

# Global symmetries of Yang-Mills squared in various dimensions

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## ABSTRACT

Tensoring two on-shell super Yang-Mills multiplets in dimensions  $D \leq 10$  yields an on-shell supergravity multiplet, possibly with additional matter multiplets. Associating a (direct sum of) division algebra(s)  $\mathbb{D}$  with each dimension  $3 \leq D \leq 10$  we obtain formulae for the algebras  $\mathfrak{g}$  and  $\mathfrak{h}$  of the U-duality group  $G$  and its maximal compact subgroup  $H$ , respectively, in terms of the internal global symmetry algebras of each super Yang-Mills theory. We extend our analysis to include supergravities coupled to an arbitrary number of matter multiplets by allowing for non-supersymmetric multiplets in the tensor product.

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# 1 Introduction

The idea of understanding aspects of quantum gravity in terms of a double copy of gauge theories has a long history going back at least to the KLT relations of string theory [1]. There has since been a wealth of developments expanding on this concept, perhaps most notably, but certainly not exclusively, in the context of gravitational and gauge scattering amplitudes. See for example [2–24]. Indeed, invoking the Bern-Carrasco-Johansson colour-kinematic duality it has been conjectured [8] that the on-mass-shell momentum-space scattering amplitudes for gravity are the “double-copy” of gluon scattering amplitudes in Yang-Mills theory to all orders in perturbation theory.

This remarkable and somewhat surprising proposal motivates the question: to what extent can one regard quantum gravity as the double copy of Yang-Mills theory? In this context it is natural to ask how the symmetries of each theory are related. In recent work [23] it was shown that the off-shell *local* transformation rules of (super)gravity (namely general covariance, local Lorentz invariance,  $p$ -form gauge invariance and local supersymmetry) may be derived from those of flat space Yang-Mills (namely local gauge invariance and global super-Poincare) at the linearised level.

Equally important in the context of M-theory are the non-compact *global* symmetries of supergravity [25], which are intimately related to the concept of U-duality [26, 27]. In this case, it was shown in [28] that tensoring two  $D = 3$ ,  $\mathcal{N} = 1, 2, 4, 8$  super Yang-Mills multiplets results in a “Freudenthal magic square of supergravity theories”, as summarised in Table 1. The corresponding Lie algebras of Table 1 are concisely summarised by the magic square formula [28, 29],

$$\mathfrak{L}_3(\mathbb{A}_{\mathcal{N}_L}, \mathbb{A}_{\mathcal{N}_R}) := \mathfrak{tri}(\mathbb{A}_{\mathcal{N}_L}) \oplus \mathfrak{tri}(\mathbb{A}_{\mathcal{N}_R}) + 3(\mathbb{A}_{\mathcal{N}_L} \otimes \mathbb{A}_{\mathcal{N}_R}), \quad (1.1)$$

which takes as its argument a pair of division algebras  $\mathbb{A}_{\mathcal{N}_L}, \mathbb{A}_{\mathcal{N}_R} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ , where we have adopted the convention that  $\dim \mathbb{A}_{\mathcal{N}} = \mathcal{N}$ . The triality algebra of  $\mathbb{A}$ , denoted  $\mathfrak{tri}(\mathbb{A})$ , is related to the total on-shell global symmetries of the associated super Yang-Mills theory [30]. This rather surprising connection, relating the magic square of Lie algebras to the square of super Yang-Mills, can be attributed to the existence of a unified  $\mathbb{A}_{\mathcal{N}} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$  description of  $D = 3$ ,  $\mathcal{N} = 1, 2, 4, 8$  super Yang-Mills theories.

$\mathbb{A}_{\mathcal{N}_L} \backslash \mathbb{A}_{\mathcal{N}_R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathcal{N} = 2, f = 4$ $G = \text{SL}(2, \mathbb{R})$ $H = \text{SO}(2)$	$\mathcal{N} = 3, f = 8$ $G = \text{SU}(2, 1)$ $H = \text{SO}(3) \times \text{SO}(2)$	$\mathcal{N} = 5, f = 16$ $G = \text{USp}(4, 2)$ $H = \text{SO}(5) \times \text{SO}(3)$	$\mathcal{N} = 9, f = 32$ $G = F_{4(-20)}$ $H = \text{SO}(9)$
$\mathbb{C}$	$\mathcal{N} = 3, f = 8$ $G = \text{SU}(2, 1)$ $H = \text{SO}(3) \times \text{SO}(2)$	$\mathcal{N} = 4, f = 16$ $G = \text{SU}(2, 1)^2$ $H = \text{SO}(3)^2 \times \text{SO}(2)^2$	$\mathcal{N} = 6, f = 32$ $G = \text{SU}(4, 2)$ $H = \text{SO}(6) \times \text{SO}(3) \times \text{SO}(2)$	$\mathcal{N} = 10, f = 64$ $G = E_{6(-14)}$ $H = \text{SO}(10) \times \text{SO}(2)$
$\mathbb{H}$	$\mathcal{N} = 5, f = 16$ $G = \text{USp}(4, 2)$ $H = \text{SO}(5) \times \text{SO}(3)$	$\mathcal{N} = 6, f = 32$ $G = \text{SU}(4, 2)$ $H = \text{SO}(6) \times \text{SO}(3) \times \text{SO}(2)$	$\mathcal{N} = 8, f = 64$ $G = \text{SO}(8, 4)$ $H = \text{SO}(8) \times \text{SO}(3) \times \text{SO}(3)$	$\mathcal{N} = 12, f = 128$ $G = E_{7(-5)}$ $H = \text{SO}(12) \times \text{SO}(3)$
$\mathbb{O}$	$\mathcal{N} = 9, f = 32$ $G = F_{4(-20)}$ $H = \text{SO}(9)$	$\mathcal{N} = 10, f = 64$ $G = E_{6(-14)}$ $H = \text{SO}(10) \times \text{SO}(2)$	$\mathcal{N} = 12, f = 128$ $G = E_{7(-5)}$ $H = \text{SO}(12) \times \text{SO}(3)$	$\mathcal{N} = 16, f = 256$ $G = E_{8(8)}$ $H = \text{SO}(16)$

Table 1:  $(\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R)$ -extended  $D = 3$  supergravities obtained by left/right super Yang-Mills multiplets with  $\mathcal{N}_L, \mathcal{N}_R = 1, 2, 4, 8$ . The algebras of the corresponding U-duality groups  $G$  and their maximal compact subgroups  $H$  are given by the magic square of Freudenthal-Rosenfeld-Tits [31–33].  $f$  denotes the total number of degrees of freedom in the resulting supergravity and matter multiplets.

This observation was subsequently generalised to  $D = 3, 4, 6$  and 10 dimensions [30, 34] by incorporating the well-known relationship between the existence of minimal super Yang-Mills theories in

$D = 3, 4, 6, 10$  and the existence of the four division algebras  $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$  [35–39]. From this perspective the  $D = 3$  magic square forms the base of a “magic pyramid” of supergravities. These constructions build on a long line of work relating division algebras and magic squares to spacetime and supersymmetry. See [36–78] for a glimpse of the relevant literature.

The magic pyramid, however, constitutes a rather special subset of supergravity theories: those given by tensoring the  $D = 3, 4, 6, 10$  division algebraic super Yang-Mills theories constructed in [30]. In the present work we address the natural question of generalisation beyond this select subclass of theories.

In section 2 we consider all tensor products of left  $\mathcal{N}_L$ -extended and right  $\mathcal{N}_R$ -extended super Yang-Mills multiplets in  $D = 3, \dots, 10$  dimensions and introduce three formulae describing the global symmetries of the resulting  $(\mathcal{N}_L + \mathcal{N}_R)$ -extended supergravity multiplets:

1. The algebra  $\mathfrak{ra}(\mathcal{N}_L + \mathcal{N}_R, D)$  of  $(\mathcal{N}_L + \mathcal{N}_R)$ -extended R-symmetry in  $D$  dimensions,

$$\mathfrak{ra}(\mathcal{N}_L + \mathcal{N}_R, D) = \mathfrak{a}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{a}(\mathcal{N}_R, \mathbb{D}) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]; \quad (1.2)$$

2. The algebra  $\mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, D)$  of  $H$ , the maximal compact subgroup of the U-duality group  $G$ ,

$$\mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, D) = [\mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{p}_L] \oplus [\mathfrak{sa}(\mathcal{N}_R, \mathbb{D}) \oplus \mathfrak{p}_R] \oplus \delta_{D,4}\mathfrak{u}(1) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]; \quad (1.3)$$

3. The algebra  $\mathfrak{g}(\mathcal{N}_L + \mathcal{N}_R, D)$  of the U-duality group  $G$  itself,

$$\mathfrak{g}(\mathcal{N}_L + \mathcal{N}_R, D) = \mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, D) + \mathbb{D}_*[\mathcal{N}_L] \otimes \mathbb{D}_*[\mathcal{N}_R] + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R] + \mathbb{R}_L \otimes \mathbb{R}_R + \delta_{D,4}\mathbb{R}_L \otimes \mathbb{R}_R. \quad (1.4)$$

Here we have used  $\oplus$  and  $+$  to distinguish the direct sum between Lie algebras and vector spaces; only if  $[\mathfrak{m}, \mathfrak{n}] = 0$  do we use  $\mathfrak{m} \oplus \mathfrak{n}$ . The meaning of these formulae and, in particular, their relation to the symmetries of the left and right super Yang-Mills factors, will be described in section 2. For the moment we simply note that they make the left/right structure manifest and uniform for all  $\mathcal{N}_L, \mathcal{N}_R$  and  $D$  and, as we shall see, each summand appearing in the three formulae has a natural *left*  $\otimes$  *right* origin. The groups  $H$  and  $G$  corresponding to (1.3) and (1.4) are given in the generalised pyramids of Figure 1 and Figure 2, respectively. For these groups, the formulae presented above can be regarded as generalised “matrix models”, in the sense of [29] (not to be confused with (M)atrix models), for classical and exceptional Lie algebras. As a matrix model, it is perhaps not as elegant as those presented in [29]. For one, we make *no* use of the octonions. However, it has the advantage, from our perspective, that it describes systematically all groups obtained by squaring super Yang-Mills and, moreover, makes the left and right factors manifest.

For  $\mathcal{N}_L + \mathcal{N}_R$  half-maximal or less the super Yang-Mills tensor products yield supergravity multiplets together with additional matter multiplets, as described in Table 2. They may always be obtained as consistent truncations or, in many cases, factorised orbifold truncations of the maximally supersymmetric cases, as in [21]. The type, number and coupling of these multiplets is fixed with respect to (1.3) and (1.4). However, as we shall describe in subsection 3.1, by including a non-supersymmetric factor in the tensor product these matter couplings may be generalised to include an arbitrary number of vector multiplets (thus clearly not truncations). This procedure naturally yields analogous formulae for  $\mathfrak{h}$  and  $\mathfrak{g}$ , corresponding to specific couplings. The nature of these couplings is in a certain sense as simple as possible. This follows from the symmetries assumed, which may be regarded as a consequence of simple interactions, to be present in the non-supersymmetric factor of the tensor product.

## 2 Global symmetries of super Yang-Mills squared

### 2.1 Tensoring super Yang-Mills theories in $D \geq 3$

Tensoring  $\mathcal{N}_L$ -extended and  $\mathcal{N}_R$ -extended super Yang-Mills multiplets,  $[\mathcal{N}_L]_V$  and  $[\mathcal{N}_R]_V$ , yields an  $(\mathcal{N}_L + \mathcal{N}_R)$ -extended supergravity multiplet,  $[\mathcal{N}_L + \mathcal{N}_R]_{grav}$ ,

$$[\mathcal{N}_L]_V \otimes [\mathcal{N}_R]_V \rightarrow [\mathcal{N}_L + \mathcal{N}_R]_{grav} + [\mathcal{N}_L + \mathcal{N}_R]_{matter}, \quad (2.1)$$

with additional matter multiplets,  $[\mathcal{N}_L + \mathcal{N}_R]_{matter}$ , for  $[\mathcal{N}_L + \mathcal{N}_R]_{grav}$  half-maximal or less. See Table 2 and Table 3.

We consider on-shell space-time little group super Yang-Mills multiplets with global symmetry algebra

$$\mathfrak{so}(D-2)_{ST} \oplus \mathfrak{int}(\mathcal{N}, D), \quad (2.2)$$

where  $\mathfrak{int}(\mathcal{N}, D)$  denotes the global internal symmetry algebra of the Lagrangian. For  $\mathfrak{so}(D-2)_{ST}$  the tensor products are  $\mathfrak{so}(D-2)_{ST}$ -modules, while for  $\mathfrak{int}(\mathcal{N}_L, D)$  and  $\mathfrak{int}(\mathcal{N}_R, D)$  they are  $\mathfrak{int}(\mathcal{N}_L, D) \oplus \mathfrak{int}(\mathcal{N}_R, D)$ -modules. Very schematically, the general tensor product is given by,

$$\begin{array}{c|ccc} \otimes & \tilde{A}_\nu & \tilde{\lambda}^{a'} & \tilde{\phi}^{i'} \\ \hline A_\mu & g_{\mu\nu} + B_{\mu\nu} + \phi & \psi_\mu^{a'} + \lambda^{a'} & A_\mu^{i'} \\ \lambda^a & \psi_\nu^a + \lambda^a & \phi_{RR}^{aa'} + \dots & \lambda^{ai'} \\ \phi^i & A_\nu^i & \lambda^{ia'} & \phi^{ii'} \end{array} \quad (2.3)$$

where  $a, i$  and  $a', i'$  are indices of the appropriate  $\mathfrak{int}(\mathcal{N}_L, D)$  and  $\mathfrak{int}(\mathcal{N}_R, D)$  representations, respectively. Note, we will always dualise  $p$ -forms to their lowest possible rank consistent with their little group representations, for example,  $B_{\mu\nu} \rightarrow \phi, A_\mu$  in  $D = 4, 5$ , respectively. This ensures U-duality is manifest. The particular set of Ramond-Ramond  $p$ -forms  $\phi_{RR}^{aa'} + \dots$  one obtains is dimension dependent.

