New Scaling Limit for Fuzzy Spheres

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Using a new scaling limit as well as a new cut-off procedure , we show that ϕ^4 theory on noncommutative $\mathbf{R}^4$ can be obtained from the corresponding theory on fuzzy $\mathbf{S}^2 \times \mathbf{S}^2$. The star-product on this noncommutative $\mathbf{R}^4$ is effectively local in the sense that the theory naturally has an ultra-violet cut-off Λ which is inversely proportional to the noncommutativity θ , i.e $\Lambda = \frac{2}{\theta}$. We show that the UV-IR mixing in this case is absent to one loop in the $2-$point function and also comment on the $4-$point function.

I. INTRODUCTION

Noncommutative field theories provide a rich variety of interesting conceptual phenomena. The usual focus of research has been to try to understand the quantum behavior of theories defined on certain non-compact manifolds like $\mathbf{R}^{2n}$ or compact ones like the fuzzy sphere $\mathbf{S}^2_F$, $\mathbf{CP}^n$ etc . Theories on $\mathbf{R}^{2n}_\theta$ have received considerable attention not only because they arise in string theory, but also because of their peculiar formal properties like UV-IR mixing [1] . Theories on compact noncommutative manifolds possess the attractive feature that they are simply finite-dimensional matrix models, and thus hold out the hope that they correspond to regularized versions of quantum field theories on ordinary manifolds. However, problems like UV-IR mixing are still present, although in a regularized form which is different from their noncompact counterparts [2, 3]. The fuzzy sphere $\mathbf{S}^2_F$ can be "flattened" (by scaling the radius R and the cut-off l to infinity) to give the noncommutative plane, and the UV-IR mixing re-emerges in its usual singular form [4] .

Interestingly, as we show in this letter, if we start with a low energy sector (which we will define more precisely below) of the theory on $\mathbf{S}^2_F \times \mathbf{S}^2_F$, and simultaneously use a new scaling to flatten the spheres, we obtain a theory on $\mathbf{R}^4_\theta$ that has ultra-violet cut-off $\Lambda = \frac{2}{\theta}$. Moreover, this theory shows no singular UV-IR mixing, as the noncommutative parameter θ and the UV cut-off are intimately related.

A popular strategy for studying noncommutative theories is to use a version of the star-product that is appropriate in near $\theta = 0$ [5]. This θ-expanded theory has many puzzling features (as emphasized recently in [6]), which can essentially be traced to the fact that the star-product has excellent smoothing properties (i.e. it nicely gets rid of all modes of characteristic length $L << \theta$), whereas the θ-expanded product does not cut-off high frequency modes. In this article, we propose a resolution of these puzzles by defining noncommutative theories on $\mathbf{R}^2$ or $\mathbf{R}^4$ as certain scaling limits of corresponding theories on $\mathbf{S}^2_F$ and $\mathbf{S}^2_F \times \mathbf{S}^2_F$. In the process, we also achieve a discretization of noncommutative theories that allows numerical simulations to study field-theoretic phenomena. Instead of approximating theories on nc $\mathbf{R}^2$ or $\mathbf{R}^4$ by the usual commutative lattice (but changing the rule for multiplying functions), a more natural regularization that retains the memory of underlying spacetime symmetries is to use finite matrix models on fuzzy sphere $\mathbf{S}^2_F$ and $\mathbf{S}^2_F \times \mathbf{S}^2_F$. However , and as we will show , in order that the star product on $\mathbf{R}^4$ and $\mathbf{R}^2$ has the property of cutting-off top modes of the order of $1/\theta$ and higher , as it should be , the scaling limit has to be defined precisely. It is worth noting here that in this re-definition of the Moyal plane as a sequence of fuzzy spheres , the resulting cut-off is in fact non-trivial, i.e it is a consequence of the underlying star product on $\mathbf{S}^2_F$ and thus it cannot be simply introduced into the theory . We now explain all this in some detail for the case of $\mathbf{S}^2_F$ and then in the next sections we derive explicitly all these results for the case of $\mathbf{S}^2_F \times \mathbf{S}^2_F$.

The fuzzy sphere is described by three matrices $x^F_i = \theta L_i$ where L_i's are the generators of $SU(2)$ for the spin l representation and θ has dimension of length. The radius R of the sphere is related to θ and l as $R^2 = \theta^2 l(l+1)$.

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The usual action for a matrix model on $\mathbf{S}_F^2$ is

$$S = \frac{R^2}{2l+1}\text{Tr}\ \left(\frac{[L_i, \Phi]^\dagger[L_i, \Phi]}{R^2} + \mu_l^2\Phi^2 + V[\Phi]\right),\tag{1}$$

and has the right continuum limit as $l \to \infty$. Because of the noncommutative nature of $\mathbf{S}_F^2$, there is a natural ultra-violet (UV) cut-off: the maximum energy Λ is $= 2l(2l+1)/R^2$. To get the theory on a noncommutative plane, the usual strategy [4] is to restrict to the neighbourhood of (say) the north pole (where $L_3 \simeq l$) , define the noncommutative coordinates as $x_a^{NC} \equiv x_a^F$, $a = 1, 2$, and then take both l and R to infinity with $\theta' = \frac{R}{\sqrt{l}}$ fixed, giving us the commutation relations

$$[x_1^{NC}, x_2^{NC}] = -i\theta'^2.\tag{2}$$

In this limit Λ clearly diverges .

Here , we point out another scaling limit in which $R, l \longrightarrow \infty$ while keeping $\theta \sim R/l$ fixed . The above noncommutativity relation becomes simply

$$[x_1^{NC}, x_2^{NC}] = -iR\theta\tag{3}$$

which means that x_a^{NC}'s are now strongly noncommuting coordinates ($R \to \infty$) and hence nonplanar amplitudes are expected to simply drop out in accordance with [1]. This can also be seen from the fact that in this scaling (as is obvious from the relation $R^2 = \theta^2 l(l+1)$) Λ no longer diverges: it is now of order $1/\theta$, and there are no momentum modes in the theory larger then this value. This mode expansion thus seems appropriate as a starting point for studying (or even defining perturbatively) noncommutative field theories.

Alternatively we will also show that in this limit the noncommutative coordinates can be instead identified as $X_a^{NC} = x_a^{NC}/\sqrt{l}$ with noncommutative structure

$$[X_1^{NC}, X_2^{NC}] = -i\theta^2.\tag{4}$$

While this scaling for obtaining $\mathbb{R}_\theta^2$ is simply stated, obtaining the corresponding theory with the above criteria is somewhat subtle. Indeed passing from $[x_1^{NC}, x_2^{NC}] = -iR\theta$ to (??) corresponds in the quantum theory to a re-scaling of momenta sending thus the finite cut-off $\Lambda = \frac{2}{\theta}$ to infinity. In order to bring the cut-off back to a finite value $\Lambda_x = x\Lambda$, where x is an arbitrary positive real number, we modify the Laplacian on the fuzzy sphere $\Delta = [L_i, [L_i, ..]]$ so that to project out modes of momentum greater than a certain value j given by $j = [\frac{2\sqrt{l}}{x}]$. In other words, the theory on the noncommutative plane $\mathbb{R}_\theta^2$ with UV cut-off θ^{-1} is obtained by flattening not the full theory on the fuzzy sphere but only a "low energy" sector. One can argue that only for when $\Lambda_x = \Lambda$ that the canonical UV-IR singularities become smoothen out. At this value we have $j = [2\sqrt{l}]$ which marks somehow the boundary between commutative and noncommutative field theories. This means that this scaling limit as defined above gives a theory on $\mathbf{R}_\theta^2$ where θ has now the physical interpretation of a UV cut-off .

