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Scalar and Spinor Field Actions on Fuzzy S^4: fuzzy $\mathbb{C}P^3$ as a S_F^2 bundle over S_F^4.

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Abstract: We present a manifestly Spin(5) invariant construction of squashed fuzzy $\mathbb{C}P^3$ as a fuzzy S^2 bundle over fuzzy S^4. We develop the necessary projectors and exhibit the squashing in terms of the radii of the S^2 and S^4. Our analysis allows us give both scalar and spinor fuzzy action functionals whose low lying modes are truncated versions of those of a commutative S^4.

Keywords: Differential and Algebraic Geometry, Non-Commutative Geometry, Matrix Models

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1 Introduction

Noncommutative spaces with four dimensions are an interesting way to model space-time at small length scales. Amongst the simplest four dimensional manifolds S^4 is, on account of the one-point compactification of Euclidean field theories, an important model. We focus on the fuzzy approach to noncommutative spaces, where the function algebra is replaced by a sequence of finite dimensional matrix algebras $\mathcal{A}_L$ and the metrical geometry is determined, in our case, by a Laplacian acting on "functions" Δ. We will also present a Dirac type operator that recovers the spectrum of the standard round Dirac operator on S^4 in a certain limit.

The fuzzy noncommutative 4-sphere, S_F^4, was first constructed in [11] but has been known for some time in different settings [1]-[2] along with other 4-dimensional fuzzy spaces [3]-[4]. The key feature of S_F^4 is that the algebra of functions does not form a closed associative algebra. This can be understood since the product of two "functions" takes one out of the algebra of functions of S^4_F and a projection is necessary to bring the product back [5].

Here we follow the line presented in [6] where the algebra is associative but it includes modes which do not belong to the fuzzy 4-sphere. The quantized version of S^4 can be constructed only in an indirect manner if one demands associativity of the algebra, this is a consequence of the fact that S^4 does not admit a Poisson structure. The approach taken here is based on the fuzzy complex projective spaces, first given in [7] and further explored in [8]-[10], and in the fact that $\mathbb{C}\mathrm{P}^3$ is a fibration over S^4. In this context a construction for the scalar theory on a fuzzy 4-sphere was first carried out as a Hopf fibration in [11], but without a method of suppressing the unwanted modes. The necessary suppression mechanism was supplied in [6].

A method to obtain an effective scalar field theory on S^4_F was given in [6], there, an algebraic approach was taken to eliminate the unwanted modes by constructing a positive definite operator whose kernel consists of exactly all the modes in $\mathbb{C}\mathrm{P}^3_F$ that belong to S^4_F, this operator was interpreted as a modification of the Laplacian. In the present work we give a geometrical interpretation of the suppresion mechanism in terms of the fibre bundle picture for $\mathbb{C}\mathrm{P}^3$.

In section 2 we present a brief review of the aspects needed of $\mathbb{C}\mathrm{P}^N_F$ and S^4_F, we follow essentially [7], [6]. Section 3 presents the construction of our case of interest, $\mathbb{C}\mathrm{P}^3$, first as a Spin(6) and then as a Spin(5) adjoint orbit. It continues with the calculation of the invariant line element and isotropy subgroup in both approaches using the Maurer-Cartan forms of the aforementioned groups, this is done only at a particular fiducial point that we call the "north pole", by equivariance this suffices. Section 4 presents a one-parameter dependent squashed Laplacian Δ_h which fixes the symmetry of $\mathbb{C}\mathrm{P}^3_F$ to be Spin(5) instead of the "round" Spin(6) symmetry. This Laplacian turns out to be an interpolation of Spin(5) and Spin(6) quadratic Casimir operators. Section 5 deals with the use of the $*$-product map to construct the commutative analogue of Δ_h. The metric of the squashed $\mathbb{C}\mathrm{P}^3$ is obtained from the squashed Laplacian explicitely as a combination of projectors. The line element of the bundle $\mathbb{C}\mathrm{P}^3 \to S^4$ is computed and reinterpeted in terms of the found radii of the fibre and base space. In section 6, in the spirit of [12], we present a first order operator on $\mathbb{C}\mathrm{P}^3_F$ that projects down to the Dirac operator in a certain limit and hence give a prescription to construct an action for fermions supressing the unwanted degrees of freedom. Section 7 presents our conclusions.

2 Review of $\mathbb{C}\mathrm{P}^N_F$ and S^4_F

In the usual construction, $\mathbb{C}\mathrm{P}^N$ is defined as the space of all equivalence classes $[\psi]$ of unit vectors $\psi \in \mathbb{C}^{N+1}$, $|\psi| = 1$, given by the equivalence relation: $\psi_1 \sim \psi_2$ if and only if $\psi_1 = e^{\imath\varphi}\psi_2$ for some $\varphi \in (0, 2\pi]$. We follow closely the presentation in [7] where the general details are given, and specialize later to the case under study of $\mathbb{C}\mathrm{P}^3$. It was shown in [7] that each equivalence class is associated with a hermitian rank one projector in $\mathbb{C}^{N+1}$, $\mathcal{P} = \psi \otimes \psi^\dagger$, we have then the following alternative definition of $\mathbb{C}\mathrm{P}^N$

$$\mathbb{C}\mathrm{P}^N := \{\mathcal{P} \in Mat_{N+1} : \mathcal{P}^2 = \mathcal{P} = \mathcal{P}^\dagger,\ \ \mathrm{Tr}\,\mathcal{P} = 1\}. \tag{2.1}$$

Each projector $\mathcal{P}$ is associated with a point in $\mathbb{C}\mathrm{P}^N$, a coordinate system is introduced by expanding the projector in the basis of matrices given by the identity and the generators of $su(N+1)$ in the fundamental representation, denoted by $\{\Lambda_\mu,\quad \mu = 1,\ldots,N^2+2N\}$:

$$\mathcal{P} = \frac{\mathbf{1}}{N+1} + \frac{1}{\sqrt{2}}\xi_\mu \Lambda_\mu. \tag{2.2}$$

The generators have been chosen to be orthogonal and with such normalization that their algebra is

$$\Lambda_\alpha \Lambda_\beta = \frac{2}{N+1}\delta_{\alpha\beta}\mathbf{1} + \frac{1}{2}(d_{\alpha\beta\gamma} + \imath f_{\alpha\beta\gamma})\Lambda_\gamma. \tag{2.3}$$

The conditions in (2.1) together with (2.2) and (2.3) result into a set of quadratic constraints for the real coordinates ξ_μ,

$$\xi_\mu\xi_\mu = \frac{N}{N+1}, \quad d_{\alpha\beta\gamma}\xi_\alpha\xi_\beta = \sqrt{8}\left(\frac{N-1}{N+1}\right)\xi_\gamma, \tag{2.4}$$

these constraints describe the embedding $\mathbb{C}\mathrm{P}^N \hookrightarrow \mathbb{R}^{N^2+2N}$, wherefrom the coordinates ξ_μ can be seen to be a globally well defined overcomplete coordinate system. The metric $\mathbf{P}$, complex structure $\mathbf{J}$, and Kähler structure $\mathbf{K}$ on $\mathbb{C}\mathrm{P}^N$ were found in [7] to be given as

$$\begin{aligned}
\mathbf{P}_{\alpha\beta} &= \frac{2}{N+1}\delta_{\alpha\beta} + \frac{1}{\sqrt{2}}d_{\alpha\beta\gamma}\xi_\gamma - 2\xi_\alpha\xi_\beta, \\
\mathbf{J}_{\alpha\beta} &= \frac{1}{\sqrt{2}}f_{\alpha\beta\gamma}\xi_\gamma, \\
\mathbf{K} &= \frac{1}{2}(\mathbf{P} + \imath\mathbf{J}).
\end{aligned} \tag{2.5}$$

Notice that the complex structure satisfies $\mathbf{J}^2 = -\mathbf{P}$.

