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The Flavoured BFSS Model at High Temperature

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Abstract: We study the high temperature series expansion of the Berkooz-Douglas matrix model which describes the D0/D4–brane system. At high temperature the model is weakly coupled and we develop the series to second order. We check our results against the high temperature regime of the bosonic model (without fermions) and find excellent agreement. We track the temperature dependence of the bosonic model and find backreaction of the fundamental fields lifts the zero temperature adjoint mass degeneracy. In the low temperature phase the system is well described by a gaussian model with three masses $m^t_A = 1.964 \pm 0.003$, $m^l_A = 2.001 \pm 0.003$ and $m_f = 1.463 \pm 0.001$, the adjoint longitudional and transverse masses and the mass of the fundamental fields respectively.

Contents

1 Introduction

The Berkooz-Douglas model (BD model) [1] was introduced as a non-perturbative formulation of M-theory in the presence of a background of longitudinal M5-branes with the M2-brane quantised in light-cone gauge. Its action is written as that of the BFSS model [3] with additional fundamental hypermultiplets to describe the M5-branes. The BFSS model can also be viewed as a many-body system of D0-branes of the IIA superstring. In this framework the BD model is a D0/D4 system with the massless case being the D0/D4 intersection. When the number of D0-branes far exceeds that of the D4-branes the dynamics of the D0-branes is only weakly affected by that of the D4-branes and is captured by the IIA supergravity background holographically dual to the BFSS model. In this context the D4-branes, representing the fundamental degrees of freedom of the BD model, are treated as Born-Infeld probe 4-branes. This holographic set up is a tractable realisation of gauge/gravity duality with flavour.

Both the BFSS model and the BD model are supersymmetric quantum mechanical models with an $SU(N)$ gauge symmetry. When they are put in a thermal bath they become strongly coupled at low temperature. At finite temperature their gravity duals involve a black hole whose Hawking-temperature is that of the thermal bath. These duals can be used to provide non-perturbative predictions at low temperature. The BFSS and BD models can also be studied by the standard non-perturbative field theory method of Monte Carlo simulation. These models therefore provide excellent candidates for testing gauge/gravity duality non-perturbatively and in a broken supersymmetric setting.

There are now several non-perturbative studies of the BFSS model [7–11] and several recent reviews[12–14]. Also, the BD model was recently studied non-perturbatively in [15]. In all cases the predictions from the gauge/gravity duals were found to be in excellent agreement with that of Monte Carlo simulations of the finite temperature models.

The situation is conceptually simpler at high temperature as the dimensionless inverse temperature, scaled in terms of the BD-coupling provides a natural small parameter for the model. In this paper, we obtain the first two terms in the high temperature expansion of the BD model.

In the high temperature limit only the bosonic Matsubara zero modes survive and the resulting model is a pure potential. This potential, which provides the non-perturbative aspect of our high temperature study, also plays a rôle in the ADHM construction [16]. We study the model for adjoint matrix size N between 4 and 32 for $N_f = 1$ (with N_f the number of D4-branes) and for N_f between 2 and 16 for N from 9 to 20. For $N_f \geq 2N$ we find that the system has difficulties with ergodicity. In particular, for $N_f = 2N$ and $N_f = 2N+1$ the system fails to thermalise satisfactorily. In contrast the system has no difficulties for $N_f = 2N - 1$. This condition is closely related to the singularity structure of instanton moduli space where irreducible $SU(N_f)$ instantons of Chern number N exist only for $N \geq \frac{N_f}{2}$ [17, 18]. The moduli space of such instantons is equivalent to the zero locus of the potential with $X^a = 0$ and $\mathcal{D}^A = 0$ (see equation (2.4)). This moduli space is in general singular and non-singular only when this bound is satisfied.

There is also a natural $1 + 1$ dimensional analogue of the BD model which has $\mathcal{N} = 4$ supersymmetry, associated with the D1/D5 system of [5] whose BFSS relative was discussed in [19–21]. When the Euclidean finite temperature version of this $1 + 1$ dimensional quantum field theory is considered on a torus with the spatial circle of period β and euclidean time[1] of period $1/T$, then at high temperature the fermions decouple and one is left with the purely bosonic version of the BD model. We refer to this model as the bosonic BD model and study the small period behaviour (equivalent for us to our high temperature regime) of the massless version of this model as a check on our high-temperature series. We find the high temperature series results are in excellent agreement with Monte Carlo simulations of the bosonic BD model. By fitting the dependence, of the expectation values of our observables, on the number of flavour multiplets, N_f, we find that extrapolation, to $N_f = 0$, agrees well with the corresponding observables of the BFSS model.

As β grows the bosonic BD model undergoes a set of phase transitions. These are the phase transitions of the bosonic BFSS model. We find the high temperature series expansion is valid down to $\beta \sim 1$, which is just above the phase transition region. Below the transition the bosonic BD model is well described by free massive fields where the backreaction of the

[1] In this paragraph we avoid using β for $1/T$ for simplicity of the comparison.

fundamental fields has lifted the degeneracy of the longitudinal and transverse masses.

The principal results of this paper are:

- We obtain expansions for observables of the BD model to second order in a high temperature series.
- We tabulate the coefficients of this expansion as functions of N and N_f in the range $4 \leq N \leq 32$ and $1 \leq N_f \leq 16$.
- We measure the expectation values of the composite operator $\langle r^2 \rangle_{\text{bos}}$, (see equation (2.8)), and the mass susceptibility $\langle \mathcal{C}^m \rangle_{\text{bos}}$, (see equation (4.10)), of bosonic BD model as a function of temperature down to zero and use it to check our coefficients for the high temperature series of the full BD model.
- We find that the fundamental fields of the bosonic BD model have mass $m_f = 1.463 \pm 0.001$.
- We measure the backreacted mass of the longitudinal adjoint scalars to be $m_A^l = 2.001$ and find that the transverse mass is largely unaffected by backreaction being $m_A^t = 1.964 \pm 0.003$ which should be compared with the bosonic BFSS model where the fields have mass $m_A = 1.965 \pm 0.007$.
- We use the measured masses to predict the zero temperature values of our fundamental field observables $\langle r^2 \rangle_{\text{bos}}$ and mass susceptibility $\langle \mathcal{C}^m \rangle_{\text{bos}}$ and find excellent agreement with direct measurements.

The paper is organised as follows: In section 2 we present the finite temperature BD model and describe our notation and observables. In section 3 we set up and implement the high temperature series expansion working to second order in the inverse temperature β. Section 4 describes the dependence of our observables on the coefficients in the expansion which must be determined by numerical simulation of the zero-mode model. We perform lattice simulations of the bosonic BD model and find excellent agreement with the high temperature expansion. We also find the low temperature phase of the model is well described by a system of gaussian quantum fields. Section 5 gives our concluding remarks. There are three appendices; appendix A gives details of the expansion of observables while appendix B gives tables for different N and N_f of the coefficients determined non-perturbatively. The final appendix, C, presents graphs of predictions for the high temperature behaviour of our observables for the supersymmetric model.

2 Berkooz-Douglas Model

We begin by describing the field content of the model following the notation used in [5]. The action of the BFSS model is given by

$$S_{\rm BFSS} = \frac{1}{g^2}\int dt \sum_{i=1}^{9} {\rm Tr}\left\{\frac{1}{2}(\mathcal{D}_0 X^i)^2 + \frac{1}{4}[X^i,X^j]^2 - \frac{i}{2}\Psi^T C_{10}\,\Gamma^0\mathcal{D}_0\Psi + \frac{1}{2}\Psi^T C_{10}\,\Gamma^i[X^i,\Psi]\right\}\ , \tag{2.1}$$

where $\mathcal{D}_0\,\cdot = \partial_t\ \cdot - i[A,\ \cdot\ ]$, Ψ is a thirty two component Majorana–Weyl spinor, Γ^μ are ten dimensional gamma matrices and C_{10} is the charge conjugation matrix satisfying $C_{10}\Gamma^\mu C_{10}^{-1} = -\Gamma^{\mu T}$. The fields X^i and Ψ are in the adjoint representation of the gauge symmetry group $SU(N)$ and A is the gauge field.

To describe the addition of the fundamental fields we break the $SO(9)$ vector X^i into an $SO(5)$ vector X^a and an $SO(4)$ vector which we re-express as $X_{\rho\dot\rho}$ via

$$X_{\rho\dot\rho} = \frac{i}{\sqrt{2}}\sum_{m=1}^{4}\sigma^m{}_{\rho\dot\rho}X^{10-m}, \tag{2.2}$$

where $\sigma^4 = -i\mathbf{1}_2$ and σ^A's $(A = 1,2,3)$ are the Pauli matrices. The $X_{\rho\dot\rho}$ $(\rho,\dot\rho = 1,2)$ are complex scalars which together transform as a real vector of $SO(4)$ which satisfies the reality condition $X_{\rho\dot\rho} = \varepsilon_{\rho\sigma}\varepsilon_{\dot\rho\dot\sigma}\bar{X}^{\sigma\dot\sigma}$. The indices ρ and $\dot\rho$ are those of $SU(2)_R$ and $SU(2)_L$, respectively, where $SO(4) = SU(2)_L \times SU(2)_R$.

The nine BFSS scalar fields, X^i, become X^a $(a = 1,\cdots,5)$ and $X_{\rho\dot\rho}$. The sixteen adjoint fermions of the BFSS model become λ_ρ and $\theta_{\dot\rho}$ with λ_ρ being $SO(5,1)$ symplectic Majorana-Weyl spinors of positive chirality and satisfying $\lambda_\rho = \varepsilon_{\rho\sigma}(\lambda^c)^\sigma$ while $\theta_{\dot\rho}$ are symplectic Majorana-Weyl spinors of negative chirality satisfying $\theta_{\dot\rho} = -\varepsilon_{\dot\rho\dot\sigma}(\theta^c)^{\dot\sigma}$. They combine together to form an $SO(9,1)$ Majorana-Weyl spinor in the adjoint of $SU(N)$. This $SO(9)$ symmetry is recovered only if the fundamental fields are turned off.

To describe the longitudinal M5-branes (or D4-branes), we have Φ_ρ and χ, which transform in the fundamental representations of both $SU(N)$ and the global $SU(N_f)$ flavour symmetry. Φ_ρ are complex scalar fields with hermitian conjugates $\bar\Phi^\rho$, and χ is an $SO(5,1)$ spinor of negative chirality.

After rotating to imaginary time the Euclidean action describing the model at finite temperature $T = \beta^{-1}$ becomes:

$$\begin{aligned} S = N\int_0^\beta d\tau\ \Bigg[&{\rm Tr}\left(\frac{1}{2}D_\tau X^a D_\tau X^a + \frac{1}{2}D_\tau\bar{X}^{\rho\dot\rho}D_\tau X_{\rho\dot\rho} + \frac{1}{2}\lambda^{\dagger\rho}D_\tau\lambda_\rho + \frac{1}{2}\theta^{\dagger\dot\rho}D_\tau\theta_{\dot\rho}\right) \\ &+ {\rm tr}\left(D_\tau\bar\Phi^\rho D_\tau\Phi_\rho + \chi^\dagger D_\tau\chi\right) \\ &- {\rm Tr}\left(\frac{1}{4}[X^a,X^b]^2 + \frac{1}{2}[X^a,\bar{X}^{\rho\dot\rho}][X^a,X_{\rho\dot\rho}]\right) \\ &+ \frac{1}{2}\,{\rm Tr}\sum_{A=1}^{3}\mathcal{D}^A\mathcal{D}^A + {\rm tr}\left(\bar\Phi^\rho(X^a - m^a)^2\Phi_\rho\right) \end{aligned}$$

$$- \mathrm{Tr}\left(-\frac{1}{2}\lambda^{\dagger\rho}\gamma^a[X^a,\lambda_\rho] + \frac{1}{2}\theta^{\dagger\dot\rho}\gamma^a[X^a,\theta_{\dot\rho}] - \sqrt{2}i\varepsilon^{\rho\sigma}\theta^{\dagger\dot\rho}[X_{\sigma\dot\rho},\lambda_\rho]\right)$$
$$- \mathrm{tr}\left(\chi^\dagger\gamma^a(X^a - m^a)\chi + \sqrt{2}i\varepsilon^{\rho\sigma}\chi^\dagger\lambda_\rho\Phi_\sigma + \sqrt{2}i\varepsilon_{\rho\sigma}\bar\Phi^\rho\lambda^{\dagger\sigma}\chi\right)\Bigg], \tag{2.3}$$

where

$$\mathcal{D}^A = \sigma_\rho^{A\ \sigma}\left(\frac{1}{2}[\bar X^{\rho\dot\rho}, X_{\sigma\dot\rho}] - \Phi_\sigma\bar\Phi^\rho\right), \tag{2.4}$$

with D_τ the covariant derivative which, for the fields of the fundamental multiplet, Φ_ρ and χ, acts as $D_\tau \cdot = (\partial_\tau - iA)\cdot$. The trace of $SU(N)$ is written as Tr while that of $SU(N_f)$ is denoted by tr. The diagonal matrices, m^a, correspond to the transverse positions of the D4-branes.

We will restrict our attention to $m^a = 0$ so that the D4-branes are attached to the D0-branes, and the strings between D0 and D4 are massless, i.e. the fundamental fields are massless. The factor of N in front of the integral in (2.3) is the remnant of the 't Hooft coupling $\lambda = g^2N$ which is kept fixed and absorbed into τ and the fields with $\beta = \lambda^{1/3}/T$. Note that without loss of generality we can set $\lambda = 1$.

As discussed in the introduction, the BFSS model is also the matrix regularization of a supermembrane theory [2], so the BFSS part of this model can be also interpreted as M2-brane dynamics. In this context the D4-branes lift to M5-branes and the model can describe M2-branes ending on longitudinal M5-branes.

The BD model is a version of supersymmetric quantum mechanics and could in principle be treated by Hamiltonian methods. The partition function is then

$$Z = Tr(\mathrm{e}^{-\beta H}) = \int [dX][d\lambda][d\theta][d\Phi][d\bar\Phi][d\chi][d\chi^\dagger]\mathrm{e}^{-S} \tag{2.5}$$

with Tr the trace over the Hilbert space of the Hamiltonian restricted to its gauge invariant subspace and the action S in the path integral given by equation (2.3).

The principal observable of the model is the energy[2], $E = \langle H\rangle/N^2$. To obtain the path integral version of E, note that N^2E is minus the derivative of logarithm of the partition function with respect to β and any temperature dependence of the kinetic terms must be cancelled by a corresponding temperature dependence of the measure. Once this is ensured one obtains the path integral version:

$$E = \langle\varepsilon_b\rangle + \langle\varepsilon_f\rangle\,, \quad \text{where}$$
$$\varepsilon_b = \frac{3}{N\beta}\int_0^\beta d\tau\Bigg[\mathrm{Tr}\left(-\frac{1}{4}[X^i,X^j]^2\right)$$
$$+ \mathrm{tr}\left(\bar\Phi^\rho X^{a\,2}\Phi_\rho - \bar\Phi^\rho[\bar X^{\sigma\dot\rho}, X_{\rho\dot\rho}]\Phi_\sigma - \frac{1}{2}\bar\Phi^\rho\Phi_\sigma\bar\Phi^\sigma\Phi_\rho + \bar\Phi^\rho\Phi_\rho\bar\Phi^\sigma\Phi_\sigma\right)\Bigg],$$

[2] We divide by N^2 so that E remains finite in the large-N limit.

$$\varepsilon_f = \frac{3}{2N\beta}\int_0^\beta d\tau\Big[\mathrm{Tr}\left(\frac{1}{2}\lambda^{\dagger\rho}\gamma^a[X^a,\lambda_\rho] - \frac{1}{2}\theta^{\dagger\dot\rho}\gamma^a[X^a,\theta_{\dot\rho}] + \sqrt{2}i\varepsilon^{\rho\sigma}\theta^{\dagger\dot\rho}[X_{\sigma\dot\rho},\lambda_\rho]\right) + \mathrm{tr}\left(-\chi^\dagger\gamma^a X^a\chi - \sqrt{2}i\varepsilon^{\rho\sigma}\chi^\dagger\lambda_\rho\Phi_\sigma - \sqrt{2}i\varepsilon_{\rho\sigma}\bar\Phi^\rho\lambda^{\dagger\sigma}\chi\right)\Big]. \tag{2.6}$$

We fix the static gauge: $\partial_\tau A = 0$, so the path integral requires the corresponding ghost fields c and $\bar c$ with the ghost term $N\int_0^\beta d\tau\ \mathrm{Tr}\,\partial_\tau\bar c D_\tau c$ added to the action (2.3).