II.   FUZZY SCALAR THEORY

The generalization to noncommutative $\mathbf{R}^4$ is obvious: we work on $\mathbf{S}_F^2 \times \mathbf{S}_F^2$ and then take the scaling limit with θ fixed, which is the case of most interest to us in this article. By analogy with (1), the scalar theory with quartic self-interaction on $\mathbf{S}_F^2 \times \mathbf{S}_F^2$ is

$$S = \frac{R_a^2}{2l_a+1}\frac{R_b^2}{2l_b+1}Tr_aTr_b\left(\frac{[L_i^{(a)}, \Phi]^\dagger[L_i^{(a)}, \Phi]}{R_a^2} + \frac{[L_i^{(b)}, \Phi]^\dagger[L_i^{(b)}, \Phi]}{R_b^2} + \mu_l^2\Phi^2 + \frac{\lambda_4}{4!}\Phi^4\right),\tag{5}$$

where a and b label the first and the second sphere respectively, and $L_i^{(a,b)}$'s are the generators of rotation in spin $l_{a,b}$-dimensional representation of $SU(2)$, and Φ is a $(2l_a + 1) \times (2l_a + 1) \otimes (2l_b + 1) \times (2l_b + 1)$ hermitian matrix. As l_a, l_b go to infinity, we recover the scalar theory on an ordinary $\mathbf{S}^2 \times \mathbf{S}^2$. One can argue that it is enough to set $l_a = l_b = l$ and $R_a = R_b = R$ which corresponds in the limit to a nc $\mathbf{R}^4$ with a trivial $\mathbf{R}^2 \times \mathbf{R}^2$ metric. The general case would only correspond to different deformation parameters in the two $\mathbf{R}^2$'s and the extension of all results is therefore obvious [8] .

Following [2, 7] the fuzzy field Φ can be expanded in terms of polarization operators [9] as follows

$$\Phi = (2l+1) \sum_{k_1=0}^{2l} \sum_{m_1=-k_1}^{k_1} \sum_{p_1=0}^{2l} \sum_{n_1=-p_1}^{p_1} \Phi^{k_1 m_1 p_1 n_1} T_{k_1 m_1}(l) \otimes T_{p_1 n_1}(l).$$

Imposing reality, i.e $\hat{\Phi}^+ = \hat{\Phi}$, we obtain the following conditions $\bar{\phi}^{k_1 m_1 p_1 n_1} = (-1)^{m_1+n_1} \phi^{k_1 -m_1 p_1 -n_1}$, and a canonical path integral quantization will therefore yield the propagator

$$< \phi^{k_1 m_1 p_1 n_1} \phi^{k_2 m_2 p_2 n_2} > = \frac{(-1)^{m_2+n_2}}{R^2} \frac{\delta^{k_1 k_2} \delta^{m_1,-m_2} \delta^{p_1 p_2} \delta^{n_1,-n_2}}{k_1(k_1+1) + p_1(p_1+1) + R^2 \mu_l^2}.$$

The Euclidean 4−momentum in this setting is given by $11 \equiv (k_1, m_1, p_1, n_1)$ with square $(11)^2 = k_1(k_1+1) + p_1(p_1+1)$. The vertex is given however by

$$S_l^{int} = \sum_{11} \sum_{22} \sum_{33} \sum_{44} V(11, 22, 33, 44) \phi^{11} \phi^{22} \phi^{33} \phi^{44}$$

$$V(11, 22, 33, 44) = R^4 \frac{\lambda_4}{4!} V_1(1234, km) V_2(1234, pn),$$

where

$$V_1(1234, km) = (2l+1) Tr_{H_1} \left[T_{k_1 m_1}(l) ... T_{k_4 m_4}(l) \right],$$

and a similar definition for $V_2(1234, pn)$.

Using standard perturbation theory, the one-loop correction to the 2−point function [13] is

$$\mu_l^2(k_1, p_1) = \mu_l^2 + \frac{1}{R^2} \frac{\lambda_4}{4!} \left[\delta\mu_l^P + \delta\mu_l^{NP}(k_1, p_1) \right] \tag{6}$$

with the planar contribution given by

$$\delta\mu_l^P = 4 \sum_{a=0}^{2l} \sum_{b=0}^{2l} A(a, b)$$

$$A(a, b) = \frac{(2a+1)(2b+1)}{a(a+1) + b(b+1) + R^2 \mu_l^2}, \tag{7}$$

whereas the non-planar contribution is given by

$$\delta\mu_l^{NP}(k_1, p_1) = 2 \sum_{a=0}^{2l} \sum_{b=0}^{2l} A(a, b)(-1)^{k_1+p_1+a+b} B_{k_1 p_1}(a, b)$$

$$B_{ab}(c, d) = (2l+1)^2 \left\{ \begin{matrix} a & l & l \\ c & l & l \end{matrix} \right\} \left\{ \begin{matrix} b & l & l \\ d & l & l \end{matrix} \right\}. \tag{8}$$

The symbol $\{\}$ in $B_{ab}(c, d)$ is of course the standard $6j$ symbol [9]. As one can immediately see from the analytic expressions above, both planar and non-planar graphs are empty from singularities, everything is finite and well defined for all finite values of l [14]. Indeed a measure for the fuzzy UV-IR mixing or the noncommutative anomaly will be the differences Δ between planar and non-planar contributions which can be defined by the equation

$$\delta\mu_l^P + \delta\mu_l^{NP}(k_1, p_1) = \Delta\mu_l^P + \frac{1}{2}\Delta(k_1, p_1)$$

$$\Delta\mu_l^P = 6 \sum_{a=0}^{2l} \sum_{b=0}^{2l} A(a, b),$$

where

$$\Delta(k_1, p_1) = 4 \sum_{a=0}^{2l} \sum_{b=0}^{2l} A(a, b) \left[(-1)^{k_1+p_1+a+b} B_{k_1 p_1}(a, b) - 1 \right]. \tag{9}$$

The fact that this difference is not zero in the limit of infinite points density , i.e $l \longrightarrow \infty$, is what is meant by UV-IR mixing on fuzzy spaces . (9) can also be taken as the regularized form of the UV-IR mixing on $\mathbf{R}^4$. Removing the UV cut-off $l \longrightarrow \infty$ while keeping the infrared cut-off R fixed $= 1$ one can show that Δ diverges as l^2 , i.e

$$\Delta(k_1, p_1) \; \longrightarrow \; (8l^2) \int_{-1}^{1} \int_{-1}^{1} \frac{dt_x dt_y}{2 - t_x - t_y} \Big[P_{k_1}(t_x) P_{p_1}(t_y) - 1 \Big], \tag{10}$$

where , for simplicity , we have assumed $\mu_l << l$ [4]. (10) is worse than the case of two dimensions [see equation (3.20) of [4]] , in here not only the difference survives the limit but also it diverges . This means in particular that the UV-IR mixing can be largely controlled or perhaps understood if one understands the role of the UV cut-off l in the scaling limit and its relation to the underlying star product on $\mathbf{S}_F^2$.

The computation of higher order correlation functions becomes very complicated, but for completeness we also write down the result for the $4-$point function . We get

$$\delta\lambda_4(1235) \; = \; \frac{\lambda_4}{4!} \sum_{k_4, k_6} \sum_{p_4, p_6} \frac{A(k_4, p_4) A(k_6, p_6)}{(2k_4 + 1)(2k_6 + 1)(2p_4 + 1)(2p_6 + 1)} \Big[8\eta_1^{(1)} \eta_2^{(1)}$$
$$+ \; 16\eta_1^{(2)} \eta_2^{(2)} + 4\eta_1^{(3)} \eta_2^{(3)} + 8\eta_1^{(4)} \eta_2^{(4)} \Big]. \tag{11}$$

The first graph in (11) is the usual one-loop contribution to the $4-$point function , i.e the two vertices are planar. The fourth graph contains also two planar vertices but with the exception that one of these vertices is twisted , i.e with an extra phase. The second graph contains on the other hand one planar vertex and one non-planar vertex, whereas the two vertices in the third graph are both non-planar. The analytic expressions for $\eta_i^{(a)} \equiv \eta_i^{(a)}(k_4 k_6; 1235) = \sum_{m_4=-k_4}^{k_4} \sum_{m_6=-k_6}^{k_6} \rho_i^{(a)}(k_4 k_6; 1235)$ are given by

$$\rho_i^{(1)} \; = \; (-1)^{m_4+m_6} V_i(\hat{1}\hat{2}4_f 6_f) V_i(\hat{3}\hat{5} - 4_f - 6_f), \quad \rho_i^{(2)} = (-1)^{m_4+m_6} V_i(\hat{1}\hat{2}4_f 6_f) V_i(\hat{3} - 4_f \hat{5} - 6_f)$$
$$\rho_i^{(3)} \; = \; (-1)^{m_4+m_6} V_i(\hat{1}4_f \hat{2}6_f) V_i(\hat{3} - 4_f \hat{5} - 6_f), \quad \rho_i^{(4)} = (-1)^{m_4+m_6} V_i(\hat{1}\hat{2}4_f 6_f) V_i(\hat{3}\hat{5} - 6_f - 4_f),$$

where the lower index in η's and ρ's labels the sphere whereas the upper index denotes the graph, and the notation $-4_f 4_f$ stands for $(k_4, -m_4, p_4, -n_4)$ in contrast with $4_f 4_f = (k_4, m_4, p_4, n_4)$.

