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Near commuting multi-matrix models

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Abstract: We investigate the radial extent of the eigenvalue distribution for Yang-Mills type matrix models. We show that, a three matrix Gaussian model with complex Myers coupling, to leading order in strong coupling is described by commuting matrices whose joint eigenvalue distribution is uniform and confined to a ball of radius $R = \left(\frac{3\pi}{2g}\right)^{1/3}$. We then study, perturbatively, a 3-component model with a pure commutator action and find a radial extent broadly consistent with numerical simulations.

Keywords: Matrix Models, 1/N Expansion

Contents

1 Introduction

Multi-matrix models arise in a wide variety of settings from matrix string theory [1], the IKKT model [2] (and its lower dimensional variants [3]), the BFFS and BMN models [4, 6] to the low energy dynamics of D-branes [7] and simple models of emergent geometry [8] and emergent gravity [9, 10]. More recently the BMN and BFFS models have received attention in the context of fast scrambling in [11].

In the context of numerical simulations it has been observed that the 3-component Yang Mills matrix model (with pure commutator action) [12] has an eigenvalue distribution that is parabolic with radial extent $R = 2.0$. It is tempting to suspect that this result should be derivable from first principals and this paper is an attempt in this direction. In part the motivation for the current paper was to attempt a theoretical estimate of the extent of the eigenvalue distribution in this simple model. Though we did not succeed in getting an exact result we succeeded in getting approximate results which are in broad agreement with the simulations.

The principal results of this paper are:

- A derivation of the uniform ball distribution in a previously studied 3-matrix model, some of whose observables have been solved exactly.

- A demonstration that one can obtain the exact extent of the eigenvalue distribution in this 3-matrix model from an effective potential for this radial extent.

- The derivation of an effective potential to 2-loops for the pure Yang-Mills matrix model.
- We derive the measure for a rotationally invariant gauge fixing which is natural if the matrices are approximately commuting.
- Estimates for the radial extent in the 3-matrix Yang-Mills model, our estimate is that the extent of the eigenvalue distribution is $R = 1.6 \pm 0.5$.

The structure of the paper is as follows:

Section 2 is dedicated to the mass regulated two matrix model. This model was introduced in [5] and arose again in the description of the low energy dynamics of N D-branes which are close together in 4-dimensional space-time [7]. The strong coupling properties of the model were investigated in ref. [7] and in [13] it was shown that at strong coupling the model is in a commuting phase with the joint eigenvalue distribution described by a hemisphere.

In the first part of Section 2 we review the results of ref. [13] using a sightly different approach. In particular we split the matrices into their diagonal and off-diagonal elements and consider a general axial gauge which is equivalent to diagonalizing one of the matrices. We integrate out the perpendicular modes and obtain an effective action for the diagonal components of the matrices. Next we consider a coarse grained approximation and show that the longitudinal part of the diagonal modes has a parabolic eigenvalue distribution [13].

In the second part of Section 2 we consider an $SO(3)$ invariant non-hermitian three matrix model constructed from a hermitian mass term and an anti-hermitian Myers term [14]. Integrating out one of the matrices one recovers the $SO(2)$ invariant two matrix model considered in the first part of Section 2. However, we study the model directly in the three matrix model realization by splitting the matrices into their diagonal and off-diagonal components and consider a general axial gauge for the off-diagonal modes. Next we integrate out the perpendicular modes and obtain a one-loop effective action (exact in the axial gauge) for the longitudinal part of the diagonal components. We average over all possible orientations of the constant unit vector specifying the axial gauge and obtain a three dimensional $SO(3)$ invariant effective action for the diagonal modes. Next we study the strongly coupled regime of the model and show that to leading order the ground state is described by a uniform joint eigenvalue distribution inside a solid ball. The hemisphere and parabolic distributions conjectured in ref. [13] can then be obtained by integrating the uniform distribution over one or two of the three coordinates respectively. Further, by conjecturing a uniform joint eigenvalue distribution, we obtain an effective potential for the radius of the distribution, which in the strong coupling limit recovers the exact radial extent of the distribution. It is this uniform joint eigenvalue distribution of the model which motivates the studies presented in Section 3 of the paper.

Section 3 of our work explores the properties of the $SO(p)$ invariant hermitian p-matrix model corresponding to a commutator squared term, i.e. a pure Yang-Mills type matrix model. This family of models have been studied both numerically and in a large p expansion in [15]. Special attention was given to the $p = 3$ case in ref. [16–20], including the effect of finite mass and Myers terms. In particular it has been shown that, as the strength of the Myers coupling is varied, the ground state of the model undergoes an exotic phase transition from a fuzzy two sphere to a matrix phase in an exotic transition [8]. It is this matrix phase that we explore in Section 3.

Section 3 contains four subsections. In the first part of the Section 3 we outline a change of coordinates involving splitting the matrices to diagonal and off-diagonal components. The change of coordinates that we consider is also equivalent to an $SO(p)$ invariant gauge fixing condition for the off-diagonal elements. We calculate the corresponding Jacobian to second order in the off-diagonal modes. This takes into account contribution of the corresponding ghost modes to two loops. At tree level the ghost determinant is a generalized $SO(p)$ invariant Vandermonde type determinant.

The second part of Section 3 derives the effective action for the diagonal modes. By setting up a systematic perturbative expansion we obtain a two loop effective action for the diagonal elements of the model and under the assumption of an $SO(p)$ invariant distribution for these modes within a ball of radius R we obtain an effective potential for the radial extent of their distribution. We show that the radial extent of these modes stabilizes in this approximation. For the three matrix model assuming a uniform distribution inside a solid ball we obtain an analytic estimate of the radius of the distribution.

The third part of Section 3 studies the gauge dependence of our perturbative expansion. We consider a standard Faddeev-Popov gauge fixing procedure and compare the results obtained in Feynman and Landau gauges. We argue that for a generic gauge the joint distribution of the diagonal modes will differ from the genuine joint distribution and that the Landau gauge is the most suitable for studying the model.

Section 4 of our paper is our concluding section. We summarize our results and discuss possible directions for future studies.

2 Strongly coupled mass regulated two matrix model

Let us consider the Hermitian two matrix model first considered by Hoppe [5]:

$$\mathcal{Z} = \int \mathcal{D}X\mathcal{D}Y e^{-N\mathrm{tr}(X^2+Y^2-g^2[X,Y]^2)} \ . \tag{2.1}$$

The matrix model (2.1) arose again in the work of Kazakov et al [7] where the partition function for large g and a fresh derivation of Hoppe's solution for the $< \frac{\mathrm{tr}}{N}(X^2) >$ are given. The solution for these quantities are given in a rather complex form, which makes a more complete analysis of the theory (valid for large $g \gg 1$) difficult. A more detailed understanding of the model was initiated in ref. [13] where it was suggested that the joint eigenvalue distribution of the commuting saddle is given by a hemisphere distribution. The analysis of ref. [13] shows that at large coupling the eigenvalues of a single matrix are described by parabolic eigenvalue distribution. The authors then argue that this suggests hemisphere eigenvalue distribution for the commuting saddle of the two matrix model.

2.1 Hemisphere distribution

Let us review the results of ref. [13], by following a slightly different approach which we find instructive. In particular we split the matrices to diagonal and off-diagonal components and integrate out the off-diagonal modes by imposing an axial gauge condition. To begin with let us define:

$$X_{ij} = x_i^1 \delta_{ij} + a_{ij}^1; \quad Y_{ij} = x_i^2 \delta_{ij} + a_{ij}^2; \quad \vec{x}_i = (x_i^1, x_i^2); \quad \vec{a}_{ij} = (a_{ij}^1, a_{ij}^2) \ . \tag{2.2}$$

The action in (2.1) can be written as:

$$S[X,Y] = \sum_{i=1}^{N} \vec{x}_i^2 + \sum_{i,j=1}^{N} |\vec{a}_{ij}|^2 + g^2 \sum_{i,j,=1}^{N} |(x_i^\mu - x_j^\mu) a_{ij}^\nu \epsilon_{\mu\nu} + (a^\mu a^\nu)_{ij} \epsilon_{\mu\nu}|^2; \quad \mu,\nu = 1,2 \ . \tag{2.3}$$

Let us also consider a constant unit vector $\vec{n} = (n^1, n^2)$ and define:

$$\vec{a}^{||} = \vec{n}(\vec{n}.\vec{a}); \quad \vec{a}^{\perp} = (\hat{1} - \vec{n}\vec{n}).\vec{a} \ . \tag{2.4}$$

