The MSSM Spectrum from $(0,2)$-Deformations of the Heterotic Standard Embedding

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Abstract

We construct supersymmetric compactifications of $E_8 \times E_8$ heterotic string theory which realise exactly the massless spectrum of the Minimal Supersymmetric Standard Model (MSSM) at low energies. The starting point is the standard embedding on a Calabi-Yau threefold which has Hodge numbers $(h^{1,1}, h^{2,1}) = (1,4)$ and fundamental group $\mathbb{Z}_{12}$, which gives an E_6 grand unified theory with three net chiral generations. The gauge symmetry is then broken to that of the standard model by a combination of discrete Wilson lines and continuous deformation of the gauge bundle. On eight distinct branches of the moduli space, we find stable bundles with appropriate cohomology groups to give exactly the massless spectrum of the MSSM.

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1 Introduction

The first attempts at finding realistic particle physics in string theory were based on 'standard embedding' compactifications of the $E_8 \times E_8$ heterotic string [1]. The standard embedding is a general solution, in which spacetime is taken to be a direct product of Minkowski space $\mathcal{M}_4$ and a Calabi-Yau threefold X, and the gauge fields corresponding to one E_8 are set 'equal' to the Levi-Civita connection on X (which takes values in $\mathfrak{su}(3)$) via the embedding $\mathfrak{su}(3) \subset \mathfrak{su}(3) \times \mathfrak{e}_6 \subset \mathfrak{e}_8$. The gauge fields of the other E_8 are set to zero. This leads to an $\mathcal{N} = 1$ supersymmetric E_6 GUT, with $h^{2,1}(X)$ and $h^{1,1}(X)$ chiral multiplets in the $\mathbf{27}$ and $\overline{\mathbf{27}}$ representations respectively. In order to obtain three net generations of chiral fermions, then, one must find a Calabi-Yau threefold with Euler number satisfying $\frac{1}{2}\chi(X) = h^{1,1} - h^{2,1} = \pm 3$.

Breaking of E_6 to the standard model gauge group $G_{\text{SM}} = SU(3) \times SU(2) \times U(1)$ can then be achieved by a combination of discrete Wilson lines and a four-dimensional supersymmetric Higgs mechanism, which also gives mass to some of the extraneous matter [2]. The vacuum expectation values (VEVs) are required to be very large, and this makes it hard to keep control of the theory. For example, the light spectrum after Higgsing depends heavily on not only the Yukawa couplings of the original theory, but also non-renormalisable terms, and these are unlikely to be readily calculable.

For these reasons, among others, heterotic model building has moved away from the standard embedding to more general backgrounds, in which the geometry is still Calabi-Yau, but the gauge fields are given by other solutions of the Hermitian-Yang-Mills equations. Such solutions correspond to slope-stable holomorphic vector bundles on X [3, 4]. Although establishing the stability of bundles is a difficult problem (see e.g. [5, 6, 7, 8]), this approach has led to the discovery of models with the spectrum of the MSSM [9, 10], as well as extensions thereof which include a massless gauge boson coupling to baryon number minus lepton number[1] [13].

In this paper we construct the first models which realise exactly the massless spectrum of the MSSM via deformation of the standard embedding—such models correspond to $(0, 2)$ rank-changing deformations of the worldsheet theory [14]. The compactification manifold is the recently-discovered Calabi-Yau threefold with Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 4)$ and fundamental group $\mathbb{Z}_{12}$ [15, 16]. The gauge bundles are irreducible $SU(5)$ bundles obtained as deformations of the standard embedding solution, thus preserving the characteristic classes and ensuring that we retain three chiral generations. The technique used to construct the deformed bundles is the one introduced in [17], and also applied in [18], with the added complication of retaining invariance under the quotient group $\mathbb{Z}_{12}$. Continuous deformation of the gauge bundle corresponds to the supersymmetric Higgs mechanism in the low energy theory i.e. giving VEVs to charged flat directions, but constructing the bundle directly ensures that we do in fact have a consistent compactification, and also allows a direct and reliable calculation of the spectrum.

Compactifying on one of these stable rank-five bundles leaves $SU(5)_{\text{GUT}}$ as the unbroken gauge group, and this can then be broken to G_{SM} by discrete Wilson lines. This two step approach is necessary, as the additional adjoint fields required to Higgs $SU(5)_{\text{GUT}}$ to G_{SM} are not present in heterotic compactifications, and E_6 cannot be broken directly to the G_{SM} by any choice of discrete Wilson lines [19].

Before embarking on our long journey through the technical details, we will summarise the main ideas and results here. We start with a Calabi-Yau manifold $\widetilde{X}$ which has Hodge numbers $(h^{1,1}, h^{2,1}) = (8, 44)$ and can be realised as a hypersurface inside a toric fourfold. For special choices of its complex structure, $\widetilde{X}$ admits a free action by the group $\mathbb{Z}_{12}$, such that the quotient manifold $X = \widetilde{X}/\mathbb{Z}_{12}$ has Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 4)$. The standard embedding on X would therefore yield an E_6 GUT with three generations of particles in the **27** representation,

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[1]Such models were also pursued in [11], but in that example, anomaly cancellation requires the introduction of anti-branes, and the compactification therefore breaks supersymmetry (additional examples like this appear in [12]).

and one extra vector-like generation. Note that in a previous paper [15], attention was focussed on a quotient of $\widetilde{X}$ by the non-Abelian group $\text{Dic}_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$, which also yields Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 4)$. Unfortunately, using the methods described here, we subsequently discovered that this manifold does not admit models with the MSSM spectrum. This result is explained briefly in Appendix C.

Our approach will be to treat the standard embedding as corresponding to the degenerate rank-five bundle $T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$, where $T\widetilde{X}$ is the tangent bundle, and $\mathcal{O}_{\widetilde{X}}$ is the trivial line bundle, and then deform this to an irreducible rank five bundle $\widetilde{V}$. We must ensure that $\widetilde{V}$ remains equivariant under the $\mathbb{Z}_{12}$ action, such that it still descends to a bundle V on the quotient space X [13, 20], and that it remains polystable, so that we still obtain a solution to the Hermitian-Yang-Mills equations. Since deformations cannot change the characteristic classes of a bundle, we automatically retain the desirable feature of having three net generations of particles, but the unbroken gauge group is reduced to $SU(5)_{\text{GUT}}$, which can then be further broken to exactly G_{SM} by $\mathbb{Z}_{12}$-valued Wilson lines.

There are a number of discrete choices involved in the above, which lead to many candidate models. First, note that the bundle $\mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$ is generated by two constant global sections, so $\mathbb{Z}_{12}$ can act on it through any of its two-dimensional representations, corresponding to multiplication of each factor by a twelfth root of unity. This gives many choices of equivariant structure on $\mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$, but we will see later that stable deformations will only exist if the representation is chosen as a sub-representation of that on $H^{1,1}(\widetilde{X})$, which is a sum of eight distinct one-dimensional representations. So there are $\binom{8}{2} = 28$ possibilities. Another way of saying this is that the space of stable $\mathbb{Z}_{12}$-equivariant deformations of $T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$ consists of 28 disjoint branches. The second choice is the particular $\mathbb{Z}_{12}$-valued Wilson lines which break $SU(5)_{\text{GUT}}$ to G_{SM}, for which there are 11 possibilities. Altogether, then, there are $11 \times 28 = 308$ distinct families of equivariant bundles.

As well as having a stable gauge bundle, a heterotic model must satisfy the anomaly cancellation condition

$$c_2(T\widetilde{X}) - c_2(\widetilde{V}) - c_2(\widetilde{V}_{\text{hid.}}) = [C] \ ,$$

where $\widetilde{V}_{\text{hid.}}$ is another stable bundle in the hidden sector and $[C]$ is the homology class of some complex curve in $\widetilde{X}$, on which M5-branes can be wrapped in the strongly-coupled regime. In our case, since $\widetilde{V}$ has the same characteristic classes as the tangent bundle, this is satisfied by a trivial bundle[2] in the hidden sector and $[C] = 0$ i.e. no 5-branes.

Once we have chosen the gauge bundle, we must of course find the corresponding matter spectrum. The first step is to calculate the cohomology group $H^1(\widetilde{X}, \widetilde{V})$, corresponding to massless chiral multiplets in the $\mathbf{10}$ of $SU(5)_{\text{GUT}}$. It is also necessary to keep track of the $\mathbb{Z}_{12}$ action on

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[2]A trivial gauge bundle in the hidden sector leads to a pure E_8 super-Yang-Mills sector in four dimensions, which becomes strongly coupled and breaks supersymmetry at a high scale via gaugino condensation. This scale could be lowered by turning on Wilson lines in the hidden sector, to break E_8 to some smaller gauge group.

this space, which combines with the Wilson lines to give the massless spectrum on the quotient space X. The index theorem guarantees that we will have three net copies of the **10**, but some models will also have unwanted vector-like pairs of fields originating in this representation.

Finally, we must ask whether these models solve the doublet-triplet splitting problem. The cohomology group $H^1(\widetilde{X}, \Lambda^2 \widetilde{V})$ gives rise to massless chiral $\overline{\mathbf{5}}$'s, and again it is important to keep track of the group action. Here, as above, we are guaranteed to have three *net* copies of this representation, but now we also want an extra vector-like pair of doublets, to play the role of the Higgs fields of the MSSM.

The remainder of this paper is devoted to carrying out the above procedure in all its gory detail. For the impatient reader, we reveal here that in the end, we find eight bundles which satisfy all the above constraints, and therefore give models with exactly the massless spectrum of the MSSM. In fact, these models are possibly unique in this respect among deformations of standard embedding models; we defer the explanation of this statement to Section 5, as it relies on certain details of the calculations we present.

2 The manifold

Our compactification manifold X is a quotient by the group $\mathbb{Z}_{12}$ of a manifold $\widetilde{X}$ which has Hodge numbers $(h^{1,1}, h^{2,1}) = (8, 44)$. In this section we will construct $\widetilde{X}$ and describe the action of the group. As we will see, $\widetilde{X}$ is constructed as an anticanonical hypersurface in a toric fourfold; for detailed general accounts of this method of constructing Calabi-Yau manifolds, see [21, 22]. We will therefore make use of the machinery of toric geometry; reviews for physicists can be found in [23, 24, 25], and a number of good textbooks are also available, including [26, 27].

The manifold $\widetilde{X}$ was first constructed as a CICY (complete intersection Calabi-Yau manifold in a product of projective spaces) in [28], and a free $\mathbb{Z}_3$ quotient was found in [29]. The quotient group was extended to $\mathbb{Z}_6$ in [30], and finally to the two order-twelve groups $\mathbb{Z}_{12}$ and $\mathrm{Dic}_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ as part of the effort to classify all free quotients of CICY's [16]. It is the quotients by these last two groups that give rise to manifolds with $\chi = -6$, and therefore to three-generation models via the standard embedding.

Although $\widetilde{X}$ can be represented as a CICY, we will find it more useful to construct it as an anticanonical hypersurface in the toric fourfold $Z = \mathrm{dP}_6 \times \mathrm{dP}_6$, where dP_6 is the del Pezzo surface of degree six.[3] The fan for dP_6 is shown in Figure 1; its one-dimensional cones are generated by the six vectors

$$\{\nu_a\}_{a=1}^6 = \{ (1,0),\, (1,1),\, (0,1),\, (-1,0),\, (-1,-1),\, (0,-1) \} \,, \tag{2.1}$$

which are the vertices of a hexagon. We will also have use for the vertices of the dual hexagon,

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[3]The surface referred to here as dP_6 is therefore the complex projective plane $\mathbb{P}^2$ blown up at 3 points in general position; in some other works, including papers by some of the present authors, it is referred to as dP_3.
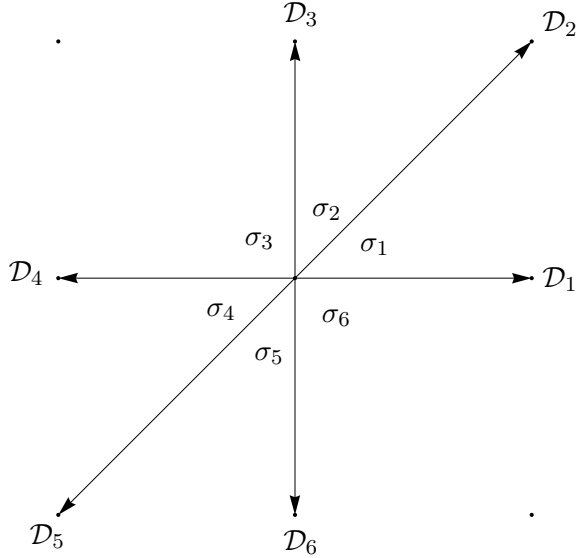
Figure 1: The fan for dP$_6$, with the rays labelled by the corresponding toric divisors. The two-dimensional cones σ_a are also labelled.

given by

$$\{\mu_a\}_{a=1}^6 = \{\,(0,1),\,(-1,1),\,(-1,0),\,(0,-1),\,(1,-1),\,(1,0)\,\}\,. \tag{2.2}$$

Our toric fourfold Z is a product of two copies of dP$_6$, so its fan is just a product of two copies of the fan for dP$_6$. Another way to say this is that it is given by cones over the faces of a four-dimensional polytope ∇, with vertices

$$\mathrm{vert}(\nabla) = \{\,(\nu_a,0)\,\}_{a=1}^6 \cup \{\,(0,\nu_a)\,\}_{a=1}^6\,.$$

Besides the origin, these twelve vertices are the only lattice points contained in the polytope ∇. They correspond to the twelve toric divisors on Z; we will write $\mathcal{D}_a$ for the divisor corresponding to $(\nu_a,0)$, and $\widetilde{\mathcal{D}}_a$ for the divisor corresponding to $(0,\nu_a)$. It will also be convenient to have notation encompassing all twelve vertices or divisors at once, so define

$$v_i = \begin{cases} (\nu_i,0) & ,\quad i=1,\ldots,6 \\ (0,\nu_{i-6}) & ,\quad i=7,\ldots,12 \end{cases}, \tag{2.3}$$

and let D_i be the divisor corresponding to v_i. We also have corresponding homogeneous coordinates $\{z_i\}_{i=1}^{12}$.