As in [6], there are two other interesting observables:

$$R^2 = \frac{1}{N\beta}\int_0^\beta d\tau\,\mathrm{Tr}\,X^{i\,2}\,, \qquad P = \frac{1}{N}\mathrm{Tr}\left(\exp\left[i\beta A\right]\right)\,. \tag{2.7}$$

Here R^2 is a hermitian operator whose expectation value is a measure of the extent of the eigenvalue distribution of the scalars X^i and P is the Polyakov loop. Note: Path-ordering is not needed here for the Polyakov loop as we consider A in the static gauge.

Since the model has new degrees of freedom it is important to consider other observables that captures properties of these new fields. The natural candidates are

$$r^2 = \frac{1}{\beta N_f}\int_0^\beta d\tau\ \mathrm{tr}\,\bar\Phi^\rho\Phi_\rho\,, \tag{2.8}$$

which is the analogue of R^2 for the fundamental degrees of freedom, and the condensate defined as

$$c^a(m) = \frac{\partial}{\partial m^a}\left(-\frac{1}{N\beta}\log Z\right) = \left\langle \frac{1}{\beta}\int_0^\beta d\tau\,\mathrm{tr}\left\{2\bar\Phi^\rho(m^a - X^a)\Phi_\rho + \chi^\dagger\gamma^a\chi\right\}\right\rangle. \tag{2.9}$$

However, for us, with $m^a = 0$, c^a will be zero. So our focus will be on the mass susceptibility

$$\langle\mathcal{C}^m\rangle := \frac{\partial c^a}{\partial m^a}(0) \tag{2.10}$$

i.e. the derivative with respect to m^a with a fixed (not summed over) and evaluated at $m^a = 0$ where

$$\mathcal{C}^m = \frac{2}{\beta}\int_0^\beta d\tau\,\mathrm{tr}\,\bar\Phi^\rho\Phi_\rho - \frac{N}{5\beta}\left(\int_0^\beta d\tau\,\mathrm{tr}\left\{-2\bar\Phi^\rho X^a\Phi_\rho + \chi^\dagger\gamma^a\chi\right\}\right)^2\,. \tag{2.11}$$

3 High Temperature Expansion

In this section, we examine the high temperature expansion of the BD model. The Fourier expansion of the fields is given by,

$$X^i(\tau) = \sum_{n\in\mathbb{Z}} X^i_n e^{2\pi i n\tau/\beta}, \quad \lambda_\rho(\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \lambda_{r\rho}e^{2\pi i r\tau/\beta}, \quad \theta_{\dot\rho}(\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}}\theta_{r\dot\rho}e^{2\pi i r\tau/\beta},$$
$$c(\tau) = \sum_{n\in\mathbb{Z},n\neq 0} c_n e^{2\pi i n\tau/\beta}, \quad \bar c(\tau) = \sum_{n\in\mathbb{Z},n\neq 0}\bar c_n e^{-2\pi i n\tau/\beta},$$

$$\Phi_\rho(\tau) = \sum_{n\in\mathbb{Z}} \Phi_{n\rho} e^{2\pi i n\tau/\beta}, \quad \chi(\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \chi_r e^{2\pi i r\tau/\beta}, \tag{3.1}$$

where thermal boundary conditions require that the bosons and ghosts are periodic in τ while the fermions are anti-periodic. In the high temperature limit, only the zero-modes play a rôle. As the temperature is lowered one can integrate out the non-zero modes perturbatively with β playing the rôle of a perturbation parameter. Using this procedure, the first two terms in the high temperature expansion of E, $\langle R^2\rangle$ and $\langle P\rangle$ for the BFSS model were obtained in [6]. We follow the same method here and obtain the corresponding expansion of these observables for the BD model and for them the novel feature will be the additional dependence on N_f, the number of flavour multiplets. In addition we have the new observable $\langle r^2\rangle$ and $\langle \mathcal{C}^m\rangle$.

In order to see the dependence on temperature, $1/\beta$, it is convenient to rescale the scalar fields as follows

$$\begin{aligned}
&X_0^i \to \beta^{-\frac{1}{4}} X_0^i\,, \qquad A \to \beta^{-\frac{1}{4}} A\,, \qquad \Phi_0 \to \beta^{-\frac{1}{4}}\Phi_0\,,\\
&X^i_{n\neq 0} \to \beta^{\frac{1}{2}} X^i_{n\neq 0}\,, \qquad \Phi_{n\neq 0} \to \beta^{\frac{1}{2}}\Phi_{n\neq 0}\,, \qquad c_n \to \beta^{\frac{1}{2}} c_n\,, \qquad \bar{c}_n \to \beta^{\frac{1}{2}}\bar{c}_n\,,
\end{aligned} \tag{3.2}$$

while the fermions remain unchanged.[3] This rescaling makes the coefficients of the zero-mode terms and the kinetic terms independent of β so that one can concentrate on the β-dependence, which now appears only in the interaction terms. The action is then written as

$$S = S_0 + S_{\text{kin}} + S_{\text{int}}\ , \tag{3.3}$$

where S_0 is a zero-mode action

$$\begin{aligned}
S_0 = -\frac{N}{4}\,\mathrm{Tr}\left([X_0^i, X_0^j]^2 + 2[A, X_0^i]^2\right) + N\,\mathrm{tr}\Big(\bar{\Phi}_0^\rho A^2 \Phi_{0\rho} + \bar{\Phi}_0^\rho (X_0^a)^2 \Phi_{0\rho}\\
- \bar{\Phi}_0^\rho [\bar{X}_0^{\sigma\dot{\rho}}, X_{0\rho\dot{\rho}}]\Phi_{0\sigma} - \frac{1}{2}\bar{\Phi}_0^\rho \Phi_{0\sigma}\bar{\Phi}_0^\sigma\Phi_{0\rho} + \bar{\Phi}_0^\rho\Phi_{0\rho}\bar{\Phi}_0^\sigma\Phi_{0\sigma}\Big),
\end{aligned} \tag{3.4}$$

S_{kin} is the kinetic part of the action for non-zero modes

$$\begin{aligned}
S_{\text{kin}} = \sum_{n\neq 0}\frac{(2\pi n)^2 N}{2}\Big[\mathrm{Tr}\left(X^a_{-n}X^a_n + \bar{X}^{\rho\dot{\rho}}_{-n}X_{n\rho\dot{\rho}} + 2\bar{c}_{-n}c_n\right) + \mathrm{tr}\left(2\bar{\Phi}^\rho_{-n}\Phi_{n\rho}\right)\Big]\\
+ \sum_r \frac{2\pi i r N}{2}\left[\mathrm{Tr}\left(\lambda^{\dagger\rho}_{-r}\lambda_{r\rho} + \theta^{\dagger\dot{\rho}}_{-r}\theta_{r\dot{\rho}}\right) + \mathrm{tr}\left(2\chi^\dagger_{-r}\chi_r\right)\right],
\end{aligned} \tag{3.5}$$

[3] This rescaling induces a change of measure in the path integral so that the partition function $Z = \beta^{-\frac{3}{4}(8(N^2-1)+4NN_f)}\bar{Z}$ where $\bar{Z}$ is the partition function in terms of the rescaled fields and the only remaining temperature dependence is in S_{int}. This can be obtained in two steps: 1) Rescale the original action so that the kinetic term, including the gauge potential, is independent of β, and the only temperature dependence is β^3 for the bosonic potential and $\beta^{3/2}$ for the fermionic potential. For this $\tau\to\beta\tau$, $X^i \to \beta^{\frac{1}{2}}X^i$, $\Phi_\rho\to\beta^{\frac{1}{2}}\Phi_\rho$, $A\to\beta^{-1}A$, $c\to\beta^{\frac{1}{2}}c$ and $\bar{c}\to\beta^{\frac{1}{2}}\bar{c}$. The fermions do not need rescaling. The path integral measure is now temperature independent. 2) Then to remove the temperature dependence from the zero mode action rescale the zero modes and gauge field as $X^i\to\beta^{-\frac{3}{4}}X^i$, $\Phi_\rho\to\beta^{-\frac{3}{4}}\Phi_\rho$ and $A\to\beta^{\frac{3}{4}}A$. The measure changes as indicated above.

and S_{int} is the interaction part of the action. The terms quadratic in non-zero modes present in S_{int} but not present in the BFSS model are

$$\Delta S_{\text{int}} = -N\beta^{\frac{3}{4}}(V_1^{(A)} + V_1^{(B)}) - N\beta^{\frac{3}{2}}(V_2^{(A)} + V_2^{(B)} + V_3) + O(\beta^{\frac{9}{4}}), \tag{3.6}$$

where

$$\begin{aligned}
V_1^{(A)} &= 4\pi \sum_{n\neq 0} n\,\text{tr}(\bar{\Phi}^{\rho}_{-n} A \Phi_{n\rho}) + \sum_r \text{tr}(i\chi^{\dagger}_{-r} A \chi_r),\\
V_1^{(B)} &= \sum_r \text{tr}(\chi^{\dagger}_{-r}\gamma^a X^a_0 \chi_r + \sqrt{2} i \varepsilon^{\rho\sigma}\chi^{\dagger}_{-r}\lambda_{r\rho}\Phi_{0\sigma} + \sqrt{2} i\varepsilon_{\rho\sigma}\bar{\Phi}^{\rho}_0 \lambda^{\dagger\sigma}_{-r}\chi_r),\\
V_2^{(A)} &= -\sum_{n\neq 0} \text{tr}(\bar{\Phi}^{\rho}_{-n} A^2 \Phi_{n\rho})\\
V_2^{(B)} &= -\sum_{n\neq 0} \text{tr}\,\big(\bar{\Phi}^{\rho}_{-n} X^{a\,2}_0 \Phi_{n\rho} + \bar{\Phi}^{\rho}_0 X^a_{-n} X^a_n \Phi_{0\rho} - \bar{\Phi}^{\rho}_{-n}[\bar{X}^{\sigma\dot{\rho}}_0, X_{0\rho\dot{\rho}}]\Phi_{n\sigma} - \bar{\Phi}^{\rho}_0[\bar{X}^{\sigma\dot{\rho}}_{-n}, X_{n\rho\dot{\rho}}]\Phi_{0\sigma}\\
&\qquad\qquad - \bar{\Phi}^{\rho}_{-n}\Phi_{0\sigma}\bar{\Phi}^{\sigma}_0\Phi_{n\rho} + 2\bar{\Phi}^{\rho}_{-n}\Phi_{0\rho}\bar{\Phi}^{\sigma}_0\Phi_{n\sigma}\big),\\
V_3 &= -\sum_{n\neq 0} \text{tr}\,\Big[(\bar{\Phi}^{\rho}_{-n} X^a_n X^a_0 \Phi_{0\rho} + \bar{\Phi}^{\rho}_{-n} X^a_0 X^a_n \Phi_{0\rho} + \bar{\Phi}^{\rho}_0 X^a_{-n} X^a_0 \Phi_{n\rho} + \bar{\Phi}^{\rho}_0 X^a_0 X^a_{-n}\Phi_{n\rho})\\
&\qquad - (\bar{\Phi}^{\rho}_{-n}[\bar{X}^{\sigma\dot{\rho}}_n, X_{0\rho\dot{\rho}}]\Phi_{0\sigma} + \bar{\Phi}^{\rho}_{-n}[\bar{X}^{\sigma\dot{\rho}}_0, X_{n\rho\dot{\rho}}]\Phi_{0\sigma}\\
&\qquad\quad + \bar{\Phi}^{\rho}_0[\bar{X}^{\sigma\dot{\rho}}_{-n}, X_{0\rho\dot{\rho}}]\Phi_{n\sigma} + \bar{\Phi}^{\rho}_0[\bar{X}^{\sigma\dot{\rho}}_0, X_{-n\rho\dot{\rho}}]\Phi_{n\sigma})\\
&\qquad - \frac{1}{2}(2\bar{\Phi}^{\rho}_{-n}\Phi_{n\sigma}\bar{\Phi}^{\sigma}_0\Phi_{0\rho} + \bar{\Phi}^{\rho}_{-n}\Phi_{0\sigma}\bar{\Phi}^{\sigma}_n\Phi_{0\rho} + \bar{\Phi}^{\rho}_0\Phi_{-n\sigma}\bar{\Phi}^{\sigma}_0\Phi_{n\rho})\\
&\qquad + (2\bar{\Phi}^{\rho}_{-n}\Phi_{n\rho}\bar{\Phi}^{\sigma}_0\Phi_{0\sigma} + \bar{\Phi}^{\rho}_{-n}\Phi_{0\rho}\bar{\Phi}^{\sigma}_n\Phi_{0\sigma} + \bar{\Phi}^{\rho}_0\Phi_{-n\rho}\bar{\Phi}^{\sigma}_0\Phi_{n\sigma})\\
&\qquad - \sum_r (\chi^{\dagger}_{-r}\gamma^a X^a_{-n}\chi_{r+n} + \sqrt{2} i\varepsilon^{\rho\sigma}\chi^{\dagger}_{-r}\lambda_{r+n\,\rho}\Phi_{-n\sigma} + \sqrt{2} i \varepsilon_{\rho\sigma}\bar{\Phi}^{\rho}_{-n}\lambda^{\dagger\sigma}_{-r}\chi_{r+n})\Big].
\end{aligned} \tag{3.7}$$

V_3 does not contribute to the expectation values of operators up to next-leading order as we will see later. The fermionic terms that involve only non-zero modes also scale as $\beta^{\frac{3}{2}}$, however, they contribute at a two and higher loop order to the expectation values of observables resulting in higher order temperature dependence and so need not be considered here.

The zero-mode action (3.4) corresponds to the bosonic part of the original model (2.3) dimensionally reduced to a point and plays an important role in the ADHM construction as the solutions to $S_0 = 0$ with $\mathcal{D}^A = 0$, where $\mathcal{D}^A$ is given in (2.4), provide the ADHM data[16]. This zero-mode model is the flavoured bosonic version of the IKKT model[4]. We use the notation $\langle \cdots \rangle_{\text{DR}}$ for the expectation value calculated with this dimensionally reduced model.

Following [6], we use $\langle\!\langle \cdots \rangle\!\rangle$ to denote expectation values obtained by integrating out the non-zero modes of (3.3) with the zero modes X^i_0, $\Phi_{0\rho}$ and A as background fields.