Now we can use the $SU(N)$ symmetry of the matrix model to fix the gauge:

$$\vec{n}.\vec{a}_{ij} = 0 \ . \tag{2.5}$$

Note that in this gauge the action is quadratic in the off-diagonal modes and hence the first loop effective action is exact! Indeed, without loss of generality we can consider $n^1 \neq 0$ and express:

$$a^1 = -\frac{n^2}{n^1} a^2 \ . \tag{2.6}$$

It is then easy to check that:

$$a^\mu a^\nu \epsilon_{\mu\nu} = [a^1, a^2] = -\frac{n^2}{n^1}[a^2, a^2] = 0 \ . \tag{2.7}$$

This simplification is not a surprise since the gauge condition (2.5) with a constant $\vec{n}$ is equivalent to diagonalizing one of the matrices. The simplified action can be written as:

$$S[X^\mu] = \sum_{i=1}^{N} \left[(\vec{n}.\vec{x}_i)^2 + \vec{x}_i^{\perp 2} \right] + \sum_{i,j=1}^{N} \left[1 + g^2 (\vec{n}.(\vec{x}_i - \vec{x}_j))^2 \right] |\vec{a}_{ij}^{\perp}|^2 + S_{\rm FP} \ . \tag{2.8}$$

Here S_{FP} is the contribution from the Faddeev-Popov determinant resulting from the gauge choice (2.5). We have also split the diagonal modes to longitudinal $\vec{n}.\vec{x}$ and perpendicular $\vec{x}_i^{\perp}$ components. It is easy to verify that:

$$S_{FP} = -\frac{1}{2} \sum_{i,j=1}^{N} \log(\vec{n}.(\vec{x}_i - \vec{x}_j))^2 \ . \tag{2.9}$$

Note that if we choose a basis in which $\vec{n} = (1,0)$ or $\vec{n} = (0,1)$ the Faddeev-Popov determinant is just the standard Vandermonde determinant. It is now straightforward to integrate out the perpendicular elements of the matrices. The resulting effective action (we divide by N^2) for the longitudinal diagonal modes is:

$$S_{\rm eff}[\vec{x}] = \frac{1}{N}\sum_{i=1}^{N}(\vec{n}.\vec{x}_i)^2 - \frac{1}{2N^2}\sum_{i,j=1}^{N}\log\left[\frac{(\vec{n}.(\vec{x}_i-\vec{x}_j))^2}{1+g^2(\vec{n}.(\vec{x}_i-\vec{x}_j))^2}\right] \ . \tag{2.10}$$

A few comments are in order: Note that the first loop effective action (2.10) is exact in this gauge. It is also valid for all g^2. However, we know that the matrices commute only in the large g^2 limit. Therefore what we derived is not an exact effective action for the eigenvalues of the matrices but for their diagonal elements. We can define a joint distribution for the diagonal elements, which in the $g^2\to\infty$ limit coincide with the joint eigenvalue distribution. In this spirit we consider a coarse grained description and extremize the following functional:

$$S_{\rm eff}[\rho(\vec{x})] = \int d^2x\rho(\vec{x})(\vec{n}.\vec{x})^2 - \frac{1}{2}\int\int d^2xd^2x'\rho(\vec{x})\rho(\vec{x}')\log\left[\frac{(\vec{n}.(\vec{x}-\vec{x}'))^2}{1+g^2(\vec{n}.(\vec{x}-\vec{x}'))^2}\right] + \tag{2.11}$$
$$+\ \mu\left(\int d^2x\rho(\vec{x})-1\right)$$

Upon variation with respect to ρ we obtain[1] the equation:

$$\mu + (\vec{n}.\vec{x})^2 = \int d^2x'\rho(\vec{x}')\log\left[\frac{(\vec{n}.(\vec{x}-\vec{x}'))^2}{1+g^2(\vec{n}.(\vec{x}-\vec{x}'))^2}\right] \ . \tag{2.12}$$

Now we can apply the following differential operator $\vec{n}.\vec{\nabla}_{\vec{x}}$ to both hand sides of equation (2.12). The result is:

$$\vec{n}.\vec{x} = \int \frac{d^2x'\rho(\vec{x}')}{(\vec{n}.(\vec{x}-\vec{x}'))(1+g^2(\vec{n}.(\vec{x}-\vec{x}'))^2)} \tag{2.13}$$

This is the equation that we can use to deduce the form of the joint eigenvalue distribution $\rho(\vec{x})$. Note that this is almost equation (5) from ref. [13]. To make the analogy complete we define $u=\vec{n}.\vec{x}$ and choose a coordinate frame in the integral along $\vec{x}'$ in which we have $\vec{n}.\vec{x}'=x'^1$.

$$u = \int\frac{dx'^1\rho_1(x'^1)}{(u-x'^1)(1+g^2(u-x'^1)^2)} \tag{2.14}$$

Note that we have defined:

$$\rho_1(x^1) = \int dx^2\rho(x^1,x^2) \tag{2.15}$$

We next observe that for large g:

$$\frac{1}{2}\log\left[\frac{g^2x^2}{1+g^2x^2}\right] = -\frac{\pi}{g}\,\delta(x)+O(1/g^2) \quad \text{and} \quad \frac{1}{x(1+g^2x^2)} = -\frac{\pi}{g}\delta'(x)+O(1/g^2)\ , \tag{2.16}$$

[1] If we average over n here we obtain

$$\mu+\frac{\vec{x}^2}{2} = \int d^3x'\rho(\vec{x}')4\pi\ln\left(\frac{|\vec{x}-\vec{x}'|}{1+\sqrt{1+g^2|\vec{x}-\vec{x}'|^2}}\right).$$

Substituting in equation (2.14) we obtain:

$$u = -\frac{\pi}{g}\int dx'^1 \rho_1(x'^1)\delta'(u - x'^1) + O(1/g^2) = -\frac{\pi}{g}\rho_1'(u) + O(1/g^2)\ . \tag{2.17}$$

Next we solve equation (2.17) to leading order in g^{-1} and normalize $\int du \rho_1(u) = 1$. The resulting distribution is given by [13]:

$$\rho_1(x^1) = \frac{3}{4R^3}(R^2 - {x^1}^2)\ , \tag{2.18}$$

with

$$R = \left(\frac{3\pi}{2g}\right)^{1/3}\ . \tag{2.19}$$

Now the definition of ρ_1 from equation (2.15) and the $SO(2)$ symmetry of the distribution suggests [13] the unique solution for $\rho(x^1, x^2)$:

$$\rho(x^1, x^2) = \frac{3\sqrt{R^2 - \vec{x}^2}}{2\pi R^3}\ . \tag{2.20}$$

Equation (2.20) is the desired hemisphere distribution reported in ref. [13]. Note that the derivation of the two-dimensional distribution (2.20) is somewhat indirect. Indeed what we derived from the effective action (2.10) was the one-dimensional parabolic distribution (2.18). In the next subsection we will consider a three matrix model equivalent to the two matrix model (2.1) and prove directly that at strong coupling the corresponding three dimensional eigenvalue distribution is an uniform distribution inside a solid ball. The two dimensional hemisphere eigenvalue distribution can be obtained by integrating out one of the eigenvalues.

2.2 Three matrix model realization and uniform distribution

Let us now consider the model (also originally introduced parenthetically by Hoppe [5] page 73 and further discussed in [7] and [13]:

$$\mathcal{Z} = \int \mathcal{D}X\mathcal{D}Y\mathcal{D}Z e^{-N\mathrm{tr}(\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 - \mathrm{i}\alpha[\mathrm{X},\mathrm{Y}]\mathrm{Z})}\ . \tag{2.21}$$

It is easy to verify that if one integrates out the Z matrix and defines $g^2 = (i\alpha)^2/4$ one recovers the two matrix model (2.1). This suggests that the model (2.21) should be as solvable as the two matrix model. Note also that there is a global $SO(3)$ symmetry rotating the X, Y and Z matrices. We find it instructive to analyze the model in the spirit described in section 2.1. To begin with let us define:

$$X_{ij} = x_i^1\delta_{ij} + a_{ij}^1;\ \ Y_{ij} = x_i^2\delta_{ij} + a_{ij}^2;\ \ Z_{ij} = x_i^3\delta_{ij} + a_{ij}^3;\ \ \vec{x}_i = (x_i^1, x_i^2, x^3);\ \ \vec{a}_{ij} = (a_{ij}^1, a_{ij}^2, a_{ij}^3)\ . \tag{2.22}$$

