frac{1}{2}\left\|k\right\|^{2}\lambda_{0}^{\prime\prime}+o\left(\left\|k\right\|^{2}\right),\tag{5.2}$$

of λ_{k} allowing us to analyze $E_{k}^{B}\left(\beta,\rho\right)$ for *small* $\left\|k\right\|$ (phonon part). Here $\lambda_{0}^{\prime\prime}\leq0$ is the second derivative for $k=0$ and $\left|\lambda_{k}\right|\leq const.\left\|k\right\|^{-1}$.
Let $\rho>\rho_{c}\left(\beta\right)$ or $\beta>\beta_{c}\left(\mu_{\beta,\rho}\right)$, i.e. $\widehat{x}\left(\beta,\rho\right)>0$ (cf. (i) of theorem 3.7). Then the collective

spectrum of excitations $E_k^B(\beta,\rho)$ in this domain of (β,ρ) verifies:

$$E_k^B(\beta,\rho) = \begin{cases} \left(\frac{\hbar^2}{m}\lambda_0\widehat{x}\right)^{1/2} \|k\| = \hbar \mathrm{w}\,\|k\|\,, & \text{for } \|k\| \to 0^+. \\ \varepsilon_k = \hbar^2 k^2/2m & \text{, for } \|k\| \to +\infty. \end{cases} \tag{5.3}$$

The gapless spectrum $E_k^B(\beta,\rho)$ is *phonon-like* for small $\|k\|$ ($\rho > \rho_c(\beta)$), whereas for *large* wave-vectors it behaves like the *single-particle excitations* ε_k.
Since λ_k attains its maximum at $k=0$, one can choose the potential $\varphi(x)$ in such a way that

$$\left(\varepsilon_k\left(\varepsilon_k + 2\widehat{x}\lambda_{\|k\|}\right)\right)' = 0 \text{ at } \|k\| = \|k_{\mathrm{rot}}\| \neq 0, \tag{5.4}$$

i.e. the spectrum $E_k^B(\beta,\rho)$ has a local ("roton") minimum at $\|k_{\mathrm{rot}}\|$. On the other hand, one gets:

$$E_k^B(\beta,\rho) \geq \|k\| \left(\frac{\hbar^2}{2m}\right)^{1/2} \left\{\min_k \left(\varepsilon_k + 2\lambda_k^2\widehat{x}\right)\right\}^{1/2} \equiv \hbar\,\|k\|\,\mathrm{v}_0(\beta,\rho)\,. \tag{5.5}$$

The Bogoliubov spectrum $E_k^B(\beta,\rho)$ is a Landau-type excitation spectrum for $\rho > \rho_c(\beta)$ or $\beta > \beta_c\left(\mu_{\beta,\rho}\right)$ and an illustration is given by figure 5.1.

Remark 5.1. *The famous Landau's criterion of superfluidity of 1941 [19,20] gives the following critical velocity:*

$$\inf_k \left\{\frac{E_k^B(\beta,\rho)}{\hbar\,\|k\|}\right\} = \mathrm{v}_0(\beta,\rho) = \left\{\begin{array}{l} 0 \quad, \text{ for } \beta \leq \beta_c\left(\mu_{\beta,\rho}\right) \text{ or } \rho \leq \rho_c(\beta)\,. \\ > 0, \text{ for } \beta > \beta_c\left(\mu_{\beta,\rho}\right) \text{ or } \rho > \rho_c(\beta)\,. \end{array}\right\}$$

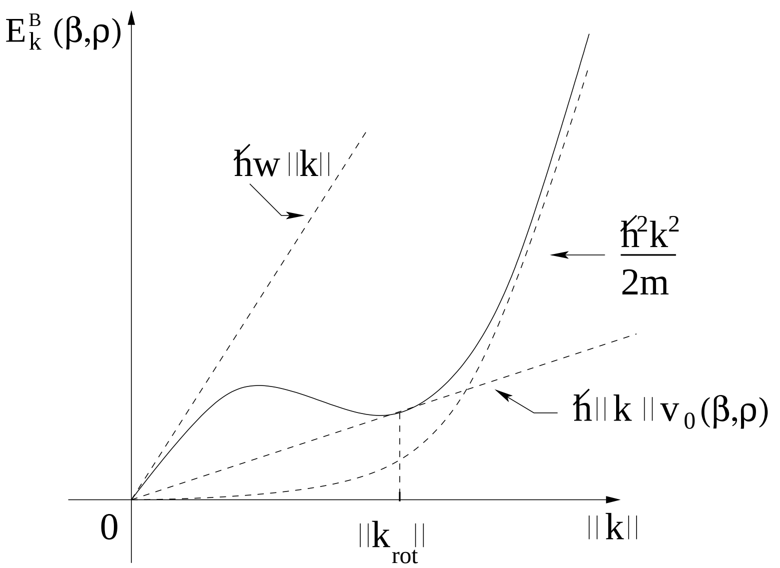

Abbildung 5.1: *The Bogoliubov spectrum $E_k^B(\beta,\rho)$ for $\beta > \beta_c\left(\mu_{\beta,\rho}\right)$ or $\rho > \rho_c(\beta)$.*

5.2. Two complementary Bose systems: Cooper pairs and gas of quasi-particles

We give here the quantum interpretation of the canonical thermodynamic properties of the model $H_{\Lambda,0}^{B}$. First note that, in terms of particle densities, we obtain (see theorem 3.7):

- A non-conventional Bose condensation appears with the density $\widehat{x}(\beta,\rho) > 0$ for $\rho > \rho_c(\beta)$, whereas at all densities $\rho > 0$ there is no Bose condensation (of any type I, II, or III [38–40]) outside the zero-mode.
- Even for zero-temperature, we have a non-zero particle density outside the zero-mode for any $\rho > 0$:

$$\begin{cases} \lim\limits_{\beta\to+\infty}\lim\limits_{\Lambda}\dfrac{1}{V}\sum\limits_{k\in\Lambda^*\backslash\{0\}}\langle a_k^* a_k\rangle_{H_{\Lambda,0}^{B}}(\beta,\rho) = \left\{\dfrac{1}{(2\pi)^3}\displaystyle\int_{\mathbb{R}^3}\dfrac{x^2\lambda_k^2}{2E_{k,0}^{B}\left[f_{k,0}+E_{k,0}^{B}\right]}d^3k\right\}\Bigg|_{\substack{x=\widehat{x}\\ \alpha=\alpha(\widehat{x})}} > 0, \\ \forall k\in\Lambda^*\backslash\{0\},\quad \lim\limits_{\beta\to+\infty}\lim\limits_{\Lambda}\langle a_k^* a_k\rangle_{H_{\Lambda,0}^{B}}(\beta,\rho) = \left\{\dfrac{x^2\lambda_k^2}{2E_{k,0}^{B}\left[f_{k,0}+E_{k,0}^{B}\right]}\right\}\Bigg|_{\substack{x=\widehat{x}\\ \alpha=\alpha(\widehat{x})}} > 0. \end{cases} \tag{5.6}$$

  In the regime $\rho > \rho_c(\beta)$, the system follows the Bogoliubov statistics (v) of theorem 3.7, whereas in the absence of the Bose condensation, i.e. for $\rho \leq \rho_c(\beta)$, the (standard) Bose statistics holds.

1. The origin of the Bogoliubov statistics and also of (5.6) is a phenomenon of interaction. Actually, it has been known since [12] that the collection of particles outside the zero-mode imposes, through the non-diagonal interaction U_{Λ}^{ND}, a *glue-like attraction* between particles in the zero-mode.

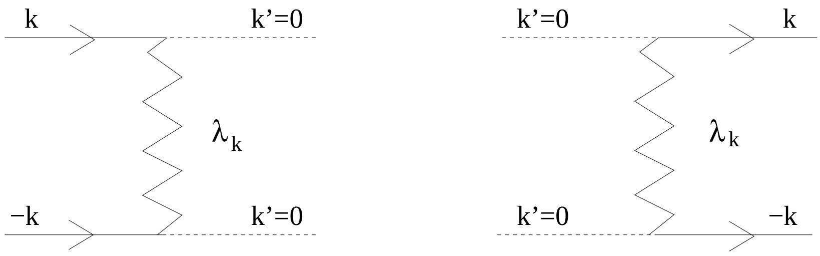

Abbildung 5.2: *Non-diagonal-interaction vertices corresponding to* U_{Λ}^{ND}.

A natural way to see this phenomenon is to remark that the non-diagonal interaction U_{Λ}^{ND} (see figure 5.2) implies an *effective interaction term* $g_{\Lambda,00}$ for bosons with $k=0$, see figure 5.3. Evaluated via a Fröhlich transformation in the second order [12] (see also the review [9]), $g_{\Lambda,00}$ is strictly negative. The corresponding thermodynamic limit

$$\lim_{\Lambda} g_{\Lambda,00} = g_{00} < 0$$

remarkably gives (2.1). In particular, this effective *attraction* term g_{00} amazingly plays a crucial rôle in the rigorous thermodynamic analysis of $H_{\Lambda,0}^{B}$ (see for example the proof of theorem 3.2 and Section 4). It is also essential in the rigorous study of the Weakly Imperfect Bose Gas, i.e. the Bogoliubov Hamiltonian $H_{\Lambda,\lambda_0>0}^{B}$, see [9–11, 13].