By using extensively the different identities in [9] we can find after a long calculation that the above 4-point function has the form

$$\delta\lambda_4(1235) \; = \; \frac{\lambda_4}{4!} \Big[8\delta\lambda_4^{(1)}(1235) + 16\delta\lambda_4^{(2)}(1235) + 4\delta\lambda_4^{(3)}(1235) + 8\delta\lambda_4^{(4)}(1235) \Big], \quad \text{where}$$
$$\delta\lambda_4^{(a)}(1235) \; = \; \sum_{k_4, k_6} \sum_{p_4, p_6} A(k_4, p_4) A(k_6, p_6) \nu_1^{(a)}(k_4 k_6; 1235) \nu_2^{(a)}(p_4 p_6; 1235), \quad a = 1 \ldots 4, \tag{12}$$

The label f stands for the shells we integrated over and hence it corresponds to $q^2 = (2l+1)^2$ for the full one-loop contribution. The planar amplitudes, in the first $\mathbb{R}^2$ factor for example, are given by

$$\nu_1^{(1)} = \sum_k (-1)^{k+k_4+k_6} \delta_k(1235) E_{k_1 k_2}^{k_4 k_6}(k) E_{k_3 k_5}^{k_4 k_6}(k), \quad \nu_1^{(4)} = \sum_k \delta_k(1235) E_{k_1 k_2}^{k_4 k_6}(k) E_{k_3 k_5}^{k_4 k_6}(k) \tag{13}$$

whereas the non-planar amplitudes are given by

$$\nu_1^{(2)} = \sum_k (-1)^{k_3+k_4} \delta_k(1235) E_{k_1 k_2}^{k_4 k_6}(k) F_{k_3 k_5}^{k_4 k_6}(k), \quad \nu_1^{(3)} = \sum_k (-1)^{k_2+k_3} \delta_k(1235) F_{k_1 k_2}^{k_6 k_4}(k) F_{k_3 k_5}^{k_4 k_6}(k) \tag{14}$$

with

$$F_{k_1 k_2}^{k_4 k_6}(k) \; = \; (2l+1)\sqrt{(2k_1+1)(2k_2+1)} \left\{ \begin{matrix} k_4 & l & l \\ k_6 & l & l \\ k & k_1 & k_2 \end{matrix} \right\}$$
$$E_{k_1 k_2}^{k_4 k_6}(k) \; = \; (2l+1)\sqrt{(2k_1+1)(2k_2+1)} \left\{ \begin{matrix} k_1 & k_2 & k \\ l & l & l \end{matrix} \right\} \left\{ \begin{matrix} k_4 & k_6 & k \\ l & l & l \end{matrix} \right\}. \tag{15}$$

The "fuzzy delta" function $\delta_k(1235)$ is defined by

$$\delta_k(1235) = (-1)^m C_{k_1 m_1 k_2 m_2}^{km} C_{k_3 m_3 k_5 m_5}^{k-m}. \tag{16}$$

III. CONTINUUM PLANAR LIMIT

We can now state with some detail the continuum limits in which the fuzzy spheres approach (in a precise sense) the noncommutative planes. There are primarily two limits of interest to us: one is the canonical large stereographic projection of the spheres onto planes, while the second is a new flattening limit which we will argue corresponds to a conventional cut-off.

For simplicity, consider a single fuzzy sphere with cut-off l and radius R, and define the fuzzy coordinates $x_i^F = \theta L_i$ (i.e. $x_\pm^F = x_1^F \pm i x_2^F$) where $\theta = R/\sqrt{l(l+1)}$. The stereographic projection onto the noncommutative plane is realized as

$$y_+^F = 2R x_+^F \frac{1}{R - x_3^F}, \quad y_-^F = 2R \frac{1}{R - x_3^F} x_-^F. \tag{17}$$

In the large l limit it is obvious that these fuzzy coordinates indeed approach the canonical stereographic coordinates. A planar limit can be defined from above as follows:

$$\theta'^2 = \frac{R^2}{\sqrt{l(l+1)}} = \text{fixed as} \quad l, R \to \infty. \tag{18}$$

In this limit, the commutation relation becomes

$$[y_+^{NC}, y_-^{NC}] = -2\theta'^2, \quad y_\pm^{NC} \equiv y_\pm^F = x_\pm^F, \tag{19}$$

where we have substituted $L_3 = -l$ corresponding to the north pole. The above commutation relation may also be put in the form

$$[x_1^{NC}, x_2^{NC}] = -i\theta'^2, \quad x_a^{NC} \equiv x_a^F, \quad a = 1, 2 \tag{20}$$

The minus sign is simply due to our convention for the coherent states on co-adjoint orbits. The extension to the case of two fuzzy spheres is trivial.

A second way to obtain the noncommutative plane is by taking the limit

$$\theta = \frac{R}{\sqrt{l(l+1)}} = \quad l, R \to \infty. \tag{21}$$

A UV cut-off is automatically built into this limit: the maximum energy a scalar mode can have on the fuzzy sphere is $2l(2l+1)/R^2$, which in this scaling limit is $4/\theta^2$. There are no modes with energy larger than this value. To understand this limit a little better, let us restrict ourselves to the north pole $\vec{n} = \vec{n}_0 = (0, 0, 1)$ where we have $\langle \vec{n}_0, l | L_3 | \vec{n}_0, l \rangle = -l$ and $\langle \vec{n}_0, l | L_a | \vec{n}_0, l \rangle = 0$, $a = 1, 2$. The commutator $[L_1, L_2] = iL_3 = -il$, so the noncommutative coordinates on this noncommutative plane "tangential to the north pole" can be given either simply by x_a^F as above. This now defines a strongly noncommuting plane, viz

$$[x_a^F, x_b^F] = -il\theta^2 \epsilon_{ab}. \tag{22}$$

Or aletrnatively one can define the noncommutative coordinate by $X_a^{NC} \equiv \sqrt{\frac{\theta}{R}} x_a^F$, satisfying

$$[X_a^{NC}, X_b^{NC}] = -i\theta^2 \epsilon_{ab}. \tag{23}$$

In the convention used here, $\epsilon_{12} = 1$ and $\epsilon_{ac}\epsilon_{cb} = -\delta_{ab}$.

Intuitively, the second scaling limit may be understood as follows. Noncommutativity introduces a short distance cut-off of the order $\delta X = \sqrt{\frac{\theta^2}{2}}$ because of the uncertainty relation $\Delta X_1^{NC} \Delta X_2^{NC} \geq \frac{\theta^2}{2}$. However, the Laplacian operators on generic noncommutative planes do not reflect this short distance cut-off, as they are generally taken to be the same as the commutative Laplacians. On the above noncommutative plane (23) the cut-off δX effectively translates into the momentum space as some cut-off $\delta P = \frac{1}{\sqrt{2\theta^2}}$. This is because of (and in accordance with) the commutation relations $[X_a^{NC}, P_b^{NC}] = i\delta_{ab}, P_a^{NC} = -\frac{1}{\theta^2}\epsilon_{ab} X_b^{NC}$, giving us the uncertainty relations $\Delta X_a^{NC} \Delta P_b^{NC} \geq \frac{\delta_{ab}}{2}$. Since one can not probe distances less than δX, energies above δP should not be accessible either, i.e. $[P_a^{NC}, P_b^{NC}] = -\frac{i}{\theta^2}\epsilon_{ab}$. The fact that the maximum energy of a mode is of order $1/\theta$ in the second scaling limit ties in nicely with this expectation.

The limit (21) may thus be thought of as a regularization prescription of the noncommutative plane which takes into account our expectation of "UV-finiteness" of noncommutative quantum field theories.

A. Field Theory in the Canonical Planar Limit

We are now in a position to study what happens to the scalar field theory in the limit (20). First we match the spectrum of the Laplacian operator on each sphere with the spectrum of the Laplacian operator on the limiting noncommutative plane as follows

$$a(a+1) = R^2 p_a^2, \tag{24}$$

where p_a is of course the modulus of the two dimensional momentum on the noncommutative plane which corresponds to the integer a, and has the correct mass dimension. However since the range of a's is from 0 to $2l$, the range of p_a^2 will be from 0 to $\frac{2l(2l+1)}{R^2} = l\Lambda'^2 \to \infty$, $\Lambda' = 2/\theta'$. In other words, all information about the UV cut-off is lost in this limit.