Next we consider a constant unit vector $\vec{n} = (n^1, n^2, n^3)$, define:

$$\vec{x}^{\perp} = (\hat{1} - \vec{n}\vec{n}).\vec{x};\quad \vec{a}^{||} = \vec{n}(\vec{n}.\vec{a});\quad \vec{a}^{\perp} = (\hat{1} - \vec{n}\vec{n}).\vec{a}\ . \tag{2.23}$$

and impose the axial gauge $\vec{n}.\vec{a}=0$. The action in (2.21) can then be written as:

$$S[\vec{x},\vec{a}]=\sum_{i=1}^{N}\left[(\vec{n}.\vec{x}_i)^2+\vec{x}_i^{\perp 2}\right]+\sum_{i,j=1}^{N}a_{ij}^{\mu\perp}\left[\delta^{\mu\nu}-i\frac{\alpha}{2}\epsilon_{\mu\nu\rho}n^{\rho}(\vec{n}.\vec{\Delta}_{ij})\right]a_{ji}^{\nu\perp}\ , \tag{2.24}$$

where we have defined $\vec{\Delta}_{ij}=\vec{x}_i-\vec{x}_j$. Note that there is no term cubic in $\vec{a}$ in the action (2.24), because of the axial gauge. This suggests that the first loop effective action is exact. Now we proceed as in section 2.1 and integrate the perpendicular matrix elements. One can show that the resulting effective action is given by:

$$S_{\text{eff}}[(\vec{n}.\vec{x})]=\frac{1}{N}\sum_{i=1}^{N}(\vec{n}.\vec{x}_i)^2-\frac{1}{2N^2}\sum_{i,j=1}^{N}\log\left[\frac{g^2(\vec{n}.(\vec{x}_i-\vec{x}_j))^2}{1+g^2(\vec{n}.(\vec{x}_i-\vec{x}_j))^2}\right]+\frac{(N-1)}{2N}\log g^2\ , \tag{2.25}$$

where $g^2=(i\alpha)^2/4$. Next we consider a coarse grained approximation an vary the corresponding distribution function ρ to obtain the equation:

$$\mu+(\vec{n}.\vec{x})^2=\int d^3x'\rho(\vec{x}')\log\left[\frac{g^2(\vec{n}.(\vec{x}-\vec{x}'))^2}{1+g^2(\vec{n}.(\vec{x}-\vec{x}'))^2}\right]\ . \tag{2.26}$$

Note that equation (2.26) is valid for any choice of $\vec{n}$. Next we average over all possible directions that $\vec{n}$ can take with a uniform weight. Or equivalently average over the unit two-sphere. It is easy to show that:

$$\frac{1}{4\pi}\int d\Omega_2(\vec{n}.\vec{x})^2=\frac{1}{3}\vec{x}^2\ . \tag{2.27}$$

The right-hand side of equation (2.26) requires a bit more careful analysis. One can show that:

$$\begin{aligned}J(g|\vec{x}-\vec{x}'|)&:=\frac{1}{4\pi}\int d\Omega_2\log\left[\frac{g^2(\vec{n}.(\vec{x}-\vec{x}'))^2}{1+g^2(\vec{n}.(\vec{x}-\vec{x}'))^2}\right]=\frac{1}{2g|\vec{x}-\vec{x}'|}\int\limits_{-g|\vec{x}-\vec{x}'|}^{g|\vec{x}-\vec{x}'|}d\eta\log\left(\frac{\eta^2}{1+\eta^2}\right)\\&=\frac{-2\arctan(g|\vec{x}-\vec{x}'|)}{g|\vec{x}-\vec{x}'|}+\log\left[\frac{g^2(\vec{x}-\vec{x}')^2}{1+g^2(\vec{x}-\vec{x}')^2}\right]=-\frac{\pi}{g}\frac{1}{|\vec{x}-\vec{x}'|}+\left(\frac{1}{g^2}\right)\ .\end{aligned} \tag{2.28}$$

To leading order in $1/g$ equation (2.26) becomes:

$$\mu+\frac{1}{3}\vec{x}^2=-\frac{\pi}{g}\int d^3x'\rho(\vec{x}')\frac{1}{|\vec{x}-\vec{x}'|}\ . \tag{2.29}$$

Next we apply the operator of Laplacian Δ_x on both hand-sides of equation (2.29) to obtain:

$$2=\frac{4\pi^2}{g}\int d^3x'\rho(\vec{x}')\delta(\vec{x}-\vec{x}')=\frac{4\pi^2}{g}\rho(\vec{x})\ . \tag{2.30}$$

Therefore we conclude that:

$$\rho(\vec{x})=\frac{g}{2\pi^2}=\text{const} \tag{2.31}$$

and hence the ground state at strong coupling corresponds to an uniform eigenvalue distribution inside a ball. In order to estimate the radius of the eigenvalue distribution R we use the normalization of $\rho(\vec{x})$:

$$\int d^3x\rho(\vec{x}) = \frac{4\pi}{3}R^3\rho = 1 \ . \tag{2.32}$$

From equations (2.31) and (2.32) we obtain:

$$R = \left(\frac{3\pi}{2g}\right)^{1/3} , \tag{2.33}$$

which is exactly the radius obtained in ref. [13] reported in equation (2.19). It is also straightforward to obtain the hemisphere distribution (2.20). Indeed:

$$\rho_2(x^1, x^2) = \int\limits_{-\sqrt{R^2-{x^1}^2-{x^2}^2}}^{\sqrt{R^2-{x^1}^2-{x^2}^2}} dx^3\rho(x^1, x^2, x^3) = \frac{3\sqrt{R^2 - {x^1}^2 - {x^2}^2}}{2\pi R^3} \ . \tag{2.34}$$

Some additional comments are worthwhile here. First one can push the analysis further by observing that if we don't assume large coupling in stead of (2.29) we obtain

$$\mu + \frac{1}{3}\vec{x}^2 = \int d^3x'\rho(\vec{x}')J(g|\vec{x} - \vec{x}'|). \tag{2.35}$$

which upon acting with the Laplacian and noting

$$\nabla^2 J(x) = \frac{2}{(1 + g^2|\vec{x} - \vec{x}'|^2)|\vec{x} - \vec{x}'|^2} \tag{2.36}$$

we obtain the integral equation

$$1 = \int d^3x' \frac{\rho(\vec{x}')}{(1 + g^2|\vec{x} - \vec{x}'|^2)|\vec{x} - \vec{x}'|^2} \tag{2.37}$$

whose large g behaviour is

$$\frac{1}{(1 + g^2|\vec{x} - \vec{x}'|^2)|\vec{x} - \vec{x}'|^2} = \frac{\pi}{2g}\frac{\delta(|\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|^2} + O(1/g^2) = \frac{2\pi^2}{g}\delta(\vec{x} - \vec{x}') + O(1/g^2) \tag{2.38}$$

So to leading order we obtain $\rho(x) = \frac{g}{2\pi^2}\theta(R - r)$.

Given that the eigenvalue distribution is concentrated in the interior of a ball of radius R we can further deduce that this radius is determined by

$$g^2 = \frac{2}{3\pi}(gR)^3 + \cdots \quad \text{and} \quad \nu = \frac{g^2}{3}\int \rho(\vec{x})\vec{x}^2 = \frac{(gR)^2}{5} + \ldots \tag{2.39}$$

we find that to leading order in large g the observable

$$\nu = \frac{(12\pi)^{2/3}}{20}g^{4/3} + \cdots \tag{2.40}$$

in agreement with [7].

Further: assuming a constant eigenvalue distribution within a sphere of radius R i.e. $\rho(r) = \frac{3\theta(R-r)}{4\pi R^3}$ and using

$$\int d^3x d^3x' \frac{\rho(x)\rho(x')}{|\vec{x}-\vec{x}'|} = \frac{6}{5R} \tag{2.41}$$

we obtain

$$V_{eff}(R) = \frac{R^2}{5} + \frac{3\pi}{5gR} \tag{2.42}$$

Varying with respect to R we find that $R^3 = \frac{3\pi}{2g}$ in agreement with the exact expression (2.19). In the next section we attempt to use such an effective potential (derived perturbatively) to estimate the extent of the eigenvalue distribution in a three matrix model with pure commutator action.