The twelve vectors v_i span a four dimensional space, so there are eight linear relations between them, which we write as

$$\sum_{i=1}^{12} Q_{\alpha i}\, v_i = 0\,,\ \alpha = 1,\ldots,8\,.$$

The matrix Q is only determined up to the left action of $GL(8,\mathbb{Z})$; one possibility is to take it to be, in block form,

$$Q_{\alpha i} = \begin{pmatrix} Q' & 0 \\ 0 & Q' \end{pmatrix}, \tag{2.4}$$

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where

$$Q' = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix} .$$

We can now follow Batyrev's procedure to construct a family of Calabi-Yau hypersurfaces in Z [21]. Let M be the dual lattice to N, and denote by $M_\mathbb{R}$ and $N_\mathbb{R}$ the real vector spaces in which the lattices lie. Then the dual polytope to ∇ is defined in $M_\mathbb{R}$ by

$$\Delta = \{ u \in M_\mathbb{R} \mid \langle u, v \rangle \geq -1 \ \forall \ v \in \nabla \} .$$

For the present example, this is simple to describe explicitly. Note that the dual hexagon we mentioned earlier, with vertices given by (2.2), contains seven lattice points including the origin. The polytope Δ therefore contains 49 lattice points, given by pairs of these. Of these points, 36 are the vertices, given by

$$\mathrm{vert}(\Delta) = \{ \ (\mu_a, \mu_b) \ \}_{a,b=1}^6 .$$

The thirteen additional lattice points consist of the origin, and points of the form $(\mu_a, 0)$ or $(0, \mu_a)$; the 12 three-faces of Δ are hexagonal prisms each with two hexagonal faces and six rectangular faces, and these extra points arise as one point interior to each of the hexagonal two-faces [15]. Altogether, the 49 lattice points of Δ correspond to a basis of the space of global sections of the anticanonical bundle $\omega_Z^{-1} \cong \mathcal{O}_Z(\sum_i D_i)$, via the prescription

$$u \ \rightarrow \ \prod_i z_i^{\langle u, v_i \rangle + 1} . \tag{2.5}$$

2.1 The group action

The group $\mathbb{Z}_{12}$, being cyclic, has a very simple representation ring. All of its irreducible representations are one-dimensional, and there are twelve altogether, corresponding to the twelfth roots of unity. We will denote by $\mathbf{n}$ the representation which sends the generator g_{12} of $\mathbb{Z}_{12}$ to $e^{\frac{\pi i}{6} n}$, where n takes the values 0 through 11.

The action of $\mathbb{Z}_{12}$ on the lattice N (in which ∇ lives) is generated by the matrix

$$A_N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} .$$

Geometrically, this exchanges the two hexagons, then rotates the first clockwise by one-sixth of a turn, and the second anti-clockwise by one third of a turn. The $\mathbb{Z}_{12}$ action is therefore transitive on the set of toric divisors, being given by

$$g_{12} \ : \ \mathcal{D}_a \rightarrow \widetilde{\mathcal{D}}_{a+2} \ , \ \widetilde{\mathcal{D}}_a \rightarrow \mathcal{D}_{a-1} \ ,$$

where the subscripts are understood modulo 6. If we instead use the notation D_i, $i = 1, \ldots, 12$ for the divisors, as in (2.3), the group action is explicitly given by the permutation

$$D_1 \to D_9 \to D_2 \to D_{10} \to D_3 \to D_{11} \to D_4 \to D_{12} \to D_5 \to D_7 \to D_6 \to D_8 \to D_1 , \qquad (2.6)$$

with the homogeneous coordinates being permuted in the same way.

To find the invariant anticanonical hypersurfaces, we need to know how the group acts on the dual lattice M, and hence on the monomials (2.5). This is determined by demanding that the duality pairing is preserved, i.e.

$$\langle A_M u, A_N v \rangle = \langle u, v \rangle \quad \forall \ u \in M , \ v \in N .$$

In matrix terms, this implies $A_M = (A_N^{-1})^T$, which is explicitly

$$A_M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

The 49 lattice points in Δ fall into five orbits under the action of $\mathbb{Z}_{12}$. Obviously the origin is fixed by the group, and therefore corresponds to an invariant monomial f_0, while the other 48 points fall into four orbits of length twelve. Each orbit yields a unique invariant polynomial, given by the sum of the twelve monomials which get permuted. This yields four more invariant polynomials $f_1, \ldots, f_4$, with the most general invariant polynomial being given by $f = \sum_{\lambda=0}^{4} a_\lambda f_\lambda$.

It can be checked (see e.g. [15]) that for a generic choice of the coefficients a_λ, the corresponding hypersurface $\widetilde{X}$ is smooth. These coefficients are therefore projective coordinates on the four-dimensional moduli space of a family of smooth quotient manifolds X. For the remainder of the paper, we will choose them to be generic integers; none of our results depend on this choice.

We can also now calculate the group action on $H^{1,1}(\widetilde{X}) \cong H^1(\widetilde{X}, \Omega^1 \widetilde{X})$, which will be important later. First we note that $H^{1,1}(\widetilde{X}) \cong H^{1,1}(Z)$, which follows from the Lefschetz hyperplane theorem after noting that X is an anticanonical hypersurface in Z, and that the anticanonical bundle of Z is very ample. We then have the general result for smooth compact toric varieties,

$$H^{1,1}(Z) \cong \frac{\bigoplus_i \mathbb{C} \cdot v_i}{M \otimes_{\mathbb{Z}} \mathbb{C}} .$$

The permutation action of $\mathbb{Z}_{12}$ on the toric divisors corresponds to the regular representation $\text{Reg}_{\mathbb{Z}_{12}} = \bigoplus_n \mathbf{n}$, and by diagonalising the matrix A_M, one sees that $\mathbb{Z}_{12}$ acts on $M \otimes_{\mathbb{Z}} \mathbb{C}$ via the faithful representation[4] $\mathbf{1} \oplus \mathbf{5} \oplus \mathbf{7} \oplus \mathbf{11}$. We conclude that $H^{1,1}(Z)$, and hence $H^{1,1}(\widetilde{X})$, transforms under the difference of these representations:

$$H^{1,1}(\widetilde{X}, \mathbb{C}) \cong H^{1,1}(Z, \mathbb{C}) \sim \mathbf{0} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{4} \oplus \mathbf{6} \oplus \mathbf{8} \oplus \mathbf{9} \oplus \mathbf{10} . \qquad (2.7)$$

We will rederive this result later by a different method.

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[4]This also follows from noting that A_M is real, and neither A_M^4 nor A_M^6 has 1 as an eigenvalue.

3 Deforming the gauge bundle

In this section we will explicitly construct our desired family of rank-five bundles on the covering space $\widetilde{X}$, and argue that they are stable. The reader wishing to avoid these details may skip to the commutative diagram in (3.11), which provides the ultimate definition of the bundles, and serves as the basis for the cohomology calculations of Section 4.

3.1 Stability and deformations

Perhaps the most difficult part of constructing appropriate vector bundles for heterotic string theory is ensuring that the bundles are slope stable, and therefore, by the Donaldson-Uhlenbeck-Yau theorem, admit a Hermitian-Yang-Mills connection [3, 4]. Here we are interested in bundles which are deformations of $T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}}^{\oplus r}$, where $\widetilde{X}$ is a Calabi-Yau manifold, and in this case Li and Yau found much simpler necessary and sufficient conditions for a solution [17]. Before turning to the details of our bundles, we will briefly review their result.

Consider a one-parameter family of bundles $\widetilde{V}_s$, such that $\widetilde{V}_0 = T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}}^{\oplus r}$. Then the 'tangent' to this family of bundles at $s = 0$ is given by the Kodaira-Spencer deformation class, which lives in $H^1(\widetilde{X}, \widetilde{V}_0 \otimes \widetilde{V}_0^*)$. Using the decomposition of $\widetilde{V}$, and the fact that $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ since $\widetilde{X}$ is Calabi-Yau, this becomes

$$H^1(\widetilde{X}, \widetilde{V}_0 \otimes \widetilde{V}_0^*) = H^1(\widetilde{X}, \Omega^1\widetilde{X} \otimes T\widetilde{X}) \oplus H^1(\widetilde{X}, T\widetilde{X})^{\oplus r} \oplus H^1(\widetilde{X}, \Omega^1\widetilde{X})^{\oplus r} \ . \tag{3.1}$$

The Li-Yau conditions are that the projections of the Kodaira-Spencer class onto each of the last two terms consist of r linearly independent elements of the respective cohomology groups.

To get some intuition for what this means, note that $H^1(\widetilde{X}, T\widetilde{X})$ is naturally isomorphic to $\mathrm{Ext}^1(\mathcal{O}_{\widetilde{X}}, T\widetilde{X})$, which parametrises extensions of $\mathcal{O}_{\widetilde{X}}$ by $T\widetilde{X}$, i.e. bundles B which fit into a short exact sequence

$$0 \longrightarrow T\widetilde{X} \longrightarrow B \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow 0 \ .$$

If $B \cong T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}}$, the extension is called *split*, and this corresponds to the zero element of $\mathrm{Ext}^1(\mathcal{O}_{\widetilde{X}}, T\widetilde{X})$. If we replace $\mathcal{O}_{\widetilde{X}}$ by $r\mathcal{O}_{\widetilde{X}} := \mathcal{O}_{\widetilde{X}}^{\oplus r}$, then we get r copies of the same Ext^1 group. An element of this group is therefore an r-tuple of elements of $\mathrm{Ext}^1(\mathcal{O}_{\widetilde{X}}, T\widetilde{X})$, and these r elements are linearly independent if and only if the corresponding extension is *completely non-split*, i.e. cannot be written as $B \cong B' \oplus \mathcal{O}_{\widetilde{X}}$ for some bundle B'. Analogous comments apply to $H^1(\widetilde{X}, \Omega^1\widetilde{X})$, which parametrises 'opposite' extensions i.e. extensions of $T\widetilde{X}$ by $\mathcal{O}_{\widetilde{X}}$.

An extension of $T\widetilde{X}$ by $\mathcal{O}_{\widetilde{X}}$ or vice versa can never be stable, but if the Kodaira-Spencer class of the family $\widetilde{V}_s$ has a non-zero piece in each of $H^1(\widetilde{X}, T\widetilde{X})^{\oplus r}$ and $H^1(\widetilde{X}, \Omega^1\widetilde{X})^{\oplus r}$, then $\widetilde{V}_s$ for $s \neq 0$ has no such simple interpretation, and may be stable. The theorem of Li and Yau asserts that $\widetilde{V}_s$ is stable for small enough s precisely if the components of each of these r-tuples are linearly independent.

3.2 Constructing the deformations

As we wish to ultimately construct a rank-five bundle on $\widetilde{X}$ which is a deformation of $T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$, it is necessary to first give an explicit description of the tangent bundle itself. We begin with the normal bundle short exact sequence, noting that the normal bundle to $\widetilde{X}$ inside Z is isomorphic to the restriction of the anticanonical bundle of Z,

$$0 \longrightarrow T\widetilde{X} \longrightarrow TZ|_{\widetilde{X}} \xrightarrow{df} \mathcal{O}_{\widetilde{X}}\left(\sum_i D_i\right) \longrightarrow 0 \ . \tag{3.2}$$

In turn, the bundle TZ is simply described by a generalisation of the Euler sequence, which applies to all smooth toric varieties:

$$0 \longrightarrow 8\mathcal{O}_Z \longrightarrow \bigoplus_{i=1}^{12} \mathcal{O}_Z(D_i) \longrightarrow TZ \longrightarrow 0 \ , \tag{3.3}$$

where the first map here is defined on global sections by a matrix with entries $Q_{\alpha,i} z_i$ (no summation on i), where $Q_{\alpha,i}$ is the toric charge matrix from (2.4). We will pause here briefly to explain this in a little more detail.

Following Cox's prescription [31], the space Z can be constructed from the affine space $\mathbb{C}^{12}$ parametrised by $\{z_i\}$ by deleting certain coordinate planes, and then imposing the equivalence relations

$$(z_1, \ldots, z_{12}) \sim (\lambda^{Q_{\alpha,1}} z_1, \ldots, \lambda^{Q_{\alpha,12}} z_{12}) \ \forall \ \lambda \in \mathbb{C}^* \ , \ \alpha = 1, \ldots, 8 \ . \tag{3.4}$$

Vector fields on $\mathbb{C}^{12}$ are of the form $\sum_i h_i(z) \frac{\partial}{\partial z_i}$, but these will only descend to Z if they are invariant under (3.4), i.e. if $h_i(z)$ transforms like z_i under all the rescalings, which is equivalent to saying that $h_i(z)$ is a section of $\mathcal{O}_Z(D_i)$. So vector fields on Z are given by sections of $\bigoplus_i \mathcal{O}_Z(D_i)$, but some of these may correspond to the zero vector field. This will be the case when the vector field on $\mathbb{C}^{12}$ is tangent to the equivalence classes defined by (3.4), i.e. when $h_i = k\, Q_{\alpha,i} z_i$ for some constant k. This argument can be made rigorous, and the result is that the sequence in (3.3) is exact.

So we have an eight-dimensional space of 'Euler vectors' spanned by

$$\left\{ E_\alpha = \sum_{i=1}^{12} Q_{i\alpha} z_i \frac{\partial}{\partial z_i} \right\}_{\alpha=1}^{8} \ , \tag{3.5}$$

which corresponds to the space of global sections of $8\mathcal{O}_{\widetilde{X}}$ in (3.3). These will be very important as we proceed.

We will construct our desired bundles by adapting the techniques of [17, 18]. First, notice that after restricting the Euler sequence (3.3) to the hypersurface $\widetilde{X}$, we can compose the last map in this sequence with the differential df from (3.2) to get a map

$$\Phi_0 : \bigoplus_i \mathcal{O}_{\widetilde{X}}(D_i) \to \mathcal{O}_{\widetilde{X}}\left(\sum_i D_i\right) \ .$$

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Then if we denote by $\mathcal{F}_0$ the bundle which is the kernel of this map,[5] a simple diagram-chasing argument shows that the dashed arrows in the following commutative diagram can be filled in such that all the rows and columns are exact

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & 8\,\mathcal{O}_{\widetilde{X}} & \xrightarrow{\ =\ } & 8\,\mathcal{O}_{\widetilde{X}} & & & & \\
& & \vdots & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{F}_0 & \longrightarrow & \displaystyle\bigoplus_{i=1}^{12}\mathcal{O}_{\widetilde{X}}(D_i) & \xrightarrow{\Phi_0} & \mathcal{O}_{\widetilde{X}}\!\left(\displaystyle\sum_{i=1}^{12}D_i\right) & \longrightarrow & 0 \\
& & \vdots & & \downarrow & & \ \parallel & & \\
0 & \longrightarrow & T\widetilde{X} & \longrightarrow & TZ|_{\widetilde{X}} & \xrightarrow{df} & \mathcal{O}_{\widetilde{X}}(\displaystyle\sum_{i=1}^{12}D_i) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

(3.6)

Note that the exactness of rows and columns here implies that $8\mathcal{O}_{\widetilde{X}} \subset \ker\Phi_0$, which says that Φ_0 annihilates all eight Euler vectors in (3.5). This is a result of the generalised Euler identities $E_\alpha(f) \propto f$, and the fact that all bundles are restricted to $\widetilde{X}$, where $f \equiv 0$.