Let us see how the other operators in the theory scales in the above planar limit. It is not difficult to show that the free action scales as

$$\sum_{a,b} \sum_{m_a,m_b} \left[R^2 a(a+1) + R^2 b(b+1) + R^4 \mu_l^2 \right] |\phi^{abm_a m_b}|^2 \simeq \int_{\sqrt{l}\Lambda'} \frac{d^2\vec{p}_a d^2\vec{p}_b}{\pi^2} \left[p_a^2 + p_b^2 + M^2 \right] |\phi_{NC}^{p_a p_b \phi_a \phi_b}|^2. \tag{25}$$

The scalar field is assumed to have the scaling property $\phi_{NC}^{p_a p_b \phi_a \phi_b} \simeq R^4 \phi^{abm_a m_b}$, which gives the momentum-space scalar field the correct mass dimension of -3 [recall that $[\phi^{abm_a m_b}] = M$]. The ϕ_a and ϕ_b above (not to be confused with the scalar field!) are the angles of the two momenta $\vec{p}_a$ and $\vec{p}_b$ respectively, i.e. $\phi_a = \frac{\pi m_a}{a+\frac{1}{2}}$ and $\phi_b = \frac{\pi m_b}{b+\frac{1}{2}}$. This formula is exact, and can be simplified further when quantum numbers a's and b's are large: the ϕ_a and ϕ_b will be in the range $[-\pi, \pi]$. It is also worth pointing out that the mass parameter M of the planar theory is exactly equal to that on the fuzzy spheres, i.e. $M = \mu_\infty$, and no scaling is required.

With these ingredients, it is not then difficult to see that the flattening limit of the planar 2−point function (7) is given by

$$\delta M^P \equiv \frac{\delta \mu_l^P}{R^2} = 16 \int \int \frac{p_a p_b dp_a dp_b}{p_a^2 + p_b^2 + M^2} \tag{26}$$

which is the 2-point function on noncommutative $\mathbb{R}^4$ with a Euclidean metric $\mathbb{R}^2 \times \mathbb{R}^2$. By rotational invariance it may be rewritten as

$$\delta M^P = \frac{4}{\pi^2} \int_{\sqrt{l}\Lambda'} \frac{d^4 k}{k^2 + M^2}. \tag{27}$$

We do now the same exercise for the non-planar 2-point function (8). Since the external momenta k_1 and p_1 are generally very small compared to l, one can use the following approximation for the $6j$-symbols [9]

$$\left\{ \begin{matrix} a & l & l \\ b & l & l \end{matrix} \right\} \approx \frac{(-1)^{a+b}}{2l} P_a \left(1 - \frac{b^2}{2l^2}\right), \quad l \to \infty, \ a << l, \ 0 \le b \le 2l, \tag{28}$$

By putting in all the ingredients of the planar limit we obtain the result

$$\delta M^{NP}(k_1, p_1) \equiv \frac{\delta \mu_l^{NP}}{R^2} = 8 \int_0^\infty \int_0^\infty \frac{p_a p_b dp_a dp_b}{p_a^2 + p_b^2 + M^2} P_{k_1} \left(1 - \frac{\theta'^4 p_a^2}{2R^2}\right) P_{p_1} \left(1 - \frac{\theta'^4 p_b^2}{2R^2}\right).$$

Although the quantum numbers k_1 and p_1 in this limit are very small compared to l, they are large themselves i.e. $1 << k_1, p_1 << l$. On the other hand, the angles ν_a defined by $\cos \nu_a = 1 - \frac{\theta'^4 p_a^2}{2R^2}$ can be considered for all practical purposes small, i.e. $\nu_a = \frac{\theta'^2 p_a}{R}$ because of the large R factor, and hence we can use the formula (see for eg [15], page 72)

$$P_n(\cos \nu_a) = J_0(\eta) + \sin^2 \frac{\nu_a}{2} \left[\frac{J_1(\eta)}{2\eta} - J_2(\eta) + \frac{\eta}{6} J_3(\eta) \right] + O(\sin^4 \frac{\nu_a}{2}), \tag{29}$$

for $n >> 1$ and small angles ν_a, with $\eta = (2n+1) \sin \frac{\nu_a}{2}$. To leading order we then have

$$P_{k_1} \left(1 - \frac{\theta'^4 p_a^2}{2R^2}\right) = J_0(\theta'^2 p_{k_1} p_a) = \frac{1}{2\pi} \int_0^{2\pi} d\phi_a e^{i\theta'^2 \cos \phi_a p_{k_1} p_a}.$$

This result becomes exact in the strict limit of $l, R \to \infty$ where all fuzzy quantum numbers diverge with R. We get then

$$\delta M^{NP}(p_{k_1}, p_{p_1}) = \frac{2}{\pi^2} \int \int \int \int \frac{(p_a dp_a d\phi_a)(p_b dp_b d\phi_b)}{p_a^2 + p_b^2 + M^2} e^{i\theta'^2 p_{k_1}(p_a \cos \phi_a)} e^{i\theta'^2 p_{p_1}(p_b \cos \phi_b)}.$$

By rotational invariance we can set $\theta'^2 B^{\mu\nu} p_{k_1\mu} p_{a\nu} = \theta'^2 p_{k_1}(p_a \cos \phi_a)$, where $B^{12} = -1$. In other words, we can always choose the two-dimensional momentum p_{k_1} to lie in the y-direction, thus making ϕ_a the angle between $\vec{p}_a$ and the x-axis. The same is also true for the other exponential. We thus obtain the canonical non-planar 2-point function on the noncommutative $\mathbb{R}^4$ (with Euclidean metric $\mathbb{R}^2 \times \mathbb{R}^2$). Again by rotational invariance, this non-planar contribution to the 2-point function may be put in the compact form

$$\delta M^{NP}(p) = \frac{2}{\pi^2} \int_{\sqrt{l}\Lambda'} \frac{d^4k}{k^2 + M^2} e^{i\theta'^2 pBk}. \tag{30}$$

The structure of the effective action in momentum space allows us to deduce the star products on the underlying noncommutative space. For example, by using the tree level action (25) together with the one-loop contributions (27) and (30) one can find that the effective action obtained in the large stereographic limit (18) is given by

$$\int_{\sqrt{l}\Lambda'} \frac{d^4\vec{p}}{(2\pi)^4} \frac{1}{2} \left[\vec{p}^2 + M^2 + \frac{g_4^2}{6} \left[2 \int_{\sqrt{l}\Lambda'} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + M^2} + \int_{\sqrt{l}\Lambda'} \frac{d^4k}{(2\pi)^4} \frac{e^{i\theta'^2 \vec{p}B\vec{k}}}{k^2 + M^2} \right] \right] |\phi_1(\vec{p})|^2 \tag{31}$$

where $g_4^2 = 8\pi^2 \lambda_4$ and $\phi_1(\vec{p}) = 4\pi\sqrt{2} \phi_{NC}^{p_a p_b \phi_a \phi_b}$ and $\sqrt{l}\Lambda' \to \infty$. This effective action can be obtained from the quantization of the action

$$\int d^4x \left[\frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2} M^2 \phi_1^2 + \frac{g_4^2}{4!} \phi_1 *' \phi_1 *' \phi_1 *' \phi_1 \right],$$

where $\phi_1 \equiv \phi_1(x^{NC}) = \int \frac{d^4p}{(2\pi)^4} \phi_1(\vec{p}) e^{-ipx^{NC}} = \phi_1^\dagger$ and $*'$ is the canonical (or Moyal-Weyl) star product

$$f *' g(x^{NC}) = e^{\frac{i}{2}\theta'^2 B^{\mu\nu} \partial_\mu^y \partial_\nu^z} f(y)g(z)|_{y=z=x^{NC}} \tag{32}$$

This is consistent with the commutation relation (20) and provides a nice check that that the canonical star product on the sphere derived in [3] (also given here by equation (??)) reduces in the limit (18) to the above Moyal-Weyl product (32). In the above, B is the antisymmetric tensor which can always be rotated such that the non vanishing components are given by $B^{12} = -B_{21} = -1$ and $B^{34} = -B_{43} = -1$.