The detailed form of these tensor products for  $D > 3$  are summarised in Table 2 and Table 3, where for a given little group representation we have collected the  $\mathfrak{int}(\mathcal{N}_L, D) \oplus \mathfrak{int}(\mathcal{N}_R, D)$  representations into the appropriate representations of  $\mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, D)$ . For example, consider the square of the  $D = 5, \mathcal{N} = 2$  super Yang-Mills multiplet, which has global symmetry algebra  $\mathfrak{so}(3)_{ST} \oplus \mathfrak{sp}(2)$ ,

$$\begin{array}{c|ccc} \otimes & \tilde{A}_\mu & \tilde{\lambda} & \tilde{\phi} \\ \hline & (\mathbf{3}; \mathbf{1}) & (\mathbf{2}; \mathbf{4}) & (\mathbf{1}; \mathbf{5}) \\ \hline A_\mu (\mathbf{3}; \mathbf{1}) & (\mathbf{5}; \mathbf{1}, \mathbf{1}) + (\mathbf{3}; \mathbf{1}, \mathbf{1}) + (\mathbf{1}; \mathbf{1}, \mathbf{1}) & (\mathbf{4}; \mathbf{1}, \mathbf{4}) + (\mathbf{2}; \mathbf{1}, \mathbf{4}) & (\mathbf{3}; \mathbf{1}, \mathbf{5}) \\ \lambda (\mathbf{2}; \mathbf{4}) & (\mathbf{4}; \mathbf{4}, \mathbf{1}) + (\mathbf{2}; \mathbf{4}, \mathbf{1}) & (\mathbf{3}; \mathbf{4}, \mathbf{4}) + (\mathbf{1}; \mathbf{4}, \mathbf{4}) & (\mathbf{2}; \mathbf{4}, \mathbf{5}) \\ \phi (\mathbf{1}; \mathbf{5}) & (\mathbf{3}; \mathbf{5}, \mathbf{1}) & (\mathbf{2}; \mathbf{5}, \mathbf{4}) & (\mathbf{1}; \mathbf{5}, \mathbf{5}) \end{array} \quad (2.4)$$

On gathering the spacetime little group representations in (2.4), the  $\mathfrak{int}(2, 5) \oplus \mathfrak{int}(2, 5) = \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$  representations they carry may be combined into irreducible  $\mathfrak{h}(4, 5) = \mathfrak{sp}(4)$  representations, as illustrated by their decomposition under  $\mathfrak{sp}(4) \supset \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$ :

$$\begin{aligned} (\mathbf{5}; \mathbf{1}) &\rightarrow (\mathbf{5}; \mathbf{1}, \mathbf{1}), \\ (\mathbf{4}; \mathbf{8}) &\rightarrow (\mathbf{4}; \mathbf{4}, \mathbf{1}) + (\mathbf{4}; \mathbf{1}, \mathbf{4}), \\ (\mathbf{3}; \mathbf{27}) &\rightarrow (\mathbf{3}; \mathbf{1}, \mathbf{1}) + (\mathbf{3}; \mathbf{5}, \mathbf{1}) + (\mathbf{3}; \mathbf{1}, \mathbf{5}) + (\mathbf{3}; \mathbf{4}, \mathbf{4}), \\ (\mathbf{2}; \mathbf{48}) &\rightarrow (\mathbf{2}; \mathbf{4}, \mathbf{1}) + (\mathbf{2}; \mathbf{1}, \mathbf{4}) + (\mathbf{2}; \mathbf{4}, \mathbf{5}) + (\mathbf{2}; \mathbf{5}, \mathbf{4}), \\ (\mathbf{1}; \mathbf{42}) &\rightarrow (\mathbf{1}; \mathbf{1}, \mathbf{1}) + (\mathbf{1}; \mathbf{4}, \mathbf{4}) + (\mathbf{1}; \mathbf{5}, \mathbf{5}). \end{aligned} \quad (2.5)$$

## 2.2 R-symmetry algebras

We begin with the simple relationship between the R-symmetry algebras of supergravity and its generating super Yang-Mills factors. While somewhat trivial this example introduces much of the notation and concepts needed later for the  $H$  and  $G$  algebras.

R-symmetry is defined here as the automorphism group of the supersymmetry algebra. Its action on the  $\mathcal{N}$ -extended supersymmetry generators  $Q$  is given schematically by

$$[T_A, Q_a] = (U_A)_a{}^b Q_b, \quad a, b = 1, \dots, \mathcal{N}. \quad (2.6)$$

The R-symmetry algebra is fixed by the reality properties of the minimal spinor representation in  $D \bmod 8$  dimensions. See, for example, [79].

$D = 4, \mathfrak{so}(2)_{ST}$	$\mathcal{N} = 4 \quad \mathfrak{su}(4)$ $A_\mu \quad (1; \mathbf{1})$ $\lambda \quad (\frac{1}{2}; \mathbf{4})$ $\phi \quad (0; \mathbf{6})$ <i>+c.c.</i>	$\mathcal{N} = 2 \quad \mathfrak{u}(2)$ $A_\mu \quad (1; \mathbf{1}(0))$ $\lambda \quad (\frac{1}{2}; \mathbf{2}(1))$ $\phi \quad (0; \mathbf{1}(2))$ <i>+c.c.</i>	$\mathcal{N} = 1 \quad \mathfrak{u}(1)$ $A_\mu \quad (1; 0)$ $\lambda \quad (\frac{1}{2}; -3)$ <i>+c.c.</i>
$\mathcal{N} = 4 \quad \mathfrak{su}(4)$ $A_\mu \quad (1; \mathbf{1}) + c.c.$ $\lambda \quad (\frac{1}{2}; \mathbf{4}) + c.c.$ $\phi \quad (0; \mathbf{6})$	$\mathcal{N} = 8 \quad \mathfrak{su}(8)$ $g_{\mu\nu} \quad (2; \mathbf{1}) + c.c.$ $\psi_\mu \quad (\frac{3}{2}; \mathbf{8}) + c.c.$ $A_\mu \quad (1; \mathbf{28}) + c.c.$ $\lambda \quad (\frac{1}{2}; \mathbf{56}) + c.c.$ $\phi \quad (0; \mathbf{70})$	$\mathcal{N} = 6 \quad \mathfrak{u}(6)$ $g_{\mu\nu} \quad (2; \mathbf{1}(0)) + c.c.$ $\psi_\mu \quad (\frac{3}{2}; \mathbf{6}(1)) + c.c.$ $A_\mu \quad (1; \mathbf{1}(-6) + \mathbf{15}(2)) + c.c.$ $\lambda \quad (\frac{1}{2}; \mathbf{6}(-5) + \mathbf{20}(3)) + c.c.$ $\phi \quad (0; \mathbf{15}(-4)) + c.c.$	$\mathcal{N} = 5 \quad \mathfrak{u}(5)$ $g_{\mu\nu} \quad (2; \mathbf{1}(0)) + c.c.$ $\psi_\mu \quad (\frac{3}{2}; \mathbf{5}(1)) + c.c.$ $A_\mu \quad (1; \mathbf{10}(2)) + c.c.$ $\lambda \quad (\frac{1}{2}; \mathbf{1}(-5) + \mathbf{10}(3)) + c.c.$ $\phi \quad (0; \mathbf{5}(-4)) + c.c.$
$\mathcal{N} = 2 \quad \mathfrak{u}(2)$ $A_\mu \quad (1; \mathbf{1}(0)) + c.c.$ $\lambda \quad (\frac{1}{2}; \mathbf{2}(1)) + c.c.$ $\phi \quad (0; \mathbf{1}(2)) + c.c.$	$\mathcal{N} = 6 \quad \mathfrak{u}(6)$ $g_{\mu\nu} \quad (2; \mathbf{1}(0)) + c.c.$ $\psi_\mu \quad (\frac{3}{2}; \mathbf{6}(1)) + c.c.$ $A_\mu \quad (1; \mathbf{1}(-6) + \mathbf{15}(2)) + c.c.$ $\lambda \quad (\frac{1}{2}; \mathbf{6}(-5) + \mathbf{20}(3)) + c.c.$ $\phi \quad (0; \mathbf{15}(-4)) + c.c.$	$\mathcal{N} = 4 \quad \mathfrak{u}(4) \oplus \mathfrak{u}(1)$ $g_{\mu\nu} \quad (2; \mathbf{1}(0)(0)) + c.c.$ $\psi_\mu \quad (\frac{3}{2}; \mathbf{4}(1)(1)) + c.c.$ $A_\mu \quad (1; \mathbf{6}(2)(2)) + c.c.$ $\lambda \quad (\frac{1}{2}; \mathbf{4}(3)(3)) + c.c.$ $\phi \quad (0; \mathbf{1}(4)(4)) + c.c.$ $2[\mathcal{N} = 4]_V$ $2 \times A_\mu \quad (1; \mathbf{1}(-6)(0)) + c.c.$ $\quad (1; \mathbf{1}(2)(-4)) + c.c.$ $2 \times \lambda \quad (\frac{1}{2}; \mathbf{4}(-5)(1)) + c.c.$ $\quad (\frac{1}{2}; \mathbf{4}(3)(-3)) + c.c.$ $2 \times \phi \quad (0; \mathbf{6}(-4)(2))$ $\quad (0; \mathbf{6}(4)(-2))$	$\mathcal{N} = 3 \quad \mathfrak{u}(3) \oplus \mathfrak{u}(1)$ $g_{\mu\nu} \quad (2; \mathbf{1}(0)(0)) + c.c.$ $\psi_\mu \quad (\frac{3}{2}; \mathbf{3}(1)(2)) + c.c.$ $A_\mu \quad (1; \mathbf{3}(2)(4)) + c.c.$ $\lambda \quad (\frac{1}{2}; \mathbf{1}(3)(6)) + c.c.$ $[N = 3]_V$ $A_\mu \quad (1; \mathbf{1}(2)(-6)) + c.c.$ $\lambda \quad (\frac{1}{2}; \mathbf{3}(3)(-4)) + c.c.$ $\phi \quad (0; \mathbf{3}(4)(-2)) + c.c.$
$\mathcal{N} = 1 \quad \mathfrak{u}(1)$ $A_\mu \quad (1; 0) + c.c.$ $\lambda \quad (\frac{1}{2}; -3) + c.c.$	$\mathcal{N} = 5 \quad \mathfrak{u}(5)$ $g_{\mu\nu} \quad (2; \mathbf{1}(0)) + c.c.$ $\psi_\mu \quad (\frac{3}{2}; \mathbf{5}(1)) + c.c.$ $A_\mu \quad (1; \mathbf{10}(2)) + c.c.$ $\lambda \quad (\frac{1}{2}; \mathbf{1}(-5) + \mathbf{10}(3)) + c.c.$ $\phi \quad (0; \mathbf{5}(-4)) + c.c.$	$\mathcal{N} = 3 \quad \mathfrak{u}(3) \oplus \mathfrak{u}(1)$ $g_{\mu\nu} \quad (2; \mathbf{1}(0)(0)) + c.c.$ $\psi_\mu \quad (\frac{3}{2}; \mathbf{3}(1)(2)) + c.c.$ $A_\mu \quad (1; \mathbf{3}(2)(4)) + c.c.$ $\lambda \quad (\frac{1}{2}; \mathbf{1}(3)(6)) + c.c.$ $[N = 3]_V$ $A_\mu \quad (1; \mathbf{1}(2)(-6)) + c.c.$ $\lambda \quad (\frac{1}{2}; \mathbf{3}(3)(-4)) + c.c.$ $\phi \quad (0; \mathbf{3}(4)(-2)) + c.c.$	$\mathcal{N} = 2 \quad \mathfrak{u}(2) \oplus \mathfrak{u}(1)$ $g_{\mu\nu} \quad (2; \mathbf{1}(0)(0)) + c.c.$ $\psi_\mu \quad (\frac{3}{2}; \mathbf{2}(1)(2)) + c.c.$ $A_\mu \quad (1; \mathbf{1}(2)(4)) + c.c.$ $[N = 2]_h$ $\lambda \quad (\frac{1}{2}; \mathbf{2}(1)(-4)) + c.c.$ $\phi \quad (0; \mathbf{2}(-1)(-2)) + c.c.$
$D = 5, \mathfrak{so}(3)_{ST}$	$\mathcal{N} = 2 \quad \mathfrak{sp}(2)$ $A_\mu \quad (3; \mathbf{1})$ $\lambda \quad (2; \mathbf{4})$ $\phi \quad (1; \mathbf{5})$	$\mathcal{N} = 1 \quad \mathfrak{sp}(1)$ $A_\mu \quad (3; \mathbf{1})$ $\lambda \quad (2; \mathbf{2})$ $\phi \quad (1; \mathbf{1})$	
$\mathcal{N} = 2 \quad \mathfrak{sp}(2)$ $A_\mu \quad (3; \mathbf{1})$ $\lambda \quad (2; \mathbf{4})$ $\phi \quad (1; \mathbf{5})$	$\mathcal{N} = 4 \quad \mathfrak{sp}(4)$ $g_{\mu\nu} \quad (5; \mathbf{1})$ $\psi_\mu \quad (4; \mathbf{8})$ $A_\mu \quad (3; \mathbf{27})$ $\lambda \quad (2; \mathbf{48})$ $\phi \quad (1; \mathbf{42})$	$\mathcal{N} = 3 \quad \mathfrak{sp}(3)$ $g_{\mu\nu} \quad (5; \mathbf{1})$ $\psi_\mu \quad (4; \mathbf{6})$ $A_\mu \quad (3; \mathbf{1} + \mathbf{14})$ $\lambda \quad (2; \mathbf{6} + \mathbf{14}')$ $\phi \quad (1; \mathbf{14})$	
$\mathcal{N} = 1 \quad \mathfrak{sp}(1)$ $A_\mu \quad (3; \mathbf{1})$ $\lambda \quad (2; \mathbf{2})$ $\phi \quad (1; \mathbf{1})$	$\mathcal{N} = 3 \quad \mathfrak{sp}(3)$ $g_{\mu\nu} \quad (5; \mathbf{1})$ $\psi_\mu \quad (4; \mathbf{6})$ $A_\mu \quad (3; \mathbf{1} + \mathbf{14})$ $\lambda \quad (2; \mathbf{6} + \mathbf{14}')$ $\phi \quad (1; \mathbf{14})$	$\mathcal{N} = 2 \quad \mathfrak{sp}(2)$ $g_{\mu\nu} \quad (5; \mathbf{1})$ $\psi_\mu \quad (4; \mathbf{4})$ $A_\mu \quad (3; \mathbf{1} + \mathbf{5})$ $\lambda \quad (2; \mathbf{4})$ $\phi \quad (1; \mathbf{1})$ $[N = 2]_V$ $A_\mu \quad (3; \mathbf{1})$ $\lambda \quad (2; \mathbf{4})$ $\phi \quad (1; \mathbf{5})$	

Table 2: Tensor products of left and right super Yang-Mills multiplets in  $D = 4, 5$ . Dimensions  $D = 6, 7, 8, 9, 10$  are given in Table 3. In  $(\mathbf{m}; \mathbf{n})$   $\mathbf{m}$  denotes the spacetime little group representation and  $\mathbf{n}$  the representation of the internal global symmetry displayed, int for the super Yang-Mills multiplets and  $\mathfrak{h}$  for the resulting supergravity + matter multiplets. Here  $V$  and  $h$  denote vector and hyper multiplets, respectively.