3 The $p \geq 3$ matrix model

In this section we consider the p-matrix model:

$$S[X] = NTr\left(-\frac{1}{4}[X^\mu, X^\nu]^2\right); \quad \mu,\nu = 1\dots p; \tag{3.1}$$

where X_a are hermitian $N \times N$ matrices. The partition function is given by:

$$\mathcal{Z} = \int dX_a e^{-S[X]} = \int \mathcal{D}X e^{N\frac{1}{4}Tr[X^\mu,X^\nu]^2}. \tag{3.2}$$

Note that in addition to the $SU(N)$ gauge invariance the model (3.2) has a global $SO(p)$ symmetry transforming the matrices X^μ. We are interested in the eigenvalue distribution of one of the matrices and in particular in the extent R of the eigenvalue distribution in the large N limit. We find it convenient to split the degrees of freedom to diagonal and off-diagonal contributions:

$$X^\mu_{ij} = x^\mu_i \delta_{ij} + a^\mu_{ij}. \tag{3.3}$$

In terms of the new variables the action in (3.2) can be written as:

$$S[X] = N\frac{1}{2}\sum_{i\neq j}|\vec{\Delta}_{ij}|^2 a^\mu_{ij}\Pi^{\mu\nu}_{ij}a^\nu_{ji} - N\sum_{i\neq j}\Delta^\mu_{ij}a^\nu_{ij}[a^\mu,a^\nu]_{ji} - N\frac{1}{4}tr[a^\mu,a^\nu]^2 \,, \tag{3.4}$$

where we have defined:

$$\Delta^\mu_{ij} = x^\mu_i - x^\mu_j; \quad n^\mu_{ij} = \Delta^\mu_{ij}/|\vec{\Delta}_{ij}|; \quad \Pi^{\mu\nu}_{ij} = \delta^{\mu\nu} - n^\mu_{ij}n^\nu_{ij}; \ . \tag{3.5}$$

A standard way to proceed would be to integrate out the off-diagonal degrees of freedom a^μ_{ij} and obtain an effective action for the diagonal components x^μ_i. Note that the quadratic term in a^μ in equation (3.4) is proportional to a projector and hence cannot be directly inverted. A gauge fixing is required. We find it natural to work in a gauge in which the longitudinal modes are removed, more precisely we impose the gauge fixing condition $\vec{n}_{ij}.\vec{a}_{ij} = 0$. In the next subsection we briefly present the change of variables necessary to implement our gauge fixing condition. We refer the reader to Appendix A for a more detailed calculation of the corresponding Jacobian.

3.1 Gauge fixing

Our goal is to perform a change of coordinates which is manifestly $SO(p)$ invariant and convenient in calculating quantum corrections to the effective potential governing the ground state of the theory.

Let us consider a set of $p-1$ hermitian matrices $a_{ij}^{m\perp}$ ($m = 1 \dots p-1$; $i, j = 1 \dots N$) with vanishing diagonal components ($a_{ii}^{m\perp} = 0$). A slightly more general way to parametrize this set of matrices is to consider a set of p linearly dependent matrices $a_{ij}^{\mu\perp}$ ($\mu = 1 \dots p$) satisfying:

$$\sum_{\mu=1}^{p} n^{\mu}.a^{\mu\perp} \equiv \vec{n}.\vec{a}^{\perp} = 0 \ , \tag{3.6}$$

where $\vec{n}$ is a constant $p-$dimensional unit vector and we have suppressed the indices i, j. Next let us consider any set of $N^2 - N$ orthogonal matrices $R_{ij} \in SO(p)$ satisfying:

$$\begin{aligned} R_{ij}.\vec{n} &= \vec{n}_{ij} \quad \text{for} \quad i > j; \\ R_{ij} &= R_{ji} \quad \text{for} \quad i < j; \ . \end{aligned} \tag{3.7}$$

Clearly such a set of matrices always exists. Now we define the hermitian matrices:

$$\vec{a}_{ij} \equiv R_{ij}.\vec{a}_{ij}^{\perp} \quad \text{for} \quad i \neq j \quad \text{and} \quad \vec{a}_{ii} \equiv 0; \ . \tag{3.8}$$

The matrices $\vec{a}_{ij}$ are linearly dependent and satisfy the properties:

$$\vec{n}_{ij}.\vec{a}_{ij} = 0 \tag{3.9}$$

$$\vec{a}_{ij}.\vec{a}_{ji} = \vec{a}_{ij}^{\perp}.\vec{a}_{ji}^{\perp} \tag{3.10}$$

Next we define the change of coordinates:

$$X^{\mu} = U(x^{\mu} + a^{\mu})U^{-1}; \quad U \in SU(N); \ . \tag{3.11}$$

Note that on the left hand-side of equation (3.11) we have a set of p $N \times N$ hermitian matrices spanning a pN^2 dimensional linear space. On the other side by construction there are only $p-1$ linearly independent matrices labeled by a^{μ} and hence the dimension of the linear space spanned by $x^{\mu} + a^{\mu}$ is equal to $(p-1)(N^2 - N) + pN$. This suggests that in order to have a well defined change of coordinates in equation (3.11) we need $N^2 - N$ degrees of freedom, parameterizing the orbit of $x^{\mu} + a^{\mu}$ under the $SU(N)$ group. However a general element of $SU(N)$ has $N^2 - 1$ degrees of freedom. The $N - 1$ degrees of freedom that are left out correspond to the stabilizer of the action of $SU(N)$ and as one can check are generated by a Cartan subalgebra of $su(N)$.

It is a straightforward exercise to compute the corresponding Jacobian. The expression that one obtains is given by (we refer the reader to Appendix A for a detailed computation):

$$J = \left(\prod_{i \neq j} |\vec{\Delta}_{ij}| \right) \det \left|\left| \delta_i^l \delta_j^m + Y_{ij}^{lm} \right|\right| \det \left|\left| \frac{\delta \theta_{rs}}{\delta u_{lm}} \right|\right| \ , \tag{3.12}$$

where Y_{ij}^{lm} is given by:

$$Y_{ij}^{lm} = \frac{\vec{n}_{ij}}{|\vec{\Delta}_{ij}|}.(\vec{a}_{il}\delta_j^m - \vec{a}_{mj}\delta_i^l) + \frac{\vec{a}_{ij}.\Pi_{ij}.\vec{a}_{ml}}{|\vec{\Delta}_{ij}|^2}.(\delta_i^m - \delta_i^l - \delta_j^m + \delta_j^l) \ . \tag{3.13}$$

Note that the last determinant in equation (3.12) is the Haar measure of $SU(N)$. Now we can write down the measure in the path integral (3.2) in terms of the new variables. To second order in the off-diagonal elements $\vec{a}_{ij}$ we have the expression:

$$\mu = DU \prod_i dx_i \left(\prod_{i\neq j} |\vec{\Delta}_{ij}|\right) \prod_{i>j} \left(d^{p-1}a_{ij}^{\perp} d^{p-1}a_{ji}^{\perp}\right) \left\{1 - 2\sum_{i,j} \vec{a}_{ij}.\hat{D}_{ij}.\vec{a}_{ji} + O(a^3)\right\} \ , \tag{3.14}$$

where DU is the Haar measure of $SU(N)$ and $\hat{D}_{ij}$ is given by:

$$D_{ij}^{\mu\nu} = \frac{\Pi_{ij}^{\mu\nu}}{|\vec{\Delta}_{ij}|^2} + \frac{1}{4}\sum_l \frac{n_{il}^{\mu} n_{jl}^{\nu} + n_{il}^{\nu} n_{jl}^{\mu}}{|\vec{\Delta}_{il}||\vec{\Delta}_{jl}|} \ . \tag{3.15}$$

Note that without loss of generality one can take $\vec{n} = \vec{e}_p$. Note also that $\vec{a}_{ij} = R_{ij}.\vec{a}_{ij}^{\perp}$. Our next goal is to develop a systematic perturbative procedure to integrate out the off-diagonal degrees of freedom $\vec{a}_{ij}^{\perp}$.

3.2 The effective action to two loops and stabilization

In this subsection we will develop a perturbative technique to integrate out the off diagonal modes $\vec{a}_{ij}^{\perp}$ and calculate the semi-classical correction to the effective action for the diagonal modes $\vec{x}_i$.

