Equivariant Dimensional Reduction and Quiver Gauge Theories

Brian P. Dolan[1] and Richard J. Szabo[2]

[1] *National University of Ireland Maynooth, Ireland*
[2] *Heriot-Watt University, Edinburgh, Scotland*

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Abstract

We review recent applications of equivariant dimensional reduction techniques to the construction of Yang-Mills-Higgs-Dirac theories with dynamical mass generation and exactly massless chiral fermions.

1   A brief history of dimensional reduction

The idea that the observed fundamental forces in 4-dimensions can be understood in terms of the dynamics of a simpler higher dimensional theory is now nearly 90 years old [1]. Starting from a 5-dimensional theory on a manifold $\mathcal{M}_5 = \mathcal{M}_4 \times S^1$, where $\mathcal{M}_4$ is a curved 4-dimensional space-time and the fifth dimension is a perfect circle with radius r, and taking the 5-dimensional line element to be ($0 \le y < 2\pi$):

$$ds_{(5)}^2 = ds_{(4)}^2 + \big(r\,dy + A(x)\big)^2,$$

where $A(x) = A_\mu(x)dx^\mu$ is a 4-dimensional vector potential, the 5-dimensional Einstein action reduces to

$$\frac{1}{2\pi r}\int_{\mathcal{M}_5}\sqrt{-g_{(5)}}\,\mathcal{R}_{(5)}\,d^4x\,dy = \int_{\mathcal{M}_4}\sqrt{-g_{(4)}}\Big(\mathcal{R}_{(4)} - \frac{1}{4}F^2\Big)d^4x,$$

where $F = dA$ is a $U(1)$ field strength in 4-dimensions and $F^2 = F_{\mu\nu}F^{\mu\nu}$.

If we now introduce extra matter, e.g. a scalar field Φ, and perform a harmonic expansion on S^1,

$$\Phi(x,y) = \sum_{n=-\infty}^{\infty} \phi_n(x)e^{\frac{iny}{r}},$$

then the 5-dimensional kinetic term for Φ gives rise to an infinite tower of massive fields in $\mathcal{M}_4$, $\phi_n(x)$, with masses $m_n = \frac{n}{r}$.

A non-abelian generalisation of the Kaluza-Klein idea uses a d-dimensional manifold $\mathcal{M}_d = \mathcal{M}_4 \times S/R$, with $R \subset S$ compact Lie groups. The co-set space S/R has isometry group S and holonomy group R. Performing the integral $\int_{S/R} d\mu$ over the internal space, with $d\mu$ the S-invariant measure on S/R, leads to Yang-Mills gauge theory in 4-dimensions with gauge group S; e.g. $S^2 \simeq SU(2)/U(1)$, with $SU(2)$ isometry and $U(1)$ holonomy, gives 4-dimensional Einstein-Yang-Mills theory with gauge group $SU(2)$, see e.g. [2].

Alternatively, one can start from d-dimensional Yang-Mills theory on $\mathcal{M}_4 \times S/R$ with gauge group G. Forgács and Manton [3] showed that interesting symmetry breaking effects can occur if $R \subset G$ and one

chooses a specific embedding $R \hookrightarrow G$. Integrating over S/R then gives a Yang-Mills-Higgs system on $\mathcal{M}_4$, with a gauge group K which is the centraliser of R in G, i.e. $K \subset G$ with $[R, K] = 0$ (see also [4]). Upon dimensional reduction the internal components of the d-dimensional gauge field $\mathcal{A}$ play the rôle of Higgs fields in 4-dimensions and a Higgs potential is generated from the d-dimensional Yang-Mills action:

$$\mathcal{A}(x, y) \quad \longrightarrow \quad \begin{cases} A_\mu(x) & \text{(4-dimensional gauge fields)} \\ \Phi_a(x) & \text{(4-dimensional Higgs fields)} \end{cases}$$