In fact one can read immediately from the above effective action that the planar contribution is quadratically divergent as it should be, i.e.

$$\Delta M^P = \frac{1}{64\pi^2} \delta M^P = \int_{\sqrt{l}\Lambda' \to \infty} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + M^2} = \frac{1}{16\pi^2} l\Lambda'^2 \to \infty, \tag{33}$$

whereas the non-planar contribution is clearly finite

$$\Delta M^{NP}(p) = \frac{1}{32\pi^2} \delta M^{NP}(p) = \int_{\sqrt{l}\Lambda' \to \infty} \frac{d^4k}{(2\pi)^4} \frac{e^{i\theta'^2 \vec{p}B\vec{k}}}{k^2 + M^2}$$

$$= \frac{1}{8\pi^2} \left[\frac{2}{E^2 \theta'^4} + M^2 \ln(\theta'^2 EM) \right], \quad \text{where} \quad E^\nu = B^{\mu\nu} P_\mu. \tag{34}$$

This is the answer of [1]: it is singular at $P = 0$ as well as at $\theta' = 0$.

B. A New Planar Limit With Strong Noncommutativity

As explained earlier, the limit (21) possesses the attractive feature that a momentum space cut-off is naturally built into it. In addition to obtaining a noncommutative plane in the strict limit, UV-IR mixing is completely absent. But while the new scaling is simply stated, obtaining the corresponding field theory is somewhat subtle. We will need to modify the Laplacian on the fuzzy sphere to project our modes with momentum greater than $2\sqrt{l}$. In other words,

the noncommutative theory on a plane with UV cut-off θ is obtained not by flattening the full theory on the fuzzy sphere, but only a "low energy" sector, corresponding to momenta upto $2\sqrt{l}$.

In order to clarify the chain of arguments, we will first implement naively the limit (21) and show that it corresponds to a strongly noncommuting plane . Finite noncommuting plane is only obtainable if we pick a specific low energy sector of the fuzzy sphere before taking the limit as we will explain in the next section.

Our rule for matching the spectrum on the fuzzy sphere with that on the noncommutative plane is the same as before, namely $a(a+1) = R^2 p_a^2$. However because of (21), the range of p_a^2 is now from 0 to $\frac{2l(2l+1)}{R^2} = \frac{4}{\theta^2}$. The kinetic part of the action will scale in the same way as in (25), only now the momenta $\vec{p}$'s in (25) are restricted such that $p \leq \Lambda$. With this scaling information, we can see that the planar contribution to the 2-point function is given by

$$\delta m^P \equiv \frac{\delta \mu_l^P}{R^2} = \frac{4}{\pi^2} \int_{k \leq \Lambda} \frac{d^4 k}{k^2 + \mu_l^2}, \quad \Lambda = \frac{2}{\theta}. \tag{35}$$

We can similarly compute the non-planar contribution to the 2-point function using (28). The motivation for using this approximation is more involved and can be explained as follows. In the planar limit $l, R \to \infty$, it is obvious that the relevant quantum numbers k_1 and p_1 are in fact much larger compared to 1, i.e. $k_1 \sim R p_{k_1} >> 1$ and $p_1 \sim R p_{p_1} >> 1$, since $R \simeq \theta l$. However (28) can be used only if $k_1, p_1 << l$, or equivalently $\frac{k_1}{l} = \frac{2p_{k_1}}{\Lambda} << 1$ and $\frac{p_1}{l} = \frac{2p_{p_1}}{\Lambda} << 1$. This is clearly true for small external momenta p_{k_1} and p_{p_1}, which is exactly the regime of interest in order to see if there is UV-IR mixing. The condition for the reliability of the approximation (28) is then $\theta p_{external} << 1$. We will sometimes refer to this condition as "θ small", the precise meaning of this phrase being "momentum scale of interest is much smaller than $1/\theta$". We thus obtain

$$\delta m^{NP}(k_1, p_1) \equiv \frac{\delta \mu_l^{NP}}{R^2} = 8 \int_0^\Lambda \int_0^\Lambda \frac{p_a p_b dp_a dp_b}{p_a^2 + p_b^2 + \mu_l^2} P_{k_1}(1 - \frac{\theta^2 p_a^2}{2}) P_{p_1}(1 - \frac{\theta^2 p_b^2}{2}). \tag{36}$$

Now the angles ν_a's of (29) are defined by $\cos \nu_a = 1 - \frac{\theta^2 p_a^2}{2}$, and since $\theta p << 1$, these angles are still small. They are therefore given to the leading order in θp by $\nu_a = \theta p_a + \cdots$ where the ellipsis indicate terms third order and higher in θp. By using (29) we again have

$$P_{R p_{k_1}}(1 - \frac{\theta^2 p_a^2}{2}) = J_0(R \theta p_{k_1} p_a) = \frac{1}{2\pi} \int_0^{2\pi} d\phi_a e^{iR\theta \cos \phi_a p_{k_1} p_a}. \tag{37}$$

Using rotational invariance we can rewrite this as

$$\delta m^{NP}(p) = \frac{2}{\pi^2} \int_{k \leq \Lambda} \frac{d^4 k}{k^2 + \mu_l^2} e^{iR\theta p B k}. \tag{38}$$

One immediate central remark is in order: the noncommutative phase contains now a factor $R\theta$ instead of the naively expected factor of θ^2. This is in contrast with the previous case of canonical planar limit, where the strength of the noncommutativity θ'^2 defined by the commutation relation (20) is exactly what appears in the noncommutative phase of (30). In other words this naive implementation of (21) yields in fact the strongly noncommuting plane (22) instead of (23). Also we can similarly to the previous case put together the tree level action (25) with the one-loop contributions (35) and (38) to obtain the effective action

$$\int_\Lambda \frac{d^4 \vec{p}}{(2\pi)^4} \frac{1}{2} \left[\vec{p}^2 + \mu_l^2 + \frac{g_4^2}{6} \left[2 \int_\Lambda \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + \mu_l^2} + \int_\Lambda \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + \mu_l^2} e^{iR\theta \vec{p} B \vec{k}} \right] \right] |\phi_3(\vec{p})|^2. \tag{39}$$

As before $g_4^2 = 8\pi^2 \lambda_4$, whereas $\phi_3(\vec{p}) = l^{3/2} \phi_2(\sqrt{l}\vec{p})$, $\phi_2(\vec{p}) = 4\pi \sqrt{\frac{2}{l^3}} \phi_{NC}(\frac{\vec{p}}{\sqrt{l}})$ with $\phi_{NC}(\vec{p}) \equiv \phi_{NC}^{p_a p_b \phi_a \phi_b} = R^4 \phi^{abm_a m_b}$ (in the metric $\mathbb{R}^2 \times \mathbb{R}^2$). It is not difficult to see that the one-loop contributions δm^P and $\delta m^{NP}(p)$ given in (35) and (38) can also be given by the equations

$$\bar{\Delta} m^P = \frac{l}{64\pi^2} \delta m^P = \int_{\sqrt{l}\Lambda \to \infty} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + l\mu_l^2}$$

$$\bar{\Delta} m^{NP}(p) = \frac{l}{32\pi^2} \delta m^{NP}(\frac{p}{\sqrt{l}}) = \int_{\sqrt{l}\Lambda \to \infty} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + l\mu_l^2} e^{i\theta^2 \vec{p} B \vec{k}}. \tag{40}$$

We have already computed that the leading terms in $\bar{\Delta} m^P$ and $\bar{\Delta} m^{NP}(p)$ are given by

$$\bar{\Delta} m^P = \frac{l}{16\pi^2} \left[\Lambda^2 - \mu_l^2 \ln(1 + \frac{\Lambda^2}{\mu_l^2}) \right], \quad \bar{\Delta} m^{NP}(p) = \frac{1}{8\pi^2} \left[\frac{2}{E^2 \theta^4} + l\mu_l^2 \ln(\theta^2 \sqrt{l} E \mu_l) \right], \quad \text{where} \quad E^\nu = B^{\mu\nu} p_\mu.$$

Obviously then we obtain

$$\delta m^P = 4\left[\Lambda^2 - \mu_l^2 ln(1 + \frac{\Lambda^2}{\mu_l^2})\right], \quad \delta m^{NP}(p) = 4\mu_l^2 \ln(l\theta^2 E\mu_l). \tag{41}$$

If we now require the mass μ_l in (25) to scale as $\mu_l^2 = \frac{m^2}{l}$ (the reason will be clear shortly), then one can deduce immediately that the planar contribution δm^P is exactly finite equal to $4\Lambda^2$, whereas the non-planar contribution $\delta m^{NP}(p)$ vanishes in the limit $l\rightarrow\infty$.