3.1 Leading order

The expectation values of our observables to leading order are determined solely by the zero modes. The energy obtains its leading contribution from the change of measure described above and behaves as

$$E = \frac{3}{N}\beta^{-1}\left(\langle s_0\rangle_{\rm DR} + O(\beta^{\frac{3}{2}})\right)$$
$$= \frac{3}{4}\beta^{-1}\left\{8\left(1-\frac{1}{N^2}\right)+\frac{4N_f}{N}+O(\beta^{\frac{3}{2}})\right\}, \tag{3.8}$$

where

$$s_0 = {\rm Tr}\left(-\frac{1}{4}[X_0^i, X_0^j]^2\right) + {\rm tr}\left(\bar\Phi_0^\rho X_0^{a\,2}\Phi_{0\rho} - \bar\Phi_0^\rho[\bar X_0^{\sigma\dot\rho}, X_{0\rho\dot\rho}]\Phi_{0\sigma} - \frac{1}{2}\bar\Phi_0^\rho\Phi_{0\sigma}\bar\Phi_0^\sigma\Phi_{0\rho} + \bar\Phi_0^\rho\Phi_{0\rho}\bar\Phi_0^\sigma\Phi_{0\sigma}\right). \tag{3.9}$$

The expression in (3.8) can be equally derived using the Dyson-Schwinger equations and scales with the number of physical zero modes. Also, the leading terms in the β-expansion of $\langle R^2\rangle$ and the expectation value of the Polyakov loop are

$$\langle R^2\rangle = \frac{1}{N}\beta^{-\frac{1}{2}}\left\langle {\rm Tr}\, X_0^{i\,2}\right\rangle_{\rm DR} + O(\beta)\,, \quad \langle P\rangle = 1 - \frac{1}{N}\beta^{\frac{3}{2}}\frac{1}{2}\left\langle {\rm Tr}\, A^2\right\rangle_{\rm DR} + O(\beta^3)\,. \tag{3.10}$$

For our new observables $\langle r^2\rangle$ and $\langle \mathcal{C}^m\rangle$ we have the leading contributions

$$\langle r^2\rangle = \frac{1}{N_f}\beta^{-\frac{1}{2}}\left\langle {\rm tr}\,\bar\Phi_0^\rho\Phi_{0\,\rho}\right\rangle_{\rm DR} + O(\beta) \tag{3.11}$$

and

$$\langle \mathcal{C}^m\rangle = 2\beta^{-\frac{1}{2}}\left(\left\langle {\rm tr}\,\bar\Phi_0^\rho\Phi_{0\rho}\right\rangle_{\rm DR} - \frac{2N}{5}\left\langle({\rm tr}\,\bar\Phi_0^\rho X_0^a\Phi_{0\rho})^2\right\rangle_{\rm DR,c}\right) + O(\beta)\,. \tag{3.12}$$

Note: All the leading order contributions are purely bosonic, since fermions decouple at high temperature. The necessary expectation values are computed numerically via Monte Carlo simulation with the action S_0 of equation (3.4) and given in the tables in appendix B for different values of N and N_f.

3.2 Next-leading order

The higher order contributions in the high temperature expansion come from integrating out the non-zero modes in (3.3). The first subleading order is obtained by performing the gaussian integrals over the non-zero modes, where the potential is truncated as in (3.6), and expanding the resulting exponential and ratio of determinants in terms of β. Alternatively one can simply perform direct perturbation theory in β. When the latter is done there are two contributions to the expectation value of an observable at subleading order: the next-leading terms of an operator itself and the contribution from the expansion of $e^{-S_{\rm int}}$. To clarify the latter contribution, let us define $\mathcal{O}$ and $\mathcal{Q}$ by

$$\left\langle\!\!\left\langle e^{-S_{\rm int}}\right\rangle\!\!\right\rangle = 1 + \beta^{\frac{3}{2}}\mathcal{O} + O(\beta^3), \qquad e^{-S_{\rm int}} = 1 + \beta^{\frac{3}{4}}(\mathcal{Q}^b + \mathcal{Q}^f) + O(\beta^{\frac{3}{2}}), \tag{3.13}$$

where $\mathcal{O}$, $\mathcal{Q}^b$ and $\mathcal{Q}^f$ are independent of β. The terms with only bosonic fields are collected in $\mathcal{Q}^b$, while the rest of the terms are contained in $\mathcal{Q}^f$. They vanish after one takes the expectation value over the non-zero modes, but $\mathcal{Q}^f$ is necessary for the calculation of an operator that contains fermionic fields. Let us denote by $\mathcal{A}^b$ an arbitrary operator that contains only bosonic fields or a multiple of four fermionic fields satisfying $\langle\!\langle \mathcal{A}^b Q^f \rangle\!\rangle = 0$ and another by $\mathcal{A}^f$. Those operators are expanded as $\mathcal{A}^{b,f} = \beta^{\alpha_0} \sum_j \beta^{\frac{3}{2}j} \mathcal{A}_j^{b,f}$, where $\mathcal{A}_j^{b,f}$ are independent of β and the index j is an integer running from 0. Then, given that we are working to the order of $O(\beta^{\frac{3}{2}})$, we have

$$\langle \mathcal{A}^b \rangle = \beta^{\alpha_0} \left\{ \langle \mathcal{A}_0^b \rangle_{\rm DR} + \beta^{\frac{3}{2}} \left(\langle \langle\!\langle \mathcal{A}_1^b \rangle\!\rangle \rangle_{\rm DR} + \langle \mathcal{A}_0^b \cdot \mathcal{O} \rangle_{\rm DR,c} \right) + O(\beta^3) \right\} , \tag{3.14}$$

and

$$\langle \mathcal{A}^f \rangle = \beta^{\alpha_0} \left\{ \langle \langle\!\langle \mathcal{A}_0^f \rangle\!\rangle \rangle_{\rm DR} + \beta^{\frac{3}{4}} \langle \langle\!\langle \mathcal{A}_0^f \cdot \mathcal{Q}^f \rangle\!\rangle \rangle_{\rm DR,c} + \beta^{\frac{3}{2}} \left(\langle \langle\!\langle \mathcal{A}_1^f \rangle\!\rangle \rangle_{\rm DR} + \langle \langle\!\langle \mathcal{A}_0^f \rangle\!\rangle \cdot \mathcal{O} \rangle_{\rm DR,c} \right) + O(\beta^{\frac{9}{4}}) \right\} , \tag{3.15}$$

where $\langle \cdots \rangle_{\rm DR,c}$ means the connected part of the expectation value in terms of S_0. The contributions to $\mathcal{A}_0^{b,f}$, $\mathcal{A}_1^{b,f}$ and $\mathcal{O}$ for E, R^2 and P from the pure BFSS model were derived in [6]. For these observables it is therefore necessary to calculate only the new contributions from the flavour fields. For the convenience of the reader we gather the contributing $\mathcal{A}_i^{b,f}$ for our different observables in appendix A. The BFSS part of $\mathcal{O}$ and $\mathcal{Q}^f$ is

$$\mathcal{O}_{\rm BFSS} = \frac{4}{3} N \left({\rm Tr}\, X_0^{i\,2} - {\rm Tr}\, A^2 \right) , \qquad \mathcal{Q}^f_{\rm BFSS} = \frac{N}{2} \sum_r {\rm Tr} \left(\Psi^T_{-r} C_{10} \Gamma^i [X_0^i, \Psi_r] \right) , \tag{3.16}$$

and the new flavour contribution $\varDelta\mathcal{O} = \mathcal{O} - \mathcal{O}_{\rm BFSS}$ and $\varDelta\mathcal{Q}^f = \mathcal{Q}^f - \mathcal{Q}^f_{\rm BFSS}$ is

$$\varDelta\mathcal{O} = N \left\langle\!\!\left\langle V_2^{(A)} + V_2^{(B)} + \frac{1}{2} N \left(V_1^{(A)\,2} + V_1^{(B)\,2} \right) \right\rangle\!\!\right\rangle , \tag{3.17}$$

and

$$\varDelta\mathcal{Q}^f = N \left(\sum_r {\rm tr} (i \chi^\dagger_{-r} A \chi_r) + V_1^{(B)} \right) . \tag{3.18}$$

$\langle\!\langle V_3 \rangle\!\rangle$ is zero, and the cross-term between V_1 and the interaction term originally contained in the BFSS model vanishes to this order. A one-loop order calculation of (3.17) results in

$$\begin{aligned} \langle\!\langle V_1^{(A)\,2} \rangle\!\rangle &= -\frac{N_f}{3N^2} {\rm Tr}\, A^2 , \qquad \langle\!\langle V_1^{(B)\,2} \rangle\!\rangle = \frac{1}{N^2} \left(N_f \,{\rm Tr}\, X_0^{a\,2} + 4N \,{\rm tr}\, \bar\Phi_0^\rho \Phi_{0\rho} \right) , \\ \langle\!\langle V_2^{(A)} \rangle\!\rangle &= -\frac{N_f}{6N} {\rm Tr}\, A^2 , \qquad \langle\!\langle V_2^{(B)} \rangle\!\rangle = -\frac{1}{6N} \left(N_f \,{\rm Tr}\, X_0^{a\,2} + 4N \,{\rm tr}\, \bar\Phi_0^\rho \Phi_{0\rho} \right) . \end{aligned} \tag{3.19}$$

Hence, at one-loop, $\mathcal{O}$ is

$$\mathcal{O} = \frac{4}{3} N \left({\rm Tr}\, X_0^{i\,2} - {\rm Tr}\, A^2 + {\rm tr}\, \bar\Phi_0^\rho \Phi_{0\rho} \right) + \frac{1}{3} N_f \left({\rm Tr}\, X_0^{a\,2} - {\rm Tr}\, A^2 \right) . \tag{3.20}$$

Let us pause and focus on the purely bosonic model, i.e. the model (2.3) without the fermions. The action is presented explicitly below (4.4) and we will denote expectation values with this action with a subscript 'bos'. For the bosonic BFSS model, using [6], we have the counterpart of $\mathcal{O}$, which is $O_{\text{BFSS,bos}} = -2N/3\,\text{Tr}(X_0^{i\,2} - A^2)$. For $\mathit{\Delta}\mathcal{O}$, its bosonic terms come from $V_2^{(A)}$, $V_2^{(B)}$ and $V_1^{(A)}$. While $V_2^{(A)}$ and $V_2^{(B)}$ are all purely bosonic, $V_1^{(A)}$ has both bosonic and fermionic contributions. The bosonic contribution is the first term of $V_1^{(A)}$ in (3.7) and taking the expectation value of its square gives $\langle\!\langle (4\pi \sum_{n\neq 0} n\,\text{tr}(\bar{\Phi}^{\rho}_{-n} A \Phi_{n\rho}))^2 \rangle\!\rangle = \frac{2N_f}{3N^2}\,\text{Tr}\,A^2$. Therefore, $\mathcal{O}$ for the bosonic model is

$$\mathcal{O}_{\text{bos}} = -\frac{2N}{3}\left\{\text{Tr}(X_0^{i\,2} - A^2) + \text{tr}(\bar{\Phi}^{\rho}_0 \Phi_{0\rho})\right\} - \frac{N_f}{6}\,\text{Tr}\left(X_0^{a\,2} - A^2\right) = -\frac{1}{2}\mathcal{O}\,, \tag{3.21}$$

and we conclude that the purely bosonic contribution to $\mathcal{O}$ is exactly $-1/2$ times the full $\mathcal{O}$.

Let us now find the subleading correction to the energy E. If one turns off the flavour degrees of freedom, that is, in the pure BFSS model, $\langle\!\langle \mathcal{A}_1^{\varepsilon_b} \rangle\!\rangle$ becomes $\frac{2}{N}\,\text{Tr}\,X_0^{i\,2}$ and $\langle\!\langle \mathcal{A}_0^{\epsilon_f} \mathcal{Q}^f \rangle\!\rangle$ becomes $-\frac{6}{N}\,\text{Tr}\,X_0^{i\,2}$ [6]. The new part added to the BFSS contribution is

$$-\frac{3}{N}\left\langle\!\!\left\langle V_2^{(B)} \right\rangle\!\!\right\rangle - \frac{3}{2}\left\langle\!\!\left\langle V_1^{(B)\,2} \right\rangle\!\!\right\rangle = -\frac{1}{N^2}\left(N_f\,\text{Tr}\,X_0^{a\,2} + 4N\,\text{tr}\,\bar{\Phi}^{\rho}_0\Phi_{0\rho}\right). \tag{3.22}$$

Putting these into (3.14), one obtains the high-temperature expansion of the energy as

$$E = \frac{3}{4}\beta^{-1}\left\{8\left(1 - \frac{1}{N^2}\right) + \frac{4N_f}{N}\right\} + \beta^{1/2}\Bigg[3\left\langle \frac{1}{N} s_0 \cdot \mathcal{O} \right\rangle_{\text{DR,c}} - 4\left\langle \frac{1}{N}\,\text{Tr}\,X_0^{i\,2} \right\rangle_{\text{DR}} - \frac{N_f}{N}\left\langle \frac{1}{N}\,\text{Tr}\,X_0^{a\,2} \right\rangle_{\text{DR}} - \frac{4N_f}{N}\left\langle \frac{1}{N_f}\,\text{tr}\,\bar{\Phi}^{\rho}_0\Phi_{0\rho} \right\rangle_{\text{DR}}\Bigg] + O(\beta^2)\,. \tag{3.23}$$

The fermionic contributions to the energy are contained in $\mathcal{O}$ and $\langle\!\langle \mathcal{A}_0^{\varepsilon_f} \mathcal{Q}^f \rangle\!\rangle$. The bosonic contribution to $\langle\!\langle \mathcal{A}_1^{\varepsilon_b} \rangle\!\rangle$ from flavours is $-\frac{3}{N}\langle\!\langle V_2^{(B)} \rangle\!\rangle$. Hence, the energy of the bosonic model is given by

$$E_{\text{bos}} = \frac{3}{4}\beta^{-1}\left\{8\left(1 - \frac{1}{N^2}\right) + \frac{4N_f}{N}\right\} - \frac{1}{2}\beta^{1/2}\Bigg[3\left\langle \frac{1}{N} s_0 \cdot \mathcal{O} \right\rangle_{\text{DR,c}} - 4\left\langle \frac{1}{N}\,\text{Tr}\,X_0^{i\,2} \right\rangle_{\text{DR}} - \frac{N_f}{N}\left\langle \frac{1}{N}\,\text{Tr}\,X_0^{a\,2} \right\rangle_{\text{DR}} - \frac{4N_f}{N}\left\langle \frac{1}{N_f}\,\text{tr}\,\bar{\Phi}^{\rho}_0\Phi_{0\rho} \right\rangle_{\text{DR}}\Bigg] + O(\beta^2)\,. \tag{3.24}$$

Turning to the high-temperature behaviour of R^2 and the Polyakov loop, we see there are no new contributions to either $\langle\!\langle \mathcal{A}_0 \rangle\!\rangle$ or $\langle\!\langle \mathcal{A}_1 \rangle\!\rangle$, since these are built from the bosonic sector of the BFSS model. The resulting expectation values are given by

$$\langle R^2 \rangle = \beta^{-\frac{1}{2}}\left\langle \frac{1}{N}\,\text{Tr}\,X_0^{i\,2} \right\rangle_{\text{DR}} + \beta\left(\frac{3}{4} + \left\langle \frac{1}{N}(\text{Tr}\,X_0^{i\,2})\mathcal{O} \right\rangle_{\text{DR,c}}\right) + O(\beta^{\frac{5}{2}})\,, \tag{3.25}$$

and

$$\langle P \rangle = 1 - \beta^{\frac{3}{2}}\left[\frac{1}{2}\left\langle \frac{1}{N}\,\text{Tr}\,A^2 \right\rangle_{\text{DR}} - \beta^{\frac{3}{2}}\left\{\frac{1}{4!}\left\langle \frac{1}{N}\,\text{Tr}\,A^4 \right\rangle_{\text{DR}} - \frac{1}{2}\left\langle \frac{1}{N}(\text{Tr}\,A^2)\mathcal{O} \right\rangle_{\text{DR,c}}\right\} + O(\beta^3)\right]. \tag{3.26}$$

These appear exactly as in BFSS model, however, they are computed with S_0 and $\mathcal{O}$ of equations (3.4) and (3.20) which depend on the fundamental fields. One can obtain their purely bosonic contribution simply by replacing $\mathcal{O}$ with $\mathcal{O}_{\rm bos}$ so that

$$\langle R^2\rangle_{\rm bos} = \beta^{-\frac{1}{2}}\left\langle \frac{1}{N}\,\mathrm{Tr}\,X_0^{i\,2}\right\rangle_{\rm DR} + \beta\left(\frac{3}{4} - \frac{1}{2}\left\langle \frac{1}{N}(\mathrm{Tr}\,X_0^{i\,2})\mathcal{O}\right\rangle_{\rm DR,c}\right) + O(\beta^{\frac{5}{2}})\,, \tag{3.27}$$

and

$$\langle P\rangle_{\rm bos} = 1 - \beta^{\frac{3}{2}}\left[\frac{1}{2}\left\langle\frac{1}{N}\,\mathrm{Tr}\,A^2\right\rangle_{\rm DR} - \beta^{\frac{3}{2}}\left\{\frac{1}{4!}\left\langle\frac{1}{N}\,\mathrm{Tr}\,A^4\right\rangle_{\rm DR} + \frac{1}{4}\left\langle\frac{1}{N}(\mathrm{Tr}\,A^2)\mathcal{O}\right\rangle_{\rm DR,c}\right\} + O(\beta^3)\right]. \tag{3.28}$$