(here x^μ are co-ordinates on $\mathcal{M}_4$, y^a co-ordinates on S/R). The full d-dimensional Yang-Mills action, with field strength $\mathcal{F}$, reduces as

$$-\frac{1}{4} \int_{\mathcal{M}_d} \sqrt{-g_{(d)}} \, \mathrm{Tr} \, (\mathcal{F}^2) d^4x \, d^{d-4}y = \mathrm{vol}(S/R) \int_{\mathcal{M}_4} \sqrt{-g_{(4)}} \, \mathrm{tr} \, \Big(-\frac{1}{4} F^2 + \big(D\Phi\big)^\dagger D\Phi - V(\Phi) \Big) d^4x,$$

where Tr denotes trace over the d-dimensional gauge group G and tr is over the 4-dimensional gauge group K. Furthermore the Higgs potential can break K dynamically. In particular if $S \subset G$, then $V(\Phi)$ breaks K spontaneously to K', the centraliser of S in G, $[S, K'] = 0$.

Consider again the simplest case $S^2 \simeq SU(2)/U(1)$, where $S \cong SU(2)$ and $R \cong U(1)$. For example if $G = SU(3)$ then indeed $S \subset G$ and in the first step $R \hookrightarrow G$: $U(1) \hookrightarrow SU(3)$ breaking $SU(3)$ to $K = SU(2) \times U(1)$. Upon reduction the 4-dimensional Higgs doublet, Φ_a, $a = 1, 2$, dynamically breaks $SU(2) \times U(1) \to K' \cong U(1)$, which is the centraliser of $S = SU(2)$ in $G = SU(3)$. Going beyond $SU(2)$ symmetry on the co-set space, a harmonic expansion of, for example, a scalar field Φ on $S^2 \simeq SU(2)/U(1)$,

$$\Phi(x, y) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_{l;m}(x) Y_l^m(y),$$

generates a tower of higher modes, $\phi_{l;m}(x)$, which have masses $M_l^2 = \frac{l(l+1)}{r^2}$ in 4-dimensions.

Much of the steam was taken out of the co-set space dimensional reduction programme with Witten's proof that spinors on $\mathcal{M}_4 \times S/R$ *cannot* give a chiral theory on $\mathcal{M}_4$ [5].

Reviews of co-set space dimensional reduction are given in [6] and [7].

2 Equivariant dimensional reduction

2.1 General construction

Equivariant dimensional reduction is a systematic procedure for including internal fluxes on S/R (instantons and/or monopoles of R-fields) which are 'symmetric' (equivariant) under S [8, 9]. It relies on the fact that, with suitable restrictions on S and R, there is a one-to-one correspondence between S-equivariant complex vector bundles over $\mathcal{M}_d$

$$B \longrightarrow \mathcal{M}_d = \mathcal{M}_4 \times S/R,$$

and R-equivariant bundles over $\mathcal{M}_4$,

$$E \longrightarrow \mathcal{M}_4,$$

where S acts on the space $\mathcal{M}_d$ via the trivial action on $\mathcal{M}_4$ and by the standard left translation action on S/R (we shall restrict ourselves to the case where S and R are compact and the embedding $R \hookrightarrow S$ is maximal). If B and E are $\mathbf{C}^k$ vector bundles there is a commutative diagram of bundle maps

$$\mathbf{C}^k \xrightarrow{\quad R \quad} E \xrightarrow{\text{induce}} B \xleftarrow{\quad S \quad} \mathbf{C}^k$$
$$\downarrow \qquad\qquad \downarrow$$
$$\mathcal{M}_4 \xleftarrow{\text{restrict}} \mathcal{M}_d$$

where the induction map is defined by

$$h \in R, \quad (g, e) \in S \times E, \qquad h \cdot (g, e) = (gh^{-1}, he) \mapsto B.$$

In general the reduction gives rise to quiver gauge theories on $\mathcal{M}_4$. Including spinor fields, coupling to background equivariant fluxes, can give rise to chiral theories on $\mathcal{M}_4$. One expects zero modes of the Dirac operator on S/R to manifest themselves as massless chiral fermions in $\mathcal{M}_4$ but, as we shall see, Yukawa couplings are induced and the dimensional reduction can give masses to some zero modes [10, 11].