$D = 6, \mathfrak{su}(2) \oplus \mathfrak{su}(2)_{ST}$	$\mathcal{N} = (1, 1) \quad \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ $A_\mu \quad (2, 2; 1, 1) \quad \lambda \quad (2, 1; 1, 2) + (1, 2; 2, 1) \quad \phi \quad (1, 1; 2, 2)$	$\mathcal{N} = (1, 0) \quad \mathfrak{sp}(1) \oplus \emptyset$ $A_\mu \quad (2, 2; 1) \quad \lambda \quad (1, 2; 2)$
$\mathcal{N} = (1, 1) \quad \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ $A_\mu \quad (2, 2; 1)$ $\lambda \quad (2, 1; 1, 2) + (1, 2; 2, 1)$ $\phi \quad (1, 1; 2, 2)$	$\mathcal{N} = (2, 2) \quad \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$ $g_{\mu\nu} \quad (3, 3; 1, 1)$ $\psi_\mu \quad (2, 3; 4, 1) + (3, 2; 1, 4)$ $A_\mu \quad (2, 2; 4, 4)$ $A_{\mu\nu} \quad (3, 1; 1, 5) + (1, 3; 5, 1)$ $\lambda \quad (2, 1; 4, 5) + (1, 2; 5, 4)$ $\phi \quad (1, 1; 5, 5)$	$\mathcal{N} = (2, 1) \quad \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ $g_{\mu\nu} \quad (3, 3; 1, 1)$ $\psi_\mu \quad (2, 3; 4, 1) + (3, 2; 1, 2)$ $A_\mu \quad (2, 2; 4, 2)$ $A_{\mu\nu} \quad (3, 1; 1, 1) + (1, 3; 5, 1)$ $\lambda \quad (2, 1; 4, 1) + (1, 2; 5, 2)$ $\phi \quad (1, 1; 5, 1)$
$\mathcal{N} = (0, 1) \quad \emptyset \oplus \mathfrak{sp}(1)$ $A_\mu \quad (2, 2; 1)$ $\lambda \quad (2, 1; 2)$	$\mathcal{N} = (1, 2) \quad \mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$ $g_{\mu\nu} \quad (3, 3; 1, 1)$ $\psi_\mu \quad (2, 3; 2, 1) + (3, 2; 1, 4)$ $A_\mu \quad (2, 2; 2, 4)$ $A_{\mu\nu} \quad (3, 1; 1, 5) + (1, 3; 1, 1)$ $\lambda \quad (2, 1; 2, 5) + (1, 2; 1, 4)$ $\phi \quad (1, 1; 1, 5)$	$\mathcal{N} = (1, 1) \quad \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ $g_{\mu\nu} \quad (3, 3; 1, 1)$ $\psi_\mu \quad (2, 3; 2, 1) + (3, 2; 1, 2)$ $A_\mu \quad (2, 2; 2, 2)$ $A_{\mu\nu} \quad (3, 1; 1, 1) + (1, 3; 1, 1)$ $\lambda \quad (2, 1; 2, 1) + (1, 2; 1, 2)$ $\phi \quad (1, 1; 1, 1)$
$D = 7, \mathfrak{so}(5)_{ST}$	$\mathcal{N} = 1 \quad \mathfrak{sp}(1)$ $A_\mu \quad (5; 1) \quad \lambda \quad (4; 2) \quad \phi \quad (1; 3)$	
$\mathcal{N} = 1 \quad \mathfrak{sp}(1)$ $A_\mu \quad (5; 1)$ $\lambda \quad (4; 2)$ $\phi \quad (1; 3)$	$\mathcal{N} = 4 \quad \mathfrak{sp}(4)$ $g_{\mu\nu} \quad (14; 1)$ $\psi_\mu \quad (16; 4)$ $A_\mu, A_{\mu\nu} \quad (5; 10) + (10; 5)$ $\lambda \quad (4; 16)$ $\phi \quad (1; 14)$	
$D = 8, \mathfrak{so}(6)_{ST}$	$\mathcal{N} = 1 \quad \mathfrak{u}(1)$ $A_\mu \quad (6; 0) \quad \lambda \quad (4; -1) + (\bar{4}; 1) \quad \phi \quad (1; 2) + (1; -2)$	
$\mathcal{N} = 1 \quad \mathfrak{u}(1)$ $A_\mu \quad (6; 0)$ $\lambda \quad (4; -1) + (\bar{4}; 1)$ $\phi \quad (1; 2) + (1; -2)$	$\mathcal{N} = 2 \quad \mathfrak{u}(2)$ $g_{\mu\nu} \quad (20; 1(0))$ $\psi_\mu \quad (20'; 2(-1)) + (2\bar{0}'; 2(1))$ $A_\mu \quad (6; 3(2) + 3(-2))$ $A_{\mu\nu} \quad (15; 3(0))$ $A_{\mu\nu\rho} \quad (10; 1(-2)) + (\bar{10}; 1(2))$ $\lambda \quad (4; 2(-3) + 4(1)) + (\bar{4}; 2(-3) + 4(1))$ $\phi \quad (1; 1(4) + 1(-4) + 5(0))$	
$D = 9, \mathfrak{so}(7)_{ST}$	$\mathcal{N} = 1 \quad \emptyset \quad A_\mu \quad \mathbf{7} \quad \lambda \quad \mathbf{8} \quad \phi \quad \mathbf{1}$	
$\mathcal{N} = 1 \quad \emptyset$ $A_\mu \quad \mathbf{7}$ $\lambda \quad \mathbf{8}$ $\phi \quad \mathbf{1}$	$\mathcal{N} = 2 \quad \mathfrak{so}(2)$ $g_{\mu\nu} \quad (27; 0)$ $\psi_\mu \quad (48; (1) + (-1))$ $A_\mu \quad (7; (2) + (0) + (-2))$ $A_{\mu\nu} \quad (21; (2) + (-2))$ $A_{\mu\nu\rho} \quad (35; 0)$ $\lambda \quad (8; (3) + (1) + (-1) + (-3))$ $\phi \quad (1; (4) + (0) + (-4))$	
$D = 10, \mathfrak{so}(8)_{ST}$	$\mathcal{N} = (1, 0) \quad \emptyset \quad A_\mu \quad \mathbf{8}_v \quad \lambda \quad \mathbf{8}_s$	
$\mathcal{N} = (0, 1) \quad \emptyset$ $A_\mu \quad \mathbf{8}_v$ $\lambda \quad \mathbf{8}_c$	$\mathcal{N} = (1, 1) \quad \emptyset$ $g_{\mu\nu} \quad \mathbf{35}_v$ $\psi_\mu \quad \mathbf{56}_s + \mathbf{56}_c$ $A_\mu, A_{\mu\nu}, A_{\mu\nu\rho} \quad \mathbf{8}_v + \mathbf{28}_v + \mathbf{56}_v$ $\lambda \quad \mathbf{8}_s + \mathbf{8}_c$ $\phi \quad \mathbf{1}$	

Table 3: Tensor products of left and right super Yang-Mills multiplets in  $D = 6, 7, 8, 9, 10$ . Dimensions  $D = 4, 5$  are given in Table 2.

Making use of the super-Jacobi identities, it can be shown that the  $U_A$ 's form a representation of the algebra  $\mathfrak{so}(\mathcal{N}), \mathfrak{u}(\mathcal{N}), \mathfrak{sp}(\mathcal{N})$  for  $Q$  real, complex, quaternionic (pseudoreal), respectively. Note, since  $\mathbb{R}$ -symmetry commutes with the Lorentz algebra, only the reality properties of the spinor representation and  $\mathcal{N}$  are relevant. Consequently, we may associate a (direct sum of) division algebra(s), denoted  $\mathbb{D}$ , to every dimension, as given in Table 4, which will then dictate the  $\mathbb{R}$ -symmetry algebra. The identification of  $\mathbb{D}$  for each  $D = 3, \dots, 10$  follows from the close relationship between Clifford and division algebras. For a survey of this important correspondence see [38, 80] and the references therein.

$D$	$\text{Cliff}(D-3) \cong \text{Cliff}_0(D-2)$	$\mathbb{D}$	$D-2$ spinor representation $\cong D-3$ pinor representation	$\mathbb{R}$ -symmetry algebra
10	$\mathbb{R}[8] \oplus \mathbb{R}[8]$	$\mathbb{R}^+ \oplus \mathbb{R}^-$	$\mathbb{R}^8 \oplus \mathbb{R}^8$	$\mathfrak{so}(\mathcal{N}_+) \oplus \mathfrak{so}(\mathcal{N}_-)$
9	$\mathbb{R}[8]$	$\mathbb{R}$	$\mathbb{R}^8$	$\mathfrak{so}(\mathcal{N})$
8	$\mathbb{C}[4]$	$\mathbb{C}$	$\mathbb{C}^4$	$\mathfrak{u}(\mathcal{N})$
7	$\mathbb{H}[2]$	$\mathbb{H}$	$\mathbb{H}^2$	$\mathfrak{sp}(\mathcal{N})$
6	$\mathbb{H}[1] \oplus \mathbb{H}[1]$	$\mathbb{H}^+ \oplus \mathbb{H}^-$	$\mathbb{H} \oplus \mathbb{H}$	$\mathfrak{sp}(\mathcal{N}_+) \oplus \mathfrak{sp}(\mathcal{N}_-)$
5	$\mathbb{H}[1]$	$\mathbb{H}$	$\mathbb{H}$	$\mathfrak{sp}(\mathcal{N})$
4	$\mathbb{C}[1]$	$\mathbb{C}$	$\mathbb{C}$	$\mathfrak{u}(\mathcal{N})$
3	$\mathbb{R}[1]$	$\mathbb{R}$	$\mathbb{R}$	$\mathfrak{so}(\mathcal{N})$

Table 4: The Clifford (sub)algebras,  $\mathbb{D}$ , spinor representation and  $\mathbb{R}$ -symmetry algebra for dimensions  $D = 3, \dots, 10$ .

For a unital algebra  $A$  let  $A[m, n]$  denote the set of  $m \times n$  matrices with entries in  $A$ . When  $m = n$  we will also write  $A[n]$ . For  $D = 3, \dots, 10$  the Euclidean Clifford algebra  $\text{Cliff}(D-3)$  can be mapped to the (direct sum of) matrix algebras  $\mathbb{A}[n]$ , as given in Table 4. Up to equivalence, the unique non-trivial irreducible representations<sup>1</sup> of  $\mathbb{A}[n]$  and  $\mathbb{A}[n] \oplus \mathbb{A}[n]$  are  $\mathbb{A}^n$  and  $\mathbb{A}^n \oplus \mathbb{A}^n$ , respectively. These representations restrict to the *pinors* of  $\text{Pin}(D-3)$ , the double cover of  $\text{O}(D-3)$ , as it is generated by the subset of unit vectors in  $\mathbb{R}^{D-3}$ . There is a canonical isomorphism from  $\text{Cliff}(D-3)$  to  $\text{Cliff}_0(D-2)$ , where  $\text{Cliff}_0(m)$  denotes the subalgebra generated by products of an even number of vectors in  $\mathbb{R}^m$ . Since  $\text{Spin}(D-2)$ , the double cover of the spacetime little group, sits inside  $\text{Cliff}_0(D-2)$  as the set of all elements that are a product of an even number of unit vectors in  $\mathbb{R}^{D-2}$ , the pinors of  $\text{Pin}(D-3)$  are precisely the spinors of  $\text{Spin}(D-2)$ , as given in Table 4. The supersymmetry algebra generators,  $Q$ , transform according as these representations under  $\text{Spin}(D-2)$ . Hence, we may identify  $\mathbb{D}$  as the appropriate algebra for each spacetime dimension  $D$ . Note that in dimensions 6 and 10 the direct sum structure of  $\mathbb{D}$  corresponds to the existence of  $\mathcal{N} = (\mathcal{N}_+, \mathcal{N}_-)$  chiral theories.

Let us now briefly recall some of the standard relations between  $\mathbb{R}, \mathbb{C}, \mathbb{H}$  and the classical Lie algebras. Denote by  $\mathfrak{a}(n, \mathbb{A})$  the set of anti-Hermitian elements in  $\mathbb{A}[n]$ ,

$$\mathfrak{a}(n, \mathbb{A}) := \{x \in \mathbb{A}[n] : x^\dagger = -x\}. \quad (2.8)$$

Using the standard matrix commutator these constitute the classical Lie algebras

$$\mathfrak{a}(n, \mathbb{A}) = \begin{cases} \mathfrak{so}(n), & \mathbb{A} = \mathbb{R}; \\ \mathfrak{u}(1) \oplus \mathfrak{su}(n), & \mathbb{A} = \mathbb{C}; \\ \mathfrak{sp}(n), & \mathbb{A} = \mathbb{H}. \end{cases} \quad (2.9)$$

<sup>1</sup>Note that in Table 2 and Table 3 we work with the more familiar complex representations. However for  $D = 5, 6, 7$  one could map from a complex to a quaternionic representation via,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow 1, \quad \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \rightarrow i, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \rightarrow j, \quad \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \rightarrow k, \quad \text{and} \quad \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow q, \quad (2.7)$$

for  $a, b \in \mathbb{C}$  and  $q \in \mathbb{H}$ .