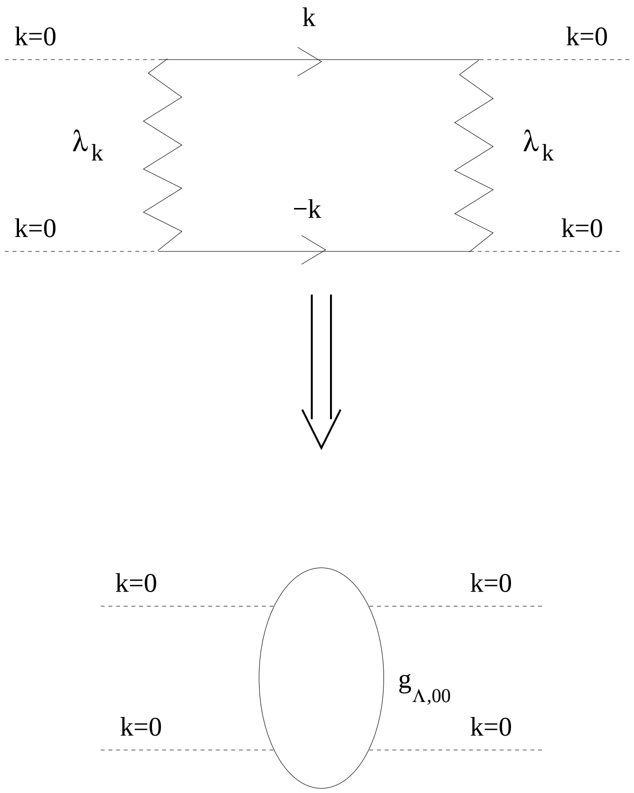

Abbildung 5.3: *Effective interaction for the zero-mode induced by the non-diagonal interaction* U_Λ^{ND}

The Bose condensate with the density $\widehat{x}(\beta,\rho)$ and the remaining system with the density $\{\rho-\widehat{x}>0\}$, *called here the Bogoliubov condensate*, only exist via this glue-like attraction g_{00} (figure 5.3). In fact, the particles inside the condensate pair up via the Bogoliubov condensate to form *"Cooper pairs"*. This Bose condensation constituted by Cooper pairs is then non-conventional [9, 11, 12, 14, 17, 18], i.e. completely transformed by the non-diagonal interaction U_Λ^{ND}.

2. As it was claimed by Bogoliubov [4–6], the *coherency* due to the presence of the Bose condensation is *not* enough to make the Perfect Bose Gas superfluid. The spectrum of elementary excitations is not collective in this case: it corresponds to individual movements of particles. In the Bose gas $H_{\Lambda,0}^B$, following Landau's criterion of superfluidity [19, 20] (remark 5.1), the Bogoliubov condensate is here *superfluid* due to phenomena of interactions which change, in the presence of the Bose condensate, the behavior of individual particles into a ideal Bose gas of *"quasi-particles"* with the given spectrum $E_k^B(\beta,\rho)$. Indeed, through the Bose condensate, the non-diagonal interaction U_Λ^{ND} *combined with* the diagonal interaction U_Λ^D creates quasi-particles from two particles respectively of modes k and $-k$ ($k\neq 0$). This can be seen via the Bogoliubov u-v transformation applied to $\left\{\widehat{H}_{\Lambda,0}^B(c)\right\}_{|c|^2=\widehat{x}>0}$, cf. (A.4) with $\{a_k\}_{k\in\Lambda^*\setminus\{0\}}\to\{\zeta_k\}_{k\in\Lambda^*\setminus\{0\}}$. This gas of quasi-particles, i.e. the Bogoliubov condensate, exists *if and only if* the non-conventional Bose condensate exists too.

3. Also for high densities $\rho > 0$ we have

$$\lim_{\rho \to +\infty} \{\rho - \widehat{x}(\beta, \rho)\} = 0, \tag{5.7}$$

cf. theorem 3.7 when (C1) holds. Actually, the non-diagonal interaction U_Λ^{ND} implies an effective *repulsion* term

$$g_{pq} \equiv \lim_{\Lambda} g_{\Lambda,pq} = \frac{\lambda_p \lambda_q}{4} \widehat{x}(\beta, \rho) \left(\frac{1}{\varepsilon_p} + \frac{1}{\varepsilon_q} \right) \geq 0, \tag{5.8}$$

inside each quasi-particle [9, 12], i.e. inside each couple of particles respectively with modes q and $-q$ ($q \neq 0$) (figure 5.4). The larger the Bose condensate density $\widehat{x}(\beta, \rho)$, the stronger the

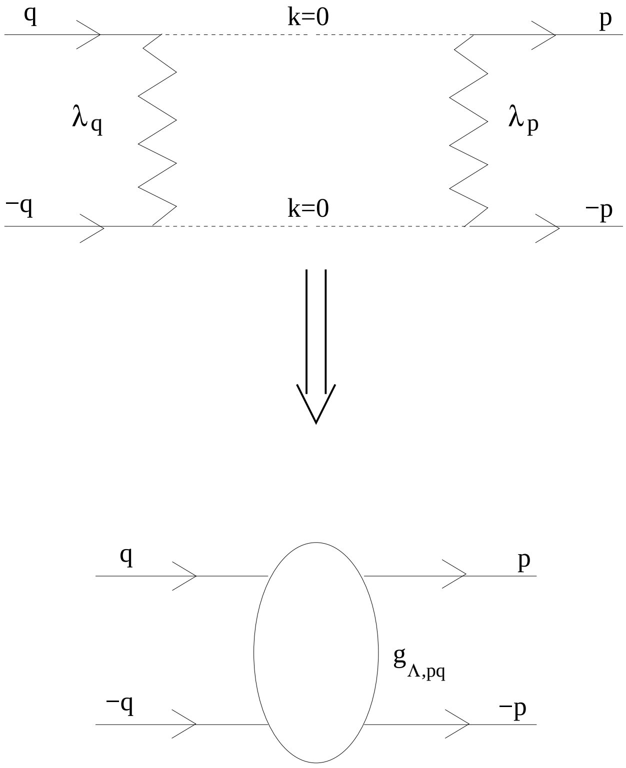

Abbildung 5.4: *Effective interaction outside the zero-mode induced by the non-diagonal interaction* U_Λ^{ND}

effective repulsion term g_{pq}. The raise of the non-conventional Bose condensate progressively destroys the Bogoliubov one, see (5.7). The Bose and Bogoliubov condensates still remain *in competition with each other.*

5.3. Microscopic theory of superfluidity of ^{4}He?

1. A microscopic interpretation at all temperatures $\mathrm{T} = (\mathrm{k_B}\beta)^{-1} \geq 0$ of Landau's theory of superfluidity follows from the Landau-type excitation spectrum $E_k^B(\beta, \rho)$ (5.1)-(5.5) (cf. figure 5.1). Note that Landau's theory of superfluidity of quantum liquids [4, 6, 7, 45–48] is based on the following principles:

- quantum liquid is still fluid even for zero-temperature;
- at low temperatures, apart translations (flow), the state of this liquid is entirely described by the spectrum of collective (elementary) excitations;
- through thermodynamic data [48, 49] (e.g. specific heat capacity) this spectrum for ^{4}He should be a phonon-like for the long-wave length collective excitations and should be above a straight line with positive slope with ("roton") minimum in the vicinity of $\|k_{\text{rot}}\| \simeq 2\ \mathring{A}^{-1}$ (figure 5.1).

2. The thermodynamic behavior of the Bose gas $H^B_{\Lambda,0}$ is also close to the liquid ^{4}He. This helium liquid is a Bose system with strong interactions. The interaction potential $U_{\text{th}}(r)$ is of Lennard-Jones type [24] and was found by Slater et Kirkwood [51] using the electronic structure of ^{4}He (see figure 5.5 with $U_{\text{th}}(r)$ in Kelvin and also [29]).

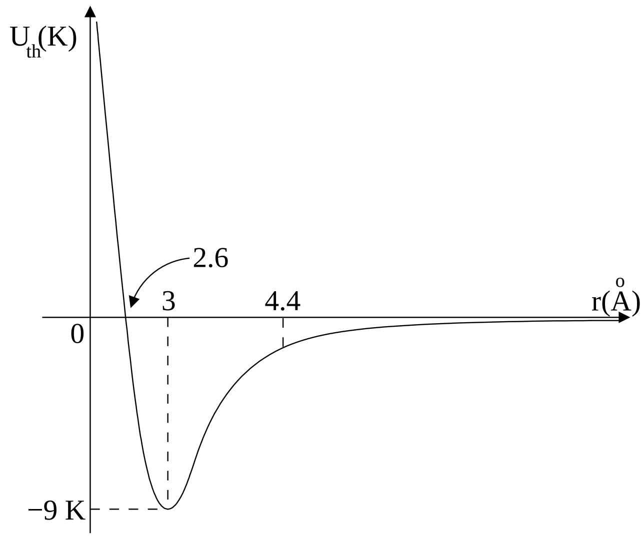

Abbildung 5.5: *The theoretical interaction potential of* ^{4}*He*

The exact formula for the interaction potential $U_{\text{th}}(r)$ given in [29] is valid only for strictly positive r, whereas close to zero it is given by a polynomial interaction like in figure 5.5. A caricature of this interaction is the hard sphere interaction potential [52,53]. This approximation gives surprisingly good estimates of the experimental condensate fraction: 9% at T = 0 K [2,3]. In our model $H^B_{\Lambda,0}$ we have to mimic an interaction potential $\varphi(x)$ close to $U_{\text{th}}(r)$. In particular, in contrast with the hard sphere potential the value of $\varphi(x)$ for $x = 0$ has to be given and has not to be infinite (see for example figure 3.1). A standard way to do it is to cut $U_{\text{th}}(r)$ when $r \to 0^+$ as follows:

$$\varphi_{\text{He}}(x) = \varphi_{\text{He}}(r = \|x\|) = \begin{cases} U_{\text{th}}(r) & \text{for } r > r_{\min}. \\ U_{\text{th}}(r_{\min}) & \text{for } 0 \leq r \leq r_{\min}. \end{cases}$$