Remark finally that despite the presence of the cut-off Λ in the effective action (39), this effective action can still be obtained from quantizing

$$\int d^4x\left[\frac{1}{2}(\partial_\mu\phi_3)^2 + \frac{1}{2}\mu_l^2\phi_3^2 + \frac{g_4^2}{4!}\phi_3 * \phi_3 * \phi_3 * \phi_3\right], \tag{42}$$

only we have to regularize all integrals in the quantum theory with a cut-off $\Lambda = 2/\theta$. $[\phi_3 \equiv \phi_3(x^F) = \int \frac{d^4p}{(2\pi)^4}\phi_3(\vec{p})e^{-ipx^F} = \phi_3^\dagger$, and the star product $*$ is the Moyal-Weyl product given in (32) with the obvious substitution $\theta'\rightarrow R\theta$].

C. A New Planar Limit With Finite Noncommutativity

Neverthless, the action (39) can also be understood in some way as the effective action on the noncommutative plane (23) with finite noncommutativity equal to θ^2. Indeed by performing the rescaling $\vec{p}\rightarrow\frac{\vec{p}}{\sqrt{l}}$ we get

$$\int_{\sqrt{l}\Lambda} \frac{d^4\vec{p}}{(2\pi)^4}\frac{1}{2}\left[\vec{p}^2 + m^2 + \frac{g_4^2}{6}\left[2\int_{\sqrt{l}\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2} + \int_{\sqrt{l}\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}e^{i\theta^2\vec{p}B\vec{k}}\right]\right]|\phi_2(\vec{p})|^2. \tag{43}$$

We have already the correct noncommutativity θ^2 in the phase and the only thing which needs a new reintepretation is the fact that the cut-off is actually given by $\sqrt{l}\Lambda\rightarrow\infty$ and not by the finite cut-off Λ. [Remark that if we do not reduce the cut-off $\sqrt{l}\Lambda$ again to the finite value Λ, the physics of (43) is then essentially that of canonical noncommutativity, i.e the limit (21) together with the above rescaling of momenta is equivalent to the limit (18)].

Now having isolated the l-dependence in the range of momentum space integrals in the effective action (43), we can argue that it is not possible to get rid of this l-dependence merely by changing variables. Actually, to correctly reproduce the theory on the noncommutative $\mathbb{R}^4$ given by (21) and (23), we will now show that one must start with a modified Laplacian (or alternately propagator) on the fuzzy space [14]. For this, we replace the Laplacian $\Delta = [L_i^{(a)}, [L_i^{(a)}, ..]]$ on each fuzzy sphere which has the canonical obvious spectrum $k(k + 1)$, $k = 0, ..., 2l$, with the modified Laplacian

$$\Delta_j = \Delta + \frac{1}{\epsilon}(1 - P_j). \tag{44}$$

Here P_j is the projector on all the modes associated with the eigenvalues $k = 0, ..., j$, i.e.

$$P_j = \sum_{k=0}^{j}\sum_{m=-k}^{k}|k,m\rangle\langle k,m|,$$

The integer j thus acts as an intermediate scale, and using the modified propagator gives us a low energy sector of the full theroy. We will fix the integer j shortly.

With this modified Laplacian, modes with momenta larger than j do not propagate: as a result, they make no contribution in momentum sums that appear in internal loops. In other words, summations like $\sum_0^{2l}$ (which go over to integrals with range $\int_0^{\Lambda}$) now collapse to $\sum_0^{j}$ (where the integrals now are of the range $\int_0^{\Lambda_j}$, with $\Lambda_j = \frac{j}{2l}\Lambda$).

The new flattening limit is now defined as follows: start with the theory on $S_F^2 \times S_F^2$, but with the modified propagator (44). First take $\epsilon \rightarrow 0$, then $R, l \rightarrow \infty$ with $\theta = R/l$ fixed. This gives us the effective action (43) but with with momentum space cut-off $\sqrt{l}\Lambda_j = \frac{j}{2\sqrt{l}}\Lambda$,i.e

$$\int_{\sqrt{l}\Lambda_j} \frac{d^4\vec{p}}{(2\pi)^4}\frac{1}{2}\left[\vec{p}^2 + m^2 + \frac{g_4^2}{6}\left[2\int_{\sqrt{l}\Lambda_j}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2} + \int_{\sqrt{l}\Lambda_j}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}e^{i\theta^2\vec{p}B\vec{k}}\right]\right]|\phi_2(\vec{p})|^2 \tag{45}$$

This also tells us that the correct choice of the intermediate scale is $j = [2\sqrt{l}]$ for which $\sqrt{l}\Lambda_j = \Lambda$. For this value of the intermediate cut-off, we obtain the noncommutative $\mathbb{R}^4$ given by (21) and (23) .

By looking at the product of two functions of the fuzzy sphere, we can understand better the role of the intermediate scale $j(= [2\sqrt{l}])$. The fuzzy spherical harmonics $T_{l_a m_a}$ go over to the usual spherical harmonics $Y_{l_a m_a}$ in the limit of large l, and so does their product, provided their momenta are fixed. Alternately, the product of two fuzzy spherical harmonics T's is "almost commutative" (i.e. almost the same as that of the corresponding Y's) if their angular momentum is small compared to the maximum angular momentum l, whereas it is "strongly noncommutative" (i.e. far from the commutative regime) if their angular momenta are sufficiently large and comparable to l. The intermediate cut-off tells us precisely where the product goes from one situation to the other: Working with fields having momenta much less than $[2\sqrt{l}]$ leaves us in the approximately commutative regime, while fields with momenta much larger than $[2\sqrt{l}]$ take us in the strongly noncommutative regime. In other words, the intermediate cut-off tells us where commutativity and noncommutativity are in delicate balance. Indeed by writing (45) in the form

$$\int_{\sqrt{l}\Lambda_j} \frac{d^4\vec{p}}{(2\pi)^4} \frac{1}{2} \left[\vec{p}^2 + m^2 + \frac{g_4^2}{6} \left[2 \int_{\sqrt{l}\Lambda_j} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} + \int_{\sqrt{l}\Lambda_j} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} e^{i\theta^2 \vec{p}B\vec{k}} \right] \right] |\phi_2(\vec{p})|^2 \equiv$$

$$\int_{\Lambda} \frac{d^4\vec{p}}{(2\pi)^4} \frac{1}{2} \left[\vec{p}^2 + \mu_{l,j}^2 + \frac{g_4^2}{6} \left[2 \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+\mu_{l,j}^2} + \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+\mu_{l,j}^2} e^{i(\frac{j}{2\sqrt{l}})^2 \theta^2 \vec{p}B\vec{k}} \right] \right] |\phi_3^{(j)}(\vec{p})|^2.$$

$[\mu_{l,j}^2 = l\mu_l^2(\frac{2\sqrt{l}}{j})^2$, $\phi_3^{(j)}(\vec{p}) = (\frac{j}{2\sqrt{l}})^3 \phi_2(\frac{j}{2\sqrt{l}}\vec{p})$, $\phi_3^{(2l)} \equiv \phi_3]$. For $j << [2\sqrt{l}]$, $(\frac{j}{2\sqrt{l}})^2 \theta^2 \to 0$ and this is the effective action on a commutative $\mathbb{R}^4$ with cut-off $\Lambda = 2/\theta$. For $j >> [2\sqrt{l}]$ this effective action corresponds to canonical noncommutativity if we insist on the first line above as our effective action or to strongly noncommuting $\mathbb{R}^4$ if we consider instead the effective action to be given by the second line. For the value $j = [2\sqrt{l}]$, where we obtain the noncommutative $\mathbb{R}^4$ given by (21) and (23), there seems to be a balance between the above two situations and one can also expect the UV-IR mixing to be smoothen out.

To show this we write first the one-loop planar and non-planar contributions for $j = [2\sqrt{l}]$, viz

$$\Delta m^P = \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} , \quad \Delta m^{NP}(p) = \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} e^{i\theta^2 \vec{p}B\vec{k}}.$$

We can evaluate these integrals by introducing a Schwinger parameter $(k^2+m^2)^{-1} = \int d\alpha \, exp\left(-\alpha(k^2+m^2)\right)$. Explicitly, we obtain for the planar contribution

$$\Delta m^P = \frac{1}{16\pi^2} \left[-\Lambda^2 \int \frac{d\alpha}{\alpha} e^{-\alpha(m^2+\Lambda^2)} + \int \frac{d\alpha}{\alpha^2} e^{-\alpha m^2} \left(1 - e^{-\alpha\Lambda^2}\right) \right]$$

$$= \frac{1}{16\pi^2} \left[\Lambda^2 + m^2 \ln \frac{m^2}{m^2+\Lambda^2} \right]. \tag{46}$$

Obviously the above planar function diverges quadratically as Λ^2 when $\theta \to 0$, i.e. the noncommutativity acts effectively as a cut-off.