Exactness of the leftmost column of (3.6) means that $\mathcal{F}_0$ is an extension of $T\widetilde{X}$ by the rank-eight trivial bundle $8\mathcal{O}_{\widetilde{X}}$, and it is not too hard to argue that it is a completely non-split extension. Indeed, suppose that it split, so that $\mathcal{F}_0 \cong \mathcal{F}_0' \oplus \mathcal{O}_{\widetilde{X}}$. Then the projection onto the second summand would correspond to a non-trivial element of $\mathrm{Hom}(\mathcal{F}_0, \mathcal{O}_{\widetilde{X}}) \cong H^0(\widetilde{X}, \mathcal{F}_0^* \otimes \mathcal{O}_{\widetilde{X}}) \cong H^0(\widetilde{X}, \mathcal{F}_0^*)$, but the calculations of Appendix A.3 show that this last group is trivial. We conclude that no such split can exist.

The bundle $\mathcal{F}_0$ we have just defined has vanishing first Chern class, but has rank eleven and, being an extension by $8\mathcal{O}_{\widetilde{X}}$, is unstable. Let us solve the first of these problems, and obtain a rank five bundle instead. We need only select a trivial rank-six sub-bundle[6] $6\mathcal{O}_{\widetilde{X}} \subset 8\mathcal{O}_{\widetilde{X}} \subset \mathcal{F}_0$, and define $\widetilde{V}_e = \mathcal{F}_0/6\mathcal{O}_{\widetilde{X}}$. Then $\widetilde{V}_e$ is a completely non-split extension[7] of $T\widetilde{X}$ by $2\mathcal{O}_{\widetilde{X}}$.

---

[5]The kernel of a map between vector bundles may fail to be a bundle itself, but instead be a more general coherent sheaf. This does not occur here or in similar situations later in the paper, so we can ignore this subtlety.

[6]We will be more specific about the choice of $6\mathcal{O}_{\widetilde{X}}$ inside $8\mathcal{O}_{\widetilde{X}}$ in Section 3.3 when we discuss group equivariance of the various bundles here.

[7]If the extension split, then, as above, it would imply a non-trivial morphism $\widetilde{V}_e \to \mathcal{O}_{\widetilde{X}}$, but composing this with the surjection $\mathcal{F}_0 \to \widetilde{V}_e$ would give a non-trivial element of $H^0(\widetilde{X}, \mathcal{F}_0^*)$, which we have already stated does not exist.

The bundle $\widetilde{V}_e$ can be considered to be a deformation of $T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$, and the argument above shows that its Kodaira-Spencer class is a non-degenerate element of $H^1(\widetilde{X}, \Omega^1 \widetilde{X})^{\oplus 2}$, thereby satisfying half of the Li-Yau conditions. We now need to further deform $\widetilde{V}_e$ so that the part of the Kodaira-Spencer class living in $H^1(\widetilde{X}, T\widetilde{X})^{\oplus 2}$ is also non-degenerate. We can do this by deforming the map Φ_0, which is defined by the first derivatives of the polynomial f, to a more general morphism

$$\Phi \; : \; \bigoplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i) \to \mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i) \; ,$$

and defining $\mathcal{F} = \ker \Phi$. If we demand that Φ still annihilates our chosen sub-bundle $6\mathcal{O}_{\widetilde{X}} \subset 8\mathcal{O}_{\widetilde{X}}$, then we can still define a rank-five bundle $\widetilde{V} = \mathcal{F}/6\mathcal{O}_{\widetilde{X}}$; this can be expressed by the following short exact sequence,

$$0 \longrightarrow 6\mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{F} \longrightarrow \widetilde{V} \longrightarrow 0 \; . \tag{3.7}$$

The general $\widetilde{V}$ defined this way is a deformation of $\widetilde{V}_e$, and therefore of $\widetilde{V}_0 = T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$. Note that the map Φ is an element of

$$\mathrm{Hom}\Big(\bigoplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i) \; , \; \mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i) \Big) \; \cong \; H^0\Big(\widetilde{X}, \; \bigoplus_{j=1}^{12} \mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i - D_j) \Big)$$

$$\cong \; \bigoplus_{j=1}^{12} H^0\Big(\widetilde{X}, \; \mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i - D_j) \Big) \; \cong \; \mathbb{C}^{12 \times 35} \; = \; \mathbb{C}^{420} \; , \tag{3.8}$$

where the dimension follows from results of Appendix A. Demanding that Φ annihilate six specific global sections restricts it to a 132-dimensional subspace.

We must now ask whether the new Φ is sufficient to satisfy the other half of the Li-Yau conditions. This could be demonstrated along the lines of the original paper [17], by constructing a smooth moduli space which interpolates between non-split extensions of $T\widetilde{X}$ by $2\mathcal{O}_{\widetilde{X}}$ and vice versa, and contains our more general bundles $\widetilde{V}$. Here though, we will simply note that any stable bundle with vanishing first Chern class has no global sections, and this condition is also sufficient in this case. We give an alternative argument for this, and a simpler proof of stability, in Appendix B.

So we need only arrange that $H^0(\widetilde{X}, \widetilde{V}) = 0$. Consider the long exact sequence in cohomology which follows from (3.7). Since $\widetilde{X}$ is Calabi-Yau, we have $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$, so that the first few terms are

$$0 \longrightarrow \mathbb{C}^6 \longrightarrow H^0(\widetilde{X}, \mathcal{F}) \longrightarrow H^0(\widetilde{X}, \widetilde{V}) \longrightarrow 0 \longrightarrow \ldots \quad .$$

To obtain $H^0(\widetilde{X}, \widetilde{V}) = 0$, we therefore need $H^0(\widetilde{X}, \mathcal{F}) \cong \mathbb{C}^6$. In words, we must demand that, when applied to global sections of $\bigoplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i)$, Φ annihilates *precisely* our chosen six-dimensional subspace. If we make a specific choice of the parameters defining Φ, this can be checked by computer, and in all examples, a general enough Φ exists such that it is true. This ensures that $\widetilde{V}$ is stable.

11

There is a different, equivalent way to define $\widetilde{V}$, which will prove useful for later calculations. We start with (3.3), which defines the tangent bundle of the ambient space, TZ, as a quotient $\oplus_{i=1}^{12} \mathcal{O}_Z(D_i)/8\mathcal{O}_Z$, and instead define a rank-six bundle $\mathcal{G}$ by dividing only by our chosen sub-bundle $6\mathcal{O}_Z$. This is captured by a short exact sequence,

$$0 \longrightarrow 6\mathcal{O}_Z \longrightarrow \bigoplus_{i=1}^{12} \mathcal{O}_Z(D_i) \longrightarrow \mathcal{G} \longrightarrow 0 \ , \tag{3.9}$$

which we can also restrict to $\widetilde{X}$. Since we demanded that $\ker \Phi \supset 6\mathcal{O}_{\widetilde{X}}$, Φ induces a well-defined map from $\mathcal{G}|_{\widetilde{X}}$ to $\mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i)$, and $\widetilde{V}$ is precisely the kernel of this map, giving another short exact sequence,

$$0 \longrightarrow \widetilde{V} \longrightarrow \mathcal{G}|_{\widetilde{X}} \longrightarrow \mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i) \longrightarrow 0 \ . \tag{3.10}$$

We also note in passing that $\widetilde{V}$ can in fact be defined in one step as the cohomology of the (non-exact) sequence

$$6\mathcal{O}_{\widetilde{X}} \longrightarrow \bigoplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i) \longrightarrow \mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i) \ .$$

This definition of $\widetilde{V}$ would allow a worldsheet description of these compactifications via the linear sigma model [32].

The two short exact sequences in (3.9) and (3.10) can be intertwined with those defining $\widetilde{V}$ in terms of $\mathcal{F}$, to yield a commutative diagram analogous to that in (3.6),

$$\tag{3.11}$$

When interpreting this diagram, one should consider $\mathcal{F}$ and $\mathcal{G}$ to be defined by exactness of the middle row and column respectively. The bundle $\widetilde{V}$ can then be defined by the exactness of either the leftmost column or the bottom row.

3.3 Bundle equivariance

In this section we will discuss how to make the preceding construction compatible with the action of the quotient group. In this context, the word 'equivariant' occurs frequently, and is used to mean two different things. Firstly, a bundle $\pi : B \to \widetilde{X}$ is said to be equivariant under $\mathbb{Z}_{12}$ if the group generator g_{12} can be made to act on B via a map $\Psi : B \to B$, such that the following diagram commutes,

$$
\begin{array}{ccc}
B & \xrightarrow{\ \Psi\ } & B \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\
\widetilde{X} & \xrightarrow{\ g_{12}\ } & \widetilde{X}
\end{array} \ ,
$$

and Ψ^{12} is the identity map on B. The second use of the word equivariant is to describe a map between two equivariant bundles. Let B_1 and B_2 be two bundles, each equivariant under $\mathbb{Z}_{12}$, with the equivariant structures being given by Ψ_1 and Ψ_2 respectively. Then a bundle morphism $\Xi : B_1 \to B_2$ is said to be $\mathbb{Z}_{12}$-equivariant if $\Xi \circ \Psi_1 = \Psi_2 \circ \Xi$. One might also say that Ξ 'commutes with the group action'.

So far we have constructed a rank-five bundle $\widetilde{V}$ on the manifold $\widetilde{X}$. If we want this to correspond to a bundle on the quotient space X, we must ensure that it is equivariant under the action of the quotient group $\mathbb{Z}_{12}$. The easiest way to guarantee this is to ensure that all the other bundles appearing in (3.11) are equivariant, and that the maps between them commute with the group action. Indeed, it is not hard to convince oneself that if two bundles B_1 and B_2 are equivariant under the action of some group, and a morphism $\Xi : B_1 \to B_2$ commutes with the group action, then the kernel and cokernel of Ξ are also equivariant [13].

Begin with the bundle $\bigoplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i)$, which is central to the whole construction. Since $\mathbb{Z}_{12}$ permutes the twelve toric divisors, it naturally acts on this bundle by permuting the twelve direct summands in the same way. The bundle $8\mathcal{O}_{\widetilde{X}}$ is embedded in this rank-twelve bundle via the eight Euler vectors in (3.5). These eight sections generate $8\mathcal{O}_{\widetilde{X}}$, and one can check that the eight-dimensional space they span is mapped to itself by the $\mathbb{Z}_{12}$ action, thus defining an action on $8\mathcal{O}_{\widetilde{X}}$ in such a way that the map $8\mathcal{O}_{\widetilde{X}} \to \bigoplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i)$ is automatically equivariant. By diagonalising the 8×8 matrix which acts on the space of Euler vectors, we find that in terms of $\mathbb{Z}_{12}$ representations,

$$
H^0(\widetilde{X}, 8\mathcal{O}_{\widetilde{X}}) \ \sim \ \mathbf{0} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{4} \oplus \mathbf{6} \oplus \mathbf{8} \oplus \mathbf{9} \oplus \mathbf{10} \ . \tag{3.12}
$$

We note in passing that using this result in the long exact cohomology sequence following from the leftmost column of (3.6), we can quickly rederive (2.7). In words, such a calculation shows that $H^{1,1}(\widetilde{X})$ transforms under the same real representation as the space of Euler vectors.

The steps above are actually fairly trivial; we have simply defined the action of $\mathbb{Z}_{12}$ on $8\mathcal{O}_{\widetilde{X}}$ via its embedding in $\bigoplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i)$, which immediately makes the embedding map equivariant. The more involved step is ensuring that the map Φ is equivariant.

First note that $\mathbb{Z}_{12}$ acts on the line bundle $\mathcal{O}_{\widetilde{X}}(\sum_i D_i)$ simply by the permutation of homogeneous coordinates given by (2.6). This permutation acts on the entire homogeneous coordinate ring, and we will denote it by

$$\sigma : \mathbb{C}[z_1, \ldots, z_{12}] \to \mathbb{C}[z_1, \ldots, z_{12}] \ .$$

If we want to explicitly represent the group action on, say, the global sections of $\bigoplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i)$, we need to apply σ as well as permute the components. For example, letting P be the matrix which permutes the components, we have

$$g_{12}\big((z_1, 0, \ldots, 0)^T\big) \ = \ \sigma\big(P \cdot (z_1, 0, \ldots, 0)^T\big) \ = \ (0, \underbrace{\ldots}_{6}, 0, z_9, 0, 0, 0)^T \ .$$

Had we not applied σ here, we would have obtained something nonsensical, since z_1 is not a section of $\mathcal{O}_{\widetilde{X}}(D_9)$.

So if we represent Φ by a 12×1 matrix, the condition for equivariance is that for any section τ of $\bigoplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i)$, represented as a column vector as above, we have

$$\Phi \cdot \big(P \cdot \sigma(\tau)\big) = \sigma\big(\Phi \cdot \tau\big) \ .$$

This may look a little technical, but it simply means, for example, that $\Phi_1 = \sigma\big(\Phi_9\big)$, so the twelve components of Φ are all related by permuting the coordinates. Before imposing equivariance, each component had 35 parameters, so we find a 35-dimensional space of equivariant maps Φ.

Recall that the original map Φ_0, induced by df, annihilates all eight Euler vectors, and that we want Φ to still annihilate a six-dimensional subspace. It is clear that, due to equivariance, this must correspond to a six-dimensional sub-representation of (3.12), of which there are 28, corresponding to the choice of two of the eight distinct charges. It is easy enough to check that in every case, there is an 11-dimensional family[8] of equivariant maps Φ which annihilate *exactly* the six chosen Euler vectors, thus guaranteeing stability of the corresponding bundle $\widetilde{V}$, as discussed at the end of Section 3.2. Therefore there are 28 distinct branches of the moduli space of stable $\mathbb{Z}_{12}$-equivariant deformations of $T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$.