This implies a Fourier transformation $\lambda_0(r_{\min})$ of $\varphi_{\text{He}}(x)$ for the mode $k = 0$ which drastically depends on $r_{\min}$ (specially when $r_{\min} \to 0^+$), i.e., $\lim\limits_{r_{\min}\to 0^+} \lambda_0(r_{\min}) = +\infty$, but it has *no* influence

on the canonical thermodynamic behavior of $H_{\Lambda,0}^{B}$. Moreover, for $k \neq 0$ the influence of $r_{\min}$ corresponds only to a small (specially when $r_{\min} \rightarrow 0^{+}$) perturbation of the Fourier transformation of $U_{\text{th}}(r)$. In fact one should choose $r_{\min} << r_{mean}$ where $r_{mean} \sim \rho^{-1/3}$ is the average length of the inter-particle distance at density $\rho > 0$.
Then, the thermodynamics of the theoretical Bose gas $H_{\Lambda,0}^{B}$ is *qualitatively* quite similar to the one of the liquid ^{4}He:

- for small densities $\rho \leq \rho_c(\beta)$ or high temperatures $\mathrm{T} \geq \mathrm{T}_c \equiv \left(\mathrm{k_B}\beta_c\left(\mu_{\beta,\rho}\right)\right)^{-1}$ the thermodynamic behavior corresponds to the Mean-Field gas (3.13),
- a non-conventional Bose condensation constituted of Cooper pairs appears via a second order transition (no discontinuity of the Bose condensate density) and the spectrum of excitations becomes a Landau-type excitation spectrum in this regime, i.e. for high densities $\rho > \rho_c(\beta)$ or small temperatures $\mathrm{T} < \mathrm{T}_c$,
- a coexistence of particles inside and outside the Bose condensate, even at zero-temperature as it is experimentally found in [2, 3].

As explained above, note that the Bose condensation becomes non-conventional with the formation of Cooper pairs via the term of attraction g_{00}, i.e. because of quantum fluctuations, see figures 5.3 and 5.4. The importance of quantum fluctuations in helium systems corresponds also to the qualitative explanation for a liquid state at such extreme temperatures [29].

Quantitatively, the critical density $\rho_c(\beta)$ is approximately given by $\rho_c(\beta) \approx \rho^{PBG}(\beta, 0)$, cf. remark 3.5. The theoretical temperature of the phase transition T_c always verifies $\mathrm{T}_c \geq \mathrm{T}_c^{PBG} = 3.14$ K (critical temperature evaluated for a Perfect Gas of helium particles) but is quite close to T_c^{PBG}:

$$\mathrm{T}_c \approx 3.14 \text{ K}.$$

(In fact we are not able to prove an exact equality at very *high* densities). The experimental transition of the *normal* liquid ^{4}He (called ^{4}He I) to *superfluid* phase ^{4}He II (discovered by Kapitza [54] and Allen, Misener [55] in 1938) takes place at a lower temperature $\mathrm{T}_\lambda = 2.17$ K (along the vapor pressure curve), which is not so far from the one of the model $H_{\Lambda,0}^{B}$. However, note that the Henshaw-Woods spectrum (experimental Landau-type excitation spectrum) *does not change* drastically when the temperature crosses T_λ, whereas there is *no* superfluidity for these temperatures.

Remark 5.2. *This means that there is a temperature $\widetilde{\mathrm{T}}_\lambda > \mathrm{T}_\lambda$ such that the experimental "quasi-particle" system still exists for $\mathrm{T} < \widetilde{\mathrm{T}}_\lambda$ even if Landau's criterion of superfluidity (remark 5.1) experimentally fails at these temperatures $\mathrm{T}_\lambda < \mathrm{T} < \widetilde{\mathrm{T}}_\lambda$.*

3. To resume, this analysis is not a complete theory of "real superfluidity". In particular, the Bogoliubov phonon-maxon-roton dispersion branch is only a part of the spectrum of the full quantum-mechanical Hamiltonian of the helium system. Therefore, this theory fails in being a complete description of all thermodynamics of liquid helium. For example, at temperatures $\mathrm{T}_\lambda < \mathrm{T} < \mathrm{T}_c$, a Bose condensation still exists in $H_{\Lambda,0}^{B}$ but not for liquid helium even if the system of "quasi-particles" resists in liquid helium for $\mathrm{T}_\lambda < \mathrm{T} < \widetilde{\mathrm{T}}_\lambda$ (remark 5.2). However, this theory is an interesting mathematical approach to a microscopic theory of many-body interacting boson systems leading to a better understanding of such superfluid systems.

5.4. Additionnal interpretations of this microscopic theory of superfluidity

Let us examine other interpretations of the Bose system $H^B_{\Lambda,0}$ in relation with the liquid ^{4}He. In fact, we give here two interpretations of the Bose gas $H^B_{\Lambda,0}$ obtained by following or not Landau's criterion of superfluidity [19, 20] (remark 5.1). As explained above, note that the model $H^B_{\Lambda,0}$ is a caricature and may contain only a small part of the physical properties of real liquid helium. The sole purpose of these discussions is to give some new directions in light of the Bose gas $H^B_{\Lambda,0}$.

1. It is known [54, 55] that below the critical temperature T_λ of the λ-transition, two fluids (^{4}He II phase) coexist: the normal fluid and the superfluid liquid. Later justified within the framework of phenomenological Landau's theory [19, 20, 48], the picture suggested by Tisza and London was to interpret the condensate of frozen in momentum space bosons with $p = 0$ as a "superfluid component", and the rest of particles as a "normal component" which is the carrier of the total entropy of the system. Experimentally, a Bose condensate was discovered in ^{4}He II. The apparition of this Bose condensate transition and the one of the superfluid liquid are strongly correlated to each other. Indeed, from [56–58] if γ_s is the fraction of superfluid liquid and γ_0 the one of the condensate, one has

$$\gamma_s(\mathrm{T}) \sim (\mathrm{T}_\lambda - \mathrm{T})^\eta \sim \gamma_0(\mathrm{T}), \text{ for } \mathrm{T} \to \mathrm{T}_\lambda^-, \tag{5.9}$$

see figure 5.6. However, even for zero-temperature the superfluid liquid is not in a full Bose

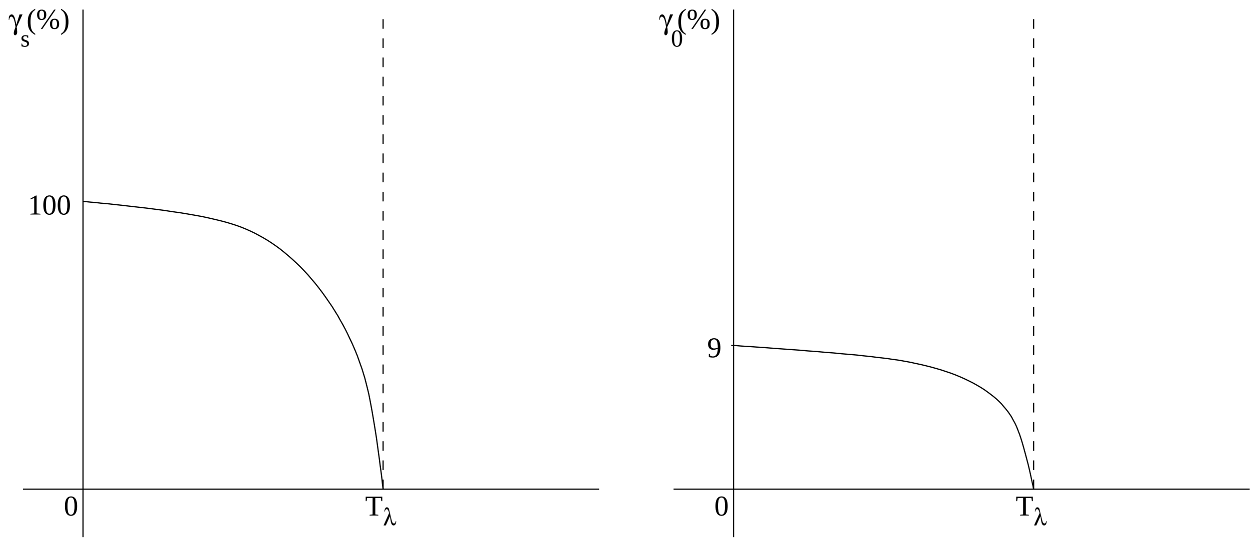

Abbildung 5.6: *The fractions, γ_s of superfluid liquid and γ_0 of the Bose condensate, as a function of the temperature* T *for ^{4}He*

condensate phase which is in contradiction with the assumption of Tisza and London.