Our observable $\langle r^2\rangle$ is similarly given by

$$\langle r^2\rangle = \beta^{-\frac{1}{2}}\left\langle\frac{1}{N_f}\,\mathrm{tr}\,\bar\Phi_0^\rho\Phi_{0\,\rho}\right\rangle_{\rm DR} + \beta\left(\frac{1}{6} + \left\langle\frac{1}{N_f}\left(\mathrm{tr}\,\bar\Phi_0^\rho\Phi_{0\,\rho}\right)\mathcal{O}\right\rangle_{\rm DR,c}\right) + O(\beta^{\frac{5}{2}}). \tag{3.29}$$

and its bosonic version is

$$\langle r^2\rangle_{\rm bos} = \beta^{-\frac{1}{2}}\left\langle\frac{1}{N_f}\,\mathrm{tr}\,\bar\Phi_0^\rho\Phi_{0\,\rho}\right\rangle_{\rm DR} + \beta\left(\frac{1}{6} - \frac{1}{2}\left\langle\frac{1}{N_f}\left(\mathrm{tr}\,\bar\Phi_0^\rho\Phi_{0\,\rho}\right)\mathcal{O}\right\rangle_{\rm DR,c}\right) + O(\beta^{\frac{5}{2}}). \tag{3.30}$$

In terms of Fourier modes we have

$$c^a(m) = \Bigg\langle \mathrm{tr}\Bigg(2\beta^{-\frac{1}{2}}m^a\bar\Phi_0^\rho\Phi_{0\rho} + 2\beta m^a\sum_{n\neq0}\bar\Phi_{-n}^\rho\Phi_{n\rho} + \sum_r\chi_r^\dagger\gamma^a\chi_r - 2\beta^{-\frac{3}{4}}\bar\Phi_0^\rho X_0^a\Phi_{0\rho} - 2\beta^{\frac{3}{4}}\sum_n(\bar\Phi_{-n}^\rho X_0^a\Phi_{n\rho} + \bar\Phi_0^\rho X_{-n}^a\Phi_{n\rho} + \bar\Phi_{-n}^\rho X_n^a\Phi_{0\rho}) - 2\beta^{\frac{3}{2}}\sum_{n,m}\bar\Phi_{-n}^\rho X_{n-m}^a\Phi_{m\rho}\Bigg)\Bigg\rangle. \tag{3.31}$$

However, we will restrict ourselves to the massless case and as discussed $SO(5)$ invariance guarantees that this observable is zero so we focus on the mass susceptibility, $\langle\mathcal{C}^m\rangle$.

Calculating $\mathcal{C}^m$ in the high temperature expansion to the next to leading order yields

$$\langle\mathcal{C}^m\rangle = 2\beta^{-\frac{1}{2}}\left(\langle\mathrm{tr}\,\bar\Phi_0^\rho\Phi_{0\rho}\rangle_{\rm DR} - \frac{2N}{5}\langle(\mathrm{tr}\,\bar\Phi_0^\rho X_0^a\Phi_{0\rho})^2\rangle_{\rm DR,c}\right) + 2\beta\left(-\frac{N_f}{3} + \left\langle(\mathrm{tr}\,\bar\Phi_0^\rho\Phi_{0\rho})\mathcal{O}\right\rangle_{\rm DR,c} - \frac{2N}{5}\left\langle(\mathrm{tr}\,\bar\Phi_0^\rho X_0^a\Phi_{0\rho})^2\mathcal{O}\right\rangle_{\rm DR,c}\right) + O(\beta^{\frac{5}{2}}). \tag{3.32}$$

The contribution $-N_f/3$ in the second parentheses in (3.32) contains both bosonic and fermionic contributions with the fermionic one being $-\frac{N}{10}\langle\!\langle(\sum_r\mathrm{tr}\,\chi_r^\dagger\gamma^a\chi_r)^2\rangle\!\rangle = -N_f/2$, while the bosonic contribution is given by $\langle\!\langle \sum_{n\neq 0} \mathrm{tr}\, \bar{\Phi}_0^\rho \Phi_{0\rho} \rangle\!\rangle = N_f/6$. Therefore, the derivative of the condensate operator for the bosonic model is given by

$$\langle \mathcal{C}^m \rangle_{\mathrm{bos}} = 2\beta^{-\frac{1}{2}} \left(\langle \mathrm{tr}\, \bar{\Phi}_0^\rho \Phi_{0\rho} \rangle_{\mathrm{DR}} - \frac{2N}{5} \langle (\mathrm{tr}\, \bar{\Phi}_0^\rho X_0^a \Phi_{0\rho})^2 \rangle_{\mathrm{DR,c}} \right) - \beta \left(-\frac{N_f}{3} + \left\langle (\mathrm{tr}\, \bar{\Phi}_0^\rho \Phi_{0\rho}) \mathcal{O} \right\rangle_{\mathrm{DR,c}} - \frac{2N}{5} \left\langle (\mathrm{tr}\, \bar{\Phi}_0^\rho X_0^a \Phi_{0\rho})^2 \mathcal{O} \right\rangle_{\mathrm{DR,c}} \right) + O(\beta^{\frac{5}{2}}). \tag{3.33}$$

4 Numerical simulations

In summary, we have the following expressions for flavoured bosonic IKKT-model observables:

$$\begin{aligned}
\Xi_1 &= \left[3 \left\langle \frac{1}{N} s_0 \cdot \mathcal{O} \right\rangle_{\mathrm{DR,c}} \right. \\
&\quad \left. - 4 \left\langle \frac{1}{N} \mathrm{Tr}\, X_0^{i\,2} \right\rangle_{\mathrm{DR}} - \frac{N_f}{N} \left\langle \frac{1}{N} \mathrm{Tr}\, X_0^{a\,2} \right\rangle_{\mathrm{DR}} - \frac{4N_f}{N} \left\langle \frac{1}{N_f} \mathrm{tr}\, \bar{\Phi}_0^\rho \Phi_{0\rho} \right\rangle_{\mathrm{DR}} \right], \\
\Xi_2 &= \left\langle \frac{1}{N} \mathrm{Tr}\, X_0^{i\,2} \right\rangle_{\mathrm{DR}}, \\
\Xi_3 &= \frac{3}{4} + \left\langle \frac{1}{N} (\mathrm{Tr}\, X_0^{i\,2}) \mathcal{O} \right\rangle_{\mathrm{DR,c}}, \\
\Xi_4 &= \frac{1}{2} \left\langle \frac{1}{N} \mathrm{Tr}\, A^2 \right\rangle_{\mathrm{DR}}, \\
\Xi_5 &= \frac{1}{4!} \left\langle \frac{1}{N} \mathrm{Tr}\, A^4 \right\rangle_{\mathrm{DR}} - \frac{1}{2} \left\langle \frac{1}{N} (\mathrm{Tr}\, A^2) \mathcal{O} \right\rangle_{\mathrm{DR,c}}, \\
\Xi_6 &= 2 \left(\langle \mathrm{tr}\, \bar{\Phi}_0^\rho \Phi_{0\rho} \rangle_{\mathrm{DR}} - \frac{2N}{5} \langle (\mathrm{tr}\, \bar{\Phi}_0^\rho X_0^a \Phi_{0\rho})^2 \rangle_{\mathrm{DR,c}} \right), \\
\Xi_7 &= 2 \left(-\frac{N_f}{3} + \left\langle (\mathrm{tr}\, \bar{\Phi}_0^\rho \Phi_{0\rho}) \mathcal{O} \right\rangle_{\mathrm{DR,c}} \right), \\
\Xi_8 &= -\frac{4N}{5} \left\langle (\mathrm{tr}\, \bar{\Phi}_0^\rho X_0^a \Phi_{0\rho})^2 \mathcal{O} \right\rangle_{\mathrm{DR,c}}, \\
\Xi_9 &= \left\langle \frac{1}{N_f} \mathrm{tr}\, \bar{\Phi}_0^\rho \Phi_{0\,\rho} \right\rangle_{\mathrm{DR}}.
\end{aligned} \tag{4.1}$$

For the bosonic BD model we have

$$\begin{aligned}
E_{\mathrm{bos}} &= \frac{3}{4} \beta^{-1} \left\{ 8 \left(1 - \frac{1}{N^2} \right) + \frac{4N_f}{N} \right\} - \frac{1}{2} \beta^{1/2} \Xi_1 + O(\beta^2), \\
\langle R^2 \rangle_{\mathrm{bos}} &= \beta^{-\frac{1}{2}} \Xi_2 + \beta \left(\frac{9}{8} - \frac{1}{2} \Xi_3 \right) + O(\beta^{\frac{5}{2}}), \\
\langle P \rangle_{\mathrm{bos}} &= 1 - \beta^{\frac{3}{2}} \left[\Xi_4 - \beta^{\frac{3}{2}} \left(\frac{3}{2} \Xi_{10} - \frac{1}{2} \Xi_5 \right) + O(\beta^3) \right],
\end{aligned}$$

$$\langle r^2\rangle_{\rm bos} = \beta^{-\frac{1}{2}}\Xi_9 - \beta\frac{1}{4N_f}\Xi_7 + O(\beta^{\frac{5}{2}})$$
$$\langle \mathcal{C}^m\rangle_{\rm bos} = \beta^{-\frac{1}{2}}\Xi_6 - \frac{1}{2}\beta(\Xi_7+\Xi_8) + O(\beta^{\frac{5}{2}}). \tag{4.2}$$

The observables of interest for the high temperature expansion are all expressed in terms of Ξ_i listed above. As discussed they are temperature independent and depend only on N, the matrix dimension of the BFSS fields and N_f the number of flavour multiplets. We computed their values for a range of N and N_f by hybrid Monte Carlo simulation with the action S_0 given in (3.4). We tabulate our results for different N and N_f. We choose $N = 4, 6, 8, 9, 10, 12, 14, 16, 18, 20, 32$ for $N_f = 1$ and tabulate them in Table 1.

From the results of Table 1 we extrapolate the N-dependence of the Ξ's by fitting them with a function[4], $a + b/N + c/N^2$ (see Figure 3 and 4). The limiting extrapolated values are included as the row $N = \infty$ in Table 1.

$\Xi_{i=1,\cdots,5,\text{and}10}$ naturally reduce to counterparts in the BFSS model when the fundamental fields are removed. We extrapolate $\Xi_{i=1,\cdots,5,\text{and}10}$ for $N_f = 1, 2, 4, 6, 8, 10, 12, 14, 16$ for fixed $N = 12, 14, 16, 18$ and 20 to $N_f = 0$ and find good agreement, to within the quoted errors, with the measured values for their BFSS counterparts as quoted in [6] .

Figure 5 shows plots of the Ξ_i's against N_f for each N and we fit the dependence on N_f with a quartic polynomial for $\Xi_{i=1,2,4 \text{ and } 9}$, however, we find that higher order terms contribute for the other Ξ_i and by using the fitting function $a + bN_f + ce^{dN_f}$ we are able to capture the dependence on N_f over the range considered.

In terms of the Ξ_i the observables of the full BD model (2.3) become:

$$E = \frac{3}{4}\beta^{-1}\left\{8\left(1-\frac{1}{N^2}\right)+\frac{4N_f}{N}\right\} + \beta^{1/2}\Xi_1 + O(\beta^2),$$
$$\langle R^2\rangle = \beta^{-\frac{1}{2}}\Xi_2 + \beta\Xi_3 + O(\beta^{\frac{5}{2}})\,,$$
$$\langle P\rangle = 1 - \beta^{\frac{3}{2}}\left[\Xi_4 - \beta^{\frac{3}{2}}\Xi_5 + O(\beta^3)\right],$$
$$\langle r^2\rangle = \beta^{-\frac{1}{2}}\Xi_9 + \beta\left(\frac{1}{2}+\frac{\Xi_7}{2N_f}\right) + O(\beta^{\frac{5}{2}})\,,$$
$$\langle \mathcal{C}^m\rangle = \beta^{-\frac{1}{2}}\Xi_6 + \beta(\Xi_7+\Xi_8) + O(\beta^{\frac{5}{2}})\,. \tag{4.3}$$

We are in the process of making a direct comparison of both the high temperature regime of the BD model as determined by the above predictions and the low temperature regime as predicted by gauge/gravity with results from a rational hybrid Monte Carlo simulation using the code used in [15]. We will present those results in a separate paper as, apart from their value as a check on the code and the computations presented here, they have additional physics that merits a separate discussion.

[4]Note that as expected we find it necessary to include a linear fall of in $1/N$ for large N. This is in contrast to the BFSS model where the fall off is $1/N^2$

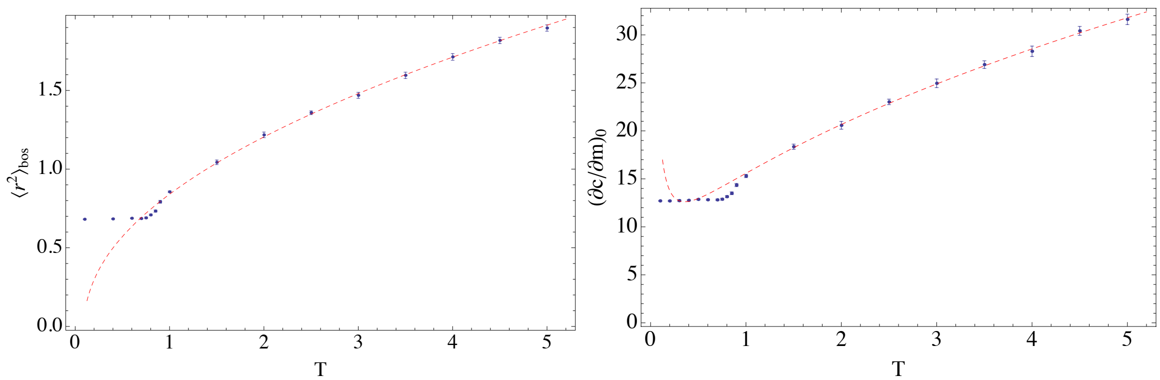

Figure 1: Comparison of the high temperature predictions for the fundamental observable $\langle r^2\rangle_{\rm bos}$ and the derivative of the condensate at zero mass, $(\partial c/\partial m)_0 = \langle \mathcal{C}^m\rangle_{\rm bos}$, with a Monte Carlo simulation of the bosonic BD model. The simulation is for $N_f = 1$ and $N = 10$.

For this paper we restrict our considerations to a comparison of the results obtained here with those obtained from the bosonic Berkooz-Douglas model given by (2.3) when the fermions are excluded and whose Euclidean action is given by

$$S_{\rm bos} = N\int_0^\beta d\tau \Bigg[\mathrm{Tr}\left(\frac{1}{2}D_\tau X^a D_\tau X^a + \frac{1}{2}D_\tau \bar{X}^{\rho\dot\rho} D_\tau X_{\rho\dot\rho} - \frac{1}{4}[X^a, X^b]^2 + \frac{1}{2}[X^a, \bar{X}^{\rho\dot\rho}][X^a, X_{\rho\dot\rho}]\right) \\ + \mathrm{tr}\left(D_\tau \bar\Phi^\rho D_\tau \Phi_\rho + \bar\Phi^\rho (X^a - m^a)^2 \Phi_\rho\right) + \frac{1}{2}\mathrm{Tr}\sum_{A=1}^{3} \mathcal{D}^A\mathcal{D}^A\Bigg]. \qquad (4.4)$$

Our comparison is presented in Figure 1 where we restrict our considerations to a high precision test with $N = 10$ and $N_f = 1$. As one can see from the figure the agreement is excellent. Furthermore, the high T expansion remains valid at temperatures as low as $T \sim 1.0$. Below this temperature the figure shows evidence of a phase transition. This is the phase transition of the bosonic BFSS model.