Our experience with the two matrix model from section 2 and in particular its three matrix model realization suggests that the ground state of the theory corresponds to a uniform joint eigenvalue distribution inside a ball of radius R. Furthermore numerical studies of the three matrix model [12] give radius $R \approx 2.0 > 1$. This suggests a perturbative expansion in powers of $1/R$ may prove useful. In order to proceed systematically we first rescale our variables $\vec{x}_i$ and $\vec{a}_{ij}$ in the following way:

$$\vec{x} = R\tilde{\vec{x}}_i; \quad \vec{a} = R\tilde{\vec{a}}_{ij}; \ . \tag{3.16}$$

Next we write the action (3.4) as:

$$S[\tilde{\vec{x}}_i, \tilde{\vec{a}}_{ij}^{\perp}] = NR^4 \sum_{i\neq j} \left(-\frac{1}{2}|\tilde{\vec{\Delta}}_{ij}|^2 \tilde{\vec{a}}_{ij}^{\perp}.\tilde{\vec{a}}_{ji}^{\perp} + \tilde{\Delta}_{ij}^{\mu}\tilde{a}_{ji}^{\nu}[\tilde{a}^{\mu}, \tilde{a}^{\nu}]_{ji} + \frac{1}{4}[\tilde{a}^{\mu}, \tilde{a}^{\nu}]_{ij}[\tilde{a}^{\mu}, \tilde{a}^{\nu}]_{ji}\right) \tag{3.17}$$

Note that from field theory point of view the parameter $1/NR^4$ can be interpreted as a loop counting parameter. Next we define the correlation function:

$$\langle \mathcal{O} \rangle_0 = \frac{\int \prod_i d\tilde{x}_i \left(\prod_{i\neq j} |\tilde{\vec{\Delta}}_{ij}|\right) \prod_{i>j} \left(d^{p-1}\tilde{a}_{ij}^{\perp} d^{p-1}\tilde{a}_{ji}^{\perp}\right) \mathcal{O} e^{-NR^4 \sum_{i\neq j} \frac{1}{2}|\tilde{\vec{\Delta}}_{ij}|^2 \tilde{\vec{a}}_{ij}^{\perp}.\tilde{\vec{a}}_{ji}^{\perp}}}{\int \prod_i d\tilde{x}_i \left(\prod_{i\neq j} |\tilde{\vec{\Delta}}_{ij}|\right) \prod_{i>j} \left(d^{p-1}\tilde{a}_{ij}^{\perp} d^{p-1}\tilde{a}_{ji}^{\perp}\right) e^{-NR^4 \sum_{i\neq j} \frac{1}{2}|\tilde{\vec{\Delta}}_{ij}|^2 \tilde{\vec{a}}_{ij}^{\perp}.\tilde{\vec{a}}_{ji}^{\perp}}} \ . \tag{3.18}$$

For the propagator of $\tilde{\vec{a}}_{ij}^{\perp}$ we obtain:

$$\langle \tilde{a}_{ij}^{\mu\perp} \tilde{a}_{lm}^{\nu\perp} \rangle_0 = \frac{1}{NR^4} \frac{(\delta^{\mu\nu} - n^\mu n^\nu)}{|\tilde{\vec{\Delta}}_{ij}|^2} \delta_i^m \delta_j^l \ . \tag{3.19}$$

In deriving (3.19) one could use a frame in which $\vec{n} = \vec{e}_p$. Note that the cubic and quartic contributions to the action (3.17) as well as the Jacobian in the measure (5.13) depend on $\tilde{\vec{a}}_{ij}^{\perp}$ through the relation $\tilde{\vec{a}}_{ij} = R_{ij}.\tilde{\vec{a}}_{ij}^{\perp}$. This is why it is convenient to calculate the two-point function:

$$\langle \tilde{a}_{ij}^{\mu} \tilde{a}_{lm}^{\nu} \rangle_0 = \frac{1}{NR^4} \frac{1}{|\tilde{\vec{\Delta}}_{ij}|^2} \Pi_{ij}^{\mu\nu} \delta_i^m \delta_j^l \ , \tag{3.20}$$

where we have used that:

$$R_{ij}^{\mu\mu'}.(\delta^{\mu'\nu'} - n^{\mu'} n^{\nu'}).R_{ij}^{\nu\nu'} = \delta^{\mu\nu} - n_{ij}^{\mu} n_{ij}^{\nu} = \Pi_{ij}^{\mu\nu} \ . \tag{3.21}$$

We now have all the machinery required for a perturbative calculation in powers of $1/NR^4$. To leading order we obtain the following first loop effective action for the diagonal modes:

$$S_{\text{eff}}^{(1)}(R, \tilde{\vec{x}}) = [(p-2)N^2 - 2(p-1)N + p] \log R + \frac{(p-2)}{2} \sum_{i \neq j} \log |\tilde{\vec{\Delta}}_{ij}|^2 \ . \tag{3.22}$$

As one can see from the first term in equation (3.22) at large N and for $p \geq 2$ at one loop the effective action gives an attractive potential $V_{\text{eff}}(R)$ and is therefore not sufficient to stabilize the radius of the distribution. On the other hand the second loop corrections has an overall factor of $1/NR^4$ and could balance the $\log R$ attractive potential in (3.22).

At second loop the effective action has contributions from the cubic and quartic vertices in (3.17) as well as from the quadratic term in the measure (5.13) ("ghost's contribution"). The corresponding correlation functions are: $\langle (NR^4 \tilde{\Delta}_{ij}^{\mu} \tilde{a}_{ji}^{\nu} [\tilde{a}^\mu, \tilde{a}^\nu]_{ji})^2 \rangle_0$, $\langle NR^4 \frac{1}{4} tr[\tilde{a}^\mu, \tilde{a}^\nu]^2 \rangle_0$ and $\langle -2 \sum_{i,j} \tilde{\vec{a}}_{ij}.D_{ij}.\tilde{\vec{a}}_{ji} \rangle_0$. Using Wick contractions and the two-point function (3.20) we can calculate the second loop contribution. After somewhat tedious but straightforward calculations we obtain:

$$\langle (NR^4 \tilde{\Delta}_{ij}^{\mu} \tilde{a}_{ji}^{\nu} [\tilde{a}^\mu, \tilde{a}^\nu]_{ji})^2 \rangle_0 = \frac{1}{2NR^4} \sum_{i,j \neq l} \frac{\{(4p-6) \sin^2 \theta_{i,jl} + \sin^2 \theta_{l,ij}\}}{\tilde{\vec{\Delta}}_{il}^2 \tilde{\vec{\Delta}}_{jl}^2} \ , \tag{3.23}$$

$$\langle NR^4 \frac{1}{4} tr[\tilde{a}^\mu, \tilde{a}^\nu]^2 \rangle_0 = \frac{-1}{2NR^4} \sum_{i,j \neq l} \frac{\{(p-1)(p-2) + \sin^2 \theta_{l,ij}\}}{\tilde{\vec{\Delta}}_{il}^2 \tilde{\vec{\Delta}}_{jl}^2} \ , \tag{3.24}$$

$$\langle -2 \sum_{i,j} \tilde{\vec{a}}_{ij}.D_{ij}.\tilde{\vec{a}}_{ji} \rangle_0 = \frac{-1}{2NR^4} \sum_{i,j \neq l} \frac{2 \sin^2 \theta_{i,jl}}{\tilde{\vec{\Delta}}_{il}^2 \tilde{\vec{\Delta}}_{jl}^2} + O\left(\frac{1}{N}\right) \ . \tag{3.25}$$

Were the angles $\theta_{l,ij}, \theta_{i,jl}$ are defined via $\cos \theta_{i,jl} = \vec{n}_{ij}.\vec{n}_{il}$ and the last term in equation (3.25) corresponds to non-planar diagrams contribution subleading in the large N limit. For the total second loop contribution to the effective potential we obtain:

$$S_{\text{eff}}^{(2)}(R, \tilde{\vec{x}}) = \frac{(p-2)}{2NR^4} \sum_{i,j \neq l} \frac{p - 1 - 4 \sin^2 \theta_{i,jl}}{\tilde{\vec{\Delta}}_{il}^2 \tilde{\vec{\Delta}}_{jl}^2} \ . \tag{3.26}$$

The full large N effective action for to one loop is then

$$S_{\text{eff}}(R, \tilde{\vec{x}}) = \frac{(p-2)}{2} \left(\sum_{i \neq j} \ln(R^2 \Delta_{i,j}^2) + \frac{1}{NR^4} \sum_{i,j \neq l} \frac{p - 1 - 4\sin^2\theta_{i,j,l}}{\tilde{\vec{\Delta}}_{il}^2 \tilde{\vec{\Delta}}_{jl}^2} \right) . \tag{3.27}$$