2. *Following Landau's criterion of superfluidity* [19, 20], the theory based on $H^B_{\Lambda,0}$ might be understood as a microscopic theory of the *superfluid liquid*. Within this framework, it allows us to understand the close connection between the Bose condensate with density $\widehat{x}(\beta, \rho)$ and the Bogoliubov one with density $\{\rho - \widehat{x} > 0\}$. These two systems may compose together the *superfluid liquid*, which coexists with the normal liquid for non-zero temperature at any positive velocity.

Note that Landau's criterion of superfluidity [19, 20] confronts an initial problem expressed by remark 5.2 and also a second one: the application of this criterion to the Henshaw-Woods spectrum gives for the critical velocity $\mathrm{v}_0 \approx 60\ m/s$ (remark 5.1), whereas superfluidity in capillaries disappears when velocity is of the order of *few* cm/s. Moreover, it depends sensitively on the diameter of the channel.
The attempts to explain these "misfittings" are concentrated around the idea that the Landau-type spectrum experimentally discovered by Henshaw and Woods is only a part of a plethora of other types of "elementary" excitations not covered by the Bogoliubov theory, see [3, 58].
Within the framework of the model $H_{\Lambda,0}^{B}$, we have seen in Section 5.2 that the Bose condensate has to exist in order to have the superfluidity property via the Bogoliubov condensate. Indeed, as soon as the non-conventional Bose condensate disappears, the collective phenomenon involved in the formation of the superfluid gas (Bogoliubov condensate) also vanishes. The introduction of a velocity in an inhomogeneous gas (in capillaries) may change the individual spectrum ε_k by increasing it. Then, the effective attraction g_{00} ((2.1), figure 5.3) becomes smaller, i.e. the *non-conventional* Bose condensate and the (*superfluid*) Bogoliubov one may be destroyed for velocities sufficiently large but smaller than v_0 (remark 5.1). Note that the non-conventional Bose condensate constituted of Cooper pairs may be changed into a conventional Bose-Einstein condensation as it exists for the Mean-Field Bose gas. An experimental study of the spectrum of excitations and also of the Bose condensation phenomenon should be interesting at different velocities.

Actually, the collective behavior of this system should be quite delicate to save. A velocity may destroy the Cooper pairs and the quasi-particles. The important point is the following: the bigger the density of non-conventional Bose condensate, the stronger the robustness of Cooper pairs and quasi-particles to any perturbations.
At temperatures $\mathrm{T} < \mathrm{T}_\lambda$ even if the Bose condensate exists, its density may be not sufficiently important to keep the collective behavior for any positive velocities: some quasi-particles and Cooper pairs may be destroyed and a fraction of normal fluid appears. At temperatures $\mathrm{T}_\lambda < \mathrm{T} < \widetilde{\mathrm{T}}_\lambda$ (remark 5.2) the thermic fluctuations become sufficiently strong to destroy the non-conventional Bose condensate. Consequently, even if the quasi-particle gas resists in liquid helium for $\mathrm{T}_\lambda < \mathrm{T} < \widetilde{\mathrm{T}}_\lambda$ (remark 5.2), it is quite unstable and any perturbation of the quasi-particles (like any positive velocity) may quickly destroy the collective system and switch it to a standard liquid where no superfluidity exists.

3. Note that this last conjecture may seem a little naive since the value T_λ is very specific. Actually, the previous discussions are just phenomenological interpretations. Therefore, to conclude we examine *another* interpretation of the Bose gas $H_{\Lambda,0}^{B}$ *without taking into account Landau's criterion of superfluidity* [19, 20], which is a *phenomenological* explanation of superfluidity.
If $\gamma_0^B(\mathrm{T}) \sim (\mathrm{T}_c - \mathrm{T})^{\eta^B}$ at temperatures $\mathrm{T} = (\mathrm{k_B}\beta)^{-1} \rightarrow \mathrm{T}_c^-$ is the fraction of Bose condensate for a fixed density $\rho > 0$, then via theorem 3.7, the fraction $\gamma_s^B(\mathrm{T}) = 1 - \rho_n/\rho$ satisfies:

$$\gamma_s^B(\mathrm{T}) \sim (\mathrm{T}_c - \mathrm{T})^{\eta^B} \sim \gamma_0^B(\mathrm{T}), \ \text{for } \mathrm{T} \rightarrow \mathrm{T}_c^-, \tag{5.10}$$

where

$$\rho_n(\mathrm{T}) = \left\{ \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} \frac{f_{k,0}}{E^B_{k,0}\left[e^{E^B_{k,0}/\mathrm{T}} - 1\right]} d^3k \right\} \Big|_{x=\widehat{x},\alpha=\alpha(\widehat{x})}. \tag{5.11}$$

The relation (5.10) is strangely similar to (5.9), see figure 5.6. The fraction $\gamma^B_s(\mathrm{T})$ may be considered as the superfluid fraction of the Bose gas $H^B_{\Lambda,0}$. Therefore, at a fixed density $\rho > 0$, the superfluid density ρ_s equals

$$\rho_s(\mathrm{T}) = \left\{ x + \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} \frac{x^2 \lambda_k^2}{2E^B_{k,0}\left[f_{k,0} + E^B_{k,0}\right]} d^3k \right\} \Big|_{x=\widehat{x},\alpha=\alpha(\widehat{x})},$$

whereas ρ_n (5.11) is the density of normal fluid which is the carrier of the total entropy of the system. Note that $\lim\limits_{\mathrm{T}\to 0^+} \rho_n = 0$ and within this framework there is 100% of superfluid liquid at zero-temperature with a density $\rho_s > \widehat{x} = \widehat{x}(\mathrm{T})$. See (i) of theorem 3.7 to see the Bose condensate density at a fixed density $\rho > 0$, i.e., for $\mu = \mu_{\beta,\rho}$ (3.57).
In fact, this conjecture has to be analyzed via the corresponding Hamiltonian with an external velocity field as it has been *recently* performed with dilute trapped Bose gases at zero-temperature [59].

5.5. Concluding remarks: superfluidity of Fermi systems

The superfluidity of a Fermi system, i.e. the ^{3}He liquid, was discovered in 1972 for sufficiently low temperatures [60,61]. All the previous theories concern Bose systems. However, it is remarkable to see that, via the effective coupling constant $g_{00} < 0$ (figure 5.3), the non-diagonal interaction U^{ND}_Λ of the model $H^B_{\Lambda,0}$ implies an attraction between particles in the zero-mode.
By analogy, it is well-known that the phenomenon of superconductivity comes from the effective electron-electron interaction in the BCS theory which results from the electron-phonon (non-diagonal) interaction in the second order of perturbation theory, see e.g. [62, 63]. Thus, in a superconductor, electrons can pair up in the metal crystal via phonons to form Cooper pairs which can then condense into a superconducting state. This phenomenon corresponds also to the explanation given for the existence of superfluidity in ^{3}He [21, 22]. Indeed, by cooling the liquid to a low enough temperature, ^{3}He atoms can pair up, making it a boson, and therefore superfluidity can be achieved.
In the Bose gas $H^B_{\Lambda,0}$, the effective attraction characterized by $g_{00} < 0$ plays exactly the same rôle on bosons by creating Cooper pairs and may also work for Fermi systems. Therefore, it should be interesting to study a similar Hamiltonian within the framework of Fermi systems.

Of course, the main difference comes from the Fermi statistics. In particular, the critical density $\rho^{PBG}(\beta, 0)$ for the Perfect Bose Gas does *not* exist for the Perfect Fermi Gas. For the Bose system $H^B_{\Lambda,0}$, the corresponding *kinetic* part only *turns on* the Bose condensation phenomenon via the Bose statistics, see for example remark 3.5. Indeed, the corresponding "chemical potential" $\alpha(\widehat{x})$ of the variational problem (3.16) for a Bose condensate density $\widehat{x}(\beta, \rho)$ becomes zero when we reach the critical density as for the Perfect Bose Gas, *but* switches again to strictly negative values for $\widehat{x} > 0$. As soon as the Bose condensate appears, the non-diagonal interaction U^{ND}_Λ becomes sufficiently important to drastically change all thermodynamic properties

of the system by instantly switching the usual Perfect Bose gas to a gas of quasi-particles: the Bose-Einstein condensation becomes non-conventional in correlation with the creation of the Bogoliubov condensate and the formation of Cooper pairs (Section 5.2).
Whereas the non-diagonal interaction U_{Λ}^{ND} is *not strong enough* to imply *alone* the Bose-condensation at the critical temperature or density of the Perfect Bose Gas, for very small temperatures it strongly dominates all thermodynamics of the system. The non-diagonal interaction U_{Λ}^{ND} obviously has a strong impact on the system (see for example the divergence of the grandcanonical pressure of $H_{\Lambda,0}^{B}$ [1]). It would have implied the non-conventional Bose condensation without the Bose statistics at sufficiently low temperatures or high densities.
In particular, if the Fermi statistics now holds, a similar non-diagonal interaction characterizing by an effective attraction g_{00} (2.1) (figure 5.3) drastically opposes the repulsion from the Pauli exclusion principle and would finally become strong enough at *sufficiently low temperatures* to imply *alone* the *non-conventional* Bose condensation (Cooper pairs) and the superfluid gas of quasi-particles explained above. This means of course that the critical temperature for the corresponding Fermi system should be quite lower than that of the Bose model $H_{\Lambda,0}^{B}$. Experimentally, the critical temperature of ^{3}He is very low in comparison with that of ^{4}He (2.17 K) : it is only two milli Kelvin for ^{3}He [60, 61].