From studies of the bosonic BFSS model [10, 20, 21] we know that it undergoes two phase transitions at $T_{c2} = 0.905 \pm 0.002$ and $T_{c1} = 0.8761 \pm 0.0003$. These are driven by the gauge field A which at high temperatures behaves as one of the X^i while at low temperatures it effectively disappears from the system and can be gauged away at zero temperature. As the temperature is increased through T_{c1} there is a deconfining phase transition with the symmetry $A(t) \to A(t) + \alpha\mathbf{1}$ broken and the distribution of eigenvalues of the holonomy[5] becomes non-uniform. When the temperature reaches T_{c2} the spectrum of the holonomy becomes gapped and above this temperature the eigenvalues no longer cover the entire $[0, 2\pi]$ range. In the low temperature phase the bosonic BFSS model becomes a set of massive

[5] The Polyakov loop, $P = \frac{1}{N}\,\mathrm{Tr}(U)$, where U is the holonomy.

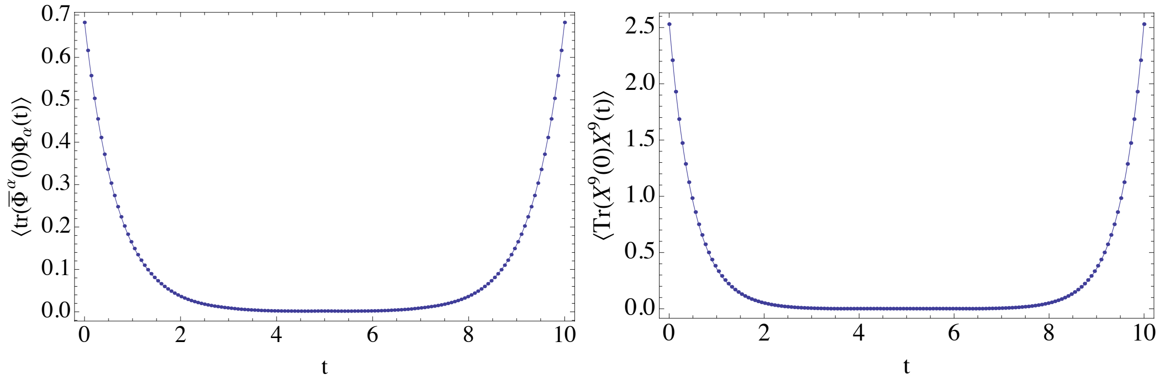

Figure 2: Plots of the Green's functions equations (4.7) and (4.8) for $\beta = 10$, $\Lambda = 144$, $N = 10$ and $N_f = 1$. The fits correspond to $m_f = 1.461$ and $m_A^l = 2.001$.

gaussian matrix models with Euclidean action

$$S_{eff} = N \int_0^\beta d\tau \,\mathrm{Tr}\left(\frac{1}{2}(\mathcal{D}_\tau X^i)^2 + m_A^2 (X^i)^2\right) , \tag{4.5}$$

with the mass $m_A = 1.965 \pm 0.007$.

For the flavoured model the BFSS transition is still present and when the X^i become massive they induce a mass for the fundamental scalars and the induced bare mass for these is estimated by integrating out the adjoint fields and expanding it to quadratic order in Φ_ρ. This gives a mass $m_f^0 = \sqrt{\frac{5}{2m_A}} \sim 1.128$. However, the fundamental scalars are still strongly interacting as they have a selfcoupling of order one and we expect the bare mass to become significantly dressed. We therefore estimate the physical mass of the scalars at sufficiently low temperature by assuming that they also can be described by a massive gaussian with mass m_f, in which case

$$\langle r^2\rangle_{\mathrm{bos}} = \left\langle \frac{1}{\beta N_f}\int d\tau \;\mathrm{tr}\,\bar{\Phi}^\rho \Phi_\rho \right\rangle_{\mathrm{bos}} \simeq \frac{1}{m_f} . \tag{4.6}$$

Note that the right-hand side of equation (4.6) is independent of β and from Figure 1 we see that $\langle r^2\rangle_{\mathrm{bos}}$ is more or less constant below the transition. A direct measurement of the expectation value (4.6) at $\beta = 0.5$ gives $0.68618 \simeq \frac{1}{m_f}$ which gives the estimate $m_f \simeq 1.4667$.

However, at zero temperature we can extract the masses for the different fields by measuring their Green's function. To this end we set the holonomy to zero, the parameter β is now just the length of the time circle and not an inverse temperature. Because the $SO(9)$ symmetry of the bosonic BFSS model is broken down to $SO(5) \times SO(4)$ there are now two adjoint masses, a longitudinal mass, m_A^l, for the four $X_{\rho\dot{\rho}}$ and a transverse mass, m_A^t , for the five matrices X^a. In figure 2 we present results for the Green's functions:

$$\langle \mathrm{tr}\,\bar{\Phi}^\rho(0)\Phi_\rho(\tau)\rangle = \frac{N_f}{m_f}\,\frac{e^{-m_f\tau} + e^{-m_f(\beta-\tau)}}{1 - e^{-\beta m_f}} , \tag{4.7}$$

$$\langle \mathrm{Tr} X^9(0) X^9(\tau)\rangle = \frac{N}{2m_A^l}\,\frac{e^{-m_A^l\tau}+e^{-m_A^l(\beta-\tau)}}{1-e^{-\beta\, m_A^l}}\ , \tag{4.8}$$

where we have chosen the last of the four $SO(4)$ adjoint scalars. We have also measured the longitudinal mass m_A^l by measuring the correlator for X^1 defined similarly to (4.8). The results for $\beta = 10$, $\Lambda = 144$, $N = 10$ and $N_f = 1$ are $m_A^l = 2.001 \pm 0.003$, $m_A^t = 1.964 \pm 0.003$ and $m_f = 1.463 \pm 0.001$. The prediction from assuming that the adjoint fields are described by an action of the form (4.5) with different masses for the transverse and longitudinal matrices is:

$$\langle R^2\rangle_{\mathrm{bos}} \simeq \frac{5}{2m_A^t} + \frac{4}{2m_A^l} = 2.270 \pm 0.001 \tag{4.9}$$

which agrees well with the direct measurement where we find $\langle R^2\rangle_{\mathrm{bos}} = 2.261 \pm 0.005$. Also the measured value of m_f using (4.6) predicts that $\langle r^2\rangle_{\mathrm{bos}} = 0.6836 \pm 0.0006$ which is in excellent agreement with the measured value.

Note that this estimate of the mass m_f is very close to the one obtained from equation (4.6). Also the slightly different values of the adjoint masses m_A^t and m_A^l from the purely BFSS case considered in equation (4.5) reflect the presence of backreaction at $N_f/N = 0.1$. Observe also the closeness of the transverse mass to the bosonic BFSS mass, which indicates that the backreaction is strongest for the longitudinal modes as one might expect.

We can now use this information to estimate the value of $\langle \mathcal{C}^m\rangle_{\mathrm{bos}}$ at zero temperature. Assuming that both X^a and Φ_ρ are well approximated by massive gaussians and using Wick's theorem on

$$\mathcal{C}^m_{\mathrm{bos}} = \frac{2}{\beta}\int_0^\beta d\tau\, \mathrm{tr}\,\bar{\Phi}^\rho\Phi_\rho - \frac{4N}{5\beta}\left(\int_0^\beta d\tau\,\mathrm{tr}\left\{\bar{\Phi}^\rho X^a \Phi_\rho\right\}\right)^2 \tag{4.10}$$

to perform the contractions, we obtain

$$\langle \mathcal{C}^m\rangle_{\mathrm{bos}}\Big|_{T=0} = \frac{2N_f}{m_f} - \frac{2N_f}{m_f^2 m_A^t(2m_f + m_A^t)} = 1.270 \pm 0.001\,. \tag{4.11}$$

Finally, a direct measurement of the measured condensate shown in Figure 1 for $T = 0.4, 0.3, 0.2$ and $T = 0.1$ extrapolated to $T = 0$ gives $\langle \mathcal{C}^m\rangle_{\mathrm{bos}}\big|_{T=0} = 1.268 \pm 0.003$ which is very close to the predicted value and confirms the validity of the gaussian approximation for both the adjoint and fundamental scalars.

5 Conclusions

We have obtained the first two terms in the high temperature series expansion for the Berkooz-Douglas model (BD model) for general adjoint matrix size N and fundamental multiplet dimension, N_f. These results should prove useful for future studies of this model. The model is an ideal testing ground for many ideas of gauge/gravity duality. The system is strongly coupled at low temperature while at high temperature it is weakly coupled, aside from the Matsubara zero-modes which remain strongly coupled even at high temperature. It is these modes that provide the residual non-perturbative aspect of the current study. Their effect can be captured in numerical coefficients that depend only on N and N_f.

Once the coefficients are determined and tabulated (see appendix B) they can be used as input for the high-temperature expansion of the observables of the BD model. We have checked these coefficients by comparing with a high precision simulation of the bosonic version of the BD model which we simulated using the Hybrid Monte Carlo approach. Its observables depend on the same numerical coefficients as the supersymmetric model and we find excellent agreement. In fact the observable $\langle r^2 \rangle_{\rm bos}$ (see equation (2.8)) and mass susceptibility (4.10) of the model, shown in Figure 1, show that the agreement is excellent even down to temperature one. Below this temperature the system undergoes a set of phase transitions. These are essentially the two phase transitions of the bosonic BFSS model.

We found that for $N_f/N = 0.1$ our measurements were sensitive to the backreaction of the fundamental fields on the adjoint fields. This backreaction lifted the mass degeneracy of the transverse and longitudinal adjoint fields. The transverse mass was essentially unaffected by the backreaction being $m_A^t = 1.964 \pm 0.003$ while the longitudinal mass was lifted to $m_A^l = 2.001 \pm 0.003$,

We found that using our understanding of the low temperature phase of the BFSS model as a system of massive gaussian quantum matrix models we could predict the zero temperature value of the mass susceptibility (4.10). The additional input that was required was the mass of the fundamental fields which we found by direct measurement to be $m_f = 1.463 \pm 0.001$.

The zero-mode model used to obtain the high-temperature coefficients is of independent interest as it is the potential that captures the ADHM data in the theory of Yang-Mills instantons on the four-sphere, S^4. It is the bosonic sector of the dimensional reduction of the BD model to zero dimensions and is equivalent to a flavoured version of the bosonic sector of the IKKT model. For this reason we refer to the model as the flavoured bosonic IKKT model. The potential is always positive semi-definite and the Higgs branch of its zero-locus is isomorphic to the instanton moduli space [16].

There was some evidence for peculiar behaviour in the zero mode model for $N_f \geq 2N$. We found that simulations required significant fine tuning for $N_f \geq 2N$, in that when using the same leapfrog step length which gave 95% acceptance rate for $N_f = 2N - 1$ the acceptance rate for $N_f \geq 2N$ fell to a fraction of this within a couple of thousand sweeps and Ward identities we use as checks on the simulations were not fulfilled. After tuning the simulation we found the generated configurations had very long auto-correlation time. Also, in fitting the dependence of the observables Ξ_i on N for a given N_f we found evidence for a simple pole at $N = 2N_f$. Furthermore, one can see from the results tabulated in appendix B that they grow rapidly when the region $N_f = 2N$ is approached. We expect that these difficulties and the growth of observables as $N_f = 2N$ is approached are related to the singular structure of the instanton moduli space which is isomorphic to the minimum of the potential in (2.3) with $X^a = 0$, $\mathcal{D}^A = 0$. We have not pursued this further in the current study as it would take us too far afield, however, we believe it merits a more careful study.

Finally, our preliminary studies of the supersymmetric BD model show [22] that, for some observables, the high temperature series expansion remains valid to lower temperatures than one might expect. This validity of the high T expansion at lower T could provide alternative quasi-analytic estimates for observables in the window where gauge/gravity duality is valid.

Acknowledgment

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A Expansion of the observables

$\mathcal{A}_0^{b,f}$ and $\mathcal{A}_1^{b,f}$ for the observables we are interested in are as follows. The energy has a part with only bosonic fields and a part that contains two fermionic fields. The purely bosonic part is

$$\begin{aligned} \varepsilon_b &= \beta^{-1}(\mathcal{A}_0^{\varepsilon_b} + \beta^{\frac{3}{2}}\mathcal{A}_1^{\varepsilon_b} + O(\beta^3)), \\ \mathcal{A}_0^{\varepsilon_b} &= \frac{3}{N}s_0, \\ \mathcal{A}_1^{\varepsilon_b} &\sim \frac{3}{N}\Bigg[\sum_{n\neq 0}\mathrm{Tr}\left(-\frac{1}{2}\left([X^i_{-n},X^j_0][X^i_0,X^j_n] + [X^i_{-n},X^j_0][X^i_n,X^j_0]\right)\right) - V_2^{(B)}\Bigg]. \end{aligned} \tag{A.1}$$

The expectation value of BFSS part in $\mathcal{A}_1^{\epsilon_b}$ is

$$\left\langle\!\!\left\langle \mathrm{Tr}\left(-\frac{1}{2}\left([X^i_{-n},X^j_0][X^i_0,X^j_n] + [X^i_{-n},X^j_0][X^i_n,X^j_0]\right)\right)\right\rangle\!\!\right\rangle = \frac{2}{N}\,\mathrm{Tr}\,X_0^{i\,2}. \tag{A.2}$$

The two-fermionic-field part of the energy is

$$\begin{aligned} \varepsilon_f &= \beta^{-1/4}(\mathcal{A}_0^{\varepsilon_f} + \beta^{\frac{3}{2}}\mathcal{A}_1^{\varepsilon_f} + O(\beta^3)), \\ \mathcal{A}_0^{\varepsilon_f} &= \frac{3}{2N}\left[\sum_r \mathrm{Tr}\left(-\frac{1}{2}\Psi^{T\rho}_{-r}C_{10}\gamma^i[X^i_0,\Psi_{r\,\rho}]\right) - V_1^{(B)}\right], \\ \mathcal{A}_1^{\varepsilon_f} &= \frac{3}{2N}\sum_{r,n}\Bigg[\mathrm{Tr}\left(-\frac{1}{2}\Psi^{T\rho}_{-r}C_{10}\gamma^i[X^i_{-n},\Psi_{r+n\,\rho}]\right) \\ &\qquad - \mathrm{tr}\left(\chi^\dagger_{-r}\gamma^a X^a_{-n}\chi_{r+n} + \sqrt{2}i\varepsilon^{\rho\sigma}\chi^\dagger_{-r}\lambda_{r+n\,\rho}\Phi_{-n\sigma} + \sqrt{2}i\varepsilon_{\rho\sigma}\bar{\Phi}^\rho_{-n}\lambda^{\dagger\sigma}_{-r}\chi_{r+n}\right)\Bigg]. \end{aligned} \tag{A.3}$$

R^2 and the Polyakov loop do not have fermionic parts. They are

$$\begin{aligned} R^2 &= \beta^{-\frac{1}{2}}(\mathcal{A}_0^{R^2} + \beta^{\frac{3}{2}}\mathcal{A}_1^{R^2} + O(\beta^3)), \\ \mathcal{A}_0^{R^2} &= \frac{1}{N}\,\mathrm{Tr}\,X_0^{i\,2}, \qquad \mathcal{A}_1^{R^2} = \frac{1}{N}\sum_{n\neq 0}\mathrm{Tr}\,X^i_{-n}X^i_n, \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} P &= 1 + \beta^{\frac{3}{2}}(\mathcal{A}_0^P + \beta^{\frac{3}{2}}\mathcal{A}_1^P + O(\beta^3)), \\ \mathcal{A}_0^P &= -\frac{1}{2}\frac{1}{N}\,\mathrm{Tr}\,A^2, \qquad \mathcal{A}_1^P = \frac{1}{4!}\frac{1}{N}\,\mathrm{Tr}\,A^4. \end{aligned} \tag{A.5}$$