One may obtain the fuzzy complex projective space $\mathbb{C}\mathrm{P}^N_F$ by considering the algebra of functions to be the full matrix algebra given as

$$Mat_{d^N_L} = \underbrace{\boxed{}\boxed{\cdot\cdot}\boxed{}}_{L} \otimes \overline{\underbrace{\boxed{}\boxed{\cdot\cdot}\boxed{}}_{L}} \tag{2.6}$$

whose decomposition into irreducible representations of $\mathrm{SU}(N+1)$ corresponds with the expansion into polarization tensors of a function on $\mathbb{C}\mathrm{P}^N_F$. The dimension of the matrix algebra (2.6) is $d^N_L = \binom{L+N}{N}$. The right-invariant vector fields induced by the action of $\mathrm{SU}(N+1)$ are $\mathcal{L}_\mu = \frac{\imath}{\sqrt{2}}J_{\mu\nu}\frac{\partial}{\partial\xi_\nu}$, and in the fuzzy realization they take the form $ad(L_\mu)$ where L_μ are the generators of the totally symmetric irreducible representation; the associated Laplacian is then the quadratic Casimir operator $\Delta = \frac{1}{R^2}(ad(L_\mu))^2$, and reflects the $\mathrm{SU}(N+1)$, hereafter called "round", symmetry of $\mathbb{C}\mathrm{P}^N_F$. The parameter R is a length scale that fixes the size of $\mathbb{C}\mathrm{P}^N$. We will analize in what follows a deformation of the Laplacian which breaks the round symmetry and corresponds to a Kaluza-Klein-type [13] fuzzy space, first constructed in [6], which effectively reduces a scalar field theory from $\mathbb{C}P^3_F$ to S^4_F through a probabilistic penalization method. To this end we shall briefly review the construction of S^4_F.

2.1 S^4_F revisited

We center our attention in the representation theory necessary to construct the S^4_F, further details can be found in [6] and [5]. Consider the Euclidean gamma matrices of $\mathbb{R}^5$, $\{\Gamma_a : a = 1, \dots, 5\}$, they satisfy the Clifford algebra relations $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}\mathbf{1}$. One may observe that by defining the operators $\chi_a := \frac{R}{\sqrt{5}}\Gamma_a$ for some real positive number R the relations

$$\chi_a \chi_a = R^2 \mathbf{1} \tag{2.7}$$

are fulfilled. These can be interpreted as the fuzzy analogue of the embedding equations for $S^4 \hookrightarrow \mathbb{R}^5$ at the lowest level of the matrix algebra sequence, that is, the defining Spin(5) representation $(\frac{1}{2}, \frac{1}{2})$.[1] Functions on S^4_F at this level are given by elements of the form $F = F_0\mathbf{1} + F_a\Gamma_a$ and even at this level they do not form a closed subalgebra. To solve this difficulty the approach that we follow, taken in [6], is to adopt the full matrix algebra Mat_4. By using the L-fold symmetrized tensor product of the defining representation, $(\frac{L}{2}, \frac{L}{2})$, and choosing as algebra of functions the sequence of matrix algebras formed by the products $(\frac{L}{2}, \frac{L}{2}) \otimes \overline{(\frac{L}{2}, \frac{L}{2})}$, the operators

$$J_a := \left(\underbrace{\Gamma_a \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{\text{L factors}} + \mathbf{1} \otimes \Gamma_a \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \Gamma_a \right)_{\text{sym}} \tag{2.8}$$

generalize Γ_a to the L-th level and satisfy the constraint

$$J_a J_a = L(L+4)\mathbf{1}. \tag{2.9}$$

We generalize the matrices χ_a by defining $X_a := \frac{R}{\sqrt{L(L+4)}} J_a$ which satisfy the constraint $X_a X_a = R^2\mathbf{1}$. In the large L limit the algebra becomes the commutative algebra $C^\infty(S^4)$ as the commutators $[X_a, X_b]$ vanish in the limit $L \to \infty$ while the constraint remains. However, at a finite level L the algebra of functions is still not closed. The procedure presented in [6] is to enlarge the algebra of functions to the full matrix algebra and then suppress the modes which are not associated with the S^4_F degrees of freedom in the (scalar) fields by giving them a very large excitation energy. The sequence of matrix algebras obtained is then $Mat_{d_L^3}$, and we can therefore conceive S^4_F effectively as a deformed $\mathbb{C}P^3_F$. In what follows we aim to give a geometrical interpretation of this procedure.

3 The orbit construction of $\mathbb{C}P^3$

In this section we present the construction of $\mathbb{C}P^3$ following [6] as Spin(6)($\cong$ SU(4) local isomorphism) and Spin(5) orbits and obtain the metric in terms of the Maurer-Cartan forms

[1] We use everywhere the highest-weight vector labeling for representations.

of these groups. Hereafter we will specialize to $N=3$, recalling the Lie algebra isomorphism $spin(6) \cong su(4)$ we find it convenient to replace the index $\mu = 1,\ldots,15$ in (2.2) by a composite index $\mu = AB$ where each index $A, B = 1, 2, \cdots, 6$ and the understanding that they appear only in antisymmetrized form. In this manner we preserve the use of Einstein's summation convention. Following [14] the algebra (2.3) of the Spin(6) generators in the fundamental representation takes the form:[2]

$$
\begin{aligned}
\Lambda_{AB}\Lambda_{CD} &= A_{AB;CD}\frac{\mathbf{1}}{2} + \frac{1}{4}\epsilon_{ABCDEF}\Lambda_{EF} \\
&\quad + \frac{\imath}{2}\left(\delta_{AC}\Lambda_{BD} + \delta_{BD}\Lambda_{AC} - \delta_{BC}\Lambda_{AD} - \delta_{AD}\Lambda_{BC}\right).
\end{aligned} \tag{3.1}
$$

$A_{AB;CD}$ is the two-index antisymmetrizer:

$$
A_{AB;CD} = \frac{1}{2}\left(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}\right). \tag{3.2}
$$

The d and f tensors in (2.3) can be read from (3.1):

$$
\begin{aligned}
d_{ABCDEF} &= \frac{1}{2}\epsilon_{ABCDEF}, \\
f_{ABCDEF} &= \delta_{AC}A_{BD;EF} - \delta_{AD}A_{BC;EF} + \delta_{BD}A_{AC;EF} - \delta_{BC}A_{AD;EF}.
\end{aligned}
$$

The projector $\mathcal{P} \in Mat_4$ in (2.2) is expanded as:[3]

$$
\mathcal{P} = \frac{1}{4}(\mathbf{1} + n_{AB}\Lambda_{AB}), \tag{3.3}
$$

the constraints (2.4) take the form:

$$
n_{AB}n_{AB} = 6, \tag{3.4}
$$

$$
\epsilon_{ABCDEF}n_{AB}n_{CD} = 8n_{EF}. \tag{3.5}
$$

By contractions of (3.4)-(3.5) we get the additional identities:

$$
n_{AC}n_{CB} = -\delta_{AB}, \tag{3.6}
$$

$$
\epsilon_{ABCDEF}n_{EF} = 2(n_{AB}n_{CD} + n_{AD}n_{BC} - n_{AC}n_{BD}), \tag{3.7}
$$

$$
\epsilon_{ABCDEF}n_{AB}n_{CD}n_{EF} = 48. \tag{3.8}
$$

In the coordinate system $\{n_{AB}\}$ the geometrical objects (2.5) are

$$
\begin{aligned}
\mathbf{P}_{AB;CD} &= \frac{1}{2}A_{AB;CD} + \frac{1}{8}\epsilon_{ABCDEF}n_{EF} - \frac{1}{4}n_{AB}n_{CD}, \\
\mathbf{J}_{AB;CD} &= \frac{1}{4}f_{ABCDEF}n_{EF}, \\
&= \frac{1}{4}(\delta_{AC}n_{BD} - \delta_{AD}n_{BC} + \delta_{BD}n_{AC} - \delta_{BC}n_{AD}), \\
\mathbf{K}_{AB;CD} &= \frac{1}{2}\left(\mathbf{P}_{AB;CD} + \imath\mathbf{J}_{AB;CD}\right).
\end{aligned} \tag{3.9}
$$

[2] The relations between gamma matrices and the Spin(6) generators of the $\mathbf{4}$ representation is $\Lambda_{AB} = \frac{1}{2}(1+\Gamma)\frac{1}{4\imath}[\Gamma_A, \Gamma_B]$, where $\Gamma = \imath\Gamma_1\cdots\Gamma_8 = \Gamma^\dagger$ is the chirality and satisfies $\Gamma^2 = \mathbf{1}$.