We reserve this analysis on Fermi systems for another paper. To conclude, notice also that the ^{3}He liquid forms, at sufficiently low temperatures, several superfluid phases (A&B), which are much richer properties than those of the superfluid ^{4}He. For a complete review of properties of ^{3}He at low temperatures, see [64, 65].

Appendix A. : the Bogoliubov u-v transformation

In this subsection we recall the Bogoliubov canonical u-v transformation by applying it on the Bogoliubov approximation [25]

$$\begin{aligned} H_{\Lambda,\lambda_0}^{B}(\alpha,c) &= \sum_{k\in\Lambda^*\setminus\{0\}} \left[\varepsilon_k - \alpha + \lambda_0 |c|^2\right] a_k^* a_k + \frac{1}{2}\sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k |c|^2 \left[a_k^* a_k + a_{-k}^* a_{-k}\right] \\ &+ \frac{1}{2}\sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k \left[c^2 a_k^* a_{-k}^* + \overline{c}^2 a_k a_{-k}\right] - \alpha |c|^2 V + \frac{\lambda_0}{2}\left(|c|^4 V - |c|^2\right) \end{aligned} \tag{A.1}$$

of $H_{\Lambda,\lambda_0}^{B}(\alpha) \equiv H_{\Lambda,\lambda_0}^{B} - \alpha N_\Lambda$ (2.4) for any $\lambda_0 \geq 0$. Then, we compute the corresponding pressure

$$p_{\Lambda,\lambda_0}^{B}(\beta,\alpha,c) = \frac{1}{\beta V} \ln Tr_{\mathcal{F}_B'} e^{-\beta H_{\Lambda,\lambda_0}^{B}(\alpha,c)}. \tag{A.2}$$

After the canonical gauge transformation to boson operators

$$a_k e^{-i\arg c},\ k \in \Lambda^* \setminus \{0\}, \tag{A.3}$$

note that $H_{\Lambda,\lambda_0}^{B}(\alpha,c)$ depends only on $x \equiv |c|^2$. Since $H_{\Lambda,\lambda_0}^{B}(\alpha,c)$ is a bilinear form in boson operators $\{a_k, a_k^*\}_{k\in\Lambda^*\setminus\{0\}}$, the Bogoliubov canonical u-v transformation diagonalizes it by using a new set of boson operators $\{b_k, b_k^*\}_{k\in\Lambda^*\setminus\{0\}}$ defined by

$$a_k = \mathbf{u}_k b_k - \mathbf{v}_k b_{-k}^*,\ a_k^* = \mathbf{u}_k b_k^* - \mathbf{v}_k b_{-k}, \tag{A.4}$$

with real coefficients $\{\mathbf{u}_k = \mathbf{u}_{-k}\}_{k\in\Lambda^*\backslash\{0\}}$ and $\{\mathbf{v}_k = \mathbf{v}_{-k}\}_{k\in\Lambda^*\backslash\{0\}}$ satisfying:

$$\mathbf{u}_k^2 - \mathbf{v}_k^2 = 1,\ 2\mathbf{u}_k\mathbf{v}_k = \frac{x\lambda_k}{E^B_{k,\lambda_0}},\ \mathbf{u}_k^2 + \mathbf{v}_k^2 = \frac{\varepsilon_k}{E^B_{k,\lambda_0}}.$$

Here

$$\begin{aligned} f_{k,\lambda_0} &= \varepsilon_k - \alpha + x\left(\lambda_0 + \lambda_k\right), \\ E^B_{k,\lambda_0} &= \sqrt{f^2_{k,\lambda_0} - x^2\lambda_k^2} = \sqrt{\left(\varepsilon_k - \alpha + x\lambda_0\right)\left(\varepsilon_k - \alpha + x\left(\lambda_0 + 2\lambda_k\right)\right)}, \end{aligned} \tag{A.5}$$

where we recall that $x \equiv |c|^2$. Thus

$$\mathbf{u}_k^2 = \frac{1}{2}\left(\frac{f_{k,\lambda_0}}{E^B_{k,\lambda_0}} + 1\right),\ \mathbf{v}_k^2 = \frac{1}{2}\left(\frac{f_{k,\lambda_0}}{E^B_{k,\lambda_0}} - 1\right).$$

Notice that $f_{k,\lambda_0} \geq x\lambda_k$ and, $|c|^2$ and α satisfy the inequality:

$$\alpha \leq |c|^2\lambda_0 + \min_{k\in\Lambda^*\backslash\{0\}} \varepsilon_k. \tag{A.6}$$

The Hamiltonian (A.1) becomes:

$$\mathrm{H}^B_{\Lambda,\lambda_0}(\alpha, c) = \sum_{k\in\Lambda^*\backslash\{0\}} E^B_{k,\lambda_0} b_k^* b_k + \frac{1}{2}\sum_{k\in\Lambda^*\backslash\{0\}}\left(E^B_{k,\lambda_0} - f_{k,\lambda_0}\right) - \alpha|c|^2 + \frac{\lambda_0}{2}\left(|c|^4 - \frac{|c|^2}{V}\right). \tag{A.7}$$

Therefore, the pressure $p^B_{\Lambda,\lambda_0}(\beta,\alpha,c)$ (A.2) equals

$$\begin{aligned} p^B_{\Lambda,\lambda_0}(\beta,\alpha,c) &= \xi_{\Lambda,\lambda_0}\left(\beta,\alpha,x\equiv|c|^2\right) + \eta_{\Lambda,\lambda_0}\left(\alpha,x\equiv|c|^2\right), \\ \xi_{\Lambda,\lambda_0}(\beta,\alpha,x) &= \frac{1}{\beta V}\sum_{k\in\Lambda^*\backslash\{0\}} \ln\left(1 - e^{-\beta E^B_{k,\lambda_0}}\right)^{-1}, \\ \eta_{\Lambda,\lambda_0}(\alpha,x) &= \frac{1}{2V}\sum_{k\in\Lambda^*\backslash\{0\}}\left(f_{k,\lambda_0} - E^B_{k,\lambda_0}\right) + \alpha x - \frac{\lambda_0}{2}\left(x^2 - \frac{x}{V}\right), \end{aligned} \tag{A.8}$$

and has the following thermodynamic limit:

$$\begin{aligned} p^B_{\lambda_0}\left(\beta,\alpha,x\equiv|c|^2\right) &\equiv \lim_{\Lambda} p^B_{\Lambda,\lambda_0}(\beta,\alpha,c) = \xi_{\lambda_0}(\beta,\alpha,x) + \eta_{\lambda_0}(\alpha,x), \\ \xi_{\lambda_0}(\beta,\alpha,x) &\equiv \lim_{\Lambda}\xi_{\Lambda,\lambda_0}(\beta,\alpha,x) = \frac{1}{(2\pi)^3\beta}\int_{\mathbb{R}^3}\ln\left(1 - e^{-\beta E^B_{k,\lambda_0}}\right)^{-1} d^3k, \\ \eta_{\lambda_0}(\alpha,x) &\equiv \lim_{\Lambda}\eta_{\Lambda,\lambda_0}(\alpha,x) = \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\left(f_{k,\lambda_0} - E^B_{k,\lambda_0}\right)d^3k + \alpha x - \frac{\lambda_0}{2}x^2, \end{aligned} \tag{A.9}$$

with $E^B_{k,\lambda_0\geq 0}$, $f_{k,\lambda_0\geq 0}$ defined by (A.5) and $\alpha \leq x\lambda_0$ (cf. (A.6)).

Appendix B. : Technical proofs

Lemma B.1. *For any sequence* $\{\mathcal{E}_\Lambda\}_{\Lambda\subset\mathbb{R}^3}$ *of subsets* $\mathcal{E}_\Lambda \subseteq \Lambda^*$ *we have*

$$\lim_\Lambda \frac{1}{V}\sum_{k\in\mathcal{E}_\Lambda\subseteq\Lambda^*} \langle a_k^* a_k\rangle_{H^{SB}_{\Lambda,\lambda_0}}(\beta,\mu) = \left\{\partial_\gamma p_0^B(\beta,\alpha,\gamma,x)\right\}\Big|_{\alpha(\widehat{x}),\gamma=0,x=\widehat{x}},$$

with

$$\begin{aligned} p_0^B(\beta,\alpha,\gamma,x) \equiv{}& \frac{1}{(2\pi)^3\beta}\int_{\mathbb{R}^3\backslash\mathcal{E}} \ln\left(1-e^{-\beta E^B_{k,0}}\right)^{-1} d^3k + \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3\backslash\mathcal{E}} \left(f_{k,0}-E^B_{k,0}\right) d^3k \\ &+\left\{\frac{1}{(2\pi)^3\beta}\int_{\mathcal{E}} \ln\left(1-e^{-\beta E^B_{k,0}}\right)^{-1} d^3k + \frac{1}{2(2\pi)^3}\int_{\mathcal{E}} \left(f_{k,0}-E^B_{k,0}\right) d^3k\right\}\Bigg|_{\alpha\to\alpha+\gamma} \\ &+\left(\alpha+\gamma\lim_\Lambda \chi_{\mathcal{E}_\Lambda}(0)\right)x, \end{aligned}$$

for any $\alpha\leq 0$ *and* $\gamma \leq 0$. *Here* $\chi_{\mathcal{E}_\Lambda}$ *denotes the characteristic function of* $\mathcal{E}_\Lambda$ *and the set* $\mathcal{E}$ *is given by* $\mathcal{E}\equiv\lim_\Lambda \mathcal{E}_\Lambda\subseteq\mathbb{R}^3$.