Let  $\mathfrak{sa}(n, \mathbb{A})$  denote their special subalgebras:

$$\begin{aligned} \mathfrak{sa}(n, \mathbb{R}) &:= \{x \in \mathbb{A}[n] : x^\dagger = -x\} &= \mathfrak{so}(n); \\ \mathfrak{sa}(n, \mathbb{C}) &:= \{x \in \mathbb{A}[n] : x^\dagger = -x, \text{tr}(x) = 0\} &= \mathfrak{su}(n); \\ \mathfrak{sa}(n, \mathbb{H}) &:= \{x \in \mathbb{A}[n] : x^\dagger = -x\} &= \mathfrak{sp}(n). \end{aligned} \quad (2.10)$$

The seemingly undemocratic definition of  $\mathfrak{sa}(n, \mathbb{A})$  follows naturally from the geometry of projective spaces since

$$\mathfrak{Isom}(\mathbb{A}\mathbb{P}^{n-1}) \cong \mathfrak{sa}(n, \mathbb{A}) \quad (2.11)$$

for  $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ . In the octonionic case only  $\mathbb{O}\mathbb{P}^1$  and  $\mathbb{O}\mathbb{P}^2$  constitute projective spaces with  $\mathfrak{Isom}(\mathbb{O}\mathbb{P}^1) \cong \mathfrak{so}(8)$  and  $\mathfrak{Isom}(\mathbb{O}\mathbb{P}^2) \cong \mathfrak{f}_{4(-52)}$ , reflecting their exceptional status.

It then follows that the  $\mathcal{N}$ -extended R-symmetry algebras in  $D$  dimensions, denoted  $\mathfrak{ra}(\mathcal{N}, D)$ , are given by

$$\mathfrak{ra}(\mathcal{N}, D) = \mathfrak{a}(\mathcal{N}, \mathbb{D}), \quad (2.12)$$

where for  $\mathcal{N} = (\mathcal{N}^+, \mathcal{N}^-)$ , as is the case for  $D = 6, 10$ , we have used the definition

$$\mathbb{D}[(\mathcal{N}^+, \mathcal{N}^-)] := \mathbb{D}^+[\mathcal{N}^+] \oplus \mathbb{D}^-[\mathcal{N}^-]. \quad (2.13)$$

Since  $\mathbb{D}[m, n] \cong \mathbb{D}^m \otimes \mathbb{D}^n$  forms a natural (but not necessarily irreducible) representation of  $\mathfrak{a}(m, \mathbb{D}) \oplus \mathfrak{a}(n, \mathbb{D})$ , it follows quite simply that the  $(\mathcal{N}_L + \mathcal{N}_R)$ -extended R-symmetry algebra is given by the  $\mathcal{N}_L$  and  $\mathcal{N}_R$  R-symmetry algebras via

$$\mathfrak{ra}(\mathcal{N}_L + \mathcal{N}_R, D) = \mathfrak{a}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D}) = \mathfrak{a}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{a}(\mathcal{N}_R, \mathbb{D}) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]. \quad (2.14)$$

The commutators for elements  $X_L \in \mathfrak{a}(\mathcal{N}_L, \mathbb{D})$ ,  $X_R \in \mathfrak{a}(\mathcal{N}_R, \mathbb{D})$  and  $M, N \in \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]$  are given by

$$\begin{aligned} [X_L, M] &= X_L M && \in \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R], \\ [X_R, M] &= -M X_R && \in \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R], \\ [M, N] &= (NM^\dagger - MN^\dagger) \oplus (NM^\dagger - MN^\dagger) && \in \mathfrak{a}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{a}(\mathcal{N}_R, \mathbb{D}). \end{aligned} \quad (2.15)$$

These commutation relations follow from the standard matrix commutators of

$$X = \begin{pmatrix} X_L & 0 \\ 0 & X_R \end{pmatrix} + \begin{pmatrix} 0 & M \\ -M^\dagger & 0 \end{pmatrix}, \quad (2.16)$$

where  $X \in \mathfrak{a}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D})$ .

Note, as a  $\mathfrak{a}(\mathcal{N}_L, \mathbb{C}) \oplus \mathfrak{a}(\mathcal{N}_R, \mathbb{C})$ -module  $\mathbb{C}[\mathcal{N}_L, \mathcal{N}_R]$  is not irreducible. For example, in the maximal  $D = 4$  case it corresponds to the  $(\mathbf{4}, \bar{\mathbf{4}}) + (\bar{\mathbf{4}}, \mathbf{4})$  representation of  $\mathfrak{su}(4) \oplus \mathfrak{su}(4) \cong \mathfrak{sa}(4, \mathbb{C}) \oplus \mathfrak{sa}(4, \mathbb{C})$ . The formula (2.14) and its commutators (2.15) amount to the well-known statement that the pairs  $[\mathfrak{so}(p+q), \mathfrak{so}(p) \oplus \mathfrak{so}(q)]$ ,  $[\mathfrak{su}(p+q), \mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathfrak{u}(1)]$  and  $[\mathfrak{sp}(p+q), \mathfrak{sp}(p) \oplus \mathfrak{sp}(q)]$  constitute type I symmetric spaces.

From the perspective of the left/right tensor product,  $\mathfrak{a}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{a}(\mathcal{N}_R, \mathbb{D})$  is generated directly by the R-symmetries of the left and right factors acting on  $Q_L$  and  $Q_R$  independently. However, together they form an irreducible doublet  $(Q_L, Q_R) \in \mathbb{D}^{\mathcal{N}_L} \oplus \mathbb{D}^{\mathcal{N}_R}$  (suppressing the spacetime little group spinor representation space), which must be rotated by an  $\mathfrak{a}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{a}(\mathcal{N}_R, \mathbb{D})$ -module. The most general consistent subset of  $\text{End}(\mathbb{D}^{\mathcal{N}_L} \oplus \mathbb{D}^{\mathcal{N}_R})$  is given by  $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]$ , which completes (2.14) as is made clear by (2.16). In the sense to be described in subsection 2.3, these additional elements can be generated by  $Q_L \otimes Q_R \in \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]$  by formally neglecting its little group representation space.

It follows from [28] that for  $D = 3$  the R-symmetry algebras admit an alternative Freudenthal magic square description. Recall that the U-duality groups in  $D = 3$  form the Freudenthal Magic square given by

$$\begin{aligned} \mathfrak{L}_3(\mathbb{A}_L, \mathbb{A}_R) &= \mathfrak{tri}(\mathbb{A}_{\mathcal{N}_L}) \oplus \mathfrak{tri}(\mathbb{A}_{\mathcal{N}_R}) + 3(\mathbb{A}_{\mathcal{N}_L} \otimes \mathbb{A}_{\mathcal{N}_R}) \\ &= \mathfrak{der} \mathbb{A}_{\mathcal{N}_L} \oplus \mathfrak{der} \mathfrak{J}_3^0(\mathbb{A}_{\mathcal{N}_R}) + \text{Im} \mathbb{A}_{\mathcal{N}_L} \otimes \mathfrak{J}_3^0(\mathbb{A}_{\mathcal{N}_R}), \end{aligned} \quad (2.17)$$

where  $\mathfrak{der}$  denotes the derivation algebra,  $\mathfrak{J}_3(\mathbb{A})$  is the Jordan algebra of  $3 \times 3$  Hermitian matrices over  $\mathbb{A}$  and  $\mathfrak{J}_3^0(\mathbb{A})$  is its traceless subspace. See for example [38]. One can generalise this construction for any rank of the Jordan algebra  $\mathfrak{J}_n(\mathbb{A})$ ,

$$\mathfrak{L}_n(\mathbb{A}_{\mathcal{N}_L}, \mathbb{A}_{\mathcal{N}_R}) = \mathfrak{der}\mathbb{A}_{\mathcal{N}_L} \oplus \mathfrak{der}\mathfrak{J}_n(\mathbb{A}_{\mathcal{N}_R}) + \text{Im}\mathbb{A}_{\mathcal{N}_L} \otimes \mathfrak{J}_n^0(\mathbb{A}_{\mathcal{N}_R}), \quad (2.18)$$

where for  $n > 3$  we must exclude the octonionic case [29]. The supergravity R-symmetry algebras in  $D = 3$  are given by  $n = 2$ ,

$$\mathfrak{L}_2(\mathbb{A}_{\mathcal{N}_L}, \mathbb{A}_{\mathcal{N}_R}) = \mathfrak{der}\mathbb{A}_{\mathcal{N}_L} \oplus \mathfrak{der}\mathfrak{J}_2(\mathbb{A}_{\mathcal{N}_R}) + \text{Im}\mathbb{A}_{\mathcal{N}_L} \otimes \mathfrak{J}_2^0(\mathbb{A}_{\mathcal{N}_R}) = \mathfrak{so}(\mathcal{N}_L + \mathcal{N}_R). \quad (2.19)$$

### 2.3 $H$ algebras

With this construction in mind we turn our attention now to the algebra  $\mathfrak{h}$  of the maximal compact subgroup  $H \subset G$  and, in particular, how it is built from the global symmetries of the left and right super Yang-Mills theories.

We will write  $\mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, D)$  in terms of  $\mathfrak{int}(\mathcal{N}_L, D)$  and  $\mathfrak{int}(\mathcal{N}_R, D)$ . First, note that  $\mathfrak{int}$  and  $\mathfrak{h}$  have a similar structure; they are both given by  $\mathfrak{sa}(\mathcal{N}, \mathbb{D})$ , possibly with an additional commuting factor, which we denote by  $\mathfrak{p}$ . Explicitly, from Table 5 we observe,

$$\mathfrak{int}(\mathcal{N}, D) = \mathfrak{sa}(\mathcal{N}, \mathbb{D}) \oplus \mathfrak{p}, \quad (2.20)$$

where  $\mathfrak{p} = \mathfrak{u}(1), \mathfrak{u}(1), \mathfrak{so}(2), \mathfrak{so}(3)$  for  $D = 4, \mathcal{N} = 1, 2$  and  $D = 3, \mathcal{N} = 2, 4$ , respectively, and is empty otherwise. In  $D = 4$  these additional factors follow from the inclusion of the CPT conjugate, whereas in  $D = 3$  they appear on dualising the gauge field into a scalar, which also enhances  $\mathfrak{so}(7) \rightarrow \mathfrak{so}(8)$  in the maximally supersymmetric case.

$D$	$Q = 16$		$Q = 8$		$Q = 4$		$Q = 2$	
	$\mathcal{N}$	$\mathfrak{int}$	$\mathcal{N}$	$\mathfrak{int}$	$\mathcal{N}$	$\mathfrak{int}$	$\mathcal{N}$	$\mathfrak{int}$
10	1	$\emptyset$	—	—	—	—	—	—
9	1	$\emptyset$	—	—	—	—	—	—
8	1	$\mathfrak{u}(1)$	—	—	—	—	—	—
7	1	$\mathfrak{sp}(1)$	—	—	—	—	—	—
6	(1, 1)	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	(1, 0)	$\mathfrak{sp}(1) \oplus \emptyset$	—	—	—	—
	(2, 0)	$\mathfrak{sp}(2) \oplus \emptyset$	(1, 0)	$\mathfrak{sp}(1) \oplus \emptyset$	—	—	—	—
5	2	$\mathfrak{sp}(2)$	1	$\mathfrak{sp}(1)$	—	—	—	—
4	4	$\mathfrak{su}(4)$	2	$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	1	$\mathfrak{u}(1)$	—	—
3	8	$\mathfrak{so}(7)$	4	$\mathfrak{so}(4)$	2	$\mathfrak{so}(2)$	1	$\emptyset$
3*	8	$\mathfrak{so}(8)$	4	$\mathfrak{so}(4) \oplus \mathfrak{so}(3)$	2	$\mathfrak{so}(2) \oplus \mathfrak{so}(2)$	1	$\emptyset$

Table 5: The internal global symmetry algebras  $\mathfrak{int}(\mathcal{N}, D)$  of super Yang-Mills theories in  $D \geq 3$ . In  $D = 6$  we have included the (2, 0) and (1, 0) tensor multiplets. Note, for  $D = 3^*$  we have dualised the vector yielding an enhanced symmetry,  $\mathfrak{int}(\mathcal{N}, 3) = \mathfrak{tri}(\mathbb{A}_{\mathcal{N}})$  for  $\mathbb{A}_{\mathcal{N}} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ . For interacting Lagrangians this symmetry is reduced to the intermediate algebra  $\mathfrak{int}(\mathbb{A}_{\mathcal{N}}) := \{(A, B, C) \in \mathfrak{tri}(\mathbb{A}_{\mathcal{N}}) | A(1) = 0\}$ , which gives  $\emptyset, \mathfrak{so}(2), \mathfrak{so}(4), \mathfrak{so}(7)$  for  $\mathbb{A}_{\mathcal{N}} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ , respectively. The enhanced  $\mathfrak{tri}(\mathbb{A}_{\mathcal{N}})$  symmetry is recovered in the infrared limit. Note, while in general the R-symmetry algebra  $\mathfrak{ra}(\mathcal{N}, D)$  and the internal global symmetry algebra  $\mathfrak{int}(\mathcal{N}, D)$  coincide, there are several exceptions such as  $\mathfrak{su}(4)$  versus  $\mathfrak{u}(4)$  for  $D = 4, \mathcal{N} = 4$ .

The commuting factors of the left and right super Yang-Mills theories are inherited in the tensor product and, hence, the resulting  $\mathfrak{h}$  also contains a commuting<sup>2</sup>  $\mathfrak{p}_L \oplus \mathfrak{p}_R$ . The algebras  $\mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, D)$

<sup>2</sup>Note that in 3 dimensions we could also work in the conventions of [28]. Here the fields of the  $\mathcal{N} = 1, 2, 4, 8$  super Yang-Mills belong to  $\mathbb{A}_{\mathcal{N}} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ . Then  $\mathfrak{p} = \mathfrak{so}(2) = \mathfrak{u}(1)$  is generated by  $i$  and  $\mathfrak{so}(3) = \mathfrak{sp}(1)$  is generated by  $i, j, k$ , the imaginary unit quaternions, which will act naturally on the other terms in (1.1). We do not adopt this convention here and work with real generators for  $\mathfrak{so}(2)$  and  $\mathfrak{so}(3)$  in  $D = 3$ .