3.4 Equivariant structures and Wilson lines

The various branches of the moduli space of $\widetilde{V}$, described previously, correspond to different choices of equivariant structure on $T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$. A non-trivial equivariant structure on $\mathcal{O}_{\widetilde{X}}$, corresponding to some $\mathbb{Z}_{12}$ representation $\mathbf{n}$, means that on traversing a non-contractible path on the quotient space X, we identify sections of $\mathcal{O}_X$ which differ by the phase $e^{\frac{\pi i}{6} n}$. So in fact we have a non-trivial flat line bundle $\mathcal{L}_n$. Since the holonomy group is identified in physics with the gauge group, this is precisely what is meant by turning on discrete Wilson lines.

---

[8]There are three common parameters, corresponding to equivariant maps which annihilate all eight Euler vectors. These represent a three-dimensional family of rank-three deformations i.e. deformations of the tangent bundle of the quotient space X.

We now come across a slight complication which we have been ignoring until now. Take the bundle $TX \oplus \mathcal{L}_{n_1} \oplus \mathcal{L}_{n_2}$, and ask about the holonomy around a non-contractible path on X. It will be given by a 5×5 matrix

$$U_5 = \mathrm{diag}(U_3, e^{\frac{\pi \mathrm{i}}{6} n_1}, e^{\frac{\pi \mathrm{i}}{6} n_2}) \ ,$$

where $U_3 \in SU(3)$ is the holonomy in TX. Although U_5 is unitary, it will not typically lie in $SU(5)$, since $\det U_5 = e^{\frac{\pi \mathrm{i}}{6}(n_1+n_2)}$. So we must multiply it by $e^{\frac{\pi \mathrm{i}}{6} \tilde{n}}$, where

$$5\tilde{n} + n_1 + n_2 = 0 \mod 12 \quad \Rightarrow \quad \tilde{n} = 7(n_1 + n_2) \mod 12 \ . \tag{3.13}$$

This corresponds to choosing an extra overall phase in the equivariant structure on $\widetilde{V}$, such that it descends to a deformation of the bundle $\mathcal{L}_{\tilde{n}} \otimes (TX \oplus \mathcal{L}_{n_1} \oplus \mathcal{L}_{n_2})$. This ensures that the holonomy group is indeed $SU(5)$.

4 Cohomology and finding the MSSM spectrum

We have constructed various families of stable $\mathbb{Z}_{12}$-equivariant bundles, a generic example of which we are denoting by $\widetilde{V}$, with structure group $SU(5)$. Embedding this $SU(5)$ in E_8 gives us a heterotic model on $\widetilde{X}$ with unbroken gauge group $SU(5)_{\mathrm{GUT}}$. The decomposition of the adjoint of E_8 is

$$E_8 \supset \frac{SU(5) \times SU(5)_{\mathrm{GUT}}}{\mathbb{Z}_5}$$

$$\mathbf{248} = (\mathbf{1}, \mathbf{24}) \oplus (\mathbf{24}, \mathbf{1}) \oplus (\mathbf{5}, \mathbf{10}) \oplus (\mathbf{10}, \overline{\mathbf{5}}) \oplus (\overline{\mathbf{5}}, \overline{\mathbf{10}}) \oplus (\overline{\mathbf{10}}, \mathbf{5}) \ . \tag{4.1}$$

The left-handed standard model fermions appear in the $\mathbf{10} \oplus \overline{\mathbf{5}}$ representation of $SU(5)_{\mathrm{GUT}}$, and therefore correspond to the fundamental $\mathbf{5}$ and the rank-two anti-symmetric $\mathbf{10}$ of the structure group of $\widetilde{V}$. Therefore the number of massless chiral superfields in each representation, before taking the quotient by $\mathbb{Z}_{12}$, is given by

$$n_{\mathbf{10}} = h^1(\widetilde{X}, \widetilde{V}) \ , \quad n_{\overline{\mathbf{5}}} = h^1(\widetilde{X}, \wedge^2 \widetilde{V})$$

$$n_{\overline{\mathbf{10}}} = h^1(\widetilde{X}, \widetilde{V}^*) \ , \quad n_{\mathbf{5}} = h^1(\widetilde{X}, \wedge^2 \widetilde{V}^*) \ ,$$

where, for example, $h^i(\widetilde{X}, \widetilde{V}) = \dim_{\mathbb{C}} H^i(\widetilde{X}, \widetilde{V})$. So it is these cohomology groups we wish to calculate. This will involve a considerable amount of algebraic gymnastics, so although we try to give complete arguments, many intermediate calculations are relegated to Appendix A.

It will also be important to keep track of the $\mathbb{Z}_{12}$ action on the various cohomology groups. We ultimately wish to break $SU(5)_{\mathrm{GUT}}$ to G_{SM} by turning on discrete Wilson lines, and the massless fields on X are those which are invariant under the combined action of $\mathbb{Z}_{12}$ on the cohomology groups, and the Wilson lines on the corresponding $SU(5)_{\mathrm{GUT}}$ representations. Note that an equivariant bundle map induces an equivariant map between cohomology groups, so all our long exact sequences actually split up into twelve independent sequences—one for each irreducible $\mathbb{Z}_{12}$ representation.

4.1 Wilson line breaking of $SU(5)_{\mathrm{GUT}}$

To break $SU(5)_{\mathrm{GUT}}$ down to G_{SM}, we turn on discrete $SU(5)_{\mathrm{GUT}}$-valued holonomy around non-contractible paths in X, such that the centraliser of this discrete holonomy group is precisely G_{SM}. This amounts to choosing a homomorphism from $\pi_1(X) \cong \mathbb{Z}_{12}$ to $SU(5)_{\mathrm{GUT}}$, so we must classify the suitable choices.

The largest subgroup of $SU(5)_{\mathrm{GUT}}$ which commutes with G_{SM} is the hypercharge group $U(1)_Y$, the embedding of which is[9]

$$U(1)_Y \ni \ e^{i\theta} \ \mapsto \ \mathrm{diag}(e^{-2i\theta}, e^{-2i\theta}, e^{-2i\theta}, e^{3i\theta}, e^{3i\theta}) \ . \tag{4.2}$$

For appropriate symmetry breaking to occur, the image of $\mathbb{Z}_{12}$ must lie in this subgroup, so there are eleven possible choices for the Wilson lines, given by $g_{12} \mapsto e^{\frac{\pi i}{6}k} \in U(1)_Y$, where $k = 1, \ldots, 11$.

Standard model matter resides in the **10** and $\overline{\mathbf{5}}$ of $SU(5)_{\mathrm{GUT}}$, which are the rank-two antisymmetric tensor and the anti-fundamental representation, respectively. Since the symmetry is broken by the discrete Wilson lines, different components of these representations will carry different $\mathbb{Z}_{12}$ 'charges', which are collected in Table 1. Massless fields on X come from cohomology classes whose $\mathbb{Z}_{12}$ transformation is conjugate to that under the Wilson line, so that they combine to form an invariant.

Field	u^c	Q	e^c	d^c	L, H_d
$SU(5)$ provenance	**10**	**10**	**10**	$\overline{\mathbf{5}}$	$\overline{\mathbf{5}}$
G_{SM} rep.	$(\overline{\mathbf{3}}, \mathbf{1})_{-4}$	$(\mathbf{3}, \mathbf{2})_1$	$(\mathbf{1}, \mathbf{1})_6$	$(\overline{\mathbf{3}}, \mathbf{1})_2$	$(\mathbf{1}, \mathbf{2})_{-3}$
$\mathbb{Z}_{12}$ charge	$8k$	k	$6k$	$2k$	$9k$

Table 1: The standard model matter representations, their origin in $SU(5)_{\mathrm{GUT}}$ representations, and their 'charges' under the discrete $\mathbb{Z}_{12}$-valued Wilson line.

We pause here briefly to clear up a possible point of confusion. In Section 3.4 we explained that choosing a non-trivial equivariant structure on the undeformed bundle $\widetilde{V}_0 = T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$ was equivalent to turning on discrete Wilson lines on X which commute with the $SU(3)$ holonomy group of the tangent bundle, and therefore must lie in E_6. In this section, we have claimed that we are free to choose discrete Wilson line values in $SU(5)_{\mathrm{GUT}} \subset E_6$. Of course, we can only choose the value of the Wilson lines once, so why is this allowed?

Note that the Wilson lines of Section 3.4 commute with $SU(5)_{\mathrm{GUT}}$, and therefore with the hypercharge group (4.2). So the full picture is that, before deforming the bundle, the discrete

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[9]We use the mathematically natural normalisation of hypercharge; conventional hypercharge assignments are a factor of three (or sometimes six) smaller.

Wilson lines lie in E_6 and are given by the product of the element specified by n_1, n_2, as described in Section 3.4 and the element of $U(1)_Y$ specified by k, described in this section. This has the effect of breaking E_6 to some smaller rank-six group such as[10] $SU(4) \times SU(2) \times U(1)^2$ or $SU(3) \times SU(2) \times U(1)^3$; the possibilities were described long ago in [33]. When we deform to an irreducible rank-five bundle, this is physically equivalent to Higgsing this extended gauge group to G_{SM}. Mathematically, the piece of the Wilson line which lies in $SU(5)$ gets mixed up with the continuous holonomy group of the bundle, whereas the piece in $U(1)_Y$ breaks $SU(5)_{\mathrm{GUT}}$ to G_{SM}. In this way, contact is made between our techniques and the older literature about heterotic standard embedding models, although we emphasise again that our top down approach of directly constructing the bundles is much more powerful than effective field theory arguments.

4.2   Exotic states from $\mathbf{10} \oplus \overline{\mathbf{10}}$

We will begin by calculating $H^1(\widetilde{X}, \widetilde{V})$, corresponding to massless chiral multiplets in the $\mathbf{10}$ of $SU(5)_{\mathrm{GUT}}$. Since $h^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ for $i = 1, 2$, the long-exact cohomology sequence following from the leftmost column of (3.11) yields in part

$$0 \longrightarrow H^1(\widetilde{X}, \mathcal{F}) \longrightarrow H^1(\widetilde{X}, \widetilde{V}) \longrightarrow 0 \;,$$

so $H^1(\widetilde{X}, \widetilde{V}) \cong H^1(\widetilde{X}, \mathcal{F})$, and we need to calculate the latter group. This fits into a long exact sequence following from the middle row of (3.11), the relevant part of which is

$$
\begin{aligned}
0 &\longrightarrow H^0(\widetilde{X}, \mathcal{F}) \longrightarrow H^0(\widetilde{X}, \oplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i)) \longrightarrow H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\textstyle\sum_i D_i)) \longrightarrow \\
&\longrightarrow H^1(\widetilde{X}, \mathcal{F}) \longrightarrow H^1(\widetilde{X}, \oplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i)) \longrightarrow \ldots
\end{aligned}
$$

Using $H^0(\widetilde{X}, \mathcal{F}) \cong \mathbb{C}^6$ and results from Appendix A, this becomes

$$
\begin{aligned}
0 &\longrightarrow \mathbb{C}^6 \longrightarrow \mathbb{C}^{12} \longrightarrow \mathbb{C}^{48} \longrightarrow \\
&\longrightarrow H^1(\widetilde{X}, \mathcal{F}) \longrightarrow 0 \longrightarrow \ldots
\end{aligned}
$$

where $\mathbb{C}^{12} \sim \mathrm{Reg}_{\mathbb{Z}_{12}}$ and $\mathbb{C}^{48} \sim 4 * \mathrm{Reg}_{\mathbb{Z}_{12}}$. Our final result is therefore

$$H^1(\widetilde{X}, \mathcal{F}) \cong \mathbb{C}^{42} \sim 3 * \mathrm{Reg}_{\mathbb{Z}_{12}} \oplus H^0(\widetilde{X}, \mathcal{F}) \;,$$

and we remind the reader that we are free to choose $H^0(\widetilde{X}, \mathcal{F})$ to transform as any sub-representation of (3.12). Finally, we recall that the equivariant structure on $\widetilde{V}$ had to be twisted by the $\mathbb{Z}_{12}$ representation $\tilde{\mathbf{n}}$, where $\tilde{n}$ is determined by (3.13), so this extra phase acts on the cohomology as well. Taking into account that $\tilde{n} \otimes \mathrm{Reg}_{\mathbb{Z}_{12}} = \mathrm{Reg}_{\mathbb{Z}_{12}}$ for any $\tilde{n}$, we get

$$H^1(\widetilde{X}, \widetilde{V}) \sim 3 * \mathrm{Reg}_{\mathbb{Z}_{12}} \oplus \left(\tilde{\mathbf{n}} \otimes H^0(\widetilde{X}, \mathcal{F}) \right) \;. \tag{4.3}$$

---

[10]Given n_1, n_2 and k, it is not too difficult to calculate the unbroken gauge group explicitly, but this is somewhat tedious, and adds nothing to the current work.

We wish to find models with massless fields filling out precisely three copies of the **10**. The regular representation, $\text{Reg}_{\mathbb{Z}_{12}}$, contains each irreducible representation of $\mathbb{Z}_{12}$ exactly once, so *any* choice of Wilson lines will lead to three massless copies of the **10** coming from $3 * \text{Reg}_{\mathbb{Z}_{12}}$. We therefore ask that no states originating in $H^0(\widetilde{X}, \mathcal{F})$ survive the projection. Referring to Table 1, we see that this means choosing k such that $(\mathbf{k} \oplus \mathbf{6k} \oplus \mathbf{8k}) \otimes \tilde{\mathbf{n}} \otimes H^0(\widetilde{X}, \mathcal{F}) \not\supset \mathbf{0}$.