Proof. Let

$$p_\Lambda^{SB}(\beta,\mu,\gamma)\equiv\frac{1}{\beta V}\ln Tr_{\mathcal{F}^B_\Lambda} e^{-\beta H^{SB}_{\Lambda,\lambda_0,\gamma}(\mu)}$$

be the pressure associated with the perturbed (superstable) Hamiltonian $H^{SB}_{\Lambda,\lambda_0,\gamma}(\mu)$ defined by:

$$H^{SB}_{\Lambda,\lambda_0,\gamma}(\mu)\equiv H^{SB}_{\Lambda,\lambda_0}-\mu N_\Lambda-\gamma\sum_{k\in\mathcal{E}_\Lambda\subseteq\Lambda^*} a_k^*a_k.$$

Since $H^{SB}_{\Lambda,\lambda_0,\gamma}(\mu)$ is superstable, its pressure is well-defined and convex for any real μ and γ. Consequently the theorem 3.2 is still valid for $\gamma\in\mathbb{R}$:

$$\begin{aligned} p^{SB}(\beta,\mu,\gamma) &\equiv \lim_\Lambda p_\Lambda^{SB}(\beta,\mu,\gamma) \\ &= \sup_{x\geq 0}\left\{\inf_{\alpha\leq 0}\left\{p_0^B(\beta,\alpha,\gamma,x)+\frac{(\mu-\alpha)^2}{2\lambda_0}\right\}\right\} \\ &= \left\{p_0^B(\beta,\alpha,\gamma,x)+\frac{(\mu-\alpha)^2}{2\lambda_0}\right\}\Bigg|_{\alpha=\alpha_\gamma(\widehat{x}_\gamma),x=\widehat{x}_\gamma}, \end{aligned} \tag{B.1}$$

with the corresponding pressure $p_0^B(\beta,\alpha,\gamma,x)$. Here $\alpha_\gamma(\widehat{x}_\gamma)$ and $\widehat{x}_\gamma$ are the corresponding solutions of the variational problems. We also have

$$\partial_\gamma p_\Lambda^{SB}(\beta,\mu,\gamma)=\frac{1}{V}\sum_{k\in\mathcal{E}_\Lambda\subseteq\Lambda^*}\langle a_k^*a_k\rangle_{H^{SB}_{\Lambda,\lambda_0,\gamma}}(\beta,\mu),$$

and, via the Griffiths lemma [36, 37], we get:

$$\lim_\Lambda\frac{1}{V}\sum_{k\in\mathcal{E}_\Lambda\subseteq\Lambda^*}\langle a_k^*a_k\rangle_{H^{SB}_{\Lambda,\lambda_0}}(\beta,\mu)=\left\{\partial_\gamma\lim_\Lambda p_\Lambda^{SB}(\beta,\mu,\gamma)\right\}\Big|_{\gamma=0}=\left\{\partial_\gamma p^{SB}(\beta,\mu,\gamma)\right\}\Big|_{\gamma=0}. \tag{B.2}$$

From remark 3.3 for $\mu \neq \mu_c(\beta)$ or $\beta \neq \beta_c(\mu)$ combined with (B.1) we get

$$\begin{aligned}\partial_\gamma p^{SB}(\beta,\mu,\gamma) &= \partial_\gamma \left\{ p_0^B(\beta,\alpha_\gamma(\widehat{x}_\gamma),\gamma,\widehat{x}_\gamma) + \frac{(\mu-\alpha_\gamma(\widehat{x}_\gamma))^2}{2\lambda_0} \right\} \\ &= \left\{ \partial_\gamma p_0^B(\beta,\alpha,\gamma,x) \right\} \Big|_{\alpha=\alpha_\gamma(\widehat{x}_\gamma),x=\widehat{x}_\gamma}\end{aligned}$$

for $|\gamma|$ sufficiently small and $\mu \neq \mu_c(\beta)$ or $\beta \neq \beta_c(\mu)$. Consequently the limit (B.2) combined with the last equation for $\gamma = 0$ gives the lemma. ■

Lemma B.2. *Let $k \in \Lambda^* \backslash \{0\}$. Since Λ is isotropic, then for Λ sufficiently large we have:*

$$\langle a_k^* a_k \rangle_{H_{\Lambda,\lambda_0}^{SB}}(\beta,\mu) \leq \frac{1}{e^{B_k(\mu)}-1} + \frac{\beta\lambda_k}{2\left(1-e^{-B_k(\mu)}\right)} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda,\lambda_0}^{SB}}(\beta,\mu)$$

with

$$B_k(\mu) \equiv \beta \left[\varepsilon_k - \mu - \frac{\lambda_k}{2V} + \lambda_0 \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_{\Lambda,\lambda_0}^{SB}}(\beta,\mu) \right] > 0,$$

cf. remark 3.3 ($\alpha(\widehat{x}) < 0$) and (3.49).

Proof. By the correlation inequalities [28, 66, 67] for the quantum Gibbs state $\omega_\Lambda^{SB}(-) = \langle - \rangle_{H_{\Lambda,\lambda_0}^{SB}}(\beta,\mu)$ we have

$$\beta\omega_\Lambda^{SB}\left(X^*\left[H_{\Lambda,\lambda_0}^{SB}(\mu),X\right]\right) \geq \omega_\Lambda^{SB}(X^*X)\ln\frac{\omega_\Lambda^{SB}(X^*X)}{\omega_\Lambda^{SB}(XX^*)},$$

where X is an observable from domain of the commutator $\left[H_{\Lambda,\lambda_0}^{SB}(\mu),.\right]$, thus we obtain

$$\beta\omega_\Lambda^{SB}\left(a_k^*\left[H_{\Lambda,\lambda_0}^{SB}(\mu),a_k\right]\right) \geq \omega_\Lambda^{SB}(N_k)\ln\frac{\omega_\Lambda^{SB}(N_k)}{\omega_\Lambda^{SB}(N_k)+1} \tag{B.3}$$

for $X = a_k$ with $k \in \Lambda^* \backslash \{0\}$. Since

$$\left[H_{\Lambda,\lambda_0}^{SB}(\mu),a_k\right] = -\left(\varepsilon_k - \mu + \lambda_k\frac{a_0^*a_0}{V} + \frac{\lambda_0}{V}\sum_{q\in\Lambda^*\backslash\{k\}} a_q^*a_q + \frac{\lambda_0}{V}a_ka_k^*\right)a_k - \frac{\lambda_k}{V}a_0^2a_{-k}^*,$$

one gets

$$\begin{aligned}\omega_\Lambda^{SB}\left(a_k^*\left[H_{\Lambda,\lambda_0}^{SB}(\mu),a_k\right]\right) &= -[\varepsilon_k-\mu]\,\omega_\Lambda^{SB}(N_k) - \lambda_k\frac{\omega_\Lambda^{SB}(N_0N_k)}{V} \\ &\quad -\lambda_0\frac{\omega_\Lambda^{SB}(N_kN_\Lambda)}{V} - \lambda_k\frac{\omega_\Lambda^{SB}\left(a_0^2a_k^*a_{-k}^*\right)}{V},\end{aligned} \tag{B.4}$$

with $N_p \equiv a_p^*a_p$. Notice that because $\omega_\Lambda^{SB}\left(a_k^*\left[H_{\Lambda,\lambda_0}^{SB}(\mu),a_k\right]\right) \in \mathbb{R}$ we get from the remaining term of (B.4) that $\omega_\Lambda^{SB}\left(a_0^2a_k^*a_{-k}^*\right) \in \mathbb{R}$. Therefore

$$2\omega_\Lambda^{SB}\left(a_0^2a_k^*a_{-k}^*\right) = \omega_\Lambda^{SB}\left(a_0^2a_k^*a_{-k}^*\right) + \omega_\Lambda^{SB}\left(a_ka_{-k}a_0^{*2}\right).$$

Moreover, since the functions ε_k and λ_k are even for $(k \to -k)$, we have

$$\omega_\Lambda^{SB}(N_0 N_k) = \omega_\Lambda^{SB}(N_0 N_{-k}).$$

Thus it follows

$$\begin{aligned}
\omega_\Lambda^{SB}\left(a_k^*\left[H_{\Lambda,\lambda_0}^{SB}(\mu), a_k\right]\right) &= -\left[\varepsilon_k - \mu\right]\omega_\Lambda^{SB}(a_k^* a_k) - \frac{\lambda_k}{2V}\omega_\Lambda^{SB}\left(a_0^2 a_k^* a_{-k}^* + a_0^{*2} a_k a_{-k}\right) \\
&\quad -\frac{\lambda_k}{2V}\omega_\Lambda^{SB}\left(N_0\left(N_k + N_{-k}\right)\right) - \frac{\lambda_0}{V}\omega_\Lambda^{SB}(N_k N_\Lambda). \qquad \text{(B.5)}
\end{aligned}$$