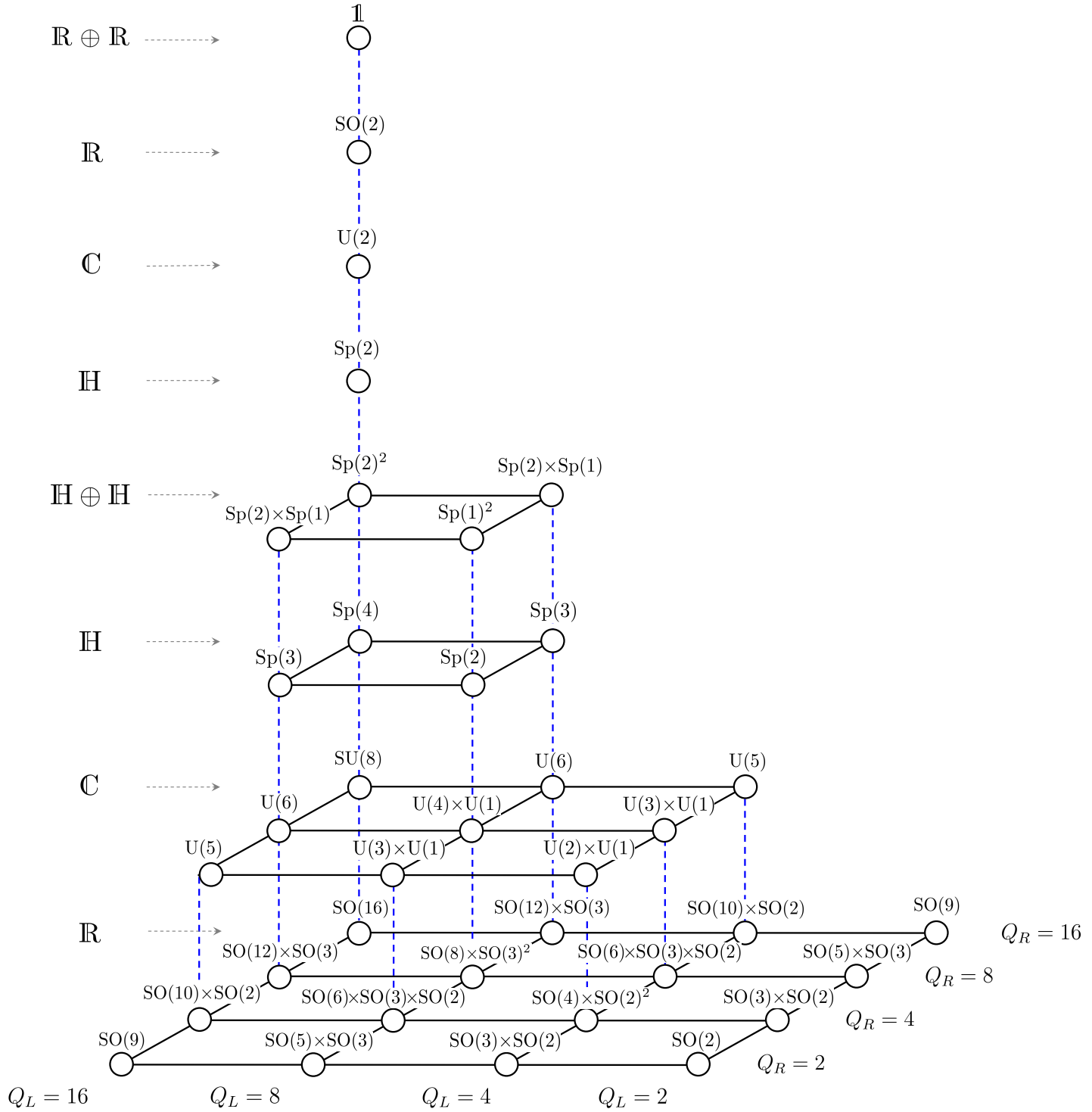


Figure 1: Pyramid of maximal compact subgroups  $H \subset G$ . The amount of supersymmetry is determined by the horizontal axes. The spacetime dimension is determined by the division algebra  $\mathbb{D}$  on the vertical axis as given in Table 4.

presented in Table 2 and Table 3 are consequently given by

$$\begin{aligned}
\mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, D) &= \text{int}(\mathcal{N}_L, D) \oplus \text{int}(\mathcal{N}_R, D) \oplus \delta_{D,4\mathbf{u}}(1) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R] \\
&= [\mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{p}_L] \oplus [\mathfrak{sa}(\mathcal{N}_R, \mathbb{D}) \oplus \mathfrak{p}_R] \oplus \delta_{D,4\mathbf{u}}(1) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R] \\
&= [\mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{sa}(\mathcal{N}_R, \mathbb{D}) \oplus \delta_{D,4\mathbf{u}}(1) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]] \oplus \mathfrak{p}_L \oplus \mathfrak{p}_R \\
&= \mathfrak{sa}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D}) \oplus \mathfrak{p}_L \oplus \mathfrak{p}_R,
\end{aligned} \tag{2.21}$$

where the non-trivial commutators are those given in (2.15). The term

$$[\mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{p}_L] \oplus [\mathfrak{sa}(\mathcal{N}_R, \mathbb{D}) \oplus \mathfrak{p}_R] \tag{2.22}$$

follows directly from the left and right super Yang-Mills symmetries. It acts on the gravitini  $\psi_L$  and  $\psi_R$  independently in the defining representation, since the left and right gauge potentials  $A_{L/R}$  are  $\text{int}(\mathcal{N}_{L/R}, D)$  singlets. However, as for the supersymmetry charges, the gravitini are collected into an irreducible doublet  $(\psi_L, \psi_R)$ , which is rotated by  $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]$ , and hence transform in the defining representation of

$$\mathfrak{sa}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D}) = \mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{sa}(\mathcal{N}_R, \mathbb{D}) \oplus \delta_{D,4\mathbf{u}}(1) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]. \tag{2.23}$$

The corresponding pyramid of  $H$  groups, which generalises the magic  $H$  pyramid of [34], is presented in Figure 1.

While the  $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]$  component of  $\mathfrak{h}$  is implied by consistency, one might also more ambitiously ask whether it can be directly generated by elementary operations acting on the left and right super Yang-Mills fields in same way  $\text{int}(\mathcal{N}_L, D) \oplus \text{int}(\mathcal{N}_R, D) \subset \mathfrak{h}$  obviously is. Having already used all left/right bosonic symmetries, only the left/right supersymmetry generators remain. The conventional infinitesimal supersymmetry variation of the *left*  $\otimes$  *right* states correctly gives the infinitesimal supersymmetry variation on the corresponding supergravity states [4, 6, 23]. Seeking, instead, internal *bosonic* transformations on the supergravity multiplet that follow from supersymmetry on the left and right Yang-Mills multiplets suggests starting from the rather unconventional tensor product of the left and right supercharges,  $Q \otimes \tilde{Q}$ . That this might work, at least formally, follows from the observation

$$Q \in \mathbb{D}^{\mathcal{N}} \quad \Rightarrow \quad Q \otimes \tilde{Q} \in \mathbb{D}^{\mathcal{N}_L} \otimes \mathbb{D}^{\mathcal{N}_R} \cong \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R], \tag{2.24}$$

where we are explicitly suppressing the spacetime indices.

Adopting the spinor-helicity formalism, a simple concrete example sufficient to illustrate the principle is given by the  $4 + 4$  positive helicity gravitini states of  $D = 4, \mathcal{N} = 8$  supergravity,

$$\psi_+^a = \lambda_+^a \otimes \tilde{A}_+, \quad \psi_+^{a'} = A_+ \otimes \tilde{\lambda}_+^{a'}, \tag{2.25}$$

where  $a, a' = 1, \dots, 4$  are the  $\mathbf{4}$  of  $\mathfrak{su}(4)_L$  and  $\mathfrak{su}(4)_R$ , respectively. Defining  $Q_-^a = -\epsilon^\alpha Q_\alpha^a$  and  $Q_+^{a'} = -\epsilon_{a'} \tilde{Q}^{a'}$ , the relevant super Yang-Mills transformations are

$$\begin{aligned}
Q_+^a A_+(p) &= 0, & Q_+^a \lambda_+^b &= \langle \epsilon p \rangle \delta_a^b A_+(p), \\
Q_-^a A_+(p) &= [p\epsilon] \lambda_+^a(p), & Q_-^a \lambda_+^b &= [p\epsilon] \phi^{[ab]}(p).
\end{aligned} \tag{2.26}$$

Applying these to (2.25) we obtain

$$\begin{aligned}
[Q_+^a \otimes \tilde{Q}_-^{a'}] \psi_+^b &= [Q_+^a \lambda_+^b] \otimes [\tilde{Q}_-^{a'} \tilde{A}] = [p\epsilon] \langle \epsilon p \rangle \delta_a^b A_+ \otimes \tilde{\lambda}_+^{a'} = [p\epsilon] \langle \epsilon p \rangle \delta_a^b \psi_+^{a'}, \\
[Q_+^a \otimes \tilde{Q}_-^{a'}] \psi_+^{b'} &= [Q_+^a A_+] \otimes [\tilde{Q}_-^{a'} \tilde{\lambda}^{b'}] = 0, \\
[Q_-^a \otimes \tilde{Q}_+^{a'}] \psi_+^b &= [Q_-^a \lambda_+^b] \otimes [\tilde{Q}_+^{a'} \tilde{A}] = 0, \\
[Q_-^a \otimes \tilde{Q}_+^{a'}] \psi_+^{b'} &= [Q_-^a A_+] \otimes [\tilde{Q}_+^{a'} \tilde{\lambda}^{b'}] = [p\epsilon] \langle \epsilon p \rangle \delta_a^{b'} \lambda_+^a \otimes \tilde{A}_+ = [p\epsilon] \langle \epsilon p \rangle \delta_a^{b'} \psi_+^a,
\end{aligned} \tag{2.27}$$

which, up to the factors of  $[p\epsilon] \langle \epsilon p \rangle$ , is precisely the action of the  $\mathfrak{su}(8)$  generators belonging to  $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]$  on the positive helicity gravitini states valued in  $\mathbb{D}^{\mathcal{N}_L} \oplus \mathbb{D}^{\mathcal{N}_R}$ , for  $D = \mathcal{N}_{L/R} = 4$ , or, in a perhaps

more familiar language, the action of generators in the  $(\mathbf{4}, \bar{\mathbf{4}}) + (\bar{\mathbf{4}}, \mathbf{4})$  component of  $\mathfrak{su}(8)$  acting on the  $\mathbf{8} = (\mathbf{4}, \mathbf{1}) + (\mathbf{1}, \mathbf{4})$  representation.

Thus, formally suppressing the spacetime components of the supercharges (and parameters) provides a definition of the elementary transformations acting on the left and right states, which correctly reproduces the action of  $\mathfrak{h}$  on their tensor product. More concretely, we have

$$Q_-^a = \int \frac{d^3 p}{(2\pi)^3 2E_p} [p\epsilon] \left[ -\lambda_+^a (A_+)^{\dagger} + \phi^{[ab]} (\lambda_+^b)^{\dagger} + 2\lambda_{b-} (\phi^{[ab]})^{\dagger} - A_- (\lambda_{a-})^{\dagger} \right], \quad (2.28)$$

which ensures the correct action of the supersymmetry operator with non-trivial equal time (anti)commutation relations:

$$\begin{aligned} [A_{\pm}(p), A_{\pm}^{\dagger}(q)] &= (2\pi)^3 2E_p \delta^3(\vec{p} - \vec{q}), \\ \{\lambda_{\pm}^a(p), \lambda_{b\pm}^{\dagger}(q)\} &= (2\pi)^3 2E_p \delta^3(\vec{p} - \vec{q}) \delta_b^a, \\ [\phi^{[ab]}(p), \phi_{[cd]}^{\dagger}(q)] &= (2\pi)^3 2E_p \delta^3(\vec{p} - \vec{q}) \delta_{[cd]}^{[ab]}, \end{aligned} \quad (2.29)$$

where  $\phi^{[ab]} = \frac{1}{4!} \varepsilon^{abcd} \phi_{[cd]}$ . The operators  $Q_L \otimes Q_R$  generating  $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]$  are then defined by simply dropping the  $[p\epsilon]$  factors in this representation of  $Q$ . For example:

$$Q_{L\mp}^a := \int \frac{d^3 p}{(2\pi)^3 2E_p} \left[ -\lambda_{\pm}^a (A_{\pm})^{\dagger} + \phi^{[ab]} (\lambda_{\pm}^b)^{\dagger} + 2\lambda_{b\mp} (\phi^{[ab]})^{\dagger} - A_{\mp} (\lambda_{a\mp})^{\dagger} \right] \quad (2.30)$$

and similarly for the remaining  $Q$ 's.

One can check this construction gives the correct action on the rest of the  $\mathcal{N} = 8$  multiplet and generalises to any dimension and number of supercharges. Note that in higher dimensions, where the little group is larger than  $\mathfrak{u}(1)$ , the tensor product of two super Yang-Mills states typically yields a direct sum of supergravity states; to pick a specific component we need to project out the desired representation. To find the action of  $Q_L \otimes Q_R$  on a state, we first act on the tensor product which contains it, and then project out the state we want. Returning to our maximal  $D = 5$  example (described in (2.4) and Table 2), we see that the gravitini states live in the  $(\mathbf{4}; \mathbf{1}, \mathbf{4}) + (\mathbf{4}; \mathbf{4}, \mathbf{1})$  representation of  $\mathfrak{so}(3)_{ST} \oplus \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$ . Focusing on the  $(\mathbf{4}; \mathbf{1}, \mathbf{4})$  states, we see that they are obtained by a projection of  $A_{\mu} \otimes \tilde{\lambda} = (\mathbf{3}; \mathbf{1}) \otimes (\mathbf{2}; \mathbf{4}) = (\mathbf{4}; \mathbf{1}, \mathbf{4}) + (\mathbf{2}; \mathbf{1}, \mathbf{4})$ . Then each of the  $Q_L$  and  $Q_R$ , both living in  $(\mathbf{2}; \mathbf{4})$  of  $\mathfrak{so}(3)_{ST} \oplus \mathfrak{sp}(2)$  will act individually on the factors in our product, yielding  $(\mathbf{4}; \mathbf{4}, \mathbf{1}) + (\mathbf{2}; \mathbf{4}, \mathbf{1}) + (\mathbf{2}; \mathbf{4}, \mathbf{5})$ . Projecting out the gravitini, we find that the  $(\mathbf{4}; \mathbf{1}, \mathbf{4})$  states have been rotated into  $(\mathbf{4}; \mathbf{4}, \mathbf{1})$  states. Finally,  $\delta_{D,4}\mathfrak{u}(1)$  is one of the Cartan generators of the  $\mathfrak{su}(n)$  algebra and will act accordingly on the other generators.