Multiplets transforming under the $\overline{\mathbf{10}}$ representation of $SU(5)_{\text{GUT}}$ come from the cohomology group $H^1(\widetilde{X}, \widetilde{V}^*)$, but there is no need to calculate this independently, as we will now explain. First we note that Serre duality on $\widetilde{X}$ implies an isomorphism $H^1(\widetilde{X}, \widetilde{V}^*) \cong H^2(\widetilde{X}, \widetilde{V})^*$, where the superscript * on the right-hand side indicates the dual vector space, and the latter is easy to calculate. Indeed, since $H^0(\widetilde{X}, \widetilde{V}) = H^3(\widetilde{X}, \widetilde{V}) = 0$ (the vanishing of H^3 is demonstrated explicitly in Appendix B), the Euler characteristic of $\widetilde{V}$ is simply

$$\chi(\widetilde{V}) = h^2(\widetilde{X}, \widetilde{V}) - h^1(\widetilde{X}, \widetilde{V}) \ ,$$

and we know that $\chi(\widetilde{V}) = -36$, since $\widetilde{V}$ is a deformation of $T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$, and χ does not change under deformation. So $h^2(\widetilde{X}, \widetilde{V})$ is determined indirectly by knowing $h^1(\widetilde{X}, \widetilde{V})$ and $\chi(\widetilde{V})$. Furthermore, since $\widetilde{V}$ is equivariant under the fixed-point-free group action, χ actually admits a simple refinement. For any irreducible representation $\mathbf{n}$ of $\mathbb{Z}_{12}$, let $h^i(\widetilde{X}, \widetilde{V})_{\mathbf{n}}$ be the number of times this representation appears in the decomposition of $H^i(\widetilde{X}, \widetilde{V})$. Then[11]

$$h^2(\widetilde{X}, \widetilde{V})_{\mathbf{n}} - h^1(\widetilde{X}, \widetilde{V})_{\mathbf{n}} = \frac{1}{12} \chi(\widetilde{V}) = -3 \ .$$

So if a particular representation occurs k times in $H^1(\widetilde{X}, \widetilde{V})$, it necessarily occurs $k - 3$ times in $H^2(\widetilde{X}, \widetilde{V})$. This means that the *conjugate* representation occurs $k - 3$ times in $H^1(\widetilde{X}, \widetilde{V}^*)$, since Serre duality involves a vector space dualisation. These fields transform in the $\overline{\mathbf{10}}$ of $SU(5)_{\text{GUT}}$, so the Wilson lines also act on them in the conjugate representation. Altogether, then, the fields projected out of $H^1(\widetilde{X}, \widetilde{V}^*)$ when taking the quotient are precisely the conjugates of those projected out of $H^1(\widetilde{X}, \widetilde{V})$, so any massless fields extraneous to the three generations of **10** will occur in vector-like pairs.

The final result is that there are 43 models which have no exotic states originating in the $\mathbf{10} \oplus \overline{\mathbf{10}}$ of $SU(5)_{\text{GUT}}$. The values of n_1, n_2 and k for these models are given in Table 2.

4.3 Doublet-triplet splitting

To find the massless states coming from the **5** and $\overline{\mathbf{5}}$ of $SU(5)_{\text{GUT}}$, we must calculate the cohomology group $H^1(\widetilde{X}, \wedge^2 \widetilde{V})$, and the $\mathbb{Z}_{12}$ representation which acts on it. It will prove more convenient to instead calculate $H^1(\widetilde{X}, \wedge^2 \widetilde{V}^*)$, and apply the arguments of the last section to relate one to the other, but this is still significantly more involved than our previous calculations. One tactic we will use repeatedly is to dualise bundles and appeal to Serre duality, in order to move as much

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[11]In the more general case of a group G and a representation $\mathbf{R}$, the fraction $\frac{1}{12}$ here is replaced by $\frac{\dim \mathbf{R}}{|G|}$.

$(n_1\,,n_2)$	k
$(0\,,3)$	2, 4, 8, 10
$(0\,,6)$	6
$(0\,,9)$	2, 4, 8, 10
$(2\,,3)$	6, 10
$(2\,,8)$	3, 6, 9
$(2\,,9)$	6, 10
$(3\,,4)$	4, 6, 7, 8, 10
$(3\,,8)$	2, 4, 6, 8, 11
$(3\,,10)$	2, 6
$(4\,,9)$	1, 4, 6, 8, 10
$(4\,,10)$	3, 6, 9
$(8\,,9)$	2, 4, 5, 6, 8
$(9\,,10)$	2, 6

Table 2: A list of the 43 models, specified by the three discrete bundle parameters n_1, n_2 and k, which have no exotic massless fields descending from the $\mathbf{\overline{10}}$ of $SU(5)_{\mathrm{GUT}}$.

non-trivial cohomology as possible into H^0 i.e. global sections. These are much easier to work with, as they are represented simply by homogeneous polynomials, perhaps evaluated modulo some ideal.

We start with the short exact sequences which form the middle column and bottom row of (3.11), defining respectively the bundles $\mathcal{G}$ on Z, and $\widetilde{V}$ on the hypersurface $\widetilde{X}$. Dualising these yields[12]

$$0 \longrightarrow \mathcal{G}^* \longrightarrow \bigoplus_{i=1}^{12} \mathcal{O}_Z(-D_i) \longrightarrow 6\mathcal{O}_Z \longrightarrow 0 \ , \tag{4.4}$$

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(-\sum_{i=1}^{12} D_i) \longrightarrow \mathcal{G}^*|_{\widetilde{X}} \longrightarrow \widetilde{V}^* \longrightarrow 0 \ . \tag{4.5}$$

To get a handle on $\wedge^2\widetilde{V}^*$, we need something more. For any short exact sequence

$$0 \longrightarrow A \xrightarrow{\psi_1} B \xrightarrow{\psi_2} C \longrightarrow 0 \ ,$$

there are associated exact sequences

$$0 \longrightarrow \wedge^2 A \longrightarrow \wedge^2 B \longrightarrow B \otimes C \longrightarrow S^2 C \longrightarrow 0 \ , \tag{4.6}$$

$$0 \longrightarrow S^2 A \longrightarrow A \otimes B \longrightarrow \wedge^2 B \longrightarrow \wedge^2 C \longrightarrow 0 \ , \tag{4.7}$$

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[12]Note that, although $\mathcal{O}_Z{}^* \cong \mathcal{O}_Z$, the dualisation has the effect of conjugating the equivariant structure. This will be important when we are calculating the group action on cohomology.

where S^2 denotes the second symmetric power of a bundle, and the maps are constructed in the obvious ways from ψ_1 and ψ_2. So from (4.5), we can find an exact sequence for $\wedge^2 \widetilde{V}^*$,

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(-2\sum_{i=1}^{12} D_i) \longrightarrow \mathcal{G}^*(-\sum_{i=1}^{12} D_i)|_{\widetilde{X}} \longrightarrow \wedge^2 \mathcal{G}^*|_{\widetilde{X}} \longrightarrow \wedge^2 \widetilde{V}^* \longrightarrow 0 \ .$$

If we introduce a bundle K_1, defined as the kernel of the map $\wedge^2 \mathcal{G}^*|_{\widetilde{X}} \to \wedge^2 \widetilde{V}^*$ (equivalently, the image of the previous map), this splits into two short exact sequences

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(-2\sum_{i=1}^{12} D_i) \longrightarrow \mathcal{G}^*(-\sum_{i=1}^{12} D_i)|_{\widetilde{X}} \longrightarrow K_1 \longrightarrow 0 \ , \tag{4.8}$$

$$0 \longrightarrow K_1 \longrightarrow \wedge^2 \mathcal{G}^*|_{\widetilde{X}} \longrightarrow \wedge^2 \widetilde{V}^* \longrightarrow 0 \ . \tag{4.9}$$

In line with our general approach of considering only non-zero H^0 if possible, we will in fact consider the dual of (4.8),

$$0 \longrightarrow K_1^* \longrightarrow \mathcal{G}(\sum_{i=1}^{12} D_i)|_{\widetilde{X}} \longrightarrow \mathcal{O}_{\widetilde{X}}(2\sum_{i=1}^{12} D_i) \longrightarrow 0 \ .$$

We know from Appendix A.3 that the second and third bundles here have $H^i = 0$ for $i > 0$, so we immediately get $H^i(\widetilde{X}, K_1^*) = 0$ for $i = 2, 3$. The other two groups fit into an exact sequence

$$0 \longrightarrow H^0(\widetilde{X}, K_1^*) \longrightarrow \mathbb{C}^{372} \longrightarrow \mathbb{C}^{312} \longrightarrow H^1(\widetilde{X}, K_1^*) \longrightarrow 0 \ .$$

The middle map here is a complicated one, induced by Φ, between large vector spaces of polynomials reduced modulo the defining polynomial of our hypersurface. Nevertheless, it is possible to show using computer algebra that this map is surjective, so that in particular, $H^1(\widetilde{X}, K_1^*) = 0$. So now we have $H^1(\widetilde{X}, K_1^*) = H^2(\widetilde{X}, K_1^*) = 0$, and Serre duality tells us that the same groups vanish for K_1. This information combines with the long exact cohomology sequence from (4.9) to give us the simple result

$$H^1(\widetilde{X}, \wedge^2 \widetilde{V}^*) \cong H^1(\widetilde{X}, \wedge^2 \mathcal{G}^*) \ .$$

We now have to compute this latter group. First we will relate it to cohomology on the ambient space Z, which can be calculated directly. The short exact sequence relating $\wedge^2 \mathcal{G}^*|_{\widetilde{X}}$ to $\wedge^2 \mathcal{G}^*$ is

$$0 \longrightarrow \wedge^2 \mathcal{G}^*\big(-\sum_{i=1}^{12} D_i\big) \longrightarrow \wedge^2 \mathcal{G}^* \longrightarrow \wedge^2 \mathcal{G}^*|_{\widetilde{X}} \longrightarrow 0 \ . \tag{4.10}$$

In Appendix A.2, we show that $H^i(Z, \wedge^2 \mathcal{G}^*(-\sum_i D_i)) = 0$ for $i < 4$. Plugging this into the long exact sequence following from (4.10) gives us the simple result that $H^1(\widetilde{X}, \wedge^2 \mathcal{G}^*) = H^1(Z, \wedge^2 \mathcal{G}^*)$, so we have succeeded in lifting the required group to the ambient space.

Now return to (4.4), which defines $\mathcal{G}^*$ in terms of line bundles. Applying the general result (4.7) to this case yields another four-term exact sequence, which we again split into two short

exact sequences by introducing a kernel K_2,

$$0 \longrightarrow \wedge^2 \mathcal{G}^* \longrightarrow \bigoplus_{i<j} \mathcal{O}_Z(-D_i - D_j) \longrightarrow K_2 \longrightarrow 0 \ , \tag{4.11}$$

$$0 \longrightarrow K_2 \longrightarrow 6 \bigoplus_i \mathcal{O}_Z(-D_i) \longrightarrow 21\mathcal{O}_Z \longrightarrow 0 \ . \tag{4.12}$$

The middle term in (4.12) has vanishing cohomology, and the third term has $H^i = 0$ for $i > 0$, so we immediately get $H^i(Z, K_2) = 0$ for $i \neq 1$, and

$$H^1(Z, K_2) \ \cong \ H^0(Z, 21\mathcal{O}_Z) \ \cong \ \mathbb{C}^{21} \ .$$

We can also take the cohomology of the middle term of (4.11) from Appendix A; the only important result is $H^1 \cong \mathbb{C}^{18}$. So the long exact cohomology sequence coming from (4.11) reads, in part,

$$0 \longrightarrow H^1(Z, \wedge^2 \mathcal{G}^*) \longrightarrow \mathbb{C}^{18} \longrightarrow \mathbb{C}^{21} \longrightarrow \ldots \quad . \tag{4.13}$$

So to summarise, we have shown that $H^1(\widetilde{X}, \wedge^2 \widetilde{V})$ is isomorphic to the kernel of the map from $\mathbb{C}^{18}$ to $\mathbb{C}^{21}$ above, where these two groups are in fact respectively $H^1(Z, \bigoplus_{i<j} \mathcal{O}_Z(-D_i - D_j))$ and $H^1(Z, K_2) \cong H^0(Z, 21\mathcal{O}_Z)$. Finding this kernel is a tedious calculation in Čech cohomology, and we will omit the details, except to note that the task is simplified by the fact that these cohomology groups are all generated by cocycles which are invariant under the torus action, and are therefore represented by constant sections on each open patch.

Of course, we also need to keep track of the $\mathbb{Z}_{12}$ representations, noting that since the equivariant structure on $\widetilde{V}$ is twisted by $\mathbf{\tilde{n}}$, the action on the cohomology of $\wedge^2 \widetilde{V}$ gets tensored by $\mathbf{2\tilde{n}}$.

Referring to Appendix A, we see that the generators of $H^1(Z, \bigoplus_{i<j} \mathcal{O}_Z(-D_i-D_j)) \cong \mathbb{C}^{18}$ come from eighteen distinct summands of the bundle. Specifically, we have $H^1(Z, \mathcal{O}_Z(-D_1 - D_3)) \cong \mathbb{C}$, and similarly for the eleven other line bundles related to this one by the $\mathbb{Z}_{12}$ action. The rest of the cohomology group originates in $H^1(Z, \mathcal{O}_Z(-D_1 - D_4)) \cong \mathbb{C}$, and the five other bundles related to this one by $\mathbb{Z}_{12}$.

So the twelve bundles related to $\mathcal{O}_Z(-D_1 - D_3)$ by $\mathbb{Z}_{12}$ obviously contribute one copy of $\text{Reg}_{\mathbb{Z}_{12}}$ to the cohomology. There is an extra subtlety associated with the bundles related to $\mathcal{O}_Z(-D_1 - D_4)$, because the order-two element of the group maps $\mathcal{O}_Z(-D_1 - D_4)$ to itself (since $g_{12}^6 : D_1 \leftrightarrow D_4$), and might do so with or without a minus sign. There are two things to consider in deciding which sign occurs. Firstly, note that the bundle we are discussing, $\bigoplus_{i<j} \mathcal{O}_Z(-D_i - D_j)$, arises as the second anti-symmetric power of $\bigoplus_i \mathcal{O}_Z(D_i)$, so there is a minus sign associated with exchanging the two factors of $\mathcal{O}_Z(-D_1 - D_4) = \mathcal{O}_Z(-D_1) \otimes \mathcal{O}_Z(-D_4)$. Secondly, we write down an explicit cocycle representing the generator of $H^1(Z, \mathcal{O}_Z(-D_1 - D_4))$, act on it with g_{12}^6, and ask whether the result is cohomologous to what we started with or its negative. It turns out that a minus sign occurs here as well, so overall, g_{12}^6 acts trivially on $H^1(Z, \mathcal{O}_Z(-D_1 - D_4))$. Therefore

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the $\mathbb{Z}_{12}$ action on the six summands of this type factors through $\mathbb{Z}_6$, and in this way corresponds to $\mathrm{Reg}_{\mathbb{Z}_6} \cong \mathbf{0} \oplus \mathbf{2} \oplus \mathbf{4} \oplus \mathbf{6} \oplus \mathbf{8} \oplus \mathbf{10}$. So altogether,

$$H^1(Z, \bigoplus_{i<j} \mathcal{O}_Z(-D_i - D_j)) \sim \mathrm{Reg}_{\mathbb{Z}_{12}} \oplus \mathrm{Reg}_{\mathbb{Z}_6} . \tag{4.14}$$

To find the representation content of $H^1(Z, K_2) \cong H^0(Z, 21\mathcal{O}_Z)$, recall that $21\mathcal{O}_Z$ occurs as the second symmetric power of $6\mathcal{O}_Z$, which appears in (4.4). The only subtlety is to remember that this sequence arose as the *dual* to the sequence in (3.11). Obviously $\mathcal{O}_Z$ is its own dual, but we must remember to dualise the action on the cohomology. This gives

$$H^1(Z, K_2) \sim S^2(H^0(\widetilde{X}, \mathcal{F})^*) . \tag{4.15}$$

So we now have the representation content of the two relevant terms in the sequence (4.13), which allows us to split it into twelve sequences, one for each $\mathbb{Z}_{12}$ irrep. This step must be done separately for each choice of the bundle parameters n_1, n_2, but if we are looking for the MSSM spectrum, we can restrict ourselves to those which appear in Table 2.