[3] We take $\xi_\mu\xi_\mu = \frac{1}{8}n_{AB}n_{AB}$. This is a more convenient normalization for our purposes.

Where, as before, $\mathbf{J}$, $\mathbf{P}$ and $\mathbf{K}$ stand for the complex structure, metric and Kähler structure on $\mathbb{C}\mathrm{P}^3$.

A more compact way to express the metric in (3.9) is

$$\mathbf{P}_{AB;CD} = \frac{1}{2}\left(A_{AB;CD} - Q_{AB;CD}\right) \tag{3.10}$$

where:[4]

$$Q_{AB;CD} = \frac{1}{2}\left(n_{AC}n_{BD} - n_{AD}n_{BC}\right). \tag{3.11}$$

The projector $\mathbf{P}_{AB;CD}$ has rank 6, it is the basic projector onto $\mathbb{CP}^3$. The orthogonal complementary projector in $\mathbb{R}^{15}$ is the rank 9 projector

$$P^{\perp}_{AB;CD} = \frac{1}{2}(A_{AB,CD} + Q_{AB,CD}). \tag{3.12}$$

We also have a rank 1 projector orthogonal to $\mathbb{C}\mathrm{P}^3$,

$$N_{AB;CD} = \frac{n_{AB}n_{CD}}{6}. \tag{3.13}$$

Notice that $\mathbf{P}_{AB;CD}$ in (3.10) and $N_{AB;CD}$ in (3.13) give a rank 7 projector. $\mathbf{P}_{AB;CD} + N_{AB;CD}$ projects $\mathbb{R}^{15}$ onto S^7 which can be viewed as an $SU(4)$ orbit over $SU(3)$. Since they are orthogonal and project onto $\mathbb{C}\mathrm{P}^3$ and $U(1)$ this S^7 admits one squashing parameter. This is a special case of the more general result that $S^{2N+1} = SU(N+1)/SU(N)$ and there is always one squashing parameter associated with the sum of the $\mathbb{C}\mathrm{P}^N$ and normal projectors.

3.1 $\mathbb{C}\mathrm{P}^3$ as an orbit under $\mathrm{Spin}(6)$

We give an explicit construction of $\mathbb{C}\mathrm{P}^3$ as a $\mathrm{Spin}(6)$ orbit and analize the induced metric. As the adjoint action of $\mathrm{Spin}(6)$ in the space of projectors (2.1) is transitive, $\mathbb{C}\mathrm{P}^3$ can be obtained as the $\mathrm{Spin}(6)$ orbit of an appropriate fiducial projector $\mathcal{P}^0$:

$$\mathcal{P} = U\mathcal{P}^0U^{-1}, \qquad U \in \mathrm{Spin}(6). \tag{3.14}$$

For $\mathcal{P}^0$ we choose:

$$\begin{aligned} \mathcal{P}^0 &= \frac{1}{4}(\mathbf{1} + n^0_{AB}\Lambda_{AB}), \\ &= \frac{1}{4}\mathbf{1} + \frac{1}{2}\left(\Lambda_{12} + \Lambda_{34} + \Lambda_{56}\right). \end{aligned} \tag{3.15}$$

We call the point corresponding to $\mathcal{P}^0$ the "north pole".

The projector (3.9) plays an essential rôle in any differential relations since

$$dn_{AB} = \mathbf{P}_{AB;CD}dn_{CD}. \tag{3.16}$$

The line element is defined as[5]

$$ds^2 := \frac{R^2}{8}dn_{AB}dn_{AB} =: \frac{R^2}{8}dn^2_{AB} =: -\frac{R^2}{8}\,\mathrm{Tr}(d\mathcal{N})^2 = \frac{R^2}{8}\mathbf{P}_{AB;CD}dn_{AB}dn_{CD}, \tag{3.17}$$

[4]One can then easily check that $AQ = QA = Q$, $Q^2 = A$ and $\mathrm{Tr}[Q] = 3$.

[5]We will typically set $R = 1$ for the round $\mathbb{C}\mathrm{P}^3$.

and justifies the appellation metric to the projector $\mathbf{P}_{AB;CD}$.

The generators Λ_{AB} transform as a rank 2 tensor under Spin(6):

$$n_{AB} = R_{AC}R_{BD}n^0_{CD}. \tag{3.18}$$

It may be shown that the line element is given as $ds^2 = -\frac{R^2}{4}Tr\left[R^{-1}dR, \mathcal{N}^0\right]^2$,where $\mathcal{N}^0$ is the matrix with entries n^0_{AB} and we rewrite $R^{-1}dR$ in terms of left invariant Maurer-Cartan forms of Spin(6) [6]: $R^{-1}dR =: -\imath e_{AB}T_{AB}$, where T_{AB} are the generators of the vector representation.[7] The line element is hence:

$$\begin{aligned} ds^2 &= 4R^2 e_{AB}\mathbf{P}^0_{AB;CD}e_{CD} = 4R^2\Big((e_{13}-e_{24})^2 + (e_{14}+e_{23})^2 + (e_{15}-e_{26})^2 + \\ &\quad (e_{16}+e_{25})^2 + (e_{35}-e_{46})^2 + (e_{36}+e_{45})^2\Big). \end{aligned} \tag{3.19}$$

It becomes apparent from (3.19) that the orbit is a six dimensional space, as expected for $\mathbb{C}\mathrm{P}^3$. It is possible to obtain the isotropy subgroup by looking at the combinations of forms e_{AB} which do not appear in the metric; the corresponding combinations of generators span the isotropy subalgebra. In the $SU(4)$ formulation the isotropy group is easily identified as $S(U(3)\times U(1))$. We obtain a coset space realization for $\mathbb{C}\mathrm{P}^3$

$$\mathbb{C}\mathrm{P}^3 = SU(4)/S(U(3)\times U(1)). \tag{3.20}$$

3.2 $\mathbb{C}\mathrm{P}^3$ as an orbit under Spin(5)

Observe that $\{\Lambda_{ab},\quad a,b=1,...,5\}$ generate the $spin(5)$ subalgebra of $spin(6)$ while Λ_{a6} transforms as a vector under Spin(5). We define $\Lambda_a := \Lambda_{a6}$ so we can write the projector (2.2) as:

$$\mathcal{P} = \frac{1}{4}\mathbf{1} + \frac{n_a}{2}\Lambda_a + \frac{n_{ab}}{4}\Lambda_{ab}, \tag{3.21}$$

the projector (3.15) takes the form:

$$\mathcal{P}^0 = \frac{1}{4}\mathbf{1} + \frac{1}{2}\Lambda_5 + \frac{1}{2}\left(\Lambda_{12}+\Lambda_{34}\right). \tag{3.22}$$

The action of Spin(5) on the space of projectors (2.1) is also transitive, hence we obtain $\mathbb{C}\mathrm{P}^3$ as an orbit of (3.15) or (3.22) under Spin(5).

The function algebra of $\mathbb{C}\mathrm{P}^3$ is now built from the two $Spin(5)$ representations n_a which carries the 5-dimensional representation and n_{ab} which carries the 10-dimensional representation. The $SO(6)$ invariant line element (3.17) can therefore be deformed to:

$$d\bar{s}^2 = \alpha dn_a^2 + \beta dn_{ab}^2. \tag{3.23}$$

[6] It would be more natural to use right-invariant Maurer-Cartan forms (see Appendix B), since these are dual to the vector fields $\mathcal{L}_{AB}$ discussed below, but both will be equivalent at the north pole, the resulting right-invariant expressions are equivalent to replacing the projectors at the north pole by those at a generic point.