Since

$$\begin{aligned}
a_0^2 a_k^* a_{-k}^* + a_0^{*2} a_k a_{-k} + a_0^* a_0 a_k^* a_k + a_0^* a_0 a_{-k}^* a_{-k} &= \left(a_0^* a_k + a_{-k}^* a_0\right)^* \left(a_0^* a_k + a_{-k}^* a_0\right) \\
&\quad - a_k^* a_k - a_0^* a_0,
\end{aligned}$$

and

$$\omega_\Lambda^{SB}(N_k N_\Lambda) \geq \omega_\Lambda^{SB}(N_k)\,\omega_\Lambda^{SB}(N_\Lambda)$$

we deduce from (B.5) the following estimate:

$$\omega_\Lambda^{SB}\left(a_k^*\left[H_{\Lambda,\lambda_0}^{SB}(\mu), a_k\right]\right) \leq -\left[\varepsilon_k - \mu - \frac{\lambda_k}{2V} + \lambda_0 \omega_\Lambda^{SB}\left(\frac{N_\Lambda}{V}\right)\right]\omega_\Lambda^{SB}(N_k) + \frac{\lambda_k}{2V}\omega_\Lambda^{SB}(N_0).$$

Therefore, (B.3) and the last inequality implies:

$$B_k(\mu)\,\omega_\Lambda^{SB}(N_k) - \frac{\beta\lambda_k}{2}\omega_\Lambda^{SB}\left(\frac{N_0}{V}\right) \leq \omega_\Lambda^{SB}(N_k)\ln\frac{\omega_\Lambda^{SB}(N_k) + 1}{\omega_\Lambda^{SB}(N_k)}.$$

Combining (3.49) with remark 3.3 we have

$$\lambda_0\left\{\lim_\Lambda \omega_\Lambda^{SB}\left(\frac{N_\Lambda}{V}\right)\right\} - \mu = -\alpha\left(\widehat{x}\right) \geq 0$$

i.e.

$$\inf_{k\in\Lambda^*\backslash\{0\}} B_k(\mu) = B_{\|k\|=\frac{2\pi}{L}}(\mu) \geq \frac{1}{L^2}\left(\frac{2\left(\pi\hbar\right)^2}{m} - \frac{\lambda_k}{2L}\right) > 0,$$

for an *isotropic* Λ sufficiently large. Hence, to estimate $x \equiv \omega_\Lambda^{SB}(N_k) \geq 0$, we have to solve the inequality

$$B_k(\mu)\,x - \frac{\beta\lambda_k}{2}\omega_\Lambda^{SB}\left(\frac{N_0}{V}\right) \leq x\ln\frac{x+1}{x}. \qquad \text{(B.6)}$$

Notice that the solution of (B.6) is the set $\{0 \leq x \leq x_1\}$, where x_1 is a solution of the equation

$$B_k(\mu)\,x_1 - \frac{\beta\lambda_k}{2}\omega_\Lambda^{SB}\left(\frac{N_0}{V}\right) = x_1\ln\frac{x_1+1}{x_1}.$$

Let

$$x_2 = \frac{1}{e^{B_k(\mu)} - 1}, \qquad \text{(B.7)}$$

be a nontrivial solution of the equation

$$B_k(\mu)\, x = x \ln \frac{x+1}{x}.$$

Then the inequality $x \le x_1$ can be rewritten as

$$x \le x_2 + (x_1 - x_2)\,. \tag{B.8}$$

Since the function $f(x) \equiv x \ln \frac{x+1}{x}$ defined for $x \ge 0$ (we put $f(x=0)=0$) is concave, we have

$$\frac{f(x_1) - f(x_2)}{f'(x_2)} \le x_1 - x_2,$$

and using (B.7) and (B.8) the lemma follows when $k \in \Lambda^* \setminus \{0\}$. ■

Lemma B.3. *Under the assumptions using in section 4 the following holds:*

(i) $\partial_{\lambda_0} \widehat{x} = 0$ *for any* $\rho \notin \left[\rho_{c,\inf}(\beta), \rho_{c,\sup}(\beta)\right]$.

(ii) For $\rho_1 = \rho - x > 0$ *with* $\rho > \rho_{c,\sup}(\beta)$ *the free-energy density* $\widehat{f}_0^B(\beta, \rho_1, x)$ *satisfies*

$$\left\{\partial_x \widehat{f}_0^B(\beta, \rho_1, x)\right\}\Big|_{\rho_1 = \rho - \widehat{x}, x = \widehat{x}} = \left\{\partial_{\rho_1} \widehat{f}_0^B(\beta, \rho_1, x)\right\}\Big|_{\rho_1 = \rho - \widehat{x}, x = \widehat{x}} = \partial_\rho f_0^B(\beta, \rho) < 0.$$

Proof. *(i)* For $\rho < \rho_{c,\inf}(\beta)$, i.e. $\mu_{\beta,\rho} < \mu_c(\beta)$ or $\beta < \beta_c\left(\mu_{\beta,\rho}\right)$, the first statement (i) is a direct consequence of remark 3.3 ($\widehat{x} = 0$). Moreover, again from remark 3.3, (3.22), (3.34)-(3.36) we find for $\rho > \rho_{c,\sup}(\beta)$

$$\left\{\partial_x \xi_0(\beta, \alpha, x) + \Omega(\alpha, x) = -\alpha(x)\right\}\Big|_{x = \widehat{x}, \alpha = \alpha(\widehat{x})},$$

which through remark 4.3 implies

$$\left\{\partial_x \xi_0(\beta, \alpha, x) + \Omega(\alpha, x)\right\}\Big|_{x = \widehat{x}, \alpha = \partial_\rho f_0^B(\beta, \rho)} = -\partial_\rho f_0^B(\beta, \rho)\,.$$

The derivative of the last expression as a function of λ_0 gives:

$$\partial_{\lambda_0} \widehat{x} \left\{\partial_x^2 \xi_0(\beta, \alpha, x) + \partial_x \Omega(\alpha, x)\right\}\Big|_{x = \widehat{x}, \alpha = \partial_\rho f_0^B(\beta, \rho)} = 0. \tag{B.9}$$

Via direct computations we have for any $\alpha \le 0$ and $x \ge 0$

$$\partial_x^2 \xi_0(\beta, \alpha, x) > 0 \text{ and } \partial_x \Omega(\alpha, x) > 0,$$

and therefore the equation (B.9) implies (i) for $\rho > \rho_{c,\sup}(\beta)$.

(*ii*) By (4.6)-(4.7), one directly gets

$$\left\{ \partial_{\rho_1} \widehat{f}_0^B (\beta, \rho_1, x) \right\} \Big|_{\rho_1 = \rho - \widehat{x}, x = \widehat{x}} = \alpha (\rho - \widehat{x}, \widehat{x}) = \alpha (\widehat{x}),$$

and

$$\left\{ \partial_x \widehat{f}_0^B (\beta, \rho_1, x) \right\} \Big|_{\rho_1 = \rho - \widehat{x}, x = \widehat{x}} = \alpha (\widehat{x}) - \left\{ \partial_x p_0^B (\beta, \alpha, x) \right\} \Big|_{\alpha = \alpha(\widehat{x}), x = \widehat{x}}.$$

Using remark 3.3 combined with remark 4.3 the last two equations imply the second statement (ii) of this lemma for $\rho > \rho_{c,\sup} (\beta)$. ■

Acknowledgments.

The work was supported by DFG grant DE 663/1-3 in the priority research program for interacting stochastic systems of high complexity. Special thanks first go to T. Dorlas and the DIAS for the very nice stay there where this work was finished. J.-B. Bru thanks Institut für Mathematik, Technische Universität Berlin, and its members for their warm hospitality during the academic year 2001-2002 and more precisely S. Adams. J.-B. Bru also wants to express his gratitude to N. Angelescu, A. Verbeure and V.A. Zagrebnov for their useful discussions. And the second author especially thanks the P. master Dukes and Dido for their help in writing/correcting this article.

Literatur

[1] S. Adams and J.-B. Bru, Critical Analysis of the Bogoliubov Theory of Superfluidity, Submitted to *J. Phys. A: Math.Gen.* (2003).

[2] Bose-Einstein condensation, ed. A. Griffin, D.W. Snoke and S. Stringari, *Cambridge Univ. Press*, Cambridge (1996).

[3] A. Griffin, Excitations in a Bose-Condensated Liquid, *Cambridge Univ. Press*, Cambridge (1993).

[4] N.N. Bogoliubov, On the theory of superfluidity, *J. Phys. (USSR)* **11**:23 (1947).

[5] N.N. Bogoliubov, About the theory of superfluidity, *Izv. Akad. Nauk USSR* **11**:77 (1947).

[6] N.N. Bogoliubov, Energy levels of the imperfect Bose-Einstein gas, *Bull. Moscow State Univ.* **7**:43 (1947).

[7] N.N. Bogoliubov, Lectures on Quantum Statistics, Vol. 1: Quantum Statistics, *Gordon and Breach Science Publishers*, New York-London-Paris (1970).

[8] N.N. Bogoliubov, Energy levels of the imperfect Bose-Einstein gas, p. 242-257 in: *Collection of papers*, *Vol. 2*, Naukova Dumka, Kiev, (1970).

[9] V.A. Zagrebnov and J.-B. Bru, The Bogoliubov Model of Weakly Imperfect Bose Gas, *Phys. Rep.* **350**:291 (2001).

[10] J.-B. Bru and V.A. Zagrebnov, Exact phase diagram of the Bogoliubov Weakly Imperfect Bose gas, *Phys. Lett. A* **244:**371 (1998).