4.4 Models with the MSSM spectrum

Let us choose one model to discuss in detail (with hindsight, we choose one which yields the correct spectrum). Let $n_1 = 3$ and $n_2 = 4$. Then from (3.13) we get $\tilde{n} = 1$, and from the discussion in Section 3.2 and Section 3.3, $H^0(\widetilde{X}, \mathcal{F}) \sim \mathbf{0} \oplus \mathbf{2} \oplus \mathbf{6} \oplus \mathbf{8} \oplus \mathbf{9} \oplus \mathbf{10}$. Plugging this into (4.3), we find

$$H^1(\widetilde{X}, \widetilde{V}) \sim 3 * \mathrm{Reg}_{\mathbb{Z}_{12}} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{7} \oplus \mathbf{9} \oplus \mathbf{10} \oplus \mathbf{11}$$

We now ask which values of k we can choose for the Wilson lines (see Section 4.1) such that we get exactly three copies of the $\mathbf{10}$ from this cohomology group. As an example, take $k = 4$. Consulting Table 1, we see that the three different components of the $\mathbf{10}$ then have $\mathbb{Z}_{12}$ charges 8, 4 and 0. Taking the tensor product of $\mathbf{0} \oplus \mathbf{4} \oplus \mathbf{8}$ with $\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{7} \oplus \mathbf{9} \oplus \mathbf{10} \oplus \mathbf{11}$ does not yield any invariants, so for this choice, there are no extra massless states originating in the $\mathbf{10} \oplus \overline{\mathbf{10}}$ of $SU(5)_{\mathrm{GUT}}$.

Now we turn to the doublet-triplet splitting. From (4.15) and $H^0(\widetilde{X}, \mathcal{F})$ above, we find that

$$H^1(\widetilde{X}, K_2) \sim 3 * \mathbf{0} \oplus \mathbf{1} \oplus 2 * \mathbf{2} \oplus \mathbf{3} \oplus 3 * \mathbf{4} \oplus \mathbf{5} \oplus 3 * \mathbf{6} \oplus \mathbf{7} \oplus 3 * \mathbf{8} \oplus \mathbf{9} \oplus 2 * \mathbf{10} .$$

Group equivariance means that the map from $\mathbb{C}^{18}$ to $\mathbb{C}^{21}$ breaks up into blocks, one for each irreducible $\mathbb{Z}_{12}$ representation. For example, we know from (4.14) that regardless of our choice of n_1, n_2, $\mathbb{C}^{18}$ contains one copy of $\mathbf{11}$. But in the present case, $\mathbb{C}^{21}$ contains no instance of this representation, so the kernel of the map must contain exactly one $\mathbf{11}$. For the other representations, we must do explicit calculations, and we find

$$H^1(Z, \wedge^2 \mathcal{G}^*) \sim \mathbf{10} \oplus \mathbf{11} .$$

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Finally, noting that $2 * (3 + 4) \equiv 2 \bmod 12$, we tensor this representation with $\mathbf{2}$ to obtain

$$H^1(\widetilde{X}, \wedge^2 \widetilde{V}^*) \sim \mathbf{0} \oplus \mathbf{1} \ . \tag{4.16}$$

Referring to Table 1, and recalling that the above corresponds to states in the $\mathbf{5}$ rather than $\overline{\mathbf{5}}$, we see that the triplets will have $\mathbb{Z}_{12}$ charge $-2k \equiv 10k$ under the Wilson line, while the doublets will have charge $-9k \equiv 3k$. For our choice $k = 4$, then, this corresponds to the representations $\mathbf{4}$ and $\mathbf{0}$ respectively. For the doublets, this $\mathbf{0}$ pairs up with the $\mathbf{0}$ in (4.16) to give us an invariant, and therefore a single massless up-type Higgs doublet (the down-type doublet is its partner from $H^1(\widetilde{X}, \wedge^2 \widetilde{V})$, guaranteed by the index theorem), while for the triplets, there is no such invariant.

We have shown that in the case $(n_1, n_2) = (3, 4)$ and $k = 4$, we obtain a model with exactly the massless spectrum of the MSSM. In total, there are eight choices which work equally well; they are listed in Table 3.

(n_1, n_2)	k
$(3, 4)$	$4, 8$
$(3, 8)$	$4, 8$
$(4, 9)$	$4, 8$
$(8, 9)$	$4, 8$

Table 3: The values of the discrete bundle parameters n_1, n_2, and corresponding Wilson line parameters k, which lead to models with exactly the light spectrum of the MSSM.

5   Discussion and conclusion

This paper achieves the long-standing goal of obtaining the MSSM spectrum from the heterotic string by deforming a three generation standard embedding solution [34]. The obvious difference to more general heterotic compactifications is the absence of M5-branes or a non-trivial gauge bundle in the hidden sector, as the anomaly cancellation condition is saturated by the second Chern class of the visible sector bundle.

It is interesting to ask whether one might be able to obtain the MSSM spectrum by supersymmetric deformation of some other three generation standard embedding solution(s). In order to break $SU(5)_{\text{GUT}}$ to G_{SM} with discrete Wilson lines, the Calabi-Yau manifold Y must have nontrivial fundamental group, and therefore be obtained as a quotient $Y = \widetilde{Y}/G$ for some finite group G, while to yield three generations, its topological Euler characteristic must satisfy $\chi(Y) = \pm 6$.

What are the possible choices for Y? In fact, there are relatively few Calabi-Yau threefolds with non-trivial fundamental group, and very few of these have $\chi = \pm 6$. Here we have taken $\widetilde{X}/\mathbb{Z}_{12}$, where the Hodge numbers of $\widetilde{X}$ are $(h^{1,1}, h^{2,1}) = (8, 44)$. This manifold admits a different free quotient by the order-twelve non-Abelian group Dic_3 with the same Hodge numbers, but as

we discuss in Appendix C, there are no MSSM models on this quotient. The mirror manifold also admits at least one free quotient with $(h^{1,1}, h^{2,1}) = (4,1)$ [15], and this might admit MSSM models, although these are probably equivalent to models on the $(1,4)$ manifolds by $(0,2)$ mirror symmetry [35].

In [36], a manifold was constructed with $(h^{1,1}, h^{2,1}) = (2,5)$ and fundamental group $\mathbb{Z}_5$. With this fundamental group, $SU(5)_{\mathrm{GUT}}$ cannot be broken to G_{SM} by Wilson lines.

The most famous three generation manifold is perhaps Yau's manifold [37], which is a $\mathbb{Z}_3$ quotient of a simply-connected manifold with Hodge numbers $(h^{1,1}, h^{2,1}) = (14,23)$. If we repeat the arguments of Section 4.2 for this example, it is easy to see that we cannot arrange for all the $\overline{\mathbf{10}}$ states to be projected out by the quotient. This is because of the relatively large value of $h^{1,1}$ for the covering space, which means that the three irreducible representations of $\mathbb{Z}_3$ each appear several times, and Wilson lines cannot be chosen to project them all out. The same problem will occur for all other known three generation manifolds, as they all have relatively small fundamental groups and/or large Hodge numbers [30, 36, 38].

So it seems that, at least if we restrict ourselves to known Calabi-Yau threefolds, the models we have found are the only way to obtain the exact massless spectrum of the MSSM from deforming the heterotic standard embedding. Of course, this is by no means enough for a *realistic* model of particle physics. In particular, we have not yet attempted to calculate the Yukawa couplings or address the problems of supersymmetry breaking and moduli stabilisation.

Acknowledgements

A Cohomology calculations

The Calabi-Yau $\widetilde{X}$ is defined as a hypersurface in Z by $f = 0$, where f is a section of the anticanonical bundle $\mathcal{O}_Z(-K_Z) \cong \mathcal{O}_Z(\sum_i D_i)$, where $\{D_i\}$ are the twelve toric divisors. As such, for any holomorphic vector bundle B, defined on Z, we have the following short exact sequence,

$$0 \longrightarrow B\big(-\sum_i D_i\big) \xrightarrow{f} B \longrightarrow B|_{\widetilde{X}} \longrightarrow 0 . \tag{A.1}$$

This allows us to 'lift' the important cohomology calculations to Z, where they are a lot simpler.

The fourfold Z is in fact a product, $Z \cong S_1 \times S_2$, where each surface is isomorphic to the del Pezzo surface dP_6. There are corresponding projections $\pi_l : Z \to S_l$, and all the relevant line bundles on Z can be written as $\pi_1^* \mathcal{L}_1 \otimes \pi_2^* \mathcal{L}_2$. In this case we have the Künneth formula

$$H^i(Z, \pi_1^* \mathcal{L}_1 \otimes \pi_2^* \mathcal{L}_2) \cong \bigoplus_{j+k=i} H^j(S_1, \mathcal{L}_1) \otimes H^k(S_2, \mathcal{L}_2) .$$

Many of our calculations therefore boil down to line bundle cohomology on dP_6, which we discuss in Appendix A.1. Cohomology of $\mathcal{G}$ and related bundles on Z is then calculated in Appendix A.2. Finally, these results are used in Appendix A.3 to calculate the required cohomology groups defined on the hypersurface $\widetilde{X}$. We will use the notation $h^\bullet(M, B) = (h^0, \dots, h^{\dim M})$ for the dimensions of the cohomology groups of a bundle B on a manifold M.

A.1 Line bundle cohomology on dP_6

The surface $S \cong \mathrm{dP}_6$ is toric, which makes it relatively easy to calculate line bundle cohomology using the Čech approach. Throughout this section, we will refer to the six vectors $\{\nu_a\}$ from (2.1), and the corresponding toric divisors $\mathcal{D}_a$. We will also use N and M to refer to the two-dimensional lattices relevant for S, rather than the four-dimensional lattices from Section 2.

Instead of thinking about line bundles in terms of local patches and transition functions, it is clearer to utilise the divisor-line bundle correspondence. So for a toric divisor $\mathcal{D} = \sum_{a=1}^6 c_a \mathcal{D}_a$, a section of $\mathcal{O}_S(\mathcal{D})$ over an open patch $\mathcal{U}$ is a meromorphic function f, defined on $\mathcal{U}$, such that $(f) + \mathcal{D}|_{\mathcal{U}} \geq 0$.

Each point u of the lattice M corresponds to an irreducible character of the torus $(\mathbb{C}^*)^2$, as well as a meromorphic function f_u on S, which transforms under the torus action according to the corresponding character. For any cone σ in the fan for dP_6, the sections of $\mathcal{D}$ over the corresponding open set are given by

$$\Gamma(\mathcal{U}_\sigma, \mathcal{O}_S(\sum_{a=1}^6 c_a \mathcal{D}_a)) = \langle f_u \mid u \in M , \ \langle u, \nu_a \rangle \geq -c_a \ \forall \ \nu_a \in \sigma \rangle . \tag{A.2}$$

In this way, the cohomology of any toric line bundle is graded by M. Note that global sections are particularly simple; since they come from points $u \in M$ satisfying the above in all open sets, f_u contributes to $H^0(S, \mathcal{O}_S(\sum_{a=1}^6 c_a \mathcal{D}_a))$ if and only if

$$\langle u, \nu_a \rangle \geq -c_a \ \forall \ a . \tag{A.3}$$

Finally, note that the fan for S has the symmetry of the hexagon, D_6, and the surface inherits a faithful action of this group. This means that any bundles related by a D_6 transformation will have the same cohomology groups (the toric weights will be related by the corresponding dual transformation of the lattice M).

The rest of this section presents the cohomology of line bundles on S which are needed in Section 4. Throughout this section, the indices $a, b, \ldots$ will run from 1 to 6, and arithmetical operations on them will be understood modulo 6. We will take an open cover of S consisting of six sets $\mathcal{U}_a$ corresponding to the six two-dimensional cones in Figure 1.

A.1.1 Explicit calculations

We will present in detail the calculation of the cohomology of one particular line bundle (or class of line bundles, related by symmetry), and then simply list the other results we need.

$\underline{\mathcal{O}_S(-\mathcal{D}_a - \mathcal{D}_{a+2})}$

To be explicit, we will consider $\mathcal{O}_S(-\mathcal{D}_1 - \mathcal{D}_3)$. It is easy to convince oneself that this has no global sections, so $H^0 = 0$. To find H^2, we Serre dualise:

$$H^2(S, \mathcal{O}_S(-\mathcal{D}_1 - \mathcal{D}_3)) \cong H^0(S, \mathcal{O}_S(\mathcal{D}_1 + \mathcal{D}_3 - \sum_a \mathcal{D}_a))^* \ .$$

Again, it is not hard to see that the group on the right is zero by considering the inequalities (A.3). So the only cohomology group which might be non-vanishing is H^1, and in this case we have simply $h^1 = -\chi$, where χ is the Euler characteristic of the bundle. This can be calculated from the Hirzebruch-Riemann-Roch formula, which gives $\chi = -1$. So $H^1(S, \mathcal{O}_S(-\mathcal{D}_1 - \mathcal{D}_3)) \cong \mathbb{C}$.

In many cases, knowing the dimension of cohomology groups is enough, but the hardest calculation in this paper is finding the kernel of the map in (4.13), and for this we need explicit representatives for the cohomology classes. Some of these live in the H^1 groups of the type we have just calculated, so our task is not complete.

First, we ask which points of the lattice M might contribute to H^1. Notice from (A.2) that the contribution of weight $u \in M$ to the Čech cochain groups depends only on whether $\langle u, \nu_a \rangle \geq 0$ or $\langle u, \nu_a \rangle < 0$, for each value of a. This divides the lattice M into various 'chambers', with the contribution of $u \in M$ to the cohomology depending only on the chamber to which it belongs. Since S is compact, all cohomology groups must be finite, so we need only consider chambers containing a finite number of points. In this case there turns out to be only one of these, and the unique point it contains is the origin, $0 \in M$, which corresponds to locally constant sections.