## 2.4 $G$ algebras

The non-compact U-duality algebras of the supergravity theories appearing in the pyramid, Figure 2, can be built straightforwardly using the tensor product of the left and right super Yang-Mills multiplets. Recall, the scalars of supergravity coupled to matter generated by squaring parametrise a  $G/H$  coset and  $T_p(G/H) \cong \mathfrak{p} = \mathfrak{g} \ominus \mathfrak{h}$ . They therefore carry the  $\mathfrak{p}$ -representation of  $H$ . Consequently, the non-compact generators  $\mathfrak{p}$ , in a manifest  $\mathfrak{int}(\mathcal{N}_L, D) \oplus \mathfrak{int}(\mathcal{N}_R, D)$  basis, can be read off from those tensor products which yield scalars, which are schematically given by:

$$A_{\mu} \otimes \tilde{A}_{\nu}, \quad \lambda^a \otimes \tilde{\lambda}^{a'}, \quad \phi^i \otimes \tilde{\phi}^{i'}. \quad (2.31)$$

To recast this observation into the language used for  $\mathfrak{h}(\mathcal{N}_L, \mathcal{N}_R, \mathbb{D})$ , we summarise here the corresponding division algebraic characterisation of the  $(D, \mathcal{N})$  super Yang-Mills multiplet  $(A_{\mu}, \lambda^a, \phi^i)$ :

1.  $A_{\mu}$ : The gauge potential is a  $\mathfrak{sa}(\mathcal{N}, \mathbb{D})$  singlet valued in  $\mathbb{R}$ .
2.  $\lambda^a$ : The  $\mathcal{N}$  gaugini transform in the defining representation of  $\mathfrak{sa}(\mathcal{N}, \mathbb{D})$  and are valued in  $\mathbb{D}^{\mathcal{N}}$ .

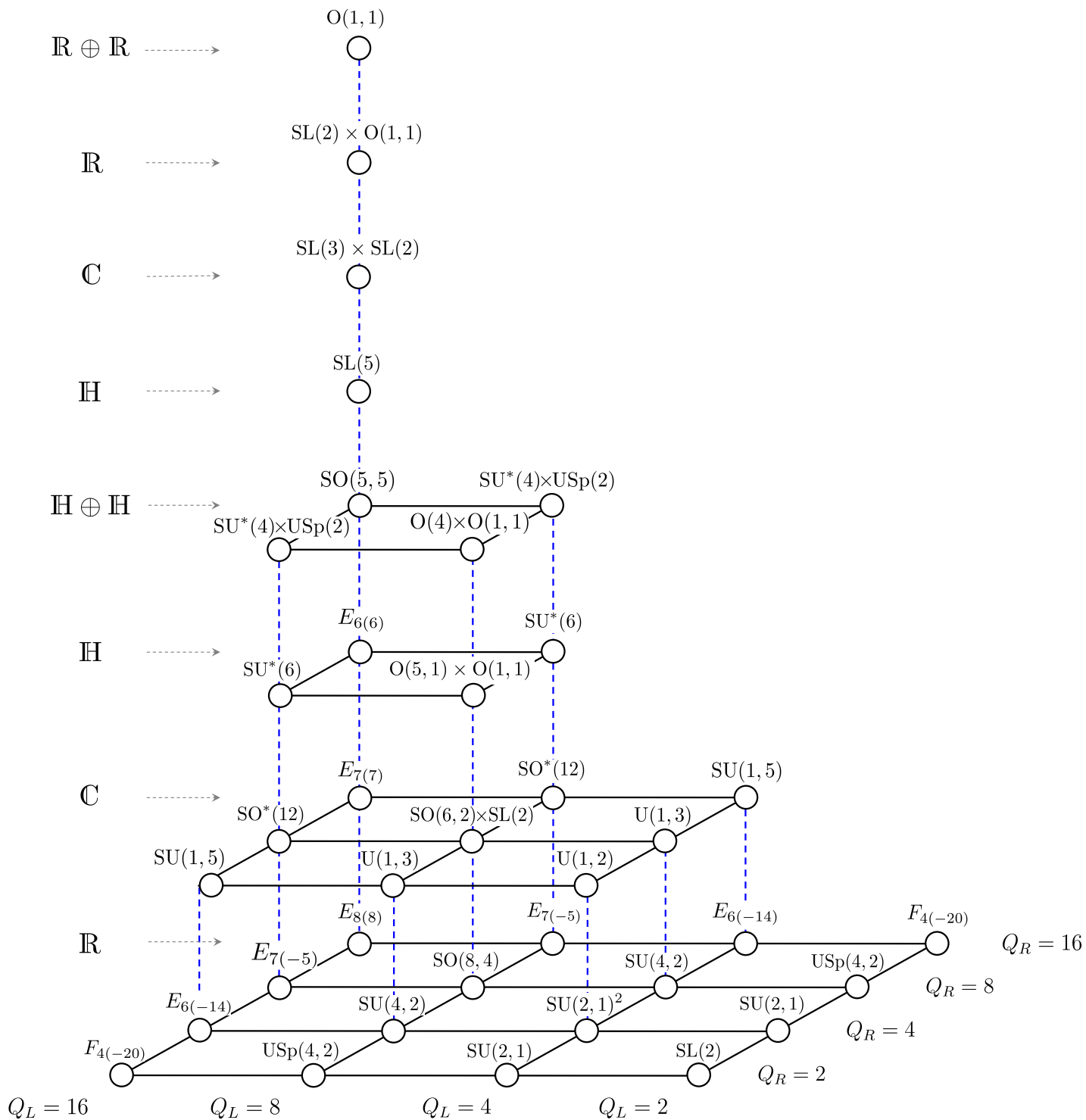


Figure 2: The U-duality group  $G$  in all dimensions.

3.  $\phi^i$ : The  $Q/2 - (D - 2)$  scalars span a subspace  $\mathbb{D}_*[\mathcal{N}] \subseteq \mathbb{D}[\mathcal{N}]$  since they are quadratic in the supersymmetry charges valued in  $\mathbb{D}^{\mathcal{N}}$  acting on the gauge potential states.

These subspaces  $\mathbb{D}_*[\mathcal{N}]$  are summarised here:

$D/Q$	16	8	4
10	$\mathfrak{a}((1, 0), \mathbb{R}^+ \oplus \mathbb{R}^-) \cong \emptyset$ $\emptyset \subseteq \mathbb{R}^+[1]$	—	—
9	$\mathfrak{a}(1, \mathbb{R}) \cong \emptyset$ $\mathbf{1} \cong \mathbb{R} \subseteq \mathbb{R}[1]$	—	—
8	$\mathfrak{a}(1, \mathbb{C}) \cong \mathfrak{u}(1)$ $(+2) + (-2) \cong \mathbb{C} \subseteq \mathbb{C}[1]$	—	—
7	$\mathfrak{a}(1, \mathbb{H}) \cong \mathfrak{sp}(1)$ $\mathbf{3} \cong \text{Im}\mathbb{H} \subseteq \mathbb{H}[1]$	—	—
6	$\mathfrak{a}((1, 1), \mathbb{H}^+ \oplus \mathbb{H}^-) \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ $(\mathbf{2}, \mathbf{2}) \cong \mathbb{H} \subseteq \mathbb{H}^+[1] \oplus \mathbb{H}^-[1]$	$\mathfrak{a}((1, 0), \mathbb{H}^+ \oplus \mathbb{H}^-) \cong \mathfrak{sp}(1)$ $\emptyset \subseteq \mathbb{H}^+[1]$	—
5	$\mathfrak{a}(2, \mathbb{H}) \cong \mathfrak{sp}(2)$ $\mathbf{5} \cong \mathfrak{J}_2^0(\mathbb{H}) \subseteq \mathbb{H}[2]$	$\mathfrak{a}(1, \mathbb{H}) \cong \mathfrak{sp}(1)$ $\mathbf{1} \cong \text{Re}\mathbb{H} \subseteq \mathbb{H}[1]$	—
4	$\mathfrak{a}(4, \mathbb{C}) \cong \mathfrak{u}(4)$ $\mathbf{6}_0 \cong \wedge_*^2 \mathbb{C}^4 \subseteq \mathbb{C}[4]$	$\mathfrak{a}(2, \mathbb{C}) \cong \mathfrak{u}(2)$ $\mathbf{1}_2 + \mathbf{1}_{-2} \cong \wedge^2 \mathbb{C}^2 \cong \mathbb{C} \subseteq \mathbb{C}[2]$	$\mathfrak{a}(1, \mathbb{C}) \cong \mathfrak{u}(1)$ $\emptyset \subseteq \mathbb{C}[1]$

Here we have listed for each  $(D, Q)$  the R-symmetry algebra  $\mathfrak{a}(\mathcal{N}, \mathbb{D})$ , the  $\mathfrak{a}(\mathcal{N}, \mathbb{D})$ -representation  $\mathfrak{n}$  carried by the scalar fields and the corresponding representation space  $\mathbb{D}_*[\mathcal{N}] \subseteq \mathbb{D}[\mathcal{N}]$ . The perhaps less familiar cases involving  $\mathbb{H}$  are given in Appendix A. Note,  $\mathfrak{J}_2^0(\mathbb{H})$  is the space of traceless  $2 \times 2$  Hermitian matrices over  $\mathbb{H}$ .

In maximal  $Q = 16$  cases this description can be easily connected back to the more familiar language of Pauli matrices (intertwiners). The  $n = 10 - D$  scalars transform in the vector representation  $V_n$  of  $\mathfrak{sa}(\mathcal{N}, \mathbb{D}) \cong \mathfrak{spin}(10 - D)$ . There is a natural inclusion  $V_n \hookrightarrow \text{End}_{\mathbb{R}}(\mathbb{D}^{\mathcal{N}}) \cong \mathbb{D}[\mathcal{N}]$  which implies that the scalars span a subspace  $\mathbb{D}_*[\mathcal{N}] \subset \mathbb{D}[\mathcal{N}]$ . As a vector space  $\mathbb{D}_*[\mathcal{N}]$  is spanned by the  $\mathfrak{spin}(n)$  Pauli matrices since  $\text{Spin}(n) \subset \text{Cliff}_0(n) \cong \text{Cliff}(n - 1) \cong \mathbb{D}[\mathcal{N}]$ . The non-maximal cases are contained in  $\mathbb{D}_*[\mathcal{N}] \subset \mathbb{D}_*[\mathcal{N}_{max}]$  subspaces.

Each component of  $\mathfrak{p}$  decomposed with respect to  $\text{int}(\mathcal{N}_L, D) \oplus \text{int}(\mathcal{N}_R, D)$  then has a direct *left*  $\otimes$  *right* origin in terms of (2.31) expressed in terms of the above representation spaces:

1.  $A_\mu \otimes \tilde{A}_\nu$ : The scalars originating from  $A_L \otimes A_R$  belong to  $\mathbb{R}_L \otimes \mathbb{R}_R \cong \mathfrak{so}(1, 1)$ . In  $D = 4$ , there is an extra  $\mathbb{R}_L \otimes \mathbb{R}_R$  term originating from the dualisation  $B_{\mu\nu} \rightarrow \phi$ . This contributes to  $\mathfrak{p}$  a term given by:

$$\mathbb{R}_L \otimes \mathbb{R}_R + i\delta_{D,4}\mathbb{R}_L \otimes \mathbb{R}_R. \quad (2.32)$$

2.  $\lambda^a \otimes \tilde{\lambda}^{a'}$ : The scalars originating from  $\lambda_L \otimes \lambda_R$  contribute a term given by

$$\mathbb{D}^{\mathcal{N}_L} \otimes \mathbb{D}^{\mathcal{N}_R} \cong \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]. \quad (2.33)$$

3.  $\phi^i \otimes \tilde{\phi}^{i'}$ : The scalars originating from  $\phi_L \otimes \phi_R$  contribute a term given by

$$\mathbb{D}_*[\mathcal{N}_L] \otimes \mathbb{D}_*[\mathcal{N}_R]. \quad (2.34)$$

Bringing these elements together, we conclude that in total  $\mathfrak{g}$  as a vector space is given by:

$$\mathfrak{g}(\mathcal{N}_L + \mathcal{N}_R, D) = \mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, D) + \mathbb{D}_*[\mathcal{N}_L] \otimes \mathbb{D}_*[\mathcal{N}_R] + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R] + \mathbb{R}_L \otimes \mathbb{R}_R + \delta_{D,4}\mathbb{R}_L \otimes \mathbb{R}_R. \quad (2.35)$$

In (2.38) we present a set of commutators which define a Lie algebra structure on (2.35), giving precisely the algebras of the generalised U-duality pyramid in Figure 2. To describe the complete set of commutators we use the left/right form of  $\mathfrak{h} \subset \mathfrak{g}$  given in (2.21),

$$\begin{aligned} \mathfrak{g}(\mathcal{N}_L + \mathcal{N}_R, D) = & \left[ \mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{sa}(\mathcal{N}_R, \mathbb{D}) \oplus \delta_{D,4}\mathbf{u}(1) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]_c \right] \oplus \mathfrak{p}_L \oplus \mathfrak{p}_R \\ & + \mathbb{D}_*[\mathcal{N}_L] \otimes \mathbb{D}_*[\mathcal{N}_R] + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]_{nc} + \mathbb{R}_L \otimes \mathbb{R}_R + i\delta_{D,4}\mathbb{R}_L \otimes \mathbb{R}_R, \end{aligned} \quad (2.36)$$

where we have distinguished the compact  $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]_c$  and non-compact  $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]_{nc}$ . The non-trivial commutators amongst the compact generators have been given in (2.15). For the generators,