[7] The normalization of the generators T_{AB} is such that they satisfy the same Lie algebra as Λ_{AB} with identical structure constants, their matrix elements being $(T_{AB})_{IJ} = -\imath(\delta_{AI}\delta_{BJ} - \delta_{AJ}\delta_{BI})$.

We will leave α, β undetermined for the moment, so that this is the most general induced Spin(5)-invariant line element, we will come back to this point at the end of this section and in section 5. It is now possible to write

$$dn_{ab}^2 = -Tr\left[R^{-1}dR, n^0\right]^2, \tag{3.24}$$

$$dn_a^2 = |R^{-1}dRn_v^0|^2, \tag{3.25}$$

where n^0 stands for the matrix of coefficients n_{ab}^0, and n_v^0 for the column vector with components n_a^0.

As before we express the line element in terms of the Spin(5) Maurer-Cartan forms $R^{-1}dR := -\imath e_{ab}T_{ab}$ where T_{ab} are the generators of the $spin(5)$ subalgebra in the vector representation.

We obtain for the line element:

$$\begin{aligned} d\bar{s}^2 &= \left(\frac{R_{S^4}^2}{2}\mathbb{P}^0_{ab,cd} + R_{S^2}^2\mathbb{X}^0_{ab,cd}\right)e_{ab}e_{cd} \\ &= (\alpha+2\beta)\left(e_{15}^2 + e_{25}^2 + e_{35}^2 + e_{45}^2\right) + 4\beta[(e_{14}+e_{23})^2 + (e_{13}-e_{24})^2] \end{aligned} \tag{3.26}$$

From (3.26) we can observe two interesting features: First, the isotropy subgroup can be constructed as before giving the following coset space realization $\mathbb{C}\mathrm{P}^3 = \mathrm{Spin}(5)/[U(1)\times SU(2)]$ and second, the space is locally of the form $S^2 \times S^4$; $\mathbb{C}\mathrm{P}^3$ is indeed a fibre bundle with base space S^4 and fibre S^2. The constants α, β can now be reinterpeted in terms of the squared radii of these spheres: $R_{S^4}^2 = \alpha + 2\beta$ and $R_{S^2}^2 = 4\beta$ and the line element (3.23) can be written in the form

$$d\bar{s}^2 = R_{S^4}^2 dn_a^2 + \frac{R_{S^2}^2}{4}(dn_{ab}^2 - 2dn_a^2). \tag{3.27}$$

Furthermore using

$$dn_a = n_{ac}n_b dn_{cb} \tag{3.28}$$

one can extract the projector $\mathbb{X}_{ab;cd}$ onto the S^2 fibre. This and related projectors are discussed in section 5.1 below. The line element can therefore be written as

$$d\bar{s}^2 = R_{S^4}^2 dn_a^2 + \frac{R_{S^2}^2}{4}\mathbb{X}_{ab;cd}dn_{ab}dn_{cd}. \tag{3.29}$$

If we restrict the Maurer-Cartan forms in (3.19) to $SO(5)$ we see that we recover the line element (3.26) with $R_{S^4}^2 = R_{S^2}^2 = R^2$.

4 Scalar field theory on S_F^4 revisited

As it was stated in section 3, $\mathbb{C}\mathrm{P}^3$ can be obtained as a Spin(6) or Spin(5) orbit. In order to specify the geometry all that is needed is to define a Laplacian. In principle we can choose the Spin(6) or Spin(5) quadratic Casimir operators, or even a more general choice: an interpolation between both of them:

$$k_1 \mathrm{C}_2^{\mathrm{Spin}(6)} + k_2 \mathrm{C}_2^{\mathrm{Spin}(5)}.$$

In [6] a prescription for a generic scalar field theory on fuzzy $\mathbb{C}\mathrm{P}^3$ was given, the expression for the action reads

$$S[\Phi] = \frac{\mathrm{Tr}}{d_L}\left(\frac{1}{2}\Phi\Delta_h\Phi + V[\Phi]\right), \tag{4.1}$$

where the full Laplacian is

$$\Delta_h = \frac{1}{8R^2}\left(\mathrm{C}_2^{\mathrm{Spin}(6)} + h(2\mathrm{C}_2^{\mathrm{Spin}(5)} - \mathrm{C}_2^{\mathrm{Spin}(6)})\right). \tag{4.2}$$

As mentioned in section 2, the algebra of functions on $\mathbb{C}\mathrm{P}^3$ is approximated by a sequence of matrix algebras of dimension $d_L^3 = \frac{(L+1)(L+2)(L+3)}{6}$.

The quadratic Casimir operators can be written using the adjoint action of the corresponding generators

$$\frac{1}{2}\mathrm{C}_2^{\mathrm{Spin}(6)} = (ad\mathcal{J}_{AB})^2\,, \tag{4.3}$$
$$\frac{1}{2}\mathrm{C}_2^{\mathrm{Spin}(5)} = (ad\mathcal{J}_{ab})^2\,. \tag{4.4}$$

The normalization of $\mathcal{J}$ in (4.3) has been chosen so that in the fundamental representation $\mathcal{J}_{AB} = \frac{1}{2}\Lambda_{AB}$. In the same manner we have in (4.4) $\mathcal{J}_{ab} = \frac{1}{2}\Lambda_{ab}$ for the fundamental representation. For the Spin(6) generators we use those in the L-fold symmetric tensor product representation $(\frac{L}{2}, \frac{L}{2}, \frac{L}{2})$ with the same dimension d_L^3, for Spin(5) we use the generators of the $(\frac{L}{2}, \frac{L}{2})$ representation, whose dimension is also d_L^3.

The choice (4.2) for the Laplacian can be understood analyzing the effect of the term

$$\Delta_I = \frac{1}{8R^2}\left(2\mathrm{C}_2^{\mathrm{Spin}(5)} - \mathrm{C}_2^{\mathrm{Spin}(6)}\right) \tag{4.5}$$

on S_F^4 modes. After an analysis of the representation content for (2.6) it was proved in [6] that Δ_I is a strictly positive operator for the non-S_F^4 modes and has as its kernel precisely the S_F^4 modes.

The mechanism is one of probabilistic penalization as the probability of a field configuration Φ can be separated into

$$\mathrm{P}[\Phi] = \frac{\mathrm{e}^{-S[\Phi]-hS_I[\Phi]}}{Z} \tag{4.6}$$

where

$$Z = \int d[\Phi]\mathrm{e}^{-S[\Phi]-hS_I[\Phi]} \tag{4.7}$$

is the partition function of the model. Taking the limit $h \to \infty$ makes the non-S_F^4 modes unreachable. The final result is that the $\mathbb{C}\mathrm{P}_F^3$ field configurations not related to S_F^4 are dynamically supressed in this limit.

5 Geometric analysis of the supression mechanism

The product of matrices together with a map to functions induces a noncommutative product on functions, this is the **-product* [7]. This useful tool allows us to access the commutative limit explicitely. Let $\widehat{M_1}$, $\widehat{M_2}$ be two matrices of dimension d_L^3 and $M_1(n)$, $M_2(n)$ be the corresponding functions obtained by the mapping:

$$M_1(n) := Tr\left(\mathcal{P}_L(n)\widehat{M_1}\right) \tag{5.1}$$

$\mathcal{P}_L(n)$ is constructed by taking the L-fold tensor product of $\mathcal{P}$ defined in (2.2), it provides a map to functions at the level L.