[11] J.-B. Bru and V.A. Zagrebnov, Exact solution of the Bogoliubov Hamiltonian for weakly imperfect Bose gas, *J. Phys. A: Math. Gen. A* **31**:9377 (1998).

[12] J.-B. Bru and V.A. Zagrebnov, Quantum interpretation of thermodynamic behaviour of the Bogoliubov weakly imperfect Bose gas, *Phys. Lett. A* **247**:37 (1998).

[13] J.-B. Bru and V.A. Zagrebnov, Thermodynamic Behavior of the Bogoliubov Weakly Imperfect Bose Gas, p. 313 in: *Mathematical Results in Statistical Mechanics*, *eds S. Miracle-Sole and al., World Scientific*, Singapore (1999).

[14] J.-B. Bru and V.A. Zagrebnov, On condensations in the Bogoliubov Weakly Imperfect Bose-Gas, *J. Stat. Phys.* **99**:1297 (2000).

[15] V.A. Zagrebnov, Generalized condensation and the Bogoliubov theory of superfluidity, *Cond. Matter Phys.* **3:**265 (2000).

[16] N. Angelescu, A. Verbeure and V.A. Zagrebnov, On Bogoliubov's model of superfluidity *J. Phys. A: Math.Gen.* **25**:3473 (1992).

[17] J.-B. Bru and V.A. Zagrebnov, Exactly soluble model with two kinds of Bose-Einstein condensations, *Physica A* **268**:309 (1999).

[18] J.-B. Bru and V.A. Zagrebnov, A model with coexistence of two kinds of Bose condensations, *J. Phys. A: Math.Gen.* **33**:449 (2000).

[19] L.D. Landau, The theory of superfluidity of Helium II, *J. Phys. (USSR)* **5**:71 (1941).

[20] L.D. Landau, On the theory of superfluidity of Helium II, *J. Phys. (USSR)* **11**:91 (1947).

[21] A.J. Leggett, p. 13 in: *Modern Trends in the Theory of Condensed Matter, eds A. Pekalski and J. Przyslawa*, Springer-Verlag, Berlin (1980).

[22] A.J. Leggett, *J. Phys. (Paris) Colloq.* **41**:C7-19 (1980).

[23] A.J. Leggett, Superfluidity, *Rev. Mod. Phys.* **71**:S318 (1999).

[24] D. Ruelle, Statistical Mechanics: Rigorous Results, *Benjamin-Reading*, New-York (1969).

[25] J. Ginibre, On the Asymptotic Exactness of the Bogoliubov Approximation for many Bosons Systems, *Commun. Math. Phys.* **8**:26 (1968).

[26] J.-B. Bru, Superstabilization of Bose Systems I: Thermodynamic Study, *J. Phys. A: Math.Gen.* **35**:8969 (2002).

[27] J.-B. Bru, Superstabilization of Bose systems II: Bose condensations and equivalence of ensembles, *J. Phys. A: Math.Gen.* **35**:8995 (2002).

[28] O. Brattelli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Vol **II**, 2nd ed. Springer-Verlag, New York (1996).

[29] K. Huang, Statistical Mechanics, *Wiley*, New York (1963).

[30] E.B. Davies, The thermodynamic limit for an imperfect boson gas, *Commun. Math. Phys.* **28**:69 (1972).

[31] M. Fannes and A. Verbeure, The condensed phase of the imperfect Bose gas, *J. Math. Phys.* **21**:1809 (1980).

[32] M. van den Berg, J.T. Lewis and Ph. de Smedt, Condensation in the Imperfect Boson Gas, *J. Stat. Phys.* **37**:697 (1984).

[33] E. Buffet and J.V. Pulè, Fluctuations Properties of the Imperfect Boson Gas, *J. Math. Phys.* **24**:1608 (1983).

[34] J.T. Lewis, J.V. Pulè and V.A. Zagrebnov, The Large Deviation Principle for the Kac Distribution, *Helv. Phys. Acta* **61**:1063 (1988).

[35] Vl.V. Papoyan and V.A. Zagrebnov, The ensemble equivalence problem for Bose systems (non-ideal Bose gas), *Theor. Math. Phys.* **69**:1240 (1986).

[36] R. Griffiths, A Proof that the Free Energy of a Spin System is extensive, *J. Math. Phys.* **5**:1215 (1964).

[37] K. Hepp E. and H. Lieb, Equilibrium Statistical Mechanics of Matter Interacting with the Quantized Radiation Field, *Phys. Rev. A* **8**:2517 (1973).

[38] M. van den Berg and J.T. Lewis, On generalized condensation in the free boson gas, *Physica A* **110**:550 (1982).

[39] M. van den Berg, On boson condensation into an infinite number of low-lying levels, *J. Math. Phys.* **23**:1159 (1982).

[40] M. van den Berg, J.T. Lewis and J.V. Pulè, A general theory of Bose-Einstein condensation, *Helv. Phys. Acta* **59**:1271 (1986).

[41] R.A. Minlos and A.Ja. Povzner, Thermodynamic limit for entropy, *Trans. Moscow Math. Soc.* **17:**269 (1967).

[42] H.O. Georgii, Large Deviations and the Equivalence of Ensembles for Gibbsian Particle Systems with Superstable Interaction, *Probab. Th. Rel. Fields* **99**:171 (1994).

[43] H.O. Georgii, The equivalence of ensembles for classical systems of particles, *Journal of Stat. Phys.* Vol. **80** :1341-1378 (1994).

[44] S. Adams, Complete Equivalence of the Gibbs ensembles for one-dimensional Markov-systems, *Journal of Stat. Phys.,* Vol. **105**, Nos. 5/6, 879-908 (2001).

[45] R.P. Feynman, Application of Quantum Mechanics to Liquid Helium, p. 199 in: *Progress in Low Temperature Physics, Vol. 1, Ch. 2. ed. J. Gorter*, North-Holland, Amsterdam (1955).

[46] R.P. Feynman and M. Cohen, Energy spectrum of the Excitations in Liquid Helium, *Phys. Rev.* **102**:1189 (1956).

[47] E.H. Lieb, The Bose fluid, in: *Lectures in Theoretical Physics, Vol. VII C, ed. W.E. Briffin*, University of Colorado, Boulder (1965).

[48] I.M. Khalatnikov, An Introduction to the Theory of Superfluidity, *Benjamin-Reading*, New York (1965).

[49] J. Wilks, An introduction to liquid helium, *Claredon*, Oxford (1970).

[50] V.N. Popov, Functional Integrals and Collective Excitations, *Cambridge Univ. Press*, Cambridge (1987).

[51] J.C. Slater and J.G. Kirkwood, The Van der Waals Forces in Gases, *Phys. Rev.* **37**:682 (1931).

[52] O. Penrose, On the Quantum Mechanics of Helium II, *Phil. Mag.* **42**:1373 (1951).

[53] O. Penrose and L. Onsager, Bose-Einstein condensation and Liquid Helium, *Phys. Rev.* **104**:576 (1956).

[54] P. Kapitza, *Nature* **141**:74 (1938) .

[55] J.F. Allen and A.D. Misener, *Nature* **141**:75 (1938).

[56] L. Aleksandrov, V.A. Zagrebnov, Zh.A. Kozlov, V.A. Parfenov and V.B. Priezzhev, High energy neutron scattering and the Bose condensate in He II, *Sov. Phys.-JETP* **41:**915 (1975).

[57] E.V. Dokukin, Zh.K. Kozlov, V.A. Parfenov and A.V. Puchkev, Investigation of the temperature dependence of the density of the Bose condensate in helium-4 in connection with the superfluidity phenomenon, Sov. *Phys.-JETP* **48**:1146 (1978).

[58] N.M. Blagoveshchenskii, I.V. Bogoyavlenskii, L.V. Karnatsevich, V.G. Kolobrodov, Zh.A. Kozlov, V.B. Priezzhev, A.V. Puchkov, A.N. Skomorokhov and V.S. Yarunin, Structure of the excitation spectrum of liquid ^{4}He, *Phys. Rev. B* **50**:16550 (1994).

[59] E.H. Lieb, R. Seiringer and J. Yngvason, Superfluidity in dilute trapped Bose gases, *Phys. Rev. B* **66**:134529 (2002).

[60] D. D. Osheroff, R. C. Richardson, and D. M. Lee, Evidence for a New Phase of Solid He-3, *Phys. Rev. Lett.* **28**:885 (1972).

[61] D. D. Osheroff, W. J. Gully, R. C. Richardson, and D. M. Lee, New Magnetic Phenomena in Liquid He-3 below 3 mK, *Phys. Rev. Lett.* **29**:920 (1972).

[62] C. Kittel, Quantum Theory of Solids, *John Wiley and Sons Inc.*, New York (1963).

[63] N.N. Bogoliubov, V.V. Tolmachev and D.V. Shirkov, A New Method in the Theory of Superconductivity, *Consultants Bureau Inc.*, New York (1959).

[64] A.J. Leggett, A theoretical description of the new phases of liquid 3He, *Rev. Mod. Phys.* **47**:331 (1975).

[65] D. Vollhardt and P. Wölfle, The Superfluid Phases of Helium 3, *Taylor and Francis*, London (1990).

[66] M. Fannes and A. Verbeure, Correlation Inequalities and Equilibrium States I, *Commun. Math. Phys.* **55**:125 (1977).

[67] M. Fannes and A. Verbeure, Correlation Inequalities and Equilibrium States II, *Commun. Math. Phys.* **57**:165 (1977).