<i>Compact</i>	<i>Non – compact</i>	
$X_L \oplus X_R \in \mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{a}(\mathcal{N}_R, \mathbb{D})$	$\gamma, \delta \in \mathbb{R}_L \otimes \mathbb{R}_R + i\delta_{D,4}\mathbb{R}_L \otimes \mathbb{R}_R$	(2.37)
$M, N \in \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]_c$	$P, Q \in \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]_{nc}$	
	$m \otimes p, n \otimes q \in \mathbb{D}_*[\mathcal{N}_L] \otimes \mathbb{D}_*[\mathcal{N}_R]$	

the non-trivial commutators (omitting those already presented for the compact subalgebra) are given by

$$\begin{aligned} [X_L \oplus X_R, P] &= (X_L P - P X_R) \\ &\in \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]_{nc} \end{aligned} \quad (2.38a)$$

$$\begin{aligned} [X_L \oplus X_R, m \otimes p] &= (X_L m - m X_L^*) \otimes p + m \otimes (X_R p - p X_R^*) \\ &\in \mathbb{D}_*[\mathcal{N}_L] \otimes \mathbb{D}_*[\mathcal{N}_R] \end{aligned} \quad (2.38b)$$

$$\begin{aligned} [M, \gamma] &= \gamma M \\ &\in \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]_{nc} \end{aligned} \quad (2.38c)$$

$$\begin{aligned} [M, P] &= (M \wedge P|_*, \text{tr}(MP)) \\ &\in \mathbb{D}_*[\mathcal{N}_L] \otimes \mathbb{D}_*[\mathcal{N}_R] + \mathbb{R}_L \otimes \mathbb{R}_R + i\delta_{D,4}\mathbb{R}_L \otimes \mathbb{R}_R \end{aligned} \quad (2.38d)$$

$$\begin{aligned} [M, m \otimes p] &= \frac{4}{3} m M p^* \\ &\in \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]_{nc} \end{aligned} \quad (2.38e)$$

$$\begin{aligned} [P, \gamma] &= \gamma P \\ &\in \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]_{nc} \end{aligned} \quad (2.38f)$$

$$\begin{aligned} [P, Q] &= (P Q^\dagger - Q P^\dagger) \oplus (P Q^\dagger - Q P^\dagger) \\ &\in \mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{sa}(\mathcal{N}_R, \mathbb{D}) \oplus \delta_{D,4}\mathbf{u}(1) \end{aligned} \quad (2.38g)$$

$$\begin{aligned} [P, m \otimes p] &= \frac{4}{3} m P p^* \\ &\in \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]_c \end{aligned} \quad (2.38h)$$

$$\begin{aligned} [m \otimes p, n \otimes q] &= (m n^\dagger - n m^\dagger) \text{tr}(p q^\dagger) \oplus (p q^\dagger - q p^\dagger) \text{tr}(m n^\dagger) \\ &\in \mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{a}(\mathcal{N}_R, \mathbb{D}) \end{aligned} \quad (2.38i)$$



Note, for the sake of brevity we have reincorporated the  $D = 4, \mathfrak{u}(1)$  factor back into  $X_L$  and  $X_R$ , which therefore have equal and opposite traces. Moreover, leaving aside  $D = 3$  for the moment, the only non-vanishing  $\alpha \oplus \beta \in \mathfrak{p}_L \oplus \mathfrak{p}_R$  occur in  $D = 4$ , the  $\mathfrak{u}(1)$  factors of  $\mathcal{N} = 2, 1$ . See Table 5. Simply regarding  $X_{L/R}$  as tracefull generators belonging to  $\mathfrak{a}(\mathcal{N}_{L/R})$  automatically accounts for their action.

In three dimensions the formula can be simplified by “dualising” the  $A_L \otimes A_R$  contributions into  $\phi_L \otimes \phi_R$  terms. We no longer have the  $\mathbb{R}_L \otimes \mathbb{R}_R$  term from tensoring the gauge fields, it is combined into a second  $\mathbb{R}[\mathcal{N}_L, \mathcal{N}_R]_{nc}$  factor resulting the simplified  $D = 3$  formula,

$$\mathfrak{g}(\mathcal{N}_L + \mathcal{N}_R) = \mathfrak{h}(\mathcal{N}_L, \mathcal{N}_R) + 2\mathbb{R}[\mathcal{N}_L, \mathcal{N}_R]_{nc}, \quad (2.39)$$

together with a simplified set of commutation relations [34].

### 3 Conclusions

We have shown that the U-duality algebras  $\mathfrak{g}$  for all supergravity multiplets obtained by tensoring two super Yang-Mills multiplets in  $D \geq 3$  can be written in a single formula with three arguments,  $\mathfrak{g}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D})$ . The formula relies on the link between the three associative normed division algebras,  $\mathbb{R}, \mathbb{C}, \mathbb{H}$ , and the representation theory of classical Lie algebras. The formula is symmetric under the interchange of  $\mathcal{N}_L$  and  $\mathcal{N}_R$  and provides another “matrix model”, in the sense of Barton and Sudbery [29], for the exceptional Lie algebras. In this language the compact subalgebra  $\mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D})$  has a simple form which makes the *left*  $\otimes$  *right* structure clear. The non-compact  $\mathfrak{p} = \mathfrak{g} - \mathfrak{h}$  generators are obtained directly by examining the division algebraic representations carried by those left/right states that produce the scalar fields of the corresponding supergravity multiplets.

Note, we are therefore implicitly assuming that the tensor product always gives supergravities with scalars parametrising a symmetric coset space. The only possible exception to this rule is given by  $\mathcal{N}_L = \mathcal{N}_R = 1$ . When there is a possible ambiguity in the coupling of the scalars it is resolved by the structure of the left and right symmetry algebras. For example, in  $D = 4$  the  $\mathcal{N}_L = \mathcal{N}_R = 1$  scalar coset manifold,

$$\frac{\mathrm{U}(1, 2)}{\mathrm{U}(1) \times \mathrm{U}(2)}, \quad (3.1)$$

is the unique possibility consistent with the left and right super Yang-Mills data.

This procedure gives all supergravity algebras with more than half-maximal supersymmetry. These cannot couple to matter, as reflected by the squaring procedure where only the fields of the supergravity multiplet are produced. However, for half-maximal and below, one can couple the theory to matter multiplets (vector or hyper). This does indeed happen when one squares; the fields obtained arrange themselves in the correct number of vector or hypermultiplets such that we fill up the entries of the generalised pyramid.

Theories with more general matter content do not naturally live in our pyramid, mainly because they lack an obvious division algebraic description. For example, the *STU* model [81] is given by  $\mathcal{N} = 2$  supergravity in four dimensions coupled to three vector multiplets, while the entry for  $\mathcal{N} = 2$  in our pyramid necessarily comes coupled to a single hypermultiplet. Can squaring accommodate more general matter couplings? All factorized orbifold projections (as defined in [21]) of  $\mathcal{N} = 8$  supergravity can be obtained from the tensor product of the corresponding left and right orbifold projections of  $\mathcal{N} = 4$  super Yang-Mills multiplets [21]. This includes a large, but still restricted, class of matter coupled supergravities with specific U-dualities.

Theories coupled to an arbitrary number of vector multiplets can be obtained by tensoring a supersymmetric multiplet with a conveniently chosen collection of bosonic fields. In particular, here we consider an  $\mathcal{N}_R = 0$  multiplet with a single gauge potential and  $n_V$  scalar fields. The symmetries of the resulting supergravity multiplet are determined by the global symmetries postulated for the  $\mathcal{N}_R = 0$  multiplet. We consider the simplest case where the  $n_V$  scalar fields transform in the vector representation of a global

$SO(n_V)$ . Following the procedure used to construct the generalised pyramid this uniquely fixes the global symmetries of the resulting supergravity multiplet and therefore, implicitly, the structure of the matter couplings. This idea is developed in the following section. We summarise the results<sup>3</sup> in Table 6.

### 3.1 $[\mathcal{N}_L]_V \times [\mathcal{N}_R = 0]$ tensor products

<i>theory</i>	<i>squaring formula</i>	$R_L$	$R_R$	$\frac{G}{H}$
$D = 3$				
$(\mathcal{N} = 8)_{SuGra} + n_v(\mathcal{N} = 8)_{vector}$	$(\mathcal{N} = 8)_V \times [n_V \phi]$	Spin(8)	$SO(n_V)$	$\frac{SO(8, n_V)}{SO(8) \times SO(n_V)}$
$(\mathcal{N} = 4)_{SuGra} + n_v(\mathcal{N} = 4)_{vector}$	$(\mathcal{N} = 4)_V \times [n_V \phi]$	Spin(4)	$SO(n_V)$	$\frac{SO(4, n_V)}{SO(4) \times SO(n_V)}$
$(\mathcal{N} = 2)_{SuGra} + n_v(\mathcal{N} = 2)_{vector}$	$(\mathcal{N} = 2)_V \times [n_V \phi]$	Spin(2)	$SO(n_V)$	$\frac{SO(2, n_V)}{SO(2) \times SO(n_V)}$
$(\mathcal{N} = 1)_{SuGra} + n_v(\mathcal{N} = 1)_{vector}$	$(\mathcal{N} = 1)_V \times [n_V \phi]$	$\emptyset$	$SO(n_V)$	$\frac{SO(1, n_V)}{SO(n_V)}$
$D = 4$				
$(\mathcal{N} = 4)_{SuGra} + n_v(\mathcal{N} = 4)_{vector}$	$(\mathcal{N} = 4)_V \times [A_\mu + n_V \phi]$	SU(4)	$SO(n_V)$	$\frac{SO(6, n_V)}{SO(6) \times SO(n_V)} \times \frac{SL(2)}{SO(2)}$
$(\mathcal{N} = 2)_{SuGra} + n_v(\mathcal{N} = 2)_{vector}$	$(\mathcal{N} = 2)_V \times [A_\mu + (n_V - 1)\phi]$	U(2)	$SO(n_V - 1)$	$\frac{SU(2) \times SO(2, n_V - 1)}{U(2) \times SO(n_V - 1)} \times \frac{SL(2)}{SO(2)}$
$D = 5$				
$(\mathcal{N} = 2)_{SuGra} + n_v(\mathcal{N} = 2)_{vector}$	$(\mathcal{N} = 2)_V \times [A_\mu + n_V \phi]$	Sp(2)	$SO(n_V)$	$\frac{SO(5, n_V)}{SO(5) \times SO(n_V)} \times O(1, 1)$
$(\mathcal{N} = 1)_{SuGra} + n_v(\mathcal{N} = 1)_{vector}$	$(\mathcal{N} = 1)_V \times [A_\mu + (n_V - 1)\phi]$	Sp(1)	$SO(n_V - 1)$	$\frac{Sp(1) \times SO(1, n_V - 1)}{Sp(1) \times SO(n_V - 1)} \times O(1, 1)$
$D = 6$				
$(\mathcal{N} = (1, 1))_{SuGra} + n_v(\mathcal{N} = (1, 1))_{vector}$	$(\mathcal{N} = (1, 1))_V \times [A_\mu + n_V \phi]$	Sp(1) $\times$ Sp(1)	$SO(n_V)$	$\frac{O(4, n_V)}{SO(4) \times O(n_V)} \times O(1, 1)$
$(\mathcal{N} = (2, 0))_{SuGra} + n_T(\mathcal{N} = (2, 0))_{tensor}$	$(\mathcal{N} = (2, 0))_{tensor} \times [B_{\mu\nu}^- + n_T \phi]$	Sp(2)	$SO(n_T)$	$\frac{O(5, n_T)}{SO(5) \times O(n_T)}$

Table 6: Matter coupling in  $D = 3, 4, 5, 6$

Note that the general form for the maximally compact subgroups in the cosets given in Table 6 is

$$H = R_L \otimes R_R \otimes \delta_{D,4} SO(2). \quad (3.2)$$

This is just the form,

$$\mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, D) = \text{int}(\mathcal{N}_L, D) \oplus \text{int}(\mathcal{N}_R, D) \oplus \delta_{D,4} \mathfrak{u}(1) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R], \quad (3.3)$$

appearing in the generalised pyramid formula (2.21) with  $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R = 0] = \emptyset$ . This is entirely consistent with the logic of the construction; we previously identified  $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]$  with the generators  $Q_L \otimes Q_R$ , which are clearly absent when  $\mathcal{N}_R = 0$ .

<sup>3</sup>Note that we have excluded  $\mathcal{N} = 1$  theories in four dimensions. It is not possible to obtain  $\mathcal{N} = 1$  supergravity coupled to only vector multiplets by squaring since one always obtains at least one chiral multiplet when tensoring  $\mathcal{N} = 1$  SYM with a non-supersymmetric multiplet. The same applies to  $\mathcal{N} = (1, 0)$  supergravity in 6 dimensions. These theories are interesting in their own right and will be analysed in forthcoming work [82].

The non-compact generators are also determined following the logic of the generalised pyramid presented in subsection 2.4, but now with only two scalar terms:  $A_\mu \otimes \tilde{A}_\nu$  and  $\phi^i \otimes \tilde{\phi}^{i'}$ , where  $\tilde{\phi}^{i'}$  are the  $n_V$  scalars transforming as a vector of  $\text{SO}(n_V)$ .

As an example, take half-maximal supergravity in five dimensions coupled to  $n_V$  vector multiplets. We obtain the field content by tensoring the maximal  $\mathcal{N} = 2$  super Yang-Mills multiplet (with R-symmetry  $\text{Sp}(2)$ ) and a non-supersymmetric multiplet consisting of a gauge field and  $n_V$  scalars transforming in the vector representations of  $\text{SO}(n_V)$ , denoted  $\mathbf{n}_V$ :

$\otimes$	$\tilde{A}_\mu$ <b>(3; 1)</b>	$\tilde{\phi}$ <b>(1; <math>\mathbf{n}_V</math>)</b>	(3.4)
$A_\mu$ <b>(3; 1)</b>	<b>(5; 1, 1) + (3; 1, 1) + (1; 1, 1)</b>	<b>(3; 1, <math>\mathbf{n}_V</math>)</b>	
$\lambda$ <b>(2; 4)</b>	<b>(4; 4, 1) + (2; 4, 1)</b>	<b>(2; 4, <math>\mathbf{n}_V</math>)</b>	
$\phi$ <b>(1; 5)</b>	<b>(3; 5, 1)</b>	<b>(1; 5, <math>\mathbf{n}_V</math>)</b>	

We therefore find,

$$\mathfrak{h} = \mathfrak{sp}(2) \oplus \mathfrak{so}(n_V), \quad (3.5)$$

and, from (3.4),

$$\mathfrak{g} \ominus \mathfrak{h} = (\mathbf{5}, \mathbf{n}_V) \oplus (\mathbf{1}, \mathbf{1}). \quad (3.6)$$

Using the commutators which follow uniquely from the transformation properties of left and right states we have

$$\mathfrak{g} = [\mathfrak{sp}(2) \oplus \mathfrak{so}(n_V) + (\mathbf{5}, \mathbf{n}_V)] \oplus (\mathbf{1}, \mathbf{1}) \cong \mathfrak{so}(5, n_V) \oplus \mathfrak{so}(1, 1). \quad (3.7)$$

This procedure applied in  $D = 3, 4, 5, 6$  yields Table 6. Note, for  $D = 4$ ,  $\mathcal{N} = 2$  and  $D = 5$ ,  $\mathcal{N} = 1$  the  $\text{SU}(2)$  and  $\text{Sp}(1)$  factors, respectively, drop out of the  $G/H$  coset. We see that the cosets admit a concise alternative description:

$$\frac{G}{H} \cong \frac{\text{SO}(\#\phi_L, \#\phi_R)}{\text{SO}(\#\phi_L) \times \text{SO}(\#\phi_R)} \times \mathcal{M}_{A_L \times A_R}. \quad (3.8)$$

where  $\#\phi_{L/R}$  is the number of scalars in the left and right multiplets we are tensoring and  $\mathcal{M}_{A_L \times A_R}$  is the coset parametrised by the scalars obtained from tensoring the gauge fields. It is given by  $\emptyset$  in  $D = 3$ , since the gauge fields (in the free theory) have been dualised to scalars,  $\text{SL}(2)/\text{SO}(2)$  in  $D = 4$  where we have two such scalars, and  $\text{O}(1, 1)$  in  $D = 5, 6$ , where we have one.