Referring again to (A.2), we can write down explicit parametrisations of the Čech cochain groups $\check{C}^0$ and $\check{C}^1$ with weight $0 \in M$,

$$\check{C}^0(\mathcal{O}_S(-\mathcal{D}_1 - \mathcal{D}_3)) = \{(0, 0, 0, \alpha_1, \alpha_2, 0) \mid \alpha_i \in \mathbb{C}\} \ ,$$

$$\check{C}^1(\mathcal{O}_S(-\mathcal{D}_1 - \mathcal{D}_3)) = \{\beta_1, \beta_2, \beta_3, \beta_4, 0, 0, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{13} \mid \beta_i \in \mathbb{C}\} \ ,$$

where the open sets $\mathcal{U}_a$ are ordered in the obvious way, and double overlaps $\mathcal{U}_{a,b} = \mathcal{U}_a \cap \mathcal{U}_b$ are ordered first by a and then by b. Acting on the latter group with the Čech differential d_1, we find a three-dimensional kernel. Obviously the image of d_0 is two-dimensional (since $\check{C}^0$ has two parameters α_1, α_2, and $H^0 = 0$), so as expected, we find a one-dimensional cohomology group, the generator of which can be taken to be the class of the cocycle

$$(-1, -1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0) \ \in \ \check{C}^1(\mathcal{O}_S(-\mathcal{D}_1 - \mathcal{D}_3)) \ .$$

By symmetry, we have $h^\bullet(S, \mathcal{O}_S(-D_a - D_{a+2})) = (0, 1, 0)$ for any value of a, and we can obtain representatives for the cohomology generators by acting on the one above with the appropriate permutation (recalling that there is a sign change associated with changing $\mathcal{U}_{a,b}$ to $\mathcal{U}_{b,a}$).

Other required cohomology dimensions (some of these results follow from others by Serre duality) are:

$$h^\bullet(S, \mathcal{O}_S) = (1, 0, 0) \ , \ \ h^\bullet(S, \mathcal{O}_S(-D_a)) = (0, 0, 0) \ ,$$

$$h^\bullet(S, \mathcal{O}_S(-D_a - D_{a+1})) = (0, 0, 0) \ , \ \ h^\bullet(S, \mathcal{O}_S(-D_a - D_{a+3})) = (0, 1, 0) \ ,$$

$$h^\bullet(S, \mathcal{O}_S(D_a - \sum_b D_b)) = (0, 0, 0) \ , \ \ h^\bullet(S, \mathcal{O}_S(-\sum_a D_a)) = (0, 0, 1) \ ,$$

$$h^\bullet(S, \mathcal{O}_S(D_a)) = (1, 0, 0) \ , \ \ h^\bullet(S, \mathcal{O}_S(D_a + D_{a+1})) = (2, 0, 0) \ ,$$

$$h^\bullet(S, \mathcal{O}_S(D_a + D_{a+2})) = (1, 0, 0) \ , \ \ h^\bullet(S, \mathcal{O}_S(D_a + D_{a+3})) = (1, 0, 0) \ ,$$

$$h^\bullet(S, \mathcal{O}_S(\sum_a D_a - D_b)) = (5, 0, 0) \ , \ \ h^\bullet(S, \mathcal{O}_S(\sum_a D_a)) = (7, 0, 0) \ ,$$

$$h^\bullet(S, \mathcal{O}_S(D_a + \sum_b D_b)) = (8, 0, 0) \ , \ \ h^\bullet(S, \mathcal{O}_S(2\sum_a D_a)) = (19, 0, 0) \ .$$

A.2 Cohomology on Z

The cohomology of line bundles on Z follows immediately from our calculations on $S \cong \mathrm{dP}_6$ above, by the Künneth formula. But we also need to know the cohomology of the bundle $\mathcal{G}$ and related bundles. We will make use of Serre duality to reduce the number of calculations we need to do, remembering that the canonical class of Z is given by $K_Z \sim -\sum_i D_i$.

$\underline{\mathcal{G} \text{ and } \mathcal{G}^*(-\sum_i D_i)}$

The bundle $\mathcal{G}$ is defined by the short exact sequence

$$0 \longrightarrow 6\mathcal{O}_Z \longrightarrow \bigoplus_i \mathcal{O}_Z(D_i) \longrightarrow \mathcal{G} \longrightarrow 0 \ .$$

Using the results of Appendix A.1 and the Künneth formula, the resulting long exact sequence in cohomology breaks up into several pieces

$$0 \longrightarrow \mathbb{C}^6 \longrightarrow \mathbb{C}^{12} \longrightarrow H^0(Z,\mathcal{G}) \longrightarrow 0 \ ,$$

$$0 \longrightarrow H^i(Z,\mathcal{G}) \longrightarrow 0 \ , \quad i > 0 \ ,$$

giving the simple result

$$h^\bullet(Z,\mathcal{G}) = (6,0,0,0,0) \ ,$$

and hence, by Serre duality,

$$h^\bullet(Z,\mathcal{G}^*(-\sum_i D_i)) = (0,0,0,0,6) \ .$$

$\underline{\mathcal{G}(\sum_i D_i)}$

Twisting the sequence which defines $\mathcal{G}$ by $\mathcal{O}_Z(\sum_i D_i)$ yields another exact sequence

$$0 \longrightarrow 6\mathcal{O}_Z(\sum_i D_i) \longrightarrow \bigoplus_i \mathcal{O}_Z(D_i + \sum_j D_j) \longrightarrow \mathcal{G}(\sum_i D_i) \longrightarrow 0 \ .$$

Once again, the associated long exact sequence breaks up very simply,

$$0 \longrightarrow \mathbb{C}^{294} \longrightarrow \mathbb{C}^{672} \longrightarrow H^0(Z,\mathcal{G}(\sum_i D_i)) \longrightarrow 0 \ ,$$

$$0 \longrightarrow H^i(Z,\mathcal{G}(\sum_i D_i)) \longrightarrow 0 \ ,$$

and we find

$$h^\bullet(Z,\mathcal{G}(\sum_i D_i)) = (378,0,0,0,0) \ .$$

$\wedge^2 \mathcal{G}$ and $\wedge^2 \mathcal{G}^*(-\sum_i D_i)$

Consider the exact sequences obtained by dualising (4.11) and (4.12)

$$0 \longrightarrow K_2^* \longrightarrow \bigoplus_{i<j} \mathcal{O}_Z(D_i + D_j) \longrightarrow \wedge^2 \mathcal{G} \longrightarrow 0 ,$$

$$0 \longrightarrow 21\mathcal{O}_Z \longrightarrow 6 \bigoplus_i \mathcal{O}_Z(D_i) \longrightarrow K_2^* \longrightarrow 0 .$$

As in the previous cases, the long exact sequence from the second of these is very simple, yielding $h^\bullet(Z, K_2^*) = (51, 0, 0, 0, 0)$. The middle term of the first sequence here is a direct sum of sixty-six line bundles. Referring to Section A.1, we see that twelve of them have $h^0 = 2$, the rest have $h^0 = 1$, while all have vanishing higher cohomology. So altogether, the first sequence gives

$$0 \longrightarrow \mathbb{C}^{51} \longrightarrow \mathbb{C}^{78} \longrightarrow H^0(Z, \wedge^2 \mathcal{G}) \longrightarrow 0 ,$$

$$0 \longrightarrow H^i(Z, \wedge^2 \mathcal{G}) \longrightarrow 0 , \quad i > 0 .$$

Our final result is therefore

$$h^\bullet(Z, \wedge^2 \mathcal{G}) = (27, 0, 0, 0, 0) \ , \quad h^\bullet\big(Z, \wedge^2 \mathcal{G}^*(-\sum_i D_i)\big) = (0, 0, 0, 0, 27) .$$

A.3 Cohomology on $\widetilde{X}$

The key to calculating the cohomology of bundles on $\widetilde{X}$ is the exact sequence (A.1), since most of the bundles of interest arise as restrictions of bundles on Z. There are several calculations needed for the arguments of Sections 3.2 and 4.

$\mathcal{O}_{\widetilde{X}}(D_i)$

From (A.1), we get the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(D_i - \sum_j D_j) \longrightarrow \mathcal{O}_Z(D_i) \longrightarrow \mathcal{O}_{\widetilde{X}}(D_i) \longrightarrow 0 .$$

The results of Appendix A.1 show that the first term here has vanishing cohomology, so the cohomology groups of $\mathcal{O}_{\widetilde{X}}(D_i)$ are the same as those of the corresponding bundle on Z,

$$h^\bullet(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(D_i)) = (1, 0, 0, 0) .$$

$\mathcal{O}_{\widetilde{X}}(\sum_i D_i - D_j)$

This time, (A.1) becomes

$$0 \longrightarrow \mathcal{O}_Z(-D_j) \longrightarrow \mathcal{O}_Z(\sum_i D_i - D_j) \longrightarrow \mathcal{O}_{\widetilde{X}}(\sum_i D_i - D_j) \longrightarrow 0 .$$

From the results of Appendix A.1, we see that the first bundle has zero cohomology, and the second only has non-vanishing H^0; in fact,

$$H^0(Z, \mathcal{O}_Z(\sum_i D_i - D_j)) \cong \mathbb{C}^{35} \,,$$

so we immediately get

$$h^\bullet(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\sum_i D_i - D_j)) = (35, 0, 0, 0) \,.$$

$\mathcal{O}_{\widetilde{X}}(\sum_i D_i)$

In this case, (A.1) gives

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_Z(\sum_i D_i) \longrightarrow \mathcal{O}_{\widetilde{X}}(\sum_i D_i) \longrightarrow 0 \,.$$

The results of Appendix A.1 show that all higher cohomology groups vanish for the first two bundles here, and

$$H^0(Z, \mathcal{O}_Z) \cong \mathbb{C} \,, \quad H^0(Z, \mathcal{O}_Z(\sum_i D_i)) \cong \mathbb{C}^{49} \,,$$

so we get

$$h^\bullet(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\sum_i D_i)) = (48, 0, 0, 0) \,.$$

$\mathcal{O}_{\widetilde{X}}(2\sum_i D_i)$

The relevant exact sequence here is

$$0 \longrightarrow \mathcal{O}_Z(\sum_i D_i) \longrightarrow \mathcal{O}_Z(2\sum_i D_i) \longrightarrow \mathcal{O}_{\widetilde{X}}(2\sum_i D_i) \longrightarrow 0 \,.$$

This is very similar to the last example; we have

$$H^0(Z, \mathcal{O}_Z(\sum_i D_i)) \cong \mathbb{C}^{49} \,, \quad H^0(Z, \mathcal{O}_Z(2\sum_i D_i)) \cong \mathbb{C}^{361} \,,$$

and all higher cohomology of both bundles vanishes. We therefore immediately obtain

$$h^\bullet(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(2\sum_i D_i)) = (312, 0, 0, 0) \,.$$

$\mathcal{F}$ and $\mathcal{F}^*$

In Section 3.2 we discussed that by choosing the map Φ appropriately, we could arrange that $H^0(\widetilde{X}, \mathcal{F}) \cong \mathbb{C}^6$ and $H^1(\widetilde{X}, \mathcal{F}) \cong \mathbb{C}^{42}$, but did not discuss the higher cohomology groups. The results above allow us to find these missing groups, and by Serre duality, those of $\mathcal{F}^*$. We have the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_i \mathcal{O}_{\widetilde{X}}(D_i) \longrightarrow \mathcal{O}_{\widetilde{X}}(\sum_i D_i) \longrightarrow 0 \,.$$

We saw above that the second and third bundles have vanishing higher cohomology, which immediately implies that $H^i(\widetilde{X}, \mathcal{F}) = 0$ for $i = 2, 3$, and hence by Serre duality that $H^i(\widetilde{X}, \mathcal{F}^*) = 0$

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for $i = 0, 1$. So in particular, $H^0(\widetilde{X}, \mathcal{F}^*) = 0$, which we used in Section 3.2 to show that $\mathcal{F}_0$ is a completely non-split extension.

$\underline{\mathcal{G}(\sum_i D_i)|_{\widetilde{X}}}$

In this case we have

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}(\sum_i D_i) \longrightarrow \mathcal{G}(\sum_i D_i)|_{\widetilde{X}} \longrightarrow 0 .$$

Referring to Appendix A.2, we see that the first two bundles again only have non-vanishing H^0, so the same is true for the third; in fact we find

$$h^\bullet(\widetilde{X}, \mathcal{G}(\sum_i D_i)|_{\widetilde{X}}) = (372, 0, 0, 0) .$$

B   A simple proof of stability

In Section 3, we outlined one argument for the stability of a generic bundle $\widetilde{V}$ in our family on $\widetilde{X}$, following the work of Li and Yau. Here we give a simple, direct argument for stability of the corresponding (still generic) bundle V on the quotient space X. By the Donaldson-Uhlenbeck-Yau (DUY) theorem on X, this implies the existence of a Hermitian-Yang-Mills (HYM) connection on V. The pullback to $\widetilde{X}$ is a HYM connection on $\widetilde{V}$, and by the DUY theorem again, this time on $\widetilde{X}$, it follows that $\widetilde{V}$ is in fact also polystable.

The reason it is easier to work on X is that its Picard group is cyclic, i.e. $h^{1,1}(X) = 1$ (and $h^{1,0}(X) = 0$). This allows us to invoke a special case of Hoppe's theorem [39, 40]: Let X be a Calabi-Yau threefold with $h^{1,1}(X) = 1$, and V a holomorphic vector bundle on X with $c_1(V) = 0$. If $H^0(X, \wedge^p V) = 0$ for $p = 1, \ldots, rk(V) - 1$, then V is stable. One fact which makes our life easier is that since $\wedge^5 V \cong \mathcal{O}_X$, we have $\wedge^p V \cong \wedge^{5-p} V^*$, and therefore by Serre duality, $H^0(X, \wedge^p V) \cong H^3(X, \wedge^{5-p} V)^*$.