The *-product is then defined through:

$$\left(M_1 * M_2\right)(n) := Tr\left(\mathcal{P}_L(n)\widehat{M_1}\widehat{M_2}\right). \tag{5.2}$$

For $\mathbb{C}\mathrm{P}^N$ the *-product can be written as a finite series of derivatives on the coordinates n_{AB}, for our purposes we will use the prescription given in [7].

$$n_{AB} := 4Tr\left(\mathcal{P}_L(n)\mathcal{J}_{AB}\right). \tag{5.3}$$

The commutator of $\mathcal{J}_{AB}$ maps into the right-invariant vector fields:

$$\mathcal{L}_{AB}M(n) := Tr\left(\mathcal{P}_L(n)\left[\mathcal{J}_{AB}, \widehat{M}\right]\right) \tag{5.4}$$

$$= 2\imath \mathbf{J}_{AB;CD}\partial_{CD}M(n) \tag{5.5}$$

The images of (4.3)-(4.4) under the *-product map are:

$$\frac{1}{2}\mathrm{C}_2^{\mathrm{Spin}(6)}\widehat{M} = \left[\mathcal{J}_{AB}, [\mathcal{J}_{AB}, \widehat{M}]\right] \longrightarrow \mathcal{C}^{(6)}M(n) = -4\kappa_6 M(n), \tag{5.6}$$

$$\frac{1}{2}\mathrm{C}_2^{\mathrm{Spin}(5)}\widehat{M} = \left[\mathcal{J}_{ab}, [\mathcal{J}_{ab}, \widehat{M}]\right] \longrightarrow \mathcal{C}^{(5)}M(n) = -4\kappa_5 M(n). \tag{5.7}$$

where

$$\kappa_6 = \mathbf{J}_{AB,CD}\partial_{CD}\left(\mathbf{J}_{AB,EF}\partial_{EF}\right) \tag{5.8}$$

$$= \mathbf{P}_{CD;EF}\partial_{CD}\partial_{EF} + \mathbf{J}_{AB;CD}(\partial_{CD}\mathbf{J}_{AB;EF})\partial_{EF} \tag{5.9}$$

$$\kappa_5 = \mathbf{J}_{ab,CD}\partial_{CD}\left(\mathbf{J}_{ab,EF}\partial_{EF}\right) \tag{5.10}$$

Now, we are interested in extracting the metric tensor comparing the relevant continuous Laplacian with the general form:

$$-\mathcal{L}^2 = \frac{1}{\sqrt{G}}\partial_\mu\left(\sqrt{G}G^{\mu\nu}\partial_\nu\right) \tag{5.11}$$

$$= G^{\mu\nu}\partial_\mu\partial_\nu + \left(\partial_\mu G^{\mu\nu}\right)\partial_\nu + \frac{1}{\sqrt{G}}G^{\mu\nu}\left(\partial_\mu\sqrt{G}\right)\partial_\nu. \tag{5.12}$$

When we retain the full Spin(6)-symmetry we have the Laplacian $\mathcal{C}^{(6)}$, the associated metric tensor is just $\mathbf{P}_{AB;CD}$ as can be seen from a straightforward calculation of κ_6:

$$\kappa_6 = \frac{1}{2}\partial_{AB}^2 + \frac{1}{2}n_{CB}n_{DA}\partial_{AB}\partial_{CD} - n_{AB}\partial_{AB}. \tag{5.13}$$

If one retains only the Spin(5) symmetry, the following expressions are found

$$\begin{aligned}\kappa_6 &= \frac{1}{2}\partial_{ab}^2 + \partial_a^2 + \frac{1}{2}n_{cb}n_{da}\partial_{ab}\partial_{cd} + 2n_a n_{bc}\partial_{ab}\partial_c - n_a n_b \partial_a\partial_b \\ &\quad - n_{ab}\partial_{ab} - 2n_a\partial_a,\end{aligned} \tag{5.14}$$

$$\begin{aligned}\kappa_5 &= \frac{1}{2}\partial_{ab}^2 + \frac{1}{2}\partial_a^2 - \frac{1}{2}n_a n_b \partial_{ac}\partial_{bc} + \frac{1}{2}n_{ca}n_{bd}\partial_{ab}\partial_{cd} - n_a n_{cb}\partial_{ab}\partial_c \\ &\quad -\frac{1}{2}n_a n_b \partial_a \partial_b - \frac{3}{4}n_{ab}\partial_{ab} - n_a\partial_a.\end{aligned} \tag{5.15}$$

5.1 The vertical and horizontal projectors

The vertical and horizontal projectors can be constructed explicitely in our coordinate system in a Spin(5)-covariant manner. The coordinates n_{AB} break up under Spin(5) as n_{ab} and n_a. These are the basic objects we will need to build projectors. From (3.6) they satisfy

$$n_{ac}n_{bc} = \delta_{ab} - n_a n_b = P_{ab}, \tag{5.16}$$

$$n_a n_{ab} = 0, \tag{5.17}$$

$$n_a n_a = 1, \tag{5.18}$$

P_{ab} is a rank 4 projector, it projects $\mathbb{R}^5 \mapsto S^4$, the usual continuum embeding of S^4 in $\mathbb{R}^5$, and its orthogonal complement is $P^{\perp}_{ab} = \delta_{ab} - P_{ab} = n_a n_b$. Defining

$$Q_{ab,c} = \frac{1}{2}(n_{ac}n_b - n_a n_{bc}), \quad Q_{ab;cd} = \frac{1}{2}(n_{ac}n_{bd} - n_{ad}n_{bc}) \tag{5.19}$$

we observe that

$$2Q_{ab,e}Q_{cd,e} = \mathbb{P}_{ab;cd} = \frac{1}{2}(\delta_{ac}P^{\perp}_{bd} - \delta_{ad}P^{\perp}_{cb} + \delta_{bd}P^{\perp}_{ac} - \delta_{bc}P^{\perp}_{ad}). \tag{5.20}$$

Where $\mathbb{P}_{ab,cd}$ projects $\mathbb{R}^{10} \mapsto S^4$, it is therefore the metric on S^4, the horizontal projector. We can then define the projectors $\mathbb{X}$ and Y:

$$\mathbb{X}_{ab;cd} = \frac{1}{2}(A_{ab;cd} - \mathbb{P}_{ab;cd} - Q_{ab;cd}), \tag{5.21}$$

$$Y_{ab;cd} = \frac{1}{2}(A_{ab;cd} - \mathbb{P}_{ab;cd} + Q_{ab;cd}). \tag{5.22}$$

Notice the ranks $\mathrm{Tr}[\mathbb{X}] = 2$ and $\mathrm{Tr}[Y] = 4$.

To see that these are orthogonal projectors one needs to observe that

$$Q_{ab;cd}Q_{cd;ef} = A_{ab;ef} - \mathbb{P}_{ab;ef}. \tag{5.23}$$

The tensor $\mathbb{X}_{ab;cd}$ is the projector onto the fibres of $\mathbb{C}\mathrm{P}^3$ as an S^2 bundle over S^4, it is the vertical projector. $\mathbb{X}, Y$ and $\mathbb{P}$ are complementary and add up to the identity in $\mathbb{R}^{10}$, $A_{ab,cd}$. It is straightforward to write the projector to the bundle, $\mathbf{P}_{ab,cd} : \mathbb{R}^{10} \mapsto \mathbb{C}\mathrm{P}^3$ as

$$\mathbf{P}_{ab,cd} := \mathbb{X}_{ab;cd} + \mathbb{P}_{ab;cd} = \frac{1}{2}(A_{ab;cd} + \mathbb{P}_{ab;cd} - Q_{ab;cd}). \tag{5.24}$$

Using these projectors we construct an ansatz for the metric of the squashed $\mathbb{C}\mathrm{P}^3$.