Next we compute the non-planar integral. To this end we introduce as above a Schwinger parameter and rewrite the integral as follows

$$\Delta m^{NP}(p) = \frac{1}{16\pi^4} \int_0^\infty d\alpha e^{-\alpha m^2 - \frac{\theta^4 E^2}{4\alpha}} \int_{\Lambda} d^4k e^{-\alpha\left[\vec{k} - \frac{i\theta^2}{2\alpha}\vec{E}\right]^2}$$

$$= \frac{1}{16\pi^4} \sum_{r=0}^{\infty} (\theta^2)^r \sum_{s=0}^{[\frac{r}{2}]} \frac{i^{r-s}}{s!(r-2s)!} \left[\int_0^\infty d\alpha (\frac{E^2}{4i\alpha})^s e^{-\alpha m^2 - \frac{\theta^4 E^2}{4\alpha}} \left[\int_{\Lambda} d^4k e^{-\alpha k^2} (\vec{k}\vec{E})^{r-2s} \right] \right] , \quad E^\nu = B^{\mu\nu} p^\mu.$$

In above we have also used the fact that θ is small in the sense we explained earlier (i.e. $E\theta << 1$) and in accordance with [6] to expand the second exponential around $\theta = 0$. This is also because the cut-off Λ is inversely proportional to θ. [In the last line we used the identity $\sum_{p=0}^{\infty} \sum_{q=0}^{p} A_{q,p-q} = \sum_{r=0}^{\infty} \sum_{s=0}^{[\frac{r}{2}]} A_{s,r-2s}$, $[\frac{r}{2}] = \frac{r}{2}$ for r even and $[\frac{r}{2}] = \frac{r-1}{2}$ for r odd] . It is not difficult to argue that the inner integral above vanishes unless r is even. Using also the fact that the cut-off Λ is rotationally invariant one can evaluate the inner integral as follows. We have

$$\int_{\Lambda} d^4k e^{-\alpha k^2} (\vec{k}\vec{E})^n = 4\pi^2 E^n (n-1)!! \left[\frac{1}{(2\alpha)^{\frac{n}{2}+2}} - \Lambda^n e^{-\alpha\Lambda^2} \sum_{q=-1}^{\frac{n}{2}} \frac{1}{(n-2q)!!} \frac{1}{(2\alpha)^{q+2}} \frac{1}{\Lambda^{2q}} \right],$$

where n is an even number given by $n = r - 2s$.

We can now put the above non-planar function in the form

$$\Delta m^{NP}(p) = \frac{1}{16\pi^2} \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{\theta^2 E}{2}\right)^{2N} \int \frac{d\alpha}{\alpha^{N+2}} e^{-\alpha m^2 - \frac{\theta^4 E^2}{4\alpha}} \sum_{M=0}^{N} C_N^M (-1)^M \left[1 - \sum_{P=0}^{M+1} \frac{(\alpha\Lambda^2)^P}{P!} e^{-\alpha\Lambda^2}\right].$$
(47)

$[C_N^M = \frac{N!}{M!(N-M)!}]$. The first term in this expansion corresponds exactly to the case of canonical noncommutativity where instead of Λ we have no cut-off, i.e.

$$\Delta m^{NP}(p) = \frac{1}{16\pi^2} \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{\theta^2 E}{2}\right)^{2N} \int \frac{d\alpha}{\alpha^{N+2}} e^{-\alpha m^2 - \frac{\theta^4 E^2}{4\alpha}} \sum_{M=0}^{N} C_N^M (-1)^M + ...$$

$$= \frac{1}{8\pi^2} \left[\frac{2}{\theta^4 E^2} + m^2 \ln(m\theta^2 E)\right] + ... \equiv \frac{1}{16\pi^2} I^{(2)}(m^2, \frac{\theta^4 E^2}{4}) + ...$$

As expected this term provides essentially the canonical UV-IR mixing. As it turns out this singular behaviour is completely regularized by the remaining $N = 0$ term in (47), i.e.

$$\Delta m^{NP}(p) = \frac{1}{16\pi^2} I^{(2)}(m^2, \frac{\theta^4 E^2}{4}) + \frac{1}{16\pi^2} \int \frac{d\alpha}{\alpha^2} e^{-\alpha m^2 - \frac{\theta^4 E^2}{4\alpha}} \left[-\sum_{P=0}^{1} \frac{(\alpha\Lambda^2)^P}{P!} e^{-\alpha\Lambda^2}\right] + ...$$

$$= \frac{1}{16\pi^2} I^{(2)}(m^2, \frac{\theta^4 E^2}{4}) - \frac{1}{16\pi^2} \left[I^{(2)}(m^2 + \Lambda^2, \frac{\theta^4 E^2}{4}) + \Lambda^2 I^{(1)}(m^2 + \Lambda^2, \frac{\theta^4 E^2}{4})\right] + ...$$
(48)

The integrals $I^{(L)}(x, y)$ are given essentially by Hankel functions , viz

$$I^{(1)}(x, y) = \int_0^{\infty} \frac{d\alpha}{\alpha} e^{-x\alpha - \frac{y}{\alpha}} = \frac{1}{2} \left[i\pi H_0^{(1)}(2i\sqrt{xy}) + h.c.\right]$$

$$I^{(L)}(x, y) = \int_0^{\infty} \frac{d\alpha}{\alpha^L} e^{-x\alpha - \frac{y}{\alpha}} = \frac{1}{2} \left[\frac{i\pi}{L-1} \left(\frac{x}{y}\right)^{\frac{L-1}{2}} \sqrt{xy} e^{\frac{iL\pi}{2}} \left[H_{L-2}^{(1)}(2i\sqrt{xy}) + H_L^{(1)}(2i\sqrt{xy})\right] + h.c.\right] , \quad L > 1.$$

Hankel functions admit the series expansion $H_0^{(1)}(z) = \frac{2i}{\pi} \ln z + ...$ and $H_\nu^{(1)}(z) = -\frac{i(\nu-1)!}{\pi} \left(\frac{2}{z}\right)^\nu + ..$ for $\nu > 0$ when $z \longrightarrow 0$. In this case the mass m and the external momentum E are both small compared to the cut-off $\Lambda = 2/\theta$ and thus the dimensionless parameters $z \equiv \sqrt{xy} = 2\frac{m}{\Lambda}\frac{E}{\Lambda}$ or $z \equiv \sqrt{xy} = 2\sqrt{1 + \frac{m^2}{\Lambda^2}}\frac{E}{\Lambda}$ are also small, in other words we can calculate for example $I^{(1)}(x, y) = -2\ln(2\sqrt{xy})$, $I^{(2)}(x, y) = 2x\ln(2\sqrt{xy}) + \frac{1}{y}$ and $I^{(L)}(x, y) = \frac{(L-2)!}{y^{L-1}}[1 - \frac{xy}{(L-2)(L-1)}]$ for $L \geq 3$. Thus the first term $N = 0$ in the above sum (i.e euqation (48)) is simply given by

$$\Delta m^{NP}(p) = -\frac{m^2}{16\pi^2} ln(1 + \frac{\Lambda^2}{m^2}) + ...$$
(49)

As one can see it does not depend on the external momentum p at all. In the commutative limit $\theta \to 0$, this diverges logarithmically as $ln\Lambda$ which is subleading compared to the quadratic divergence of the planar function. Higher corrections can also be computed and one finds essentially an expansion in $\frac{\Lambda\theta^2 E}{2} = E\theta = 2\frac{E}{\Lambda}$ given by

$$\Delta m^{NP}(p) = -\frac{m^2}{16\pi^2} \ln(1 + \frac{\Lambda^2}{m^2})$$

$$+ \frac{\Lambda^2}{16\pi^2} I^{(1)}(x, y) \sum_{p=2}^{\infty} \frac{1}{p!} \left(\frac{\Lambda\theta^2 E}{2}\right)^{2(p-1)} \eta_{p-1,p-2} + \frac{1}{16\pi^2} I^{(2)}(x, y) \sum_{p=2}^{\infty} \frac{1}{p!} \left(\frac{\Lambda\theta^2 E}{2}\right)^{2p} \eta_{p,p-2}$$

$$+ \frac{1}{16\pi^2} \sum_{N=1}^{\infty} \left(\frac{\theta^4 E^2}{4}\right)^N I^{(N+2)}(x, y) \sum_{p=2}^{\infty} \frac{1}{p!} \left(\frac{\Lambda\theta^2 E}{2}\right)^{2p} \eta_{p+N,p-2}.$$