To demonstrate that our rank-five bundles V are stable, it is therefore sufficient to show that $H^0(X, V) = H^3(X, V) = 0$, and $H^0(X, \wedge^p V) = 0$ for $p = 2, 3$. To do so, we will show that the corresponding cohomology groups of $\widetilde{V}$ already vanish on the covering space $\widetilde{X}$; to this end, consider the representation of $\widetilde{V}$ via the short exact sequence (3.7), which we repeat here

$$0 \longrightarrow 6\mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{F} \longrightarrow \widetilde{V} \longrightarrow 0 . \tag{3.7}$$

We have already explained in Section 3.2 that we can arrange for $H^0(\widetilde{X}, \widetilde{V}) = 0$. In Section A.3, we saw that $H^3(\widetilde{X}, \mathcal{F}) = 0$, and if we use this in the long exact cohomology sequence following from above, we immediately get $H^3(\widetilde{X}, \widetilde{V}) = 0$, which by Serre duality implies $H^0(\widetilde{X}, \wedge^4 \widetilde{V}) = 0$.

Calculating $H^0(\widetilde{X}, \wedge^2 \widetilde{V})$ explicitly from the exact sequences is much harder, but we can make a simple argument that it vanishes. Using the techniques we have described, we could instead construct irreducible rank-*four* bundles $\widetilde{V}^{(4)}$ as deformations of $\widetilde{V}_0^{(4)} = T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}}$. We then have $\wedge^2 \widetilde{V}_0^{(4)} = \wedge^2 T\widetilde{X} \oplus T\widetilde{X}$, and hence $h^0(\widetilde{X}, \wedge^2 \widetilde{V}_0^{(4)}) = 0$. The semi-continuity theorem for sheaf

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cohomology implies that at a general point in moduli space, $h^0(\widetilde{X}, \wedge^2 \widetilde{V}^{(4)})$ is bounded above by this value [41], and it must therefore also vanish. Now, we can consider our rank five bundles $\widetilde{V}$ to be deformations of $\widetilde{V}^{(4)} \oplus \mathcal{O}_{\widetilde{X}}$ (simply go to a point in moduli space where the map Φ annihilates seven of the Euler vectors, instead of only six), and we repeat the argument: as long as $\widetilde{V}^{(4)}$ and $\wedge^2 \widetilde{V}^{(4)}$ have no global sections, the same is true for $\wedge^2(\widetilde{V}^{(4)} \oplus \mathcal{O}_{\widetilde{X}})$, and by the semi-continuity theorem, also for $\wedge^2 \widetilde{V}$ at a general point in moduli space.

A similar argument gives $H^0(\widetilde{X}, \wedge^3 \widetilde{V}) = 0$ as long as $H^0(\widetilde{X}, \widetilde{V}) = 0$, so this is sufficient to satisfy the conditions for Hoppe's theorem, and therefore conclude that V is stable on the quotient space.

C   The non-Abelian quotient

The covering manifold $\widetilde{X}$ also admits a free quotient by the non-Abelian dicyclic group Dic_3, which yields another manifold with Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 4)$. Here we will explain briefly why this manifold does not yield MSSM models in the same way that the $\mathbb{Z}_{12}$ quotient does. For a much more detailed discussion of $\widetilde{X}/\mathrm{Dic}_3$, see [15].

First we need to understand the group Dic_3 and its representations. It can be generated by two elements, one of order three and one of order four, satisfying the additional relation

$$g_4 \, g_3 \, g_4^{-1} = g_3^2 \; ,$$

which reveals a semi-direct produce structure $\mathrm{Dic}_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$. There are therefore four distinct one-dimensional representations, in which g_3 acts trivially and g_4 corresponds to multiplication by one of the fourth roots of unity. As in [15], we will denote these representations by $R_1, R_{-1}, R_{\mathrm{i}}, R_{-\mathrm{i}}$. There are also two two-dimensional irreps, which can be distinguished by $\mathrm{Tr}(g_4^2) = \pm 2$; these we denote by $R_{\pm}^{(2)}$.

The choice of discrete Wilson line values to break $SU(5)_{\mathrm{GUT}}$ to G_{SM} is much more restricted than in the $\mathbb{Z}_{12}$ case, since there are now only three non-trivial one-dimensional representations. The corresponding Wilson lines are

$$g_4 \mapsto \mathrm{diag}(1, 1, 1, -1, -1) \; , \;\; g_4 \mapsto \mathrm{diag}(-1, -1, -1, \mathrm{i}, \mathrm{i}) \; , \;\; g_4 \mapsto \mathrm{diag}(-1, -1, -1, -\mathrm{i}, -\mathrm{i}) \; , \quad \text{(C.1)}$$

and in every case, g_3 is trivial. We note a simple feature here. Since the $\mathbf{10}$ of $SU(5)$ is the rank-two anti-symmetric tensor, in the second or third cases here it contains three distinct one-dimensional representations of Dic_3. In the first case, though, all the $\mathbf{10}$ fields transform as either R_1 or R_{-1}. We will return to this momentarily.

We know from the discussion in Section 4 that the construction of an equivariant stable deformation of $T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$ involves choosing a two-dimensional subspace of $H^{1,1}(\widetilde{X})$ which is invariant under the quotient group. After deforming, the corresponding massless fields in the $\mathbf{10} \oplus \overline{\mathbf{10}}$ of $SU(5)$ become massive via the Higgs mechanism. Under Dic_3, we have

$$H^{1,1}(\widetilde{X}) \sim R_1 \oplus R_{-1} \oplus R_{\mathrm{i}} \oplus R_{-\mathrm{i}} \oplus R_+^{(2)} \oplus R_+^{(2)} \; .$$

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We can therefore choose either $R_+^{(2)}$, or the sum of any two distinct one-dimensional representations. Either way, when we calculate $H^1(\widetilde{X}, \widetilde{V}^*)$, we will find at least two distinct one-dimensional representations of Dic$_3$. Now recall that the second and third choices in (C.1) lead to fields in the $\overline{\mathbf{10}}$ transforming under at least three distinct one-dimensional representations. Therefore in these cases we are guaranteed to have massless fields from the $\overline{\mathbf{10}}$ which survive the quotient. However, if we choose $R_1 \oplus R_{-1}$, we are left with

$$H^1(\widetilde{X}, \widetilde{V}^*) \sim R_{\mathrm{i}} \oplus R_{-\mathrm{i}} \oplus R_+^{(2)} \oplus R_+^{(2)} \ .$$

Combining these representations with the first choice of Wilson line in (C.1) gives no invariants, and therefore projects out all massless states from the $\overline{\mathbf{10}}$.

By the above reasoning, the analogue for the Dic$_3$ quotient of the 43 models in Table 2 is a unique model, in which the gauge bundle V on $X' = \widetilde{X}/\mathrm{Dic}_3$ is a deformation of

$$\mathcal{L}_{R_{-1}} \otimes \left(TX' \oplus \mathcal{L}_{R_{-1}} \oplus \mathcal{L}_{R_1} \right) \ ,$$

and the discrete Wilson line values are given by

$$g_3 \mapsto \mathbf{1}_5 \ , \ g_4 \mapsto \mathrm{diag}(1,1,1,-1,-1) \ \in \ SU(5)_{\mathrm{GUT}} \ .$$

We can then proceed to calculate $H^1(\widetilde{X}, \wedge^2 \widetilde{V}^*)$ for this model, and we find

$$H^1(\widetilde{X}, \wedge^2 \widetilde{V}^*) \sim R_{-1} \oplus R_1 \oplus 2R_-^{(2)} \ .$$

Combining this with the Wilson line, we see that although we find a massless pair of Higgs doublets, they come along with massless colour triplets. We conclude that there are no models on the Dic$_3$ quotient which yield exactly the massless spectrum of the MSSM.

References

[1] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum Configurations for Superstrings," *Nucl.Phys.* **B258** (1985) 46–74.

[2] B. R. Greene, K. H. Kirklin, P. J. Miron, and G. G. Ross, "A Three Generation Superstring Model. 2. Symmetry Breaking and the Low-Energy Theory," *Nucl. Phys.* **B292** (1987) 606.

[3] S. K. Donaldson, "Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles," *Proc. Lond. Math. Soc.* **50** (1985) 1–26.

[4] K. Uhlenbeck and S. T. Yau, "On the existence of Hermitian-Yang-Mills connections in stable vector bundles," *Commun.Pure Appl.Math.* **39** (1986) 257.

[5] V. Braun, Y.-H. He, and B. A. Ovrut, "Stability of the minimal heterotic standard model bundle," *JHEP* **0606** (2006) 032, `arXiv:hep-th/0602073 [hep-th]`.

[6] L. B. Anderson, Y.-H. He, and A. Lukas, "Monad Bundles in Heterotic String Compactifications," *JHEP* **0807** (2008) 104, `arXiv:0805.2875 [hep-th]`.

[7] L. B. Anderson, J. Gray, A. Lukas, and B. Ovrut, "The Edge Of Supersymmetry: Stability Walls in Heterotic Theory," *Phys. Lett.* **B677** (2009) 190–194, `arXiv:0903.5088 [hep-th]`.

[8] L. B. Anderson, J. Gray, A. Lukas, and B. Ovrut, "Stability Walls in Heterotic Theories," *JHEP* **09** (2009) 026, `arXiv:0905.1748 [hep-th]`.

[9] V. Bouchard and R. Donagi, "An SU(5) heterotic standard model," *Phys. Lett.* **B633** (2006) 783–791, `arXiv:hep-th/0512149`.

[10] L. B. Anderson, J. Gray, A. Lukas, and E. Palti, "Two Hundred Heterotic Standard Models on Smooth Calabi-Yau Threefolds," `arXiv:1106.4804 [hep-th]`.

[11] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, "The exact MSSM spectrum from string theory," *JHEP* **05** (2006) 043, `arXiv:hep-th/0512177`.

[12] V. Bouchard and R. Donagi, "On heterotic model constraints," *JHEP* **0808** (2008) 060, `arXiv:0804.2096 [hep-th]`.

[13] L. B. Anderson, J. Gray, Y.-H. He, and A. Lukas, "Exploring Positive Monad Bundles And A New Heterotic Standard Model," *JHEP* **02** (2010) 054, `arXiv:0911.1569 [hep-th]`.

[14] J. McOrist and I. V. Melnikov, "Old issues and linear sigma models," `arXiv:1103.1322 [hep-th]`. * Temporary entry *.

[15] V. Braun, P. Candelas, and R. Davies, "A Three-Generation Calabi-Yau Manifold with Small Hodge Numbers," *Fortsch. Phys.* **58** (2010) 467–502, `arXiv:0910.5464 [hep-th]`.

[16] V. Braun, "On Free Quotients of Complete Intersection Calabi-Yau Manifolds," arXiv:1003.3235 [hep-th].

[17] J. Li and S.-T. Yau, "The existence of supersymmetric string theory with torsion," *J.Differential Geom.* **70** (2005) 143–181, arXiv:hep-th/0411136.

[18] R. Donagi, R. Reinbacher, and S.-T. Yau, "Yukawa couplings on quintic threefolds," arXiv:hep-th/0605203.

[19] B. McInnes, "Group theoretic aspects of the Hosotani mechanism," *J. Phys.* **A22** (1989) 2309–2328.

[20] R. Donagi, B. A. Ovrut, T. Pantev, and R. Reinbacher, "SU(4) instantons on Calabi-Yau threefolds with Z(2) x Z(2) fundamental group," *JHEP* **0401** (2004) 022, arXiv:hep-th/0307273 [hep-th].

[21] V. V. Batyrev, "Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties," *J. Alg. Geom.* **3** (1994) 493–545.

[22] M. Kreuzer and H. Skarke, "Complete classification of reflexive polyhedra in four-dimensions," *Adv.Theor.Math.Phys.* **4** (2002) 1209–1230, arXiv:hep-th/0002240 [hep-th].

[23] P. S. Aspinwall, B. R. Greene, and D. R. Morrison, "Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory," *Nucl. Phys.* **B416** (1994) 414–480, arXiv:hep-th/9309097.

[24] V. Bouchard, "Lectures on complex geometry, Calabi-Yau manifolds and toric geometry," arXiv:hep-th/0702063.

[25] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*. AMS, 2003.

[26] W. Fulton, *Introduction to Toric Varieties*. Princeton University Press, 1993.

[27] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric Varieties*. AMS, 2011.

[28] P. Candelas, A. M. Dale, C. A. Lutken, and R. Schimmrigk, "Complete Intersection Calabi-Yau Manifolds," *Nucl. Phys.* **B298** (1988) 493.

[29] P. Candelas, C. A. Lutken, and R. Schimmrigk, "Complete Intersection Calabi-Yau Manifolds. 2. Three Generation Manifolds," *Nucl. Phys.* **B306** (1988) 113.

[30] P. Candelas and R. Davies, "New Calabi-Yau Manifolds with Small Hodge Numbers," *Fortsch. Phys.* **58** (2010) 383–466, arXiv:0809.4681 [hep-th].

[31] D. A. Cox, "The Homogeneous Coordinate Ring of a Toric Variety, Revised Version,"
`arXiv:alg-geom/9210008`.

[32] J. Distler, "Notes on (0,2) superconformal field theories," `arXiv:hep-th/9502012`
`[hep-th]`.

[33] E. Witten, "Symmetry Breaking Patterns in Superstring Models," *Nucl. Phys.* **B258** (1985)
75.

[34] E. Witten, "New Issues in Manifolds of SU(3) Holonomy," *Nucl. Phys.* **B268** (1986) 79.

[35] M. Kreuzer, J. McOrist, I. V. Melnikov, and M. Plesser, "(0,2) Deformations of Linear
Sigma Models," *JHEP* **1107** (2011) 044, `arXiv:1001.2104 [hep-th]`.

[36] R. Davies, "The Expanding Zoo of Calabi-Yau Threefolds," *Adv.High Energy Phys.* **2011**
(2011) Article ID 901898, `arXiv:1103.3156 [hep-th]`.

[37] S.-T. Yau, "Compact three-dimensional Kähler manifolds with zero Ricci curvature,". In
Argonne/Chicago 1985, Proceedings, Anomalies, Geometry, Topology, 395-406.

[38] P. Candelas, X. de la Ossa, Y.-H. He, and B. Szendroi, "Triadophilia: A Special Corner in
the Landscape," *Adv. Theor. Math. Phys.* **12** (2008) 2, `arXiv:0706.3134 [hep-th]`.

[39] L. B. Anderson, Y.-H. He, and A. Lukas, "Heterotic compactification, an algorithmic
approach," *JHEP* **07** (2007) 049, `arXiv:hep-th/0702210`.

[40] L. B. Anderson, "Heterotic and M-theory Compactifications for String Phenomenology,"
`arXiv:0808.3621 [hep-th]`. Ph.D.Thesis.

[41] R. Hartshorne, *Algebraic Geometry.* Graduate Texts in Mathematics. Springer, 1977.