The condensate operator has both bosonic and fermionic parts. We denote part with purely bosonic and four fermionic fields by $\mathcal{C}^{mb}$ and that with two fermionic fields by $\mathcal{C}^{mf}$. Then,

$$\mathcal{C}^{mb} = \beta^{-\frac{1}{2}}(\mathcal{A}_0^{\mathcal{C}^{mb}} + \beta^{\frac{3}{2}}\mathcal{A}_1^{\mathcal{C}^{mb}} + O(\beta^3)), \quad \mathcal{C}^{mf} = \beta^{-\frac{5}{4}}(\mathcal{A}_0^{\mathcal{C}^{mf}} + \beta^{\frac{3}{2}}\mathcal{A}_1^{\mathcal{C}^{mf}} + O(\beta^3)), \tag{A.6}$$

where

$$\mathcal{A}_0^{\mathcal{C}^{mb}} = 2\,\mathrm{tr}\,\bar{\Phi}_0^{\rho}\Phi_{0\,\rho} - \frac{4N}{5}(\mathrm{tr}\,\bar{\Phi}_0^{\rho}X_0^a\Phi_{0\,\rho})^2, \quad \text{and}$$
$$\langle\!\langle \mathcal{A}_1^{\mathcal{C}^{mb}} \rangle\!\rangle = \left\langle\!\!\left\langle 2\sum_{n\neq 0}\mathrm{tr}\,\bar{\Phi}_{-n}^{\rho}\Phi_{n\,\rho} - \frac{N}{5}\Big(\sum_r \mathrm{tr}\,\chi_{-r}^{\dagger}\gamma^a\chi_r\Big)^2\right\rangle\!\!\right\rangle,$$
$$\langle\!\langle \mathcal{A}_0^{\mathcal{C}^{mf}} \rangle\!\rangle = \left\langle\!\!\left\langle \frac{4N}{5}\sum_r \mathrm{tr}\,\bar{\Phi}_0^{\rho}X_0^a\Phi_{0\,\rho}\,\mathrm{tr}\,\chi_{-r}^{\dagger}\gamma^a\chi_r\right\rangle\!\!\right\rangle = 0,$$
$$\langle\!\langle \mathcal{A}_1^{\mathcal{C}^{mf}} \rangle\!\rangle = \left\langle\!\!\left\langle \frac{4N}{5}\sum_{n\neq 0,r} \mathrm{tr}\,\bar{\Phi}_{-n}^{\rho}X_0^a\Phi_{n\,\rho}\,\mathrm{tr}\,\chi_{-r}^{\dagger}\gamma^a\chi_r\right\rangle\!\!\right\rangle = 0.$$

r^2 is expanded as

$$r^2 = \beta^{-\frac{1}{2}}(\mathcal{A}_0^{r^2} + \beta^{\frac{3}{2}}\mathcal{A}_1^{r^2} + O(\beta^3)), \tag{A.7}$$

and

$$\mathcal{A}_0^{r^2} = \frac{1}{N_f}\,\mathrm{tr}\,\bar{\Phi}_0^{\rho}\Phi_{0\,\rho}\,,$$
$$\mathcal{A}_1^{r^2} = \frac{1}{N_f}\sum_{n\neq 0}\mathrm{tr}\,\bar{\Phi}_{-n}^{\rho}\Phi_{n\,\rho}\,.$$

B Tables for the observables Ξ_i.

In this appendix we gather the numerical data from Monte Carlo simulations for different matrix sizes, N and different numbers of flavour multiplets N_f and present it in tabular form.

$N_f = 1$					
N	Ξ_1	Ξ_2	Ξ_3	Ξ_4	Ξ_5
4	-5.3(1)	2.1492(3)	1.94(3)	0.11005(7)	0.146(2)
6	-5.3(2)	2.2201(2)	1.9(5)	0.11714(4)	0.155(4)
8	-5.3(2)	2.2475(1)	1.89(4)	0.12028(2)	0.159(3)
9	-5.3(1)	2.25562(7)	1.89(4)	0.12123(2)	0.161(3)
10	-5.36(5)	2.26162(2)	1.89(1)	0.121966(4)	0.162(4)
12	-5.4(2)	2.2699(5)	1.89(4)	0.12304(1)	0.162(1)
14	-5.4(2)	2.27509(4)	1.88(5)	0.12378(1)	0.164(6)
16	-5.4(2)	2.27883(3)	1.88(5)	0.124309(9)	0.165(5)
18	-5.4(2)	2.28149(3)	1.89(7)	0.124708(9)	0.166(6)
20	-5.4(3)	2.28353(3)	1.88(9)	0.12503(1)	0.167(9)
32	-5.4(6)	2.28974(3)	1.9(2)	0.126058(9)	0.17(2)
∞	-5.364(9)	2.29764(6)	1.894(2)	0.127633(8)	0.1707(3)

$N_f = 1$					
N	Ξ_6	Ξ_7	Ξ_8	Ξ_9	Ξ_{10}
4	1.4834(5)	0.16(3)	-0.39(2)	0.8974(2)	0.004050(5)
6	1.4389(3)	0.1(4)	-0.38(4)	0.8749(2)	0.004627(3)
8	1.4197(4)	0.08(5)	-0.37(7)	0.8644(1)	0.004893(2)
9	1.4128(4)	0.07(5)	-0.37(9)	0.8606(1)	0.004974(1)
10	1.4080(3)	0.07(2)	-0.37(3)	0.85799(3)	0.0050368(4)
12	1.4011(1)	0.05(9)	-0.4(1)	0.8539(1)	0.005129(1)
14	1.3971(3)	0.0(1)	-0.4(1)	0.8512(1)	0.005192(1)
16	1.3928(4)	0.0(1)	-0.4(2)	0.8490(1)	0.0052383(8)
18	1.3898(3)	0.0(2)	-0.4(3)	0.84732(9)	0.0052727(8)
20	1.3881(4)	0.0(3)	-0.3(4)	0.8460(1)	0.0053006(9)
32	1.3809(4)	0.0(7)	-0.3(10)	0.8415(1)	0.0053894(8)
∞	1.3700(4)	0.005(6)	-0.331(5)	0.8346(1)	0.005530(1)

Table 1: Mean values of observables $\Xi_i,\ i = 1, ..., 10$ for $N_f = 1$, were obtained from 3×10^6 Monte Carlo samples, while those for $N = 10$ had 3×10^7 samples. Errors are estimated with the Jackknife resampling. $N = \infty$ values were obtained by fitting each observable with the function $a + b/N + c/N^2$, quoted errors of this extrapolation are the fitting errors of the parameter a.

In the remaining tables we tabulate fixed $N = 9, 12, 14, 16, 18$ and 20 while we vary N_f. Mean values of observables $\Xi_i,\ i = 1, ..., 10$ were obtained from 3×10^6 samples generated by hybrid Monte Carlo simulation of flavoured bosonic IKKT model with the action specified in (3.4). Errors are estimated with the Jackknife resampling.

$N = 9$					
N_f	Ξ_1	Ξ_2	Ξ_3	Ξ_4	Ξ_5
2	0.11603(2)	0.151(3)	-6.6(2)	2.23789(7)	1.88(4)
4	0.10572(1)	0.136(3)	-8.8(2)	2.21338(7)	1.91(4)
6	0.09552(1)	0.125(3)	-10.8(2)	2.20729(7)	2.02(5)
8	0.08520(1)	0.120(3)	-12.5(3)	2.22714(8)	2.28(7)
10	0.07454(1)	0.124(3)	-14.1(5)	2.2875(1)	2.9(1)
12	0.06321(1)	0.144(4)	-15.7(8)	2.4209(2)	4.5(3)
14	0.05049(1)	0.205(4)	-18.(2)	2.7178(6)	9.8(9)
16	0.03475(2)	0.438(5)	-22.(7)	3.556(2)	4.4 (4) $\times$ 10

$N = 9$					
N_f	Ξ_6	Ξ_7	Ξ_8	Ξ_9	Ξ_{10}
2	2.9105(7)	0.4(1)	-0.8(1)	0.8894(1)	0.004557(1)
4	6.211(1)	1.9(2)	-2.2(4)	0.95582(8)	0.003788(1)
6	1.0025(2) $\times$ 10	5.8(5)	-4.5(8)	1.03843(9)	0.0030993(9)
8	1.4566(4) $\times$ 10	1.5(1) $\times$ 10	-9.(1)	1.1462(1)	0.0024753(7)
10	2.0199(7) $\times$ 10	3.5(3) $\times$ 10	-1.9(3) $\times$ 10	1.2945(2)	0.0019042(6)
12	2.772(1) $\times$ 10	8.9(6) $\times$ 10	-4.5(5) $\times$ 10	1.5183(3)	0.0013803(5)
14	3.931(2) $\times$ 10	2.7(2) $\times 10^2$	-1.3(1) $\times 10^2$	1.9094(7)	0.0008912(5)
16	6.346(6) $\times$ 10	1.5(1) $\times 10^3$	-6.4(7) $\times 10^2$	2.838(2)	0.0004317(5)

$N = 12$					
N_f	Ξ_1	Ξ_2	Ξ_3	Ξ_4	Ξ_5
2	-6.6(2)	2.25573(4)	1.88(4)	0.11914(1)	0.156(4)
4	-8.9(2)	2.23319(5)	1.89(5)	0.11139(1)	0.144(4)
6	-1.09(2) $\times$ 10	2.21952(5)	1.93(6)	0.10371(1)	0.133(4)
8	-1.27(3) $\times$ 10	2.21680(6)	2.02(8)	0.09604(1)	0.126(5)
10	-1.44(4) $\times$ 10	2.22852(8)	2.2(1)	0.0883(1)	0.121(5)
12	-1.58(5) $\times$ 10	2.26001(9)	2.5(1)	0.080412(8)	0.121(4)
14	-1.71(8) $\times$ 10	2.3206(1)	3.1(2)	0.072269(9)	0.127(5)
16	-1.8(1) $\times$ 10	2.4285(2)	4.4(4)	0.063664(9)	0.144(5)

$N = 12$					
N_f	Ξ_6	Ξ_7	Ξ_8	Ξ_9	Ξ_{10}
2	2.8635(6)	0.3(1)	-0.8(2)	0.87476(8)	0.004811(1)
4	6.000(1)	1.3(3)	-1.9(6)	0.92086(7)	0.0042087(9)
6	9.468(2)	3.5(6)	-4.(1)	0.97439(8)	0.0036540(8)
8	$1.3345(3) \times 10$	8.(1)	-6.(2)	1.0378(1)	0.0031398(7)
10	$1.7754(4) \times 10$	$1.6(2) \times 10$	$-1.(3) \times 10$	1.1149(1)	0.0026611(6)
12	$2.2888(6) \times 10$	$3.0(3) \times 10$	$-1.7(4) \times 10$	1.2121(1)	0.0022149(5)
14	$2.9051(8) \times 10$	$5.8(6) \times 10$	$-3.1(7) \times 10$	1.3391(1)	0.0017975(5)
16	$3.684(1) \times 10$	$1.2(1) \times 10^2$	$-6.(1) \times 10$	1.5145(3)	0.0014034(4)

$N = 14$					
N_f	Ξ_1	Ξ_2	Ξ_3	Ξ_4	Ξ_5
2	-6.6(2)	2.26275(4)	1.88(6)	0.12043(1)	0.158(5)
4	-8.9(2)	2.24204(4)	1.88(6)	0.113775(9)	0.147(4)
6	$-1.1(3) \times 10$	2.22748(5)	1.9(7)	0.107175(9)	0.137(5)
8	$-1.29(3) \times 10$	2.22024(4)	1.96(7)	0.100597(9)	0.130(5)
10	$-1.46(4) \times 10$	2.22179(5)	2.05(9)	0.094012(8)	0.124(5)
12	$-1.61(4) \times 10$	2.23486(5)	2.23(10)	0.087358(7)	0.121(5)
14	$-1.74(5) \times 10$	2.26318(7)	2.5(1)	0.080592(7)	0.121(5)
16	$-1.86(7) \times 10$	2.31272(8)	3.0(2)	0.073617(6)	0.126(5)

$N = 14$					
N_f	Ξ_6	Ξ_7	Ξ_8	Ξ_9	Ξ_{10}
2	2.8444(7)	0.2(2)	-0.8(3)	0.86849(8)	0.0049166(9)
4	5.917(1)	1.1(4)	-1.8(7)	0.90708(7)	0.0043920(7)
6	9.253(2)	2.8(6)	-3.(1)	0.95027(6)	0.0039014(7)
8	$1.2909(3) \times 10$	6.(1)	-5.(2)	0.99992(7)	0.0034427(6)
10	$1.6959(4) \times 10$	$1.1(2) \times 10$	-8.(3)	1.05806(8)	0.0030128(5)
12	$2.1501(6) \times 10$	$2.0(2) \times 10$	$-1.3(5) \times 10$	1.12713(8)	0.0026081(4)
14	$2.6664(7) \times 10$	$3.5(4) \times 10$	$-2.1(6) \times 10$	1.21165(9)	0.0022266(4)
16	$3.2747(10) \times 10$	$6.2(7) \times 10$	$-3.3(10) \times 10$	1.3179(1)	0.0018655(4)

$N = 16$					
N_f	Ξ_1	Ξ_2	Ξ_3	Ξ_4	Ξ_5
2	-6.6(2)	2.26780(4)	1.88(6)	0.121370(9)	0.160(6)
4	-8.9(3)	2.24878(4)	1.88(7)	0.115539(9)	0.150(6)
6	-1.11(3) × 10	2.23430(4)	1.89(7)	0.109767(8)	0.141(5)
8	-1.3(3) × 10	2.22501(3)	1.92(7)	0.103993(8)	0.134(6)
10	-1.48(3) × 10	2.22175(4)	1.99(9)	0.098248(7)	0.128(6)
12	-1.64(4) × 10	2.22610(4)	2.1(1)	0.092466(7)	0.124(6)
14	-1.78(5) × 10	2.23963(5)	2.3(1)	0.086624(7)	0.121(6)
16	-1.91(7) × 10	2.26527(6)	2.5(2)	0.080689(7)	0.121(6)

$N = 16$					
N_f	Ξ_6	Ξ_7	Ξ_8	Ξ_9	Ξ_{10}
2	2.8308(6)	0.2(2)	-0.7(4)	0.86416(7)	0.0049949(8)
4	5.856(1)	0.9(4)	-1.7(10)	0.89680(5)	0.0045293(7)
6	9.104(2)	2.3(8)	-3.(2)	0.93336(6)	0.0040921(6)
8	1.261(2) × 10	5.(1)	-5.(2)	0.97404(6)	0.0036776(6)
10	1.6429(4) × 10	9.(2)	-7.(4)	1.02062(6)	0.0032876(6)
12	2.0619(5) × 10	1.5(3) × 10	-10.(5)	1.07397(7)	0.0029176(5)
14	2.5243(7) × 10	2.5(4) × 10	-16.(8)	1.13660(9)	0.0025663(4)
16	3.0487(8) × 10	4.0(6) × 10	-2.3(9) × 10	1.2113(1)	0.0022331(4)

$N = 18$					
N_f	Ξ_1	Ξ_2	Ξ_3	Ξ_4	Ξ_5
2	-6.6(3)	2.27159(3)	1.88(7)	0.122117(9)	0.161(7)
4	-9.0(3)	2.25407(3)	1.88(8)	0.116915(9)	0.152(7)
6	-1.11(3) × 10	2.24009(3)	1.89(7)	0.111760(8)	0.144(7)
8	-1.31(3) × 10	2.22988(3)	1.91(8)	0.106645(7)	0.137(7)
10	-1.50(4) × 10	2.22424(4)	1.95(10)	0.101513(7)	0.131(7)
12	-1.66(4) × 10	2.22381(4)	2.0(1)	0.096393(6)	0.126(7)
14	-1.81(5) × 10	2.22966(5)	2.1(1)	0.091253(6)	0.123(6)
16	-1.95(7) × 10	2.24335(6)	2.3(2)	0.086042(6)	0.121(7)