For completeness we also give the complex structure of $\mathbb{C}\mathrm{P}^3$ in the Spin(5) formulation, we start by defining

$$\begin{aligned} T_{ab;cd} &= \frac{1}{4}(P_{ac}n_{bd} - P_{ad}n_{bc} + P_{bd}n_{ac} - P_{bc}n_{ad}) \\ \tilde{T}_{ab;cd} &= \frac{1}{2}(P^{\perp}_{ac}n_{bd} - P^{\perp}_{ad}n_{bc} + P^{\perp}_{bd}n_{ac} - P^{\perp}_{bc}n_{ad}) \end{aligned}$$

and noting that

$$\begin{aligned} T_{ab;ef}T_{ef;cd} &= -\mathbb{X}_{ab;cd} \ , & \qquad & T_{ab;ef}T_{ef;gh}T_{gh;cd} = -T_{ab;cd} \ , \\ \tilde{T}_{ab;ef}\tilde{T}_{ef;cd} &= -\mathbb{P}_{ab;cd} & \text{and} \quad & \tilde{T}_{ab;ef}\tilde{T}_{ef;gh}\tilde{T}_{gh;cd} = -\tilde{T}_{ab;cd} \ . \end{aligned}$$

One constructs $\mathbf{J} = T + \tilde{T}$ resulting into:[8]

$$\mathbf{J}_{ab;cd} = \frac{1}{4}(\delta_{ac}n_{bd} - \delta_{ad}n_{bc} + \delta_{bd}n_{ac} - \delta_{bc}n_{ad} + P^{\perp}_{ac}n_{bd} - P^{\perp}_{ad}n_{bc} + P^{\perp}_{bd}n_{ac} - P^{\perp}_{bc}n_{ad}). \tag{5.25}$$

It is easy to prove that

$$\mathbf{J}^2 = -\mathbf{P}. \tag{5.26}$$

We return now to the discussion regarding the Laplacian. For the deformed case, which possesses only Spin(5) symmetry, the corresponding Laplacian acting on functions is $\mathcal{L}_h^2$, we have

$$\Delta_h = \frac{1}{8R^2}\left(\mathrm{C}_2^{\mathrm{Spin}(6)} + h\left(2\mathrm{C}_2^{\mathrm{Spin}(5)} - \mathrm{C}_2^{\mathrm{Spin}(6)}\right)\right) \tag{5.27}$$

and the mapping is:

$$Tr\left(\mathcal{P}_L(n)\Delta_h\widehat{M}\right) =: \frac{1}{R^2}\mathcal{L}_h^2 M(n), \tag{5.28}$$

then

$$-\mathcal{L}_h^2 = \kappa_6 + h\left(2\kappa_5 - \kappa_6\right). \tag{5.29}$$

Our ansatz for the metric tensor related to $\mathcal{L}_h^2$ is the following:[9]

[8] It should be mentioned that since S^4 does not even admit an almost-complex structure, so $\tilde{T}$ is not a complex structure on it.

[9] Double indices in $\mathbf{P}_{ab;cd}$ and $\mathbb{X}_{ab;cd}$ are raised and lowered using $A^{ab;cd}$ and $A_{ab;cd}$, the complex structure $\mathbf{J}$ is thus an up-down tensor.

$$G^{ab;cd} = \frac{2\mathbb{P}^{ab;cd} + (h+1)\mathbb{X}^{ab;cd}}{R^2} = \frac{2\mathbb{P}^{ab;cd}}{R^2} + \frac{\mathbb{X}^{ab;cd}}{R^2_{S^2}}; \tag{5.30}$$

the tensor $\mathbb{X}_{ab;cd} = \mathbb{X}_{cd;ab}$ is recovered from the combination

$$2\kappa_5 - \kappa_6 = \frac{1}{2}\partial^2_{ab} - \frac{1}{2}n_{ab}n_{cd}\partial_{ac}\partial_{bd} - n_a n_b \partial_{ac}\partial_{bc} - \frac{1}{2}n_{ab}\partial_{ab} \tag{5.31}$$

by comparing the term in second derivatives in (5.31) against the h-dependent term in (5.30), $\mathbb{X}_{ab;cd}\partial_{ab}\partial_{cd}$, and we find that $\mathbb{X}$ thus obtained is indeed the fibre metric we had previously identified in (5.21), i.e.

$$\begin{aligned}\mathbb{X}_{ab;cd} = & \frac{1}{2}A_{ab;cd} - \frac{1}{4}\left(\delta_{ac}n_b n_d - \delta_{ad}n_b n_c + \delta_{bd}n_a n_c - \delta_{bc}n_a n_d\right) \\ & -\frac{1}{4}\left(n_{ac}n_{bd} - n_{ad}n_{bc}\right).\end{aligned} \tag{5.32}$$

In order to invert the metric tensor (5.30) we observe that $\mathbf{P}\mathbb{X} = \mathbb{X}\mathbf{P} = \mathbb{X}$, hence the covariant metric tensor is a linear combination of $\mathbb{X}$ and $\mathbf{P}$, in fact

$$G_{ab;cd} = R^2\left(\frac{\mathbb{P}_{ab,cd}}{2} + \frac{1}{h+1}\mathbb{X}_{ab;cd}\right) = \frac{R^2\mathbb{P}_{ab,cd}}{2} + R^2_{S^2}\mathbb{X}_{ab;cd} \tag{5.33}$$

satisfies the required condition: $G^{ab;cd}G_{cd;ef} = \mathbf{P}^{ab}_{ef}$.

6 Fermion fields

A fuzzy four-dimensional fermion field has the representation content:

$$\Psi \in \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \otimes \left(\frac{L}{2},\frac{L}{2},\frac{L}{2}\right) \otimes \left(\frac{L}{2},\frac{L}{2},-\frac{L}{2}\right). \tag{6.1}$$

It is shown in [6] that the algebra of fuzzy functions decomposes as

$$\left(\frac{L}{2},\frac{L}{2},\frac{L}{2}\right) \otimes \left(\frac{L}{2},\frac{L}{2},-\frac{L}{2}\right) = \bigoplus_{n=0}^{L}(n,n,0), \tag{6.2}$$

hence, the relevant decomposition is

$$\begin{aligned}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \otimes (n,n,0) \ = & \ \underbrace{\left(n+\frac{1}{2},n+\frac{1}{2},\frac{1}{2}\right)}_{n\geq 0} \oplus \underbrace{\left(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2}\right) \oplus \left(n-\frac{1}{2},n-\frac{1}{2},\frac{1}{2}\right)}_{n\geq 1} \\ := & \ D^n_+ \oplus D^n_0 \oplus D^n_-.\end{aligned} \tag{6.3}$$

The restrictions below show when these representations appear in the decomposition. The spinor field decomposes into components $\Psi = \Psi_+ \oplus \Psi_0 \oplus \Psi_-$.

For a Dirac operator on S^4_F we propose the linear spinor operator in the spirit of [12] given by the ansatz

$$\not{D}_{\tilde{h}} = \sigma_{AB}[\mathcal{J}_{AB},\cdot] + 2 + \tilde{h}\left(2\sigma_{ab}[\mathcal{J}_{ab},\cdot] - \sigma_{AB}[\mathcal{J}_{AB},\cdot]\right), \tag{6.4}$$

where σ_{AB} are the Spin(6) generators in the fundamental representation $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and σ_{ab} are the corresponding Spin(5) generators. The operator $\not{D}_{\tilde{h}}$ can be expressed in terms of the differences of Casimir operators $\mathfrak{C}_2 := 2([\mathcal{J},\cdot]+\frac{\sigma}{2})^2$ and $C_2 = 2[\mathcal{J},\cdot]^2$.