Since the ground state of the theory is in a commuting phase we can trade the discrete sums in equations (3.22),(3.26) for integrals over a joint eigenvalue distribution $\rho(\tilde{\vec{x}})$ via:

$$\frac{1}{N} \sum_i \to \int_{B^p} d^p \tilde{x} \rho(\tilde{\vec{x}}); \ , \tag{3.28}$$

where the integral is over a ball B^p of unit radius. In the large N limit we obtain the following second loop effective potential for the radius of the joint eigenvalue distribution R:

$$V_{\text{eff}}(R) = (p-2)N^2 \left(\log R + \frac{\#(p)}{R^4} \right) \ , \tag{3.29}$$

where we have defined:

$$\#(p) = \frac{1}{2} \int_{B^p} \int_{B^p} \int_{B^p} d^p \tilde{x} d^p \tilde{y} d^p \tilde{z} \rho(\tilde{\vec{x}}) \rho(\tilde{\vec{y}}) \rho(\tilde{\vec{z}}) \frac{p - 1 - 4\sin^2\theta_{x,yz}}{(\tilde{\vec{x}} - \tilde{\vec{z}})^2 (\tilde{\vec{y}} - \tilde{\vec{z}})^2} \tag{3.30}$$

and $\theta_{x,yz}$ is the analog of $\theta_{i,jl}$ defined via:

$$\cos\theta_{x,yz} = \frac{(\tilde{\vec{x}} - \tilde{\vec{y}}).(\tilde{\vec{x}} - \tilde{\vec{z}})}{|(\tilde{\vec{x}} - \tilde{\vec{y}})||(\tilde{\vec{x}} - \tilde{\vec{z}})|} \ . \tag{3.31}$$

We can now estimate the radius of the joint eigenvalue distribution corresponding to the ground state of the theory. Indeed minimizing (3.29) we obtain:

$$R_p = (4\#(p))^{1/4}; \ . \tag{3.32}$$

Note that for $p \geq 5$ we have $\#(p) > 0$ (the integrant in equation (3.30) is non-negative) and the ground state stabilizes at the finite radius estimated in equation (3.32). For $p = 3$ we evaluated analytically the integral in equation (3.30), assuming uniform joint eigenvalue distribution $\rho(\tilde{\vec{x}}) = \text{const}$ (look at Appendix B for more details). The resulting radius is:

$$R_3 = \left(9 - \frac{3}{5}\pi^2 \right)^{1/4} \approx 1.323; \ . \tag{3.33}$$

Equations (3.29)-(3.32) contain the main result of our perturbative calculation. In the next section we discuss the gauge dependence of our expression for the radius of the distribution R_p in equation (3.32).

3.3 Gauge dependence

In subsection 3.1 we outlined a change of coordinates that was equivalent to introducing the gauge $\vec{n}_{ij}.\vec{a}_{ij} = 0$. Alternatively we could have used a standard Faddeev-Popov techniques to fix our gauge. Let us consider the gauge condition:

$$f_{ij} = \vec{\Delta}_{ij}.\vec{a}_{ij} = 0; \ . \tag{3.34}$$

The corresponding Faddeev-Popov determinant is given by:

$$\Delta_{FP} = \prod_{i>j} |\vec{\Delta}_{ij}|^2 \mathrm{det}||\delta_i^l \delta_j^m + Y'^{lm}_{ij}|| \ , \tag{3.35}$$

where:

$$Y'^{lm}_{ij} = \frac{\vec{n}_{ij}}{|\vec{\Delta}_{ij}|}.(\vec{a}_{il}\delta_j^m - \vec{a}_{mj}\delta_i^l) + \frac{\vec{a}_{ij}.\vec{a}_{ml}}{|\vec{\Delta}_{ij}|^2}.(\delta_i^m - \delta_i^l - \delta_j^m + \delta_j^l) \ . \tag{3.36}$$

Notice that the gauge condition $f^{\eta}_{ij} = \vec{\Delta}_{ij}.\vec{a}_{ij} - \eta_{ij} = 0$ would result to the same Faddeev-Popov determinant (3.36). Now integrating over the family of gauge functions f^{η}_{ij} with weight $\exp(-|\eta_{ij}|^2/2\xi)$ would modify the action (3.4) to:

$$S[X,\xi] = N\frac{1}{2}\sum_{i\neq j} |\vec{\Delta}_{ij}|^2 a^{\mu}_{ij}(\Pi^{\mu\nu}_{ij} + \frac{1}{\xi}n^{\mu}_{ij}n^{\nu}_{ij})a^{\nu}_{ji} - N\sum_{i\neq j}\Delta^{\mu}_{ij}a^{\nu}_{ij}[a^{\mu},a^{\nu}]_{ji} - N\frac{1}{4}tr[a^{\mu},a^{\nu}]^2 \ , \tag{3.37}$$

Next we can go through the steps considered in section 3.2, namely rescale with the radius of the joint eigenvalue distribution R as in equation (3.16) and set up perturbative calculation in powers of $1/R$. One can show that the first loop effective action $V^{(1)}_{\text{eff}}$ is still given by equation (3.22) and is thus gauge independent. However the two-point function (3.20) (the propagator for $\tilde{a}_{ij}$) is modified to:

$$\langle \tilde{a}^{\mu}_{ij}\tilde{a}^{\nu}_{lm}\rangle_{\xi} = \frac{1}{NR^4}\frac{1}{|\tilde{\vec{\Delta}}_{ij}|^2}(\Pi^{\mu\nu}_{ij} + \xi n^{\mu}_{ij}n^{\nu}_{ij})\delta^m_i\delta^l_j \ . \tag{3.38}$$

Note that the result from equation (3.20) corresponds to the choice $\xi = 0$ (Landau gauge). Let us consider the choice $\xi = 1$ (Feynman gauge) and calculate the second loop contribution to the effective action. Going through the same steps as in section 3.2 we obtain the analog of equation (3.29):

$$V_{\text{eff}}(R) = (p-2)N^2\left(\log R + \frac{\tilde{\#}(p)}{R^4}\right) \ , \tag{3.39}$$

where $\tilde{\#}(p)$ is given by:

$$\tilde{\#}(p) = \frac{p-2}{2}\int_{B^p}\int_{B^p}\int_{B^p} d^p\tilde{x}d^p\tilde{y}d^p\tilde{z}\rho(\tilde{\vec{x}})\rho(\tilde{\vec{y}})\rho(\tilde{\vec{z}})\frac{1}{(\tilde{\vec{x}}-\tilde{\vec{z}})^2(\tilde{\vec{y}}-\tilde{\vec{z}})^2} \tag{3.40}$$

This results to the radius:

$$R'_p = (4\tilde{\#}(p))^{1/4}; \ . \tag{3.41}$$

It is clear from equation (3.40) that $\tilde{\#}(p)$ is positive for $p \geq 3$. For $p = 3$ we have evaluated analytically $\tilde{\#}(3)$ (look at Appendix B). The corresponding radius is:

$$R'_3 = \left(\frac{9}{2} + \frac{3}{5}\pi^2\right)^{1/4} \approx 1.797 \ . \tag{3.42}$$

Apparently the results obtained in Landau and Feynman gauges differ. In order to address the issue of gauge dependence let us focus on a particular representative of the family of gauge conditions $f^{\eta}_{ij} = \vec{\Delta}_{ij}.\vec{a}_{ij} - \eta_{ij} = 0$. Note that the change of coordinates (3.11) considered in section 3.2 implements the $\eta_{ij} = 0$ case. One can show that the gauge condition for general η_{ij} can be implemented along the lines of section 3.2 via the following modified change of coordinates:

$$X^{\mu} = U||x^{\mu}_i \delta_{ij} + R^{\mu\nu}_{ij} a^{\nu\perp}_{ij} + \frac{n^{\mu}_{ij}}{|\vec{\Delta}_{ij}|}\eta_{ij}||U^{-1}; \quad U \in SU(N); \ . \tag{3.43}$$

Let us suppose that the theory have settled in its ground state which is a commuting phase. There should exist unitary matrix $V \in SU(N)$ which simultaneously diagonalizes the X^{μ} matrices and hence we can write:

$$V^{-1}\lambda^{\mu}V = U||x^{\mu}_i \delta_{ij} + R^{\mu\nu}_{ij} a^{\nu\perp}_{ij} + \frac{n^{\mu}_{ij}}{|\vec{\Delta}_{ij}|}\eta_{ij}||U^{-1} \ , \tag{3.44}$$

where λ^{μ} is a diagonal matrix. Now if we square equation (3.44), take a trace over the gauge indices and sum over μ we obtain:

$$\sum_i \vec{\lambda}^2_i = \sum_i \vec{x}^2_i + \sum_{ij} |\vec{a}^{\perp}_{ij}|^2 + \sum_{ij} \frac{|\eta_{ij}|^2}{|\vec{\Delta}_{ij}|^2} \geq \sum_i \vec{x}^2_i + \sum_{ij} \frac{|\eta_{ij}|^2}{|\vec{\Delta}_{ij}|^2} \ . \tag{3.45}$$

Next we define average radii of the distribution r_λ and r_x via:

$$r^2_\lambda = \frac{1}{N}\sum_i \vec{\lambda}^2_i; \quad r^2_x = \frac{1}{N}\sum_i \vec{x}^2_i; \tag{3.46}$$

and learn that:

$$r^2_\lambda - r^2_x \geq \frac{1}{N}\sum_{ij} \frac{|\eta_{ij}|^2}{|\vec{\Delta}_{ij}|^2} \ . \tag{3.47}$$

Therefore the average radius of the eigenvalue distribution r_λ always differs from the average radius of the distribution of the diagonal modes r_x, unless $\eta_{ij} = 0$ or the eigenvalues are infinitely spread in which case there is no well defined average radius. This could explain why the gauge fixing procedure outlined above, which involved averaging over all possible values of η_{ij} failed to produce a gauge independent answer for the radius of the eigenvalue distribution. These consideration suggests that the gauge $\eta_{ij} = 0$ should be optimal for describing the almost commuting theory.