$N = 18$					
N_f	Ξ_6	Ξ_7	Ξ_8	Ξ_9	Ξ_{10}
2	2.8196(6)	0.2(3)	-0.8(6)	0.86076(7)	0.0050571(8)
4	5.810(1)	0.8(5)	-2.(1)	0.88931(5)	0.0046382(7)
6	8.993(2)	2.0(8)	-3.(2)	0.92072(5)	0.0042418(6)
8	$1.2389(3) \times 10$	4.(1)	-4.(3)	0.95536(6)	0.0038662(6)
10	$1.6045(4) \times 10$	7.(2)	-6.(5)	0.99401(6)	0.0035075(3)
12	$1.9986(5) \times 10$	$1.2(3) \times 10$	-9.(6)	1.03745(7)	0.0031671(4)
14	$2.4286(7) \times 10$	$1.9(3) \times 10$	$-1.3(8) \times 10$	1.08685(6)	0.0028438(4)
16	$2.9011(7) \times 10$	$2.9(5) \times 10$	$-1.8(10) \times 10$	1.14411(8)	0.0025339(1)

$N = 20$					
N_f	Ξ_1	Ξ_2	Ξ_3	Ξ_4	Ξ_5
2	-6.6(3)	2.27449(4)	1.88(10)	0.122677(9)	0.162(8)
4	-9.0(4)	2.25833(4)	1.9(1)	0.118009(9)	0.154(9)
6	$-1.12(4) \times 10$	2.24488(4)	1.9(1)	0.113370(1)	0.146(10)
8	$-1.32(4) \times 10$	2.23449(3)	1.9(1)	0.108747(9)	0.14(1)
10	$-1.51(4) \times 10$	2.22749(4)	1.9(1)	0.104139(8)	0.134(9)
12	$-1.69(7) \times 10$	2.22442(5)	2.0(2)	0.099529(8)	0.13(1)
14	$-1.84(8) \times 10$	2.22584(5)	2.0(2)	0.094921(8)	0.125(10)
16	$-1.99(9) \times 10$	2.23279(6)	2.1(2)	0.090267(9)	0.12(1)

$N = 20$					
N_f	Ξ_6	Ξ_7	Ξ_8	Ξ_9	Ξ_{10}
2	2.8125(6)	0.2(4)	-0.7(7)	0.85819(8)	0.0051040(8)
4	5.775(1)	0.7(7)	-2.(1)	0.88333(6)	0.0047256(7)
6	8.905(2)	2.(1)	-3.(2)	0.91089(6)	0.0043640(8)
8	$1.2223(3) \times 10$	3.(2)	-4.(3)	0.94102(7)	0.0040190(7)
10	$1.5758(4) \times 10$	6.(2)	-6.(5)	0.97400(6)	0.0036895(6)
12	$1.9535(5) \times 10$	$1.0(4) \times 10$	-8.(7)	1.01069(8)	0.0033744(6)
14	$2.3595(6) \times 10$	$1.5(6) \times 10$	$-1.(1) \times 10$	1.05149(9)	0.0030736(5)
16	$2.7992(8) \times 10$	$2.3(7) \times 10$	$-2.(1) \times 10$	1.0975(1)	0.0027844(6)

Table 2: The tables for $N = 9$ to $N = 20$ were obtained from Monte Carlo simulations with 3×10^6 samples and errors are estimated using Jackknife resampling.

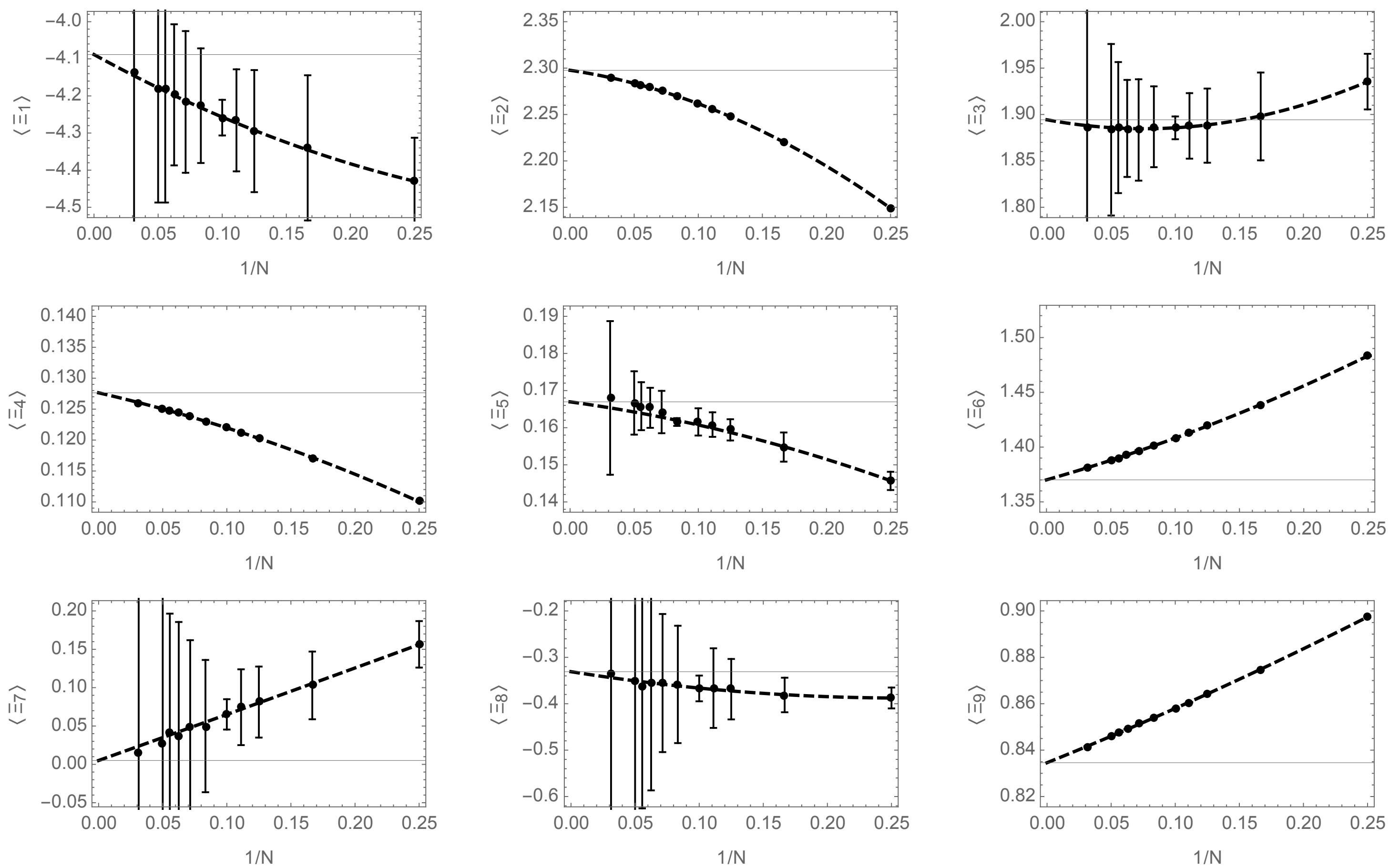

Figure 3: Mean values of observables $\Xi_i,\ i = 1, ..., 9$ plotted against N with $N_f = 1$. Data were generated using 3×10^6 (3×10^7 for the $N = 10$ case) samples generated by hybrid Monte Carlo simulation, dashed lines correspond to fits of the form $a + b/N + c/N^2$, vertical lines correspond to $N \to \infty$ values obtained from those fits. Errors are estimated with the Jackknife resampling. We have not included plots of Ξ_{10}. Its values are quite small and it is determined rather precisely in the tables.

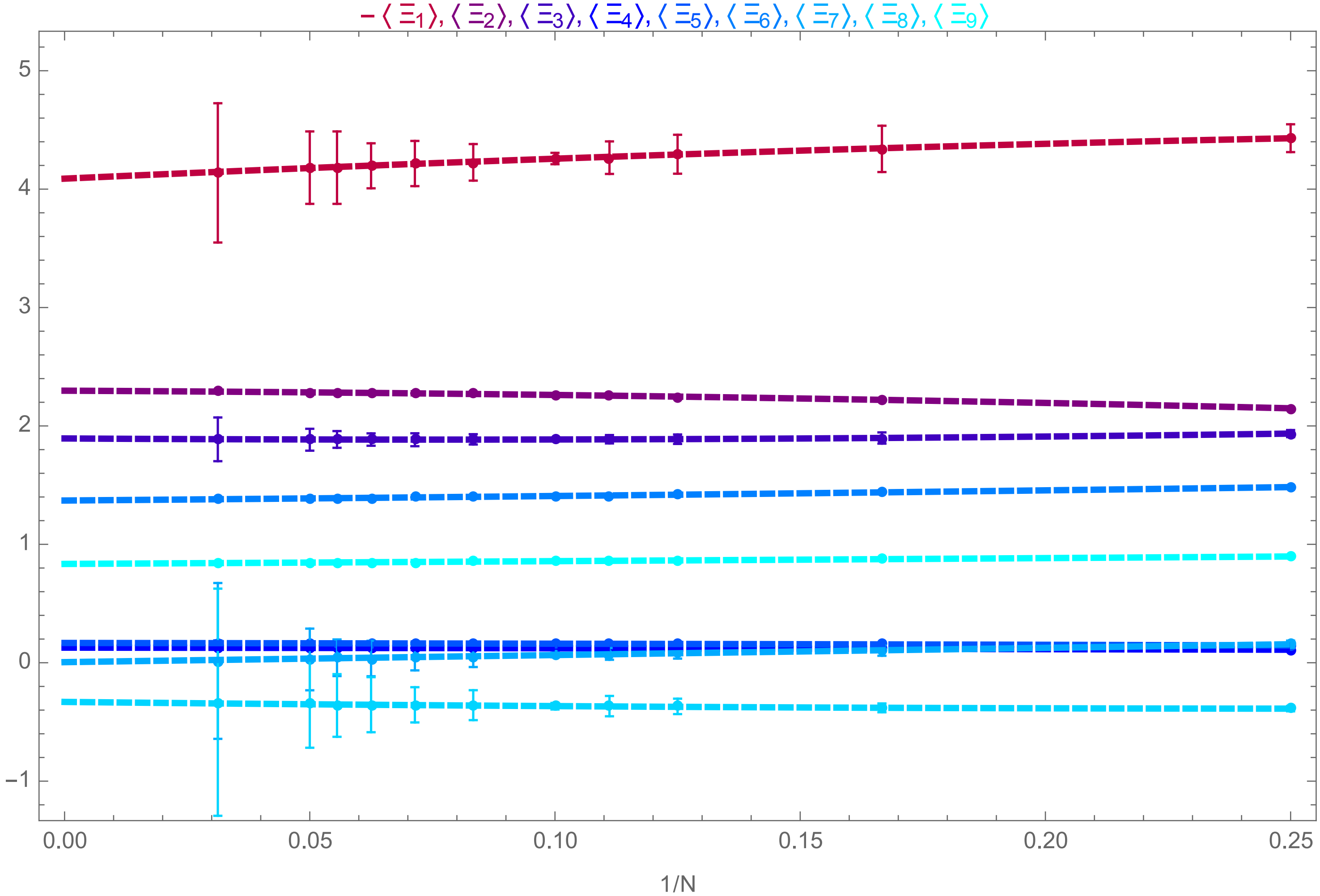

Figure 4: Mean values of observables $\Xi_i,\ i = 1, ..., 9$ plotted against N with $N_f = 1$. Data were generated using 3×10^6 (3×10^7 for the $N = 10$ case) samples generated by hybrid Monte Carlo simulation, dashed lines correspond to fits of the form $a + b/N + c/N^2$. Errors are estimated with the Jackknife resampling.

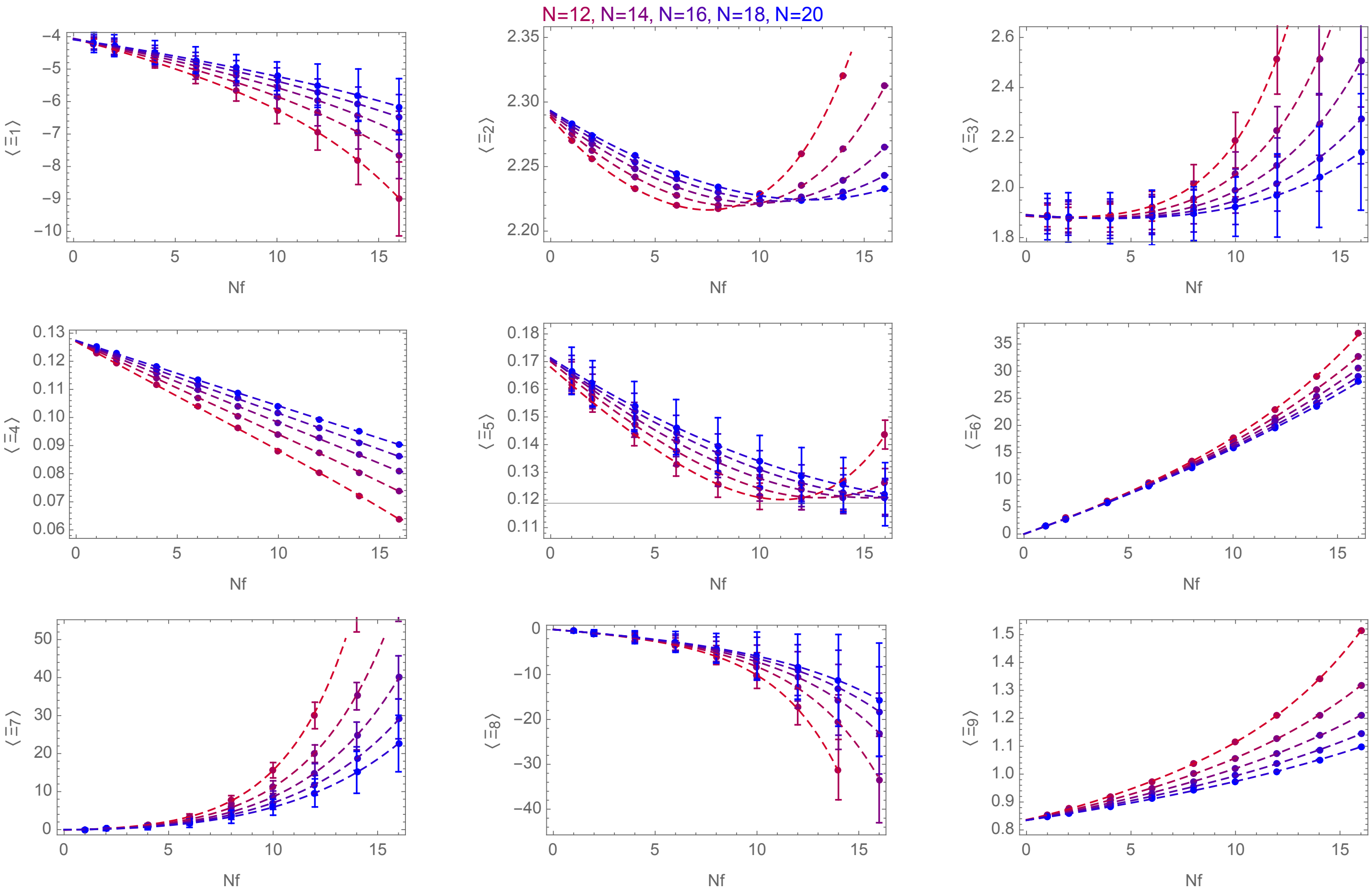

Figure 5: Mean values of observables $\Xi_i,\ i = 1, ..., 9$ plotted against N_f for different values of N. Data were generated using 3×10^6 samples generated by hybrid Monte Carlo simulation, dashed lines correspond to either fits of the form $a + bN_f + cN_f^2 + dN_f^3 + eN_f^4$ (for $i = 1, 2, 4, 9$) or $a + bN_f + ce^{dN_f}$ (for $i = 3, 5, 6, 7, 8$).