$[x = m^2 + \Lambda^2, y = \frac{\theta^4 E^2}{4}, \eta_{p+N,p-2} = \sum_{M=0}^{p-2} \frac{(-1)^M}{M!(p+N-M)!}]$. It is not difficult to find that the leading terms in the limit of small external momenta (i.e. $E/\Lambda << 1$) are effectively given by

$$\Delta m^{NP}(p) = -\frac{m^2}{16\pi^2} \ln\left(1 + \frac{\Lambda^2}{m^2}\right) - \frac{E^2}{4\pi^2} \ln\left(4\frac{E}{\Lambda}\sqrt{1 + \frac{m^2}{\Lambda^2}}\right) \left[1 + O\left(\frac{E^2}{\Lambda^2}\right)\right] + \frac{E^2}{8\pi^2} \left[1 + O'\left(\frac{E^2}{\Lambda^2}\right)\right].$$
(50)

Clearly in the strict limit of small external momenta when $E{\to}0$, we have $E^2 \ln E {\to} 0$ and the non-planar contribution does not diverge (only the first term in (50) survives this limit as it is independent of E) and hence there is no UV-IR mixing. The limit of zero noncommutativity is singular but now this divergence has the nice interpretation of being the divergence recovered in the non-planar 2−point function when the cut-off $\Lambda = \frac{2}{\theta}$ is removed. This divergence is however logarithmic and therefore is sub-leading compared to the quadratic divergence in the planar part.

The effective action (45) with $j = [2\sqrt{l}]$ can be obviously obtained from quantizing the action (42) with the replacements $\mu_l^2{\to}m^2$, $\phi_3{\to}\phi_2{\equiv}\phi_2(X^{NC}) = \int \frac{d^4p}{(2\pi)^4}\phi_2(\vec{p})e^{-ipX^{NC}} = \phi_2^\dagger$ and where as before we have to regularize all integrals in the quantum theory with a cut-off $\Lambda = 2/\theta$. The star product $*$ is the Moyal-Weyl product given in (32) with the substitutions $\theta'{\to}\theta$, $x^{NC}{\longrightarrow}X^{NC}$. This effective action can also be rewritten in the form

$$\int d^4x \left[\frac{1}{2}\partial_\mu\phi_2 *_\Lambda \partial_\mu\phi_2 + \frac{1}{2}m^2\phi_2 *_\Lambda \phi_2 + \frac{g_4^2}{4!}\phi_2 * \phi_2 * \phi_2 * \phi_2 \right], \tag{51}$$

which is motivated by the fact that the *effective* star product defined by

$$\begin{aligned} f *_\Lambda g(X^{NC}) &= \int_\Lambda \frac{d^4p}{(2\pi)^4}f(\vec{p}) \int_\Lambda \frac{d^4k}{(2\pi)^4}g(\vec{k})e^{-ipX^{NC}} * e^{-ikX^{NC}} \\ &= \int d^4y'd^4z'\delta_\Lambda^4(y')\delta_\Lambda^4(z')f(y-y') * g(z-z')|_{y=z=X^{NC}}, \end{aligned} \tag{52}$$

is such that $\int d^4x f *_\Lambda g(x) = \int_\Lambda \frac{d^4p}{(2\pi)^4}f(\vec{p})g(-\vec{p})$. The distribution $\delta_\Lambda^4(y')$ is not the Dirac delta function $\delta^4(y')$ but rather $\delta_\Lambda^4(y') = \int_\Lambda \frac{d^4p}{(2\pi)^4}e^{-ipy'}$, i.e. $\delta_\Lambda^4(y')$ tends to the ordinary delta function in the limit $\Lambda{\to}\infty$ of the commutative plane where the above product (52) also reduces to the ordinary point-wise multiplication of functions. If the cut-off Λ was not correlated with the non-commutativity parameter θ, then the limit $\Lambda{\to}\infty$ would had corresponded to the limit where the product (52) reduces to the Moyal-Weyl product given in equation (32). This way of writing the effective action (i.e. (51)) is to insist on the fact that all integrals are regularized with a cut-off $\Lambda = 2/\theta$. In other words the above new star product which appears only in the kinetic part of the action is completely equivalent to a sharp cut-off Λ and yields therefore exactly the propagator (44) with which only modes $\leq\Lambda$ can propagate.

We should also remark here regarding non-locality of the star product (52). At first sight it seems that this non-locality is more severe in (52) than in (32), but as it turns out this is not entirely true: in fact the absence of the UV-IR mixing in this product also suggests this. In order to see this more explicitly we first rewrite (52) in the form

$$f *_\Lambda g(X^{NC}) = \int d^4y'd^4z' f(y')g(z')K_\Lambda(y', z'; X^{NC})$$
$$K_\Lambda(y', z'; X^{NC}) = \delta_\Lambda^4(y - y') * \delta_\Lambda^4(z - z')|_{y=z=X^{NC}}.$$

The kernel K_Λ can be computed explicitly and is given by

$$K_\Lambda(y', z'; X^{NC}) = \int_\Lambda \frac{d^4k}{(2\pi)^4}\delta_\Lambda^4(X^{NC} - y' + \frac{\theta^2}{2}Bk)e^{ik(z'-X^{NC})}.$$

For the moment, let us say that Λ and θ are unrelated. Then, taking Λ to infinity gives [1, 8]

$$K(y', z'; X^{NC}) = \frac{16}{\theta^8 \det B} \frac{1}{(2\pi)^4}e^{\frac{2i}{\theta^2}(z'-X^{NC})B^{-1}(y'-X^{NC})}.$$

If we have for example two functions f and g given by $f(x) = \delta^4(x - p)$ and $g(x) = \delta^4(x - p)$, i.e. they are non-zero only at one point p in space-time, their star product which is clearly given by the kernel $K(p, p; X^{NC})$ is non-zero everywhere in space-time. The fact that K is essentially a phase is the source of the non-locality of (32) which leads to the UV-IR mixing.

On the other hand the kernel $K_\Lambda(p, p; X^{NC})$ with finite Λ can be found in two dimensions (say) to be given by

$$K_\Lambda(p, p; X^{NC}) = \frac{1}{\pi^2\theta^4}\int_{-\theta}^{\theta} da\delta_\Lambda(a + L_1)e^{\frac{2i}{\theta^2}L_2 a}\int_{-\theta}^{\theta} db\delta_\Lambda(b + L_2)e^{-\frac{2i}{\theta^2}L_1 b},$$

with $L_a = X_a^{NC} - p_a$, $a = 1, 2$. If we now make the approximation to drop the remaining Λ (since the effects of this cut-off were already taken anyway) one can see that the above integral is non-zero only for $-\theta + p_1{\leq}X_1^{NC}{\leq}\theta + p_1$

and $-\theta + p_2 \leq X_2^{NC} \leq \theta + p_2$ simultaneously. In other words the star product $K_\Lambda(p, p; X^{NC})$ of $f(x)$ and $g(x)$ is also localized around p within an error θ and is equal to $\frac{1}{\pi^2 \theta^4}$ there . The star product (52) is therefore effectively local.

Final remarks are in order. First we note that the effective star product (52) leads to an effective commutation relations (23) in which the parameter θ^2 is multiplied by an overall constant equal to $\int d^4 y' d^4 z' \delta_\Lambda^4(y') \delta_\Lambda^4(z')$, we simply skip the elementary proof. Remark also that this effective star product is non-associative as one should expect since it is for all practical purposes equivalent to a non-trivial sharp momentum cut-off Λ .

The last remark is to note that the prescription (44) can also be applied to the canonical limit of large stereographic projection of the spheres onto planes, and in this case one can also obtain a cut-off $\Lambda' = \frac{2}{\theta'}$ with j fixed as above such that $j = [2\sqrt{l}]$. The noncommutative plane (20) defined in this way is therefore completely equivalent to the above noncommutative plane (23).

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