Alternatively one could take the point of view that both gauge choices are equally valid and describe different approximations to the true result, they only differ due to the intrinsic errors in a perturbative calculation. If we take this point of view we can use the difference to estimate the errors in our estimate of R. If we do this we conclude that $R \sim 1.6 \pm 0.5$ which is in reasonable agreement with the numerical results.

4 Discussion

In this paper we have followed two treads, in the first we investigated the 3-matrix model of [5, 7] and find that in the large g limit the 3-matrices commute and have a joint eigenvalue distribution given by the uniform distribution within a ball of radius $R = \left(\frac{3\pi}{2g}\right)^{1/3}$. We show that a simple effective potential for the radius of the distribution reproduces the exact result.

Encouraged by the success of this effective potential calculation we develop an effective potential for the radius of the p-component Yang-Mills matrix model to two loops. We have done this by deriving an effective potential for the diagonal modes while preserving $SO(3)$ invariance and then assuming that these modes are uniformly distributed. The direct analog of the computations in earlier sections would be a two loop computation in the axial gauge (where one of the matrices is diagonalized). Unfortunately this gauge choice leads to infrared divergences at two loops and so we have not pursued this option.

We found that it is necessary to go to two loops as at one loop the effective potential is not stable since there is no classical potential and the one loop term gives an attractive potential which is a rotationally invariant version of the Vandermonde determinant. The eigenvalue repulsion arises at two loops and gives a $\frac{1}{R^4}$ hard core potential. It is easy to see that higher order terms give inverse higher powers of R and our two loop estimate can only be a very rough approximation.

Our estimate for the radius is unfortunately gauge dependent with $R = 1.323$ in the Landau gauge and $R = 1.797$ in the Feynman gauge. It is reasonable to assume that the difference between these is an indication of the errors in the method which would indicate that perhaps a reasonable estimate can be obtained by averaging the two and taking the difference as an indication of the error yielding the prediction $R = 1.6 \pm 0.5$. An alternative approach pursued by Hotta, Nishimura and Tsuchiya [15] examined similar questions in the general Yang-Mills p-matrix model[2] analysis of some observables. If we take their result

$$< \frac{\mathrm{tr}}{N}(X_a^2) >= \sqrt{\frac{p}{2}}(1 + \frac{7}{6p} + \cdots) \tag{4.1}$$

and assume that this is valid for $p = 3$ together with the assumption that the eigenvalue distribution of a single matrix is parabolic of extent R (which is consistent with a uniform joint distribution for commuting matrices) then $< \frac{\mathrm{tr}}{N}(X_a^2) >= 3\frac{R^2}{5}$ and using (4.1) we obtain the estimate $R = 1.68$, which is surprisingly close to the estimate we obtain above. In this case one can also attempt an estimate of the error by noting that if instead of (4.1) we use $< \frac{\mathrm{tr}}{N}(X_a^2) >= \sqrt{\frac{p}{2}}\frac{1}{(1-\frac{7}{6p})} + \cdots)$ which has the same leading large p expansion we obtain $R = 1.83$ and therefore estimate the error (within the assumption of a parabolic distribution) that $R = 1.75 \pm .15$ which is in surprisingly good agreement with Monte Carlo simulations [12] which give a value for $R = 2.0$

We conclude that, though the random matrices are not commuting, a useful approximation is to take the background formed by these fluctuating matrices as that commuting matrices whose joint eigenvalue distribution is approximately uniform within a ball of ra-

[2]Hotta et al [15] also developed calculated a two loop effective action for eigenvalues of the matrices X_μ by integrating out the $U(N)$ transformations that diagonalize the matrices. This yields a non-rotationally invariant effective action, which has does not lead to a stable effective potential for the radial extent of the eigenvalues.

dius R. This gives background reasonable agreement with the numerical work and serves as a reasonable starting point for further work.

5 Appendix

5.1 Appendix A

In this subsection we outline the calculation of the Jacobian (3.12). Let us begin by differentiating equation (3.11). We obtain:

$$(U^{-1}d\vec{X}U)_{ij} = d\vec{x}_i\delta_{ij} + dR_{ij}.\vec{a}_{ij}^{\perp} + R_{ij}.d\vec{a}_{ij}^{\perp} - |\vec{\Delta}_{ij}|\vec{n}_{ij}\theta_{ij} - [a,\theta]_{ij} \ , \tag{5.1}$$

where we have defined the Maurer-Cartan form $\theta \equiv U^{-1}dU$. Next we define the tetrads $\vec{E}$ via:

$$\vec{E}_i = (U^{-1}d\vec{X}U)_{ii}; \quad \vec{E}_{ij} = (U^{-1}d\vec{X}U)_{ij} \quad \text{for} \quad i \neq j; \ . \tag{5.2}$$

In matrix form we have:

$$||E|| = \begin{array}{|c|c|c|c|}\hline & d\vec{x}_k & R_{rs}.d\vec{a}_{rs}^{\perp} & du_{lm} \\ \hline \vec{E}_i & \hat{1}\delta_i^k & 0 & -\frac{\delta[\vec{a},\theta]_{ii}}{\delta u_{lm}} \\ \hline \vec{E}_{ij}^{\perp} & \Pi_{ij}.\frac{\partial R_{ij}}{\partial \vec{x}_k}.\vec{a}_{ij}^{\perp} & \hat{1}\delta_i^r\delta_j^s & -\Pi_{ij}.\frac{\delta[\vec{a},\theta]_{ij}}{\delta u_{lm}} \\ \hline E_{ij}^{||} & \frac{\partial \vec{n}_{ij}}{\partial \vec{x}_k}.\vec{a}_{ij} & 0 & |\vec{\Delta}_{ij}|\frac{\delta\theta_{ij}}{\delta u_{lm}} + \vec{n}_{ij}.\frac{\delta[\vec{a},\theta]_{ij}}{\delta u_{lm}} \\ \hline \end{array} \ ,$$

where $\Pi_{ij} = \hat{1} - \vec{n}_{ij}\vec{n}_{ij}$ and we have split the off-diagonal tetrads $\vec{E}_{ij}$ into:

$$\vec{E}_{ij}^{\perp} = \Pi_{ij}.\vec{E}_{ij}; \quad \text{and} \quad E_{ij}^{||} = -\vec{n}_{ij}.\vec{E}_{ij}; \ . \tag{5.3}$$

We have also used that:

$$\vec{n}_{ij}.\frac{\partial R_{ij}}{\partial x_k} + \frac{\partial \vec{n}_{ij}}{\partial x_k}.R_{ij} = 0 \ . \tag{5.4}$$

Now the Jacobian that we are interested J is given by the determinant of $||E||$. It is an easy exercise to show that the Jacobian is given by:

$$J = \det\Big|\Big||\vec{\Delta}_{ij}|\frac{\delta\theta_{ij}}{\delta u_{lm}} + \vec{n}_{ij}.\frac{\delta[\vec{a},\theta]_{ij}}{\delta u_{lm}} + \sum_k \frac{\partial \vec{n}_{ij}}{\partial x_k}.\vec{a}_{ij}\frac{\delta[\vec{a},\theta]_{kk}}{\delta u_{lm}}\Big|\Big| \tag{5.5}$$

One can show that the determinant in equation (5.5) factorizes:

$$J = \det\left|\left||\vec{\Delta}_{ij}|\delta_i^r\delta_j^s + \vec{n}_{ij}.\frac{\delta[\vec{a},\theta]_{ij}}{\delta\theta_{rs}} + \sum_{\mu,k}\frac{\partial\vec{n}_{ij}}{\partial x_k^\mu}.\vec{a}_{ij}\frac{\delta[a^\mu,\theta]_{kk}}{\delta\theta_{rs}}\right|\right|\det\left|\left|\frac{\delta\theta_{rs}}{\delta u_{lm}}\right|\right| \tag{5.6}$$