In some cases we reproduce cosets appearing in the generalised pyramid. For example, in  $D = 4$   $[\mathcal{N} = 2]_V \times [\mathcal{N} = 2]_V$  and  $[\mathcal{N} = 4]_V \times [\mathcal{N} = 0, n_V = 2]$  both yield  $\mathcal{N} = 4$  supergravity coupled to two vector multiplets with coset  $[\text{SL}(2) \times \text{SO}(6, 2)]/[\text{SO}(2)^2 \times \text{SO}(6)]$ . However, despite their common coset the two resulting theories have important structural differences when interpreted as truncations of  $D = 4, \mathcal{N} = 8$  supergravity. In particular, the  $\text{SL}(2)$  S-duality subgroup in  $E_{7(7)}$  can be directly identified with the  $\text{SL}(2)$  factor in  $\text{SL}(2) \times \text{SO}(6, 2) \subset E_{7(7)}$  for  $[\mathcal{N} = 4]_V \times [\mathcal{N} = 0, n_V = 2]$  whereas for  $[\mathcal{N} = 2]_V \times [\mathcal{N} = 2]_V$  it must be identified with an  $\text{SL}(2)$  subgroup of the  $\text{SO}(6, 2)$  factor, as explained in [34]. In both cases the  $8 + 8$  gauge potentials and their duals transform as the  $(\mathbf{2}, \mathbf{8})$  of  $\text{SL}(2) \times \text{SO}(6, 2)$ . Embedding the  $[\mathcal{N} = 2]_V \times [\mathcal{N} = 2]_V$  theory in  $\mathcal{N} = 8$  supergravity these  $8 + 8$  potentials and dual potentials are evenly split between the NS-NS and RR sectors, implying that the  $\text{SL}(2)$  factor mixes NS-NS and RR potentials and therefore cannot be identified with S-duality. Instead, the S-duality  $\text{SL}(2)_S$  is contained in the  $\text{SO}(6, 2)$  component:

$$\begin{aligned} \text{SL}(2) \times \text{SO}(6, 2) \supset \text{SL}(2) \times \text{SL}(2)_S \times \text{SL}(2) \times \text{SU}(2), \\ (\mathbf{2}, \mathbf{8}) \rightarrow \underbrace{(\mathbf{2}, \mathbf{2}_S, \mathbf{2}, \mathbf{1}, \mathbf{1})}_{\text{NS-NS}} + \underbrace{(\mathbf{2}, \mathbf{1}_S, \mathbf{1}, \mathbf{2}, \mathbf{2})}_{\text{RR}}. \end{aligned} \quad (3.9)$$

On the other hand, the  $\mathcal{N} = 4$  supergravity coupled to two vector multiplets obtained from  $[\mathcal{N} = 4]_V \times [\mathcal{N} = 0, n_V = 2]$  can be consistently embedded in the NS-NS sector of  $\mathcal{N} = 8$  supergravity alone:

all eight gauge potentials correspond to NS-NS states. In this scenario, the  $SL(2)$  factor in the U-duality group can be identified as the S-duality  $SL(2) \in E_{7(7)}$ :

$$\begin{aligned} SL(2) \times SO(6, 2) &\cong SL(2)_S \times SO(6, 2), \\ (\mathbf{2}, \mathbf{8}) &\equiv (\mathbf{2}_S, \mathbf{8}). \end{aligned} \tag{3.10}$$

## Acknowledgments

We would like to thank MJ Duff and A Marrani for very many useful and instructive conversations on the symmetries of supergravity. The work of LB was supported by a Schödinger Fellowship and an Imperial College Junior Research Fellowship. AA, MJH, and SN received support from STFC and EPSRC.

## A Notes on quaternionic representations

An element  $X \in \mathfrak{u}(2n) \cong \mathfrak{a}(2n, \mathbb{C})$  can be written

$$\mathbf{M} = \begin{pmatrix} a & b \\ -b^\dagger & c \end{pmatrix}, \quad \text{where } a, c \in \mathfrak{a}(n, \mathbb{C}) \quad b \in \mathbb{C}[4]. \tag{A.1}$$

For  $\mathbf{M}$  in the subalgebra  $\mathfrak{usp}(2n) := \{\mathbf{M} \in \mathfrak{u}(2n) | \mathbf{M}^T \Omega + \Omega \mathbf{M} = 0\}$

$$\mathbf{M}^T \Omega + \Omega \mathbf{M} = 0 \quad \Rightarrow \quad \mathbf{M} = \begin{pmatrix} a & b \\ -b^\dagger & a^* \end{pmatrix}, \quad \text{where } b \in \text{Sym}^2(\mathbb{C}^n). \tag{A.2}$$

The well-known Lie algebra isomorphism  $\mathfrak{usp}(2n) \cong \mathfrak{sa}(n, \mathbb{H})$  then follows from the standard algebra bijection,

$$\tau : \mathfrak{sa}_u(2, \mathbb{C}) \rightarrow \mathbb{H} \quad \text{where } \mathfrak{sa}_u(2, \mathbb{C}) := \text{Span}_{\mathbb{R}}\{\mathbf{1}, \mathfrak{sa}(2, \mathbb{C})\}, \tag{A.3}$$

given by

$$a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{H} \mapsto \begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \in \mathfrak{sa}_u(2, \mathbb{C}). \tag{A.4}$$

Note,

$$\tau(X^\dagger) = \tau(X)^*, \quad \tau(X^*) = e_2 \tau(X) e_2^*, \quad \tau(X^T) = e_2 \tau(X)^* e_2^*. \tag{A.5}$$

To set-up the isomorphism  $\mathfrak{usp}(2n) \cong \mathfrak{sa}(n, \mathbb{H})$  we introduce two maps,  $S_n$  and  $\tau_n$ . Explicitly, for  $n = 2$

$$\mathbf{M} = \begin{pmatrix} ia & b & \alpha & \beta \\ -b^* & ic & \beta & \delta \\ -\alpha^* & -\beta^* & -ia & b^* \\ -\beta^* & -\delta^* & -b & -ic \end{pmatrix} \xrightarrow{S_2} \begin{pmatrix} ia & \alpha & b & \beta \\ -\alpha^* & -ia & -\beta^* & b^* \\ -b^* & \beta & ic & \delta \\ -\beta^* & -b & -\delta^* & -ic \end{pmatrix} \xrightarrow{\tau_2} \begin{pmatrix} x & z \\ -z^* & y \end{pmatrix} \in \mathfrak{sa}(2, \mathbb{H}), \tag{A.6}$$

where

$$S_n : \mathbb{C}[2n] \rightarrow \mathbb{C}[2n]; \quad \mathbf{M} \mapsto \mathbf{S}_n \mathbf{M} \mathbf{S}_n^T \quad \text{for } \mathbf{S}_n \in SO(2n) \tag{A.7}$$

is the similarity transformation organising  $\mathbf{M}$  into  $2 \times 2$  blocks  $A_{ij}, i, j = 1, 2, \dots, n$  such that  $A_{ii} \in \mathfrak{sa}(2, \mathbb{C})$  and  $A_{ji} = -A_{ij}^\dagger \in \mathfrak{sa}_u(2, \mathbb{C})$  and

$$\tau_n : \mathbb{R}[n] \otimes \mathfrak{sa}_u(2, \mathbb{C}) \rightarrow \mathbb{H}[n]; \quad \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \mapsto \begin{pmatrix} \tau(A_{11}) & \tau(A_{12}) & \cdots & \tau(A_{1n}) \\ \tau(A_{21}) & \tau(A_{22}) & \cdots & \tau(A_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \tau(A_{n1}) & \tau(A_{n2}) & \cdots & \tau(A_{nn}) \end{pmatrix}. \tag{A.8}$$

Since  $A_{ii} \in \mathfrak{sa}(2, \mathbb{C})$  and  $A_{ji} = -A_{ij}^\dagger \in \mathfrak{sa}_u(2, \mathbb{C})$  for  $S_n(\mathbf{M}), \mathbf{M} \in \mathfrak{usp}(2n), \tau_n \circ S_n(\mathbf{M}) \in \mathfrak{sa}(n, \mathbb{H})$ .

The similarity transformation  $S_n$  is trivially a bijective matrix algebra homomorphism and therefore also a Lie algebra isomorphism. Similarly,  $\tau_n$  is an algebra isomorphism since  $\tau$  is . Therefore its restriction to  $\mathfrak{usp}(2n)$  is also a Lie algebra isomorphism since the commutators are given by matrix commutators and so  $\theta_n := \tau_n \circ S_n$ ,

$$\theta_n : \mathfrak{usp}(2n) \rightarrow \mathfrak{sa}(n, \mathbb{H}), \quad (\text{A.9})$$

is a Lie algebra isomorphism.

Returning to the specific example of  $\mathfrak{sa}(2, \mathbb{H})$ , relevant to  $D = 5$ , we can decompose with respect to the subalgebra  $\mathfrak{sa}(1, \mathbb{H}) \oplus \mathfrak{sa}(1, \mathbb{H}) \subset \mathfrak{sa}(2, \mathbb{H})$ , relevant to  $D = 6$ ,

$$\begin{aligned} \mathbf{10} &\rightarrow (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2}), \\ \begin{pmatrix} x & z \\ -z^* & y \end{pmatrix} &\rightarrow \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} + \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix}. \end{aligned} \quad (\text{A.10})$$

Hence the  $(\mathbf{2}, \mathbf{2})$  of  $\mathfrak{usp}(2) \oplus \mathfrak{usp}(2)$  can be identified with  $\mathbb{H}$ , where action of  $x \oplus y \in \mathfrak{sa}(1, \mathbb{H}) \oplus \mathfrak{sa}(1, \mathbb{H})$  on  $z \in \mathbb{H}$  is given by,

$$[(x, y), z] = xz - zy. \quad (\text{A.11})$$

The  $D = 7$  subalgebra  $\mathfrak{sa}(1, \mathbb{H})$  is obtained by identifying  $x = y$ . The  $\mathbf{3}$  of  $\mathfrak{usp}(2)$  is then given by restricting  $z$  to  $\text{Im}\mathbb{H}$ ,

$$[(x, x), z] = xz - zx = xz - (xz)^* \in \text{Im}\mathbb{H}. \quad (\text{A.12})$$

The  $\mathbf{5}$  of  $\mathfrak{usp}(4)$  can also be written in a quaternionic language using  $\theta_2$ . The  $\mathbf{6}$  of  $\mathfrak{su}(4)$  is given by a complex-self-dual 2-form  $X_{ab} \in \wedge^2 \mathbb{C}^4, (X_{ab})^* = (\star X)^{ab}$ . It can be written as a  $4 \times 4$  matrix

$$\mathbf{X} = \begin{pmatrix} 0 & \alpha & a & \beta \\ -\alpha & 0 & \beta^* & -a^* \\ -a & -\beta^* & 0 & \alpha^* \\ -\beta & a^* & -\alpha^* & 0 \end{pmatrix}. \quad (\text{A.13})$$

Under  $\mathfrak{su}(4) \supset \mathfrak{usp}(4) \cong \mathfrak{sp}(2)$ ,

$$\mathbf{6} \rightarrow \mathbf{5} + \mathbf{1}, \quad (\text{A.14})$$

where the  $\mathbf{5}$  is a symplectic traceless complex-self-dual 2-form  $X_{ab} \in \wedge_0^2 \mathbb{C}^4$ ,

$$\Omega^{ab} X_{ab} = 0, \quad \Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}. \quad (\text{A.15})$$

In terms of  $\mathbf{X}$  symplectic tracelessness implies  $a \in \mathbb{R}$

$$\mathbf{X} = \begin{pmatrix} 0 & \alpha & a & \beta \\ -\alpha & 0 & \beta^* & -a \\ -a & -\beta^* & 0 & \alpha^* \\ -\beta & a & -\alpha^* & 0 \end{pmatrix}. \quad (\text{A.16})$$

Applying  $S_2$  we obtain

$$\tilde{\mathbf{X}} = \mathbf{SXS}^T = \begin{pmatrix} 0 & a & \alpha & \beta \\ -a & 0 & -\beta^* & \alpha^* \\ -\alpha & \beta^* & 0 & -a \\ -\beta & -\alpha^* & a & 0 \end{pmatrix} \quad \text{where} \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{A.17})$$

Under  $\tau_2 : \mathbb{R}[2] \otimes \mathfrak{sa}_u(2, \mathbb{C}) \rightarrow \mathbb{H}[2]$  we find

$$\begin{aligned}
\tau_2(\tilde{\mathbf{X}}) &= \begin{pmatrix} ae_2 & \alpha_0 + \alpha_1 e_1 + \beta_0 e_2 + \beta_1 e_3 \\ -\alpha_0 - \alpha_1 e_1 + \beta_0 e_2 - \beta_1 e_3 & -ae_2 \end{pmatrix} \\
&= e_2 \begin{pmatrix} a & -\alpha_0 e_2 + \alpha_1 e_3 + \beta_0 - \beta_1 e_1 \\ \alpha_0 e_2 - \alpha_1 e_3 + \beta_0 + \beta_1 e_1 & -a \end{pmatrix} \\
&= e_2 \begin{pmatrix} a & b \\ b^* & -a \end{pmatrix} \\
&= e_2 A,
\end{aligned} \tag{A.18}$$

where  $a \in \mathbb{R}, b \in \mathbb{H}$  and  $*$  now denotes the quaternionic conjugate and  $A$  is a traceless Hermitian  $2 \times 2$  quaternionic matrix.

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