C The High temperature behaviour of energy E, Polyakov loop $\langle P \rangle$, $\langle R^2 \rangle$ and mass susceptibility $\langle \mathcal{C}^m \rangle$ for the supersymmetric model.

In this appendix we graphically present the high temperature predictions for the BD-model observables the energy E, the Polyakov loop $\langle P \rangle$, the extent of the eigenvalues of the adjoint fields X^i given by $\langle R^2 \rangle$ and the mass susceptibility $\langle \mathcal{C}^m \rangle$. Figure 6 shows the predicted high temperature behaviour of the BD-model observables.

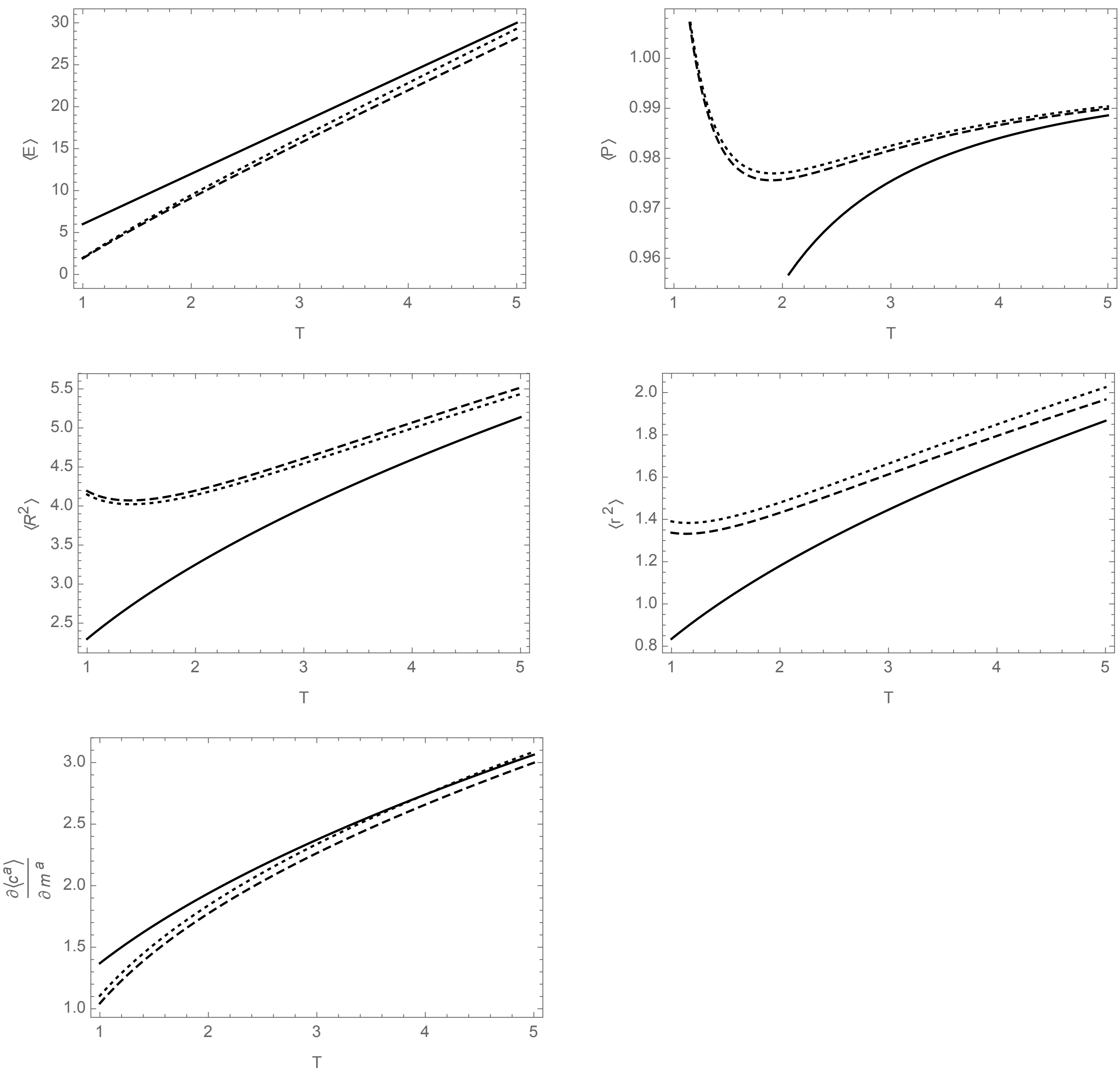

Figure 6: Temperature dependence of physical observables for the supersymmetric BD model as defined in (4.3) and with Ξ_i from table 1. The solid line is the leading order prediction for $N = \infty$, while the long dashed line has $N = \infty$ with Ξ_i taken from table 1. The third curve with short dashes is $N = 10$, $N_f = 1$. Note that in contrast to the bosonic model the high temperature dependence of the Polyakov loop turns upwards, as T decreases, between $T = 1.0$ and 2.0. This indicates that the high temperature series for $\langle P \rangle$ is not reliable in this region.

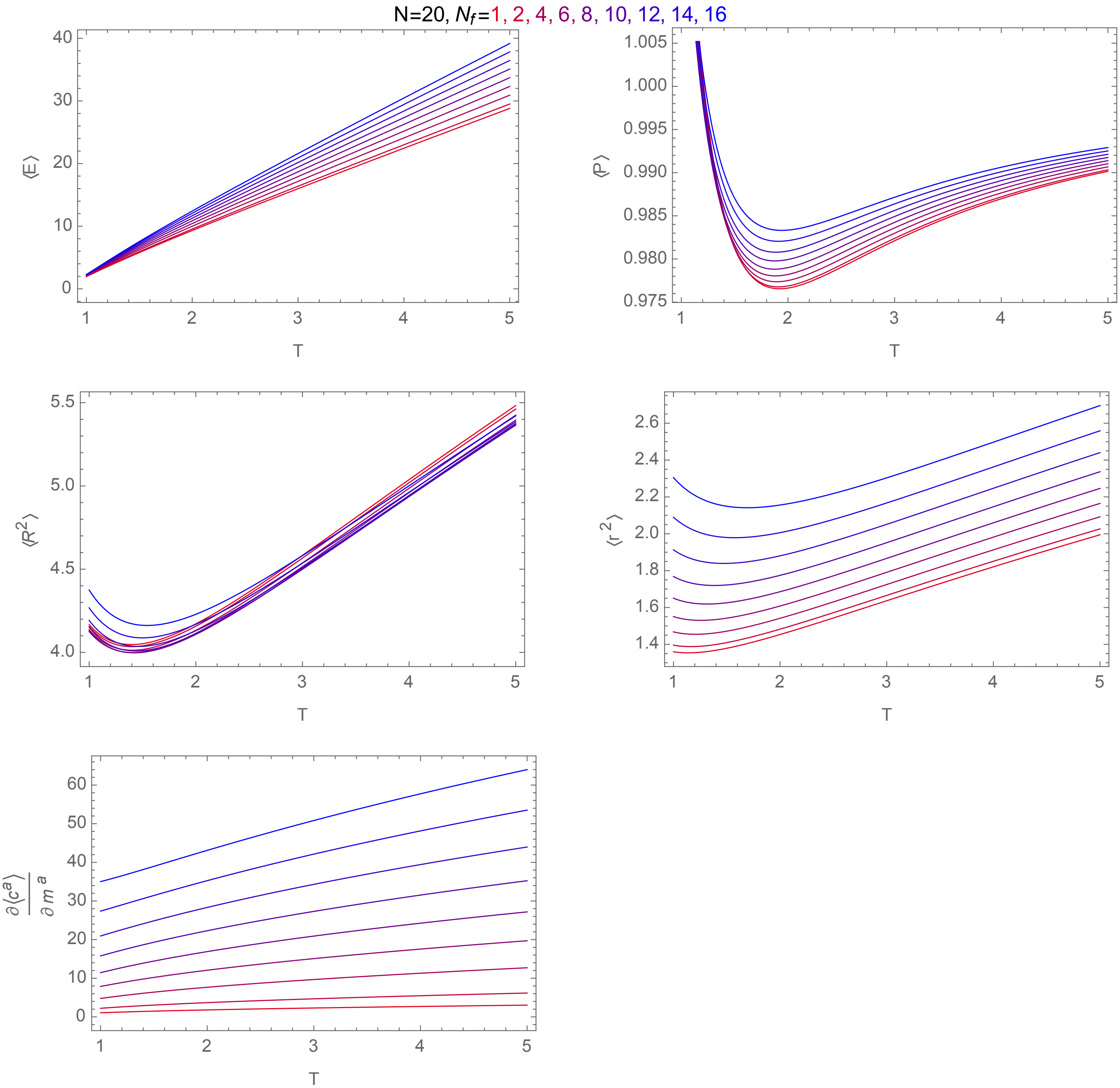

Figure 7: Temperature dependence of physical observables as defined in (4.3) with Ξ_i from table 2 for $N = 20$ and different values of N_f.

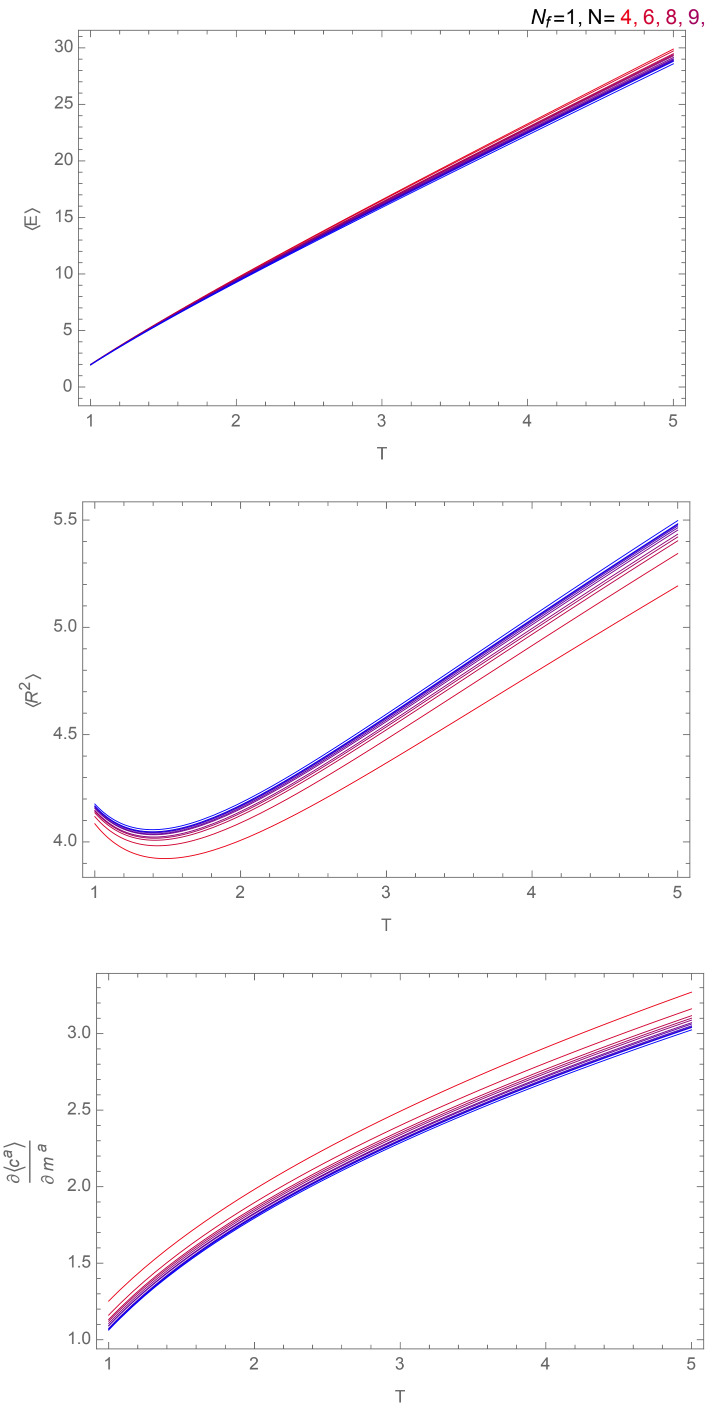

Figure 8: Temperature dependence of physical observables of the supersymmetric model as defined in (4.3) with Ξ_i from table 1 for $N_f = 1$ with different values of N.

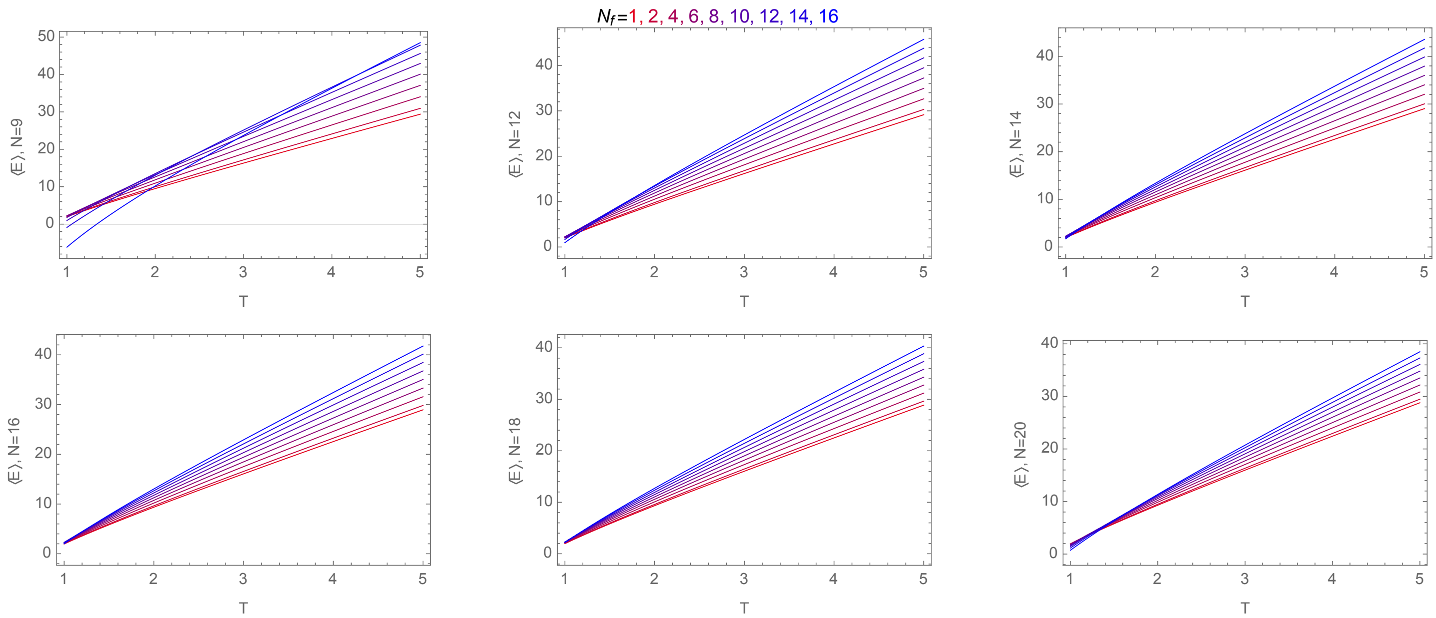

Figure 9: Dependence of the energy on the temperature for the supersymmetric model as defined in (4.3) for $N = 9, 12, 14, 16, 18, 20$ with different values of N_f. Note that for each value of N the curves (approximately) intersect at a crossing temperature T_x. At this point the energy is essentially independent of N_f. Extrapolating the crossing value to large N we find $T_x = 0.87(3)$ which is close to the observed transition region of the bosonic BFSS model.

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