By "completing the square" we may rewrite the operator $\not{D}_{\tilde{h}}$ purely in terms of quadratic Casimir operators. Note that $[\mathfrak{C}_2^{\text{Spin}(5)},\mathfrak{C}_2^{\text{Spin}(6)}] = 0$ as can be readily verified by expanding out $\mathfrak{C}_2^{\text{Spin}(6)}$ in Spin(5) indices in a Spin(5) invariant manner. It is then clear that both Casimir operators can be simultaneously diagonalized in the appropriate basis. In order to compute the spectrum of the given operator (6.4) we use the following reductions under Spin(5)

$$(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2}) = \bigoplus_{m=0}^{n-1}\left((n+\frac{1}{2},m+\frac{1}{2})\oplus(n-\frac{1}{2},m+\frac{1}{2})\right), \tag{6.5}$$

$$(n+\frac{1}{2},n+\frac{1}{2},\frac{1}{2}) = \bigoplus_{m=0}^{n}(n+\frac{1}{2},m+\frac{1}{2}). \tag{6.6}$$

Wherefrom we find the decompositions

$$\Psi_+ = \bigoplus_{m=0}^{n}\Psi^{(n+\frac{1}{2},n+\frac{1}{2},\frac{1}{2})}_{(n+\frac{1}{2},m+\frac{1}{2}),+}, \tag{6.7}$$

$$\Psi_0 = \bigoplus_{m=0}^{n-1}\left(\Psi^{(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2})}_{(n+\frac{1}{2},m+\frac{1}{2}),0}\oplus\Psi^{(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2})}_{(n-\frac{1}{2},m+\frac{1}{2}),0}\right), \tag{6.8}$$

$$\Psi_- = \bigoplus_{m=0}^{n-1}\Psi^{(n-\frac{1}{2},n-\frac{1}{2},\frac{1}{2})}_{(n-\frac{1}{2},m+\frac{1}{2}),-}\quad . \tag{6.9}$$

The spectrum of the operator $\not{D}_h$ corresponding to the component Ψ_0 has no counterpart in the known spectrum for the Dirac operator on S^4, therefore this component corresponds to degrees of freedom extraneous to the S^4 and it will, in fact, be completely supressed by our dynamical mechanism. The contributions to Ψ_+ and Ψ_- in the kernel of $\not{D}_I = 2\sigma_{ab}[\mathcal{J}_{ab},\cdot] - \sigma_{AB}[\mathcal{J}_{AB},\cdot]$ reproduce a cutoff version of the canonical spectrum of Dirac operator on the round S^4.

In detail we have the following eigenvalues, calculated with the expressions found in appendix A

$$\begin{aligned}
\not{D}_{\tilde{h}}\Psi^{(n+\frac{1}{2},n+\frac{1}{2},\frac{1}{2})}_{(n+\frac{1}{2},m+\frac{1}{2}),+} &= (n+2+\tilde{h}m)\Psi^{(n+\frac{1}{2},n+\frac{1}{2},\frac{1}{2})}_{(n+\frac{1}{2},m+\frac{1}{2}),+} \qquad n\geq 0,\\
\not{D}_{\tilde{h}}\Psi^{(n-\frac{1}{2},n-\frac{1}{2},\frac{1}{2})}_{(n-\frac{1}{2},m+\frac{1}{2}),-} &= (-n-1+\tilde{h}m)\Psi^{(n-\frac{1}{2},n-\frac{1}{2},\frac{1}{2})}_{(n-\frac{1}{2},m+\frac{1}{2}),-} \qquad n\geq 1,\\
\not{D}_h\Psi^{(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2})}_{(n+\frac{1}{2},m+\frac{1}{2}),0} &= (1+\tilde{h}(n+m+1))\Psi^{(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2})}_{(n+\frac{1}{2},m+\frac{1}{2}),0} \qquad n\geq 1,\\
\not{D}_h\Psi^{(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2})}_{(n-\frac{1}{2},m+\frac{1}{2}),0} &= (1+\tilde{h}(m-n+2))\Psi^{(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2})}_{(n-\frac{1}{2},m+\frac{1}{2}),0} \qquad n\geq 1.
\end{aligned} \tag{6.10}$$

In the large $\tilde{h}$ limit the portion of the spectrum not in the kernel of $\not{D}_I$ is sent to infinity, the remaining low lying spectrum coincides with the spectrum of the Dirac operator on S^4 up to a truncation [15], namely

$$\{\pm(n+2)\ :\ n=0,1,\cdots,L-1\}\cup\{L+2\},\ deg(n+2)=\frac{2(n+1)(n+2)(n+3)}{3}.$$

The degeneracies have been calculated using the formulae in appendix A, clearly one has $deg(n+2)=\dim(n+\frac{1}{2},\frac{1}{2})$.

A fermionic action may be now be written for a free spinor field with mass M as

$$S_\Psi=\frac{\mathrm{Tr}}{d_N^3}\left(\bar{\Psi}(\not{D}_{\tilde{h}}+M)\Psi\right). \tag{6.11}$$

We remark that the deformed spinor operator $\not{D}_{\tilde{h}}$ is not a Dirac operator on $\mathbb{C}\mathrm{P}^3$ with a squashed metric, our purpose here is to find a suitable operator for Fermions on fuzzy S^4. The operator we have found has similarities to higher spin Dirac operators introduced in [16]. As in the case of the scalar theory the statistical penalization mechanism will suppress the functional degrees of freedom in the spinor field Ψ which are not associated to S^4_F.

One can check that when maped to functions the operator D_I is mapped to $\sigma_{ab}\mathbb{X}_{ab;cd}\partial_{cd}$ and since n_e is in the kernel of this operator any function of n_e is in the kernel. It sees only the dependence on n_{ab}. The parameter $\tilde{h}$ is similarly related to the radius of the S^2 fibres and for large $\tilde{h}$ we are shrinking the fibres relative to the S^4 base.

7 Conclusions

We review the construction of fuzzy $\mathbb{C}\mathrm{P}^3$ presented in [7]. The main motivation to discretize this 6 dimentional space is due to its relation to S^4, a compactification of $\mathbb{R}^4$.

The standard construction of $\mathbb{C}\mathrm{P}^3$ involves Spin(6) symmetry, giving as result a "round" version of $\mathbb{C}\mathrm{P}^3$. We gave a different construction of $\mathbb{C}\mathrm{P}^3$ and its fuzzy version as a Spin(5) orbit where the local structure $S^2\times S^4$ is manifiest. The isotropy group was found to be $SU(2)\times U(1)$. Following the results obtained in [6] in which a convenient interpolation of the Spin(6) and Spin(5) quadratic Casimirs was introduced as the Laplacian, we interpret the deformation parameter h introduced in [6] in terms of the radii of a squashed $\mathbb{C}\mathrm{P}^3$. From the point of view of a scalar field theory this procedure can be interpreted as a Kaluza-Klein construction, where the entire space is non-trivial fibre bundle with base S^4 and fibre S^2, and in the large h limit the radius of the S^2 fibres is sent to zero.

Along the way we constructed the complex structure of $\mathbb{C}\mathrm{P}^3$ as a $Spin(5)$ orbit. The square of the complex sturcture gives minus the $\mathbb{C}\mathrm{P}^3$ projector and it naturally splits into parts which give the S^4 base and S^2 fibres.

Using *-product map techniques we have presented an explicit manner to extract the metric of the space under consideration from its Laplacian. The explicit form of the deformed metric tensor $G_{\mu\nu}$ was obtained. Examining the resulting line element ds^2 we found the ratio between radii:

$$\frac{R_{S^2}}{R}=\frac{1}{\sqrt{(1+h)}}.$$

The limit $h \to \infty$ corresponds to shrinking the S^2 fibres down to zero size, while the limit $h \to -1$ makes the fibres infinitely large.

We have also proposed a linear spinorial operator on S_F^4, based on the same geometric structure as the scalar case, and identified the relevant spinor subspaces that contain the correct spectrum of the Dirac operator on S^4, up to a truncation. This operator acts on four component spinors and does not correspond to a Dirac operator on $\mathbb{C}\mathrm{P}^3$, though it is a well defined first order operator on $\mathbb{C}\mathrm{P}^3$ and its fuzzy version. As with scalar fields, spinor fields on S_F^4 have additional degrees of freedom in the construction, however all become of arbitrarily large mass as the parameter $\tilde{h}$ is sent to infinity and so are dynamically suppressed.