The last determinant in equation (5.6) produces the correct measure of $SU(N)$ it is the first part J' that we are really interested in. One can verify that:

$$J' = \left(\prod_{i\neq j}|\vec{\Delta}_{ij}|\right)\det\left|\left|\delta_i^l\delta_j^m + Y_{ij}^{lm}\right|\right| , \tag{5.7}$$

where Y_{ij}^{lm} is given by:

$$Y_{ij}^{lm} = \frac{\vec{n}_{ij}}{|\vec{\Delta}_{ij}|}.(\vec{a}_{il}\delta_j^m - \vec{a}_{mj}\delta_i^l) + \frac{\vec{a}_{ij}.\Pi_{ij}.\vec{a}_{ml}}{|\vec{\Delta}_{ij}|^2}.(\delta_i^m - \delta_i^l - \delta_j^m + \delta_j^l) , \tag{5.8}$$

where we have used that:

$$\frac{\delta[\vec{a},\theta]_{ij}}{\delta\theta_{lm}} = (\vec{a}_{il}\delta_j^m - \vec{a}_{mj}\delta_i^l); \quad \frac{\partial n_{ij}^\mu}{\partial x_k^\nu} = \frac{\Pi_{ij}^{\mu\nu}}{|\vec{\Delta}_{ij}|}(\delta_i^k - \delta_j^k); \quad \vec{a}_{ij}.\Pi_{ij} = \vec{a}_{ij}; \ . \tag{5.9}$$

Now using that:

$$\det\left|\left|\delta_i^l\delta_j^m + Y_{ij}^{lm}\right|\right| = \exp\left\{\mathrm{tr}Y - \frac{1}{2}\mathrm{tr}Y^2 + O(Y^3)\right\} \tag{5.10}$$

One can easily verify that:

$$J' = \exp\left\{\sum_{i,j}\log|\vec{\Delta}_{ij}| - 2\sum_{i,j}\bar{\vec{a}}_{ij}.\hat{D}_{ij}.\vec{a}_{ji} + O(a^3)\right\} , \tag{5.11}$$

where $\hat{D}_{ij}$ is given by:

$$D_{ij}^{\mu\nu} = \frac{\Pi_{ij}^{\mu\nu}}{|\vec{\Delta}_{ij}|^2} + \frac{1}{4}\sum_k\frac{n_{ik}^\mu n_{jk}^\nu + n_{ik}^\nu n_{jk}^\mu}{|\vec{\Delta}_{ik}||\vec{\Delta}_{jk}|} . \tag{5.12}$$

Therefore our final expression for the measure is:

$$\mu = DU\prod_i dx_i\prod_{i>j}\left(d^{p-1}a_{ij}^\perp d^{p-1}a_{ji}^\perp\right)\exp\left\{\sum_{i,j}\log|\vec{\Delta}_{ij}| - 2\sum_{i,j}\bar{\vec{a}}_{ij}.\hat{D}_{ij}.\vec{a}_{ji} + O(a^3)\right\} , \tag{5.13}$$

5.2 Appendix B

In this subsection we will provide details about the analytic evaluation of the quantities $\#(p)$ and $\tilde{\#}(p)$ defined in equations (3.30) and (3.40) respectively. Note that one can write:

$$\#(p) = \frac{p-5}{2}A_p + 2C_p; \quad \tilde{\#}(p) = \frac{p-2}{2}A_p; \ , \tag{5.14}$$

where:

$$A_p = \int_{B^p}\int_{B^p}\int_{B^p} d^p\tilde{x}d^p\tilde{y}d^p\tilde{z}\rho(\tilde{\vec{x}})\rho(\tilde{\vec{y}})\rho(\tilde{\vec{z}})\frac{1}{(\tilde{\vec{x}}-\tilde{\vec{z}})^2(\tilde{\vec{y}}-\tilde{\vec{z}})^2} \tag{5.15}$$

$$C_p = \int_{B^p}\int_{B^p}\int_{B^p} d^p\tilde{x}d^p\tilde{y}d^p\tilde{z}\rho(\tilde{\vec{x}})\rho(\tilde{\vec{y}})\rho(\tilde{\vec{z}})\frac{\cos^2\theta_{x,yz}}{(\tilde{\vec{x}}-\tilde{\vec{z}})^2(\tilde{\vec{y}}-\tilde{\vec{z}})^2}. \tag{5.16}$$

Let us consider first the quantity A_p. Note that the integrals along $\tilde{x}$ and $\tilde{y}$ in equation (5.15) factorize and one can write:

$$A_p = \int_{B^p} d^p\tilde{z}\rho(\tilde{\vec{z}})Q_p(|\tilde{\vec{z}}|)^2 = \frac{2\pi^{p/2}}{\Gamma(p/2)}\int_0^1 d\tilde{z}\tilde{z}^{p-1}\rho(\tilde{z})Q_p(\tilde{z})^2 \ , \tag{5.17}$$

where:

$$Q_p(\tilde{z}) \equiv \int_{B^p} d^p\tilde{x}\rho(\tilde{x})\frac{1}{(\tilde{\vec{x}}-\tilde{\vec{z}})^2} \tag{5.18}$$

and we have used that the distribution is $SO(p)$ symmetric (namely $\rho(\tilde{\vec{x}}) = \rho(\tilde{x})$). Note that for a uniform distribution we have $\rho(\tilde{x}) = p\Gamma(p/2)/2\pi^{p/2}$ and we have:

$$A_p = p\int_0^1 d\tilde{z}\tilde{z}^{p-1}Q_p(\tilde{z})^2 \ . \tag{5.19}$$

One can show that for $p = 3$ and a uniform distribution $Q_3(\tilde{z})$ is given by:

$$Q_3(\tilde{z}) = \frac{3}{2}\left[1 + \frac{1-\tilde{z}^2}{z}\mathrm{Arctanh}(\tilde{z})\right] \tag{5.20}$$

and

$$A_3 = 3\int_0^1 d\tilde{z}\tilde{z}^2 Q_3(\tilde{z})^2 = \frac{3}{20}\left(15+2\pi^2\right) \ . \tag{5.21}$$

Let us now focus on the quantity C_p. After using equation (3.31) and the $SO(p)$ symmetry of the eigenvalue distribution, one can show that:

$$C_p = \frac{3}{4}A_p - \int_{B^p}\int_{B^p}\int_{B^p} d^p\tilde{x}d^p\tilde{y}d^p\tilde{z}\rho(\tilde{\vec{x}})\rho(\tilde{\vec{y}})\rho(\tilde{\vec{z}})\frac{(\tilde{\vec{x}}-\tilde{\vec{z}}).(\tilde{\vec{y}}-\tilde{\vec{z}})}{(\tilde{\vec{x}}-\tilde{\vec{z}})^2(\tilde{\vec{y}}-\tilde{\vec{z}})^4} \tag{5.22}$$

$$= \frac{3}{4}A_p + \frac{2\pi^{p/2}}{\Gamma(p/2)}\int_0^1 d\tilde{z}\tilde{z}^{p-1}\rho(\tilde{z})Q_p'(\tilde{z})\Phi_p'(\tilde{z}) \ , \tag{5.23}$$

where

$$\Phi_p(\tilde{z}) \equiv \frac{1}{2}\int_{B^p} d^p\tilde{x}\rho(\tilde{x})\log|(\tilde{\vec{x}}-\tilde{\vec{z}})| \ . \tag{5.24}$$

For an uniform distribution $\rho(\tilde{z})$ we have:

$$C_p = \frac{3}{4}A_p + p\int_0^1 d\tilde{z}\tilde{z}^{p-1}Q_p'(\tilde{z})\Phi_p'(\tilde{z}) \ . \tag{5.25}$$

One can show that for $p = 3$ and a uniform distribution $\Phi_3(\tilde{z})$ is given by:

$$\Phi_3(\tilde{z}) = \frac{1}{96\tilde{z}}\left[-34\tilde{z} + 6\tilde{z}^3 + 3(\tilde{z}-1)^3(3+\tilde{z})\log|1-\tilde{z}| - 3(\tilde{z}-3)(1+\tilde{z})^3\log|1+\tilde{z}|\right] \tag{5.26}$$

and one can calculate C_3:

$$C_3 = \frac{3}{20}\left(15 + \frac{1}{2}\pi^2\right) . \tag{5.27}$$

Now one can substitute the results from equations (5.21) and (5.27) into equation (5.14) to obtain:

$$\#(3) = \frac{3}{20}(15 - \pi^2); \quad \tilde{\#}(3) = \frac{3}{40}(15 + 2\pi^2); \; , \tag{5.28}$$

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