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A Casimir operators and dimensions

The quadratic Casimir operators for Spin(6) and Spin(5) found in [17–19] were used,

$$\mathrm{C}_2^{\mathrm{Spin}(6)}(m_1, m_2, m_3) = m_1^2 + m_2^2 + m_3^2 + 4m_1 + 2m_2, \tag{A.1}$$

$$\mathrm{C}_2^{\mathrm{Spin}(5)}(m_1, m_2) = m_1(m_1+3) + m_2(m_2+1), \tag{A.2}$$

which for the involved representations amount to

$$\mathrm{C}_2^{\mathrm{Spin}(6)}(n, n, 0) = 2n(n+3), \tag{A.3}$$

$$\mathrm{C}_2^{\mathrm{Spin}(6)}(n+\frac{1}{2}, n+\frac{1}{2}, \frac{1}{2}) = 2n(n+4) + \frac{15}{4}, \tag{A.4}$$

$$\mathrm{C}_2^{\mathrm{Spin}(6)}(n+\frac{1}{2}, n-\frac{1}{2}, -\frac{1}{2}) = 2n(n+3) + \frac{7}{4}, \tag{A.5}$$

$$\mathrm{C}_2^{\mathrm{Spin}(5)}(n+\frac{1}{2}, m+\frac{1}{2}) = n(n+4) + m(m+2) + \frac{5}{2}, \tag{A.6}$$

$$\mathrm{C}_2^{\mathrm{Spin}(5)}(n-\frac{1}{2}, m+\frac{1}{2}) = n(n+2) + m(m+2) - \frac{1}{2}. \tag{A.7}$$

We rewrite the square taking into account the following normalization:

$$(\frac{\sigma_{AB}}{2})^2 = \frac{1}{2}\mathrm{C}_2^{\mathrm{Spin}(6)}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{15}{8}, \tag{A.8}$$

$$[\mathcal{J}_{AB}, [\mathcal{J}_{AB}, \cdot]] = \frac{1}{2}\bigoplus_{n=0}^{L}\mathrm{C}_2^{\mathrm{Spin}(6)}(n, n, 0). \tag{A.9}$$

Some useful formulae for the dimensions of representations we deal with are

$$\begin{aligned}\dim(m_1,m_2,m_3) &= \frac{1}{12}(m_1^2-m_2^2+4m_1-2m_2+3) \\ &\quad\times(m_1^2-m_3^2+4m_1+4)(m_2^2-m_3^2+2m_2+1) \\ \dim(m_1,m_2) &= \frac{1}{6}(m_1^2-m_2^2+3m_1-m_2+2) \\ &\quad\times(2m_1+3)(2m_2+1).\end{aligned} \tag{A.10, A.11}$$

$$\dim(n,0) = \frac{1}{6}(n+1)(n+2)(2n+3), \tag{A.12}$$

$$\dim(n,n,0) = \frac{1}{12}(n+1)^2(n+2)^2(2n+3), \tag{A.13}$$

$$\dim(n+\frac{1}{2},n+\frac{1}{2},\frac{1}{2}) = \frac{1}{6}(n+1)(n+2)^3(n+3), \tag{A.14}$$

$$\dim(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2}) = \frac{1}{6}n(n+1)(n+2)(n+3)(2n+3), \tag{A.15}$$

$$\begin{aligned}\dim(n+\frac{1}{2},m+\frac{1}{2}) &= \frac{2}{3}(n(n+4)-m(m+2)+3) \\ &\quad\times(n+2)(m+1),\end{aligned} \tag{A.16}$$

$$\begin{aligned}\dim(n-\frac{1}{2},m+\frac{1}{2}) &= \frac{2}{3}(n(n+4)-m(m+2))(n+1) \\ &\quad\times(m+1).\end{aligned} \tag{A.17}$$

The spinor components in (6.7)-(6.9) are eigenvectors of $\left(2\mathfrak{C}_2^{\mathrm{Spin}(5)}-\mathfrak{C}_2^{\mathrm{Spin}(6)}\right)$ which appear as a part in the r.h.s of (6.4):

$$\begin{aligned}\left(2\mathfrak{C}_2^{\mathrm{Spin}(5)}-\mathfrak{C}_2^{\mathrm{Spin}(6)}\right)\Psi^{(n\pm\frac{1}{2},n\pm\frac{1}{2},\frac{1}{2})}_{(n\pm\frac{1}{2},m+\frac{1}{2}),\pm} &= \left(2m(m+2)+\frac{5}{4}\right)\Psi^{(n\pm\frac{1}{2},n\pm\frac{1}{2},\frac{1}{2})}_{(n\pm\frac{1}{2},m+\frac{1}{2}),\pm}, \\ \left(2\mathfrak{C}_2^{\mathrm{Spin}(5)}-\mathfrak{C}_2^{\mathrm{Spin}(6)}\right)\Psi^{(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2})}_{(n+\frac{1}{2},m+\frac{1}{2}),0} &= \left(2n+2m(m+2)+\frac{13}{4}\right)\Psi^{(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2})}_{(n+\frac{1}{2},m+\frac{1}{2}),0}, \\ \left(2\mathfrak{C}_2^{\mathrm{Spin}(5)}-\mathfrak{C}_2^{\mathrm{Spin}(6)}\right)\Psi^{(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2})}_{(n-\frac{1}{2},m+\frac{1}{2}),0} &= \left(-2n+2m(m+2)-\frac{11}{4}\right)\Psi^{(n+\frac{1}{2},n-\frac{1}{2},-\frac{1}{2})}_{(n-\frac{1}{2},m+\frac{1}{2}),0}.\end{aligned}$$

B Right-invariant Maurer-Cartan forms

The Spin(6) right-invariant Maurer-Cartan forms are defined by $dRR^{-1} = -\imath\mathfrak{e}_{AB}T_{AB}$, they are dual to the right-invariant vector fields

$$< \mathfrak{e}_{AB},\mathcal{L}_{CD} >= \imath\mathbf{P}_{AB;CD}, \tag{B.1}$$

and

$$\mathbf{P}_{AB;CD} = \frac{1}{16}\,\mathrm{Tr}([T_{AB},\mathcal{N}][T_{CD},\mathcal{N}]). \tag{B.2}$$

By noticing the relations

$$\frac{1}{2}\mathrm{C}_2^{\mathrm{Spin}(5)} = \mathcal{L}_{ab}\mathbf{P}_{ab;cd}\mathcal{L}_{cd}, \tag{B.3}$$

$$\frac{1}{2}\mathrm{C}_2^{\mathrm{Spin}(6)} = \mathcal{L}_{ab}(\mathbb{X}_{ab;cd} + 2\mathbb{P}_{ab;cd})\mathcal{L}_{cd}, \tag{B.4}$$

it follows that the line elements corresponding to these operators are respectively

$$ds_5^2 = \mathfrak{e}_{ab}\mathbf{P}_{ab;cd}\mathfrak{e}_{cd}, \quad ds_6^2 = \mathfrak{e}_{ab}\left(\mathbb{X}_{ab;cd} + \frac{\mathbb{P}_{ab;cd}}{2}\right)\mathfrak{e}_{cd}. \tag{B.5}$$

From here we obtain the line element (3.26) associated with Δ_h:

$$d\bar{s}^2 = 4R^2\mathfrak{e}_{ab}\left(\frac{\mathbb{P}_{ab;cd}}{2} + \frac{\mathbb{X}_{ab;cd}}{1+h}\right)\mathfrak{e}_{cd}. \tag{B.6}$$

In order to fix a normalization for the radii we define the line element (3.17) by choosing

$$ds^2 = \frac{R^2}{4}dn_{AB}dn_{AB} = 4R^2\mathfrak{e}_{AB}\mathbf{P}_{AB;CD}\mathfrak{e}_{CD} \tag{B.7}$$

and split it up under Spin(5) as:

$$ds^2 = R^2\left(dn_a^2 + \frac{dn_{ab}^2 - 2dn_a^2}{4}\right) = 4R^2\mathfrak{e}_{ab}\left(\frac{\mathbb{P}_{ab;cd}}{2} + \mathbb{X}_{ab;cd}\right)\mathfrak{e}_{cd}. \tag{B.8}$$

We can then read off from (3.23) $R_{S^4}^2 = \alpha + 2\beta$ and $R_{S^2}^2 = 4\beta$. Finally using (4.2) we find, as before,

$$R_{S^4}^2 = R^2, \qquad R_{S^2}^2 = \frac{R^2}{1+h}. \tag{B.9}$$

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