2.2 A simple example: Complex projective line

Consider once again the simplest non-trivial example with $S \cong SU(2)$ and $R \cong U(1)$, giving a 2-dimensional sphere $S^2 \simeq SU(2)/U(1)$ (or projective line $\mathbf{CP}^1$), and with $G \cong U(k)$. Choosing an embedding $S \hookrightarrow G$ gives a decomposition $U(k) \to \prod_{i=0}^{m} U(k_i)$, where $k = \sum_{i=0}^{m} k_i$, associated with the $m + 1$-dimensional irreducible representation of $SU(2)$. Let $\mathbf{g} \in G$, $\mathbf{v} \in \mathbf{C}^k$ and $\mathbf{v}_i \in \mathbf{C}^{k_i}$. Then, as a $k \times k$ matrix, $\mathbf{g}$ decomposes as

$$\mathbf{g} = m+1 \left\{ \overbrace{\begin{pmatrix} \mathbf{g}_{k_0 \times k_0} & \mathbf{g}_{k_0 \times k_1} & \cdots & \mathbf{g}_{k_0 \times k_m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}_{k_m \times k_0} & \mathbf{g}_{k_m \times k_1} & \cdots & \mathbf{g}_{k_m \times k_m} \end{pmatrix}}^{m+1}, \quad \mathbf{v} = \begin{pmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}, \right.$$

where $SU(2)$ acts on $\mathbf{g}$ as a $(m + 1) \times (m + 1)$ block matrix. Each subspace $\mathbf{v}_i$ transforms under $U(k_i) \subset U(k)$ and carries a $U(1)$ charge $p_i = m - 2i$, $-m \le p_i \le m$.

Introducing a complex co-ordinate y on S^2 (of radius r),

$$ds_{(2)}^2 = r^2 \beta \overline{\beta}, \qquad \beta = \frac{2dy}{1 + y\overline{y}},$$

we write the potential and field strength for a monopole of charge p in these co-ordinates as

$$a_p = \frac{ip(y d\overline{y} - \overline{y} dy)}{2(1 + y\overline{y})}, \quad f_p = \frac{ip}{4} \beta \wedge \overline{\beta}, \quad \frac{1}{2\pi} \int_{S^2} f_p = p.$$

The $U(k)$ gauge potential, a Lie algebra valued 1-form $\mathcal{A}$ on $\mathcal{M}_d$, now splits into $k_i \times k_j$ blocks

$$\mathcal{A}(x, y) = A(x) + a(y) + \Phi(x)\overline{\beta}(y) + \Phi^\dagger(x)\beta(y),$$

where $A = \oplus_{i=0}^{m} A^i$, $a = \oplus_{i=0}^{m} a_{m-2i}$, $A^i(x)$ is a $U(k_i)$ gauge connection on $\mathcal{M}_4$, and $\Phi(x)$ will acquire the interpretation as a set of Higgs fields. As a $(m + 1) \times (m + 1)$ block matrix

$$\mathcal{A}(x, y) = \begin{pmatrix} A^0 + a_m \mathbf{1}_{k_0} & \phi_1 \overline{\beta} & 0 & \cdots & 0 \\ \phi_1^\dagger \beta & A^1 + a_{m-2} \mathbf{1}_{k_1} & \phi_2 \overline{\beta} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_m \overline{\beta} \\ 0 & 0 & 0 & \cdots & A^m + a_{-m} \mathbf{1}_{k_m} \end{pmatrix},$$

where each ϕ_i is a $k_{i-1} \times k_i$ matrix transforming under $U(k_{i-1})_L \times U(k_i)_R$. As a $(m+1) \times (m+1)$ matrix the Higgs field is

$$\Phi = \begin{pmatrix} 0 & \phi_1 & 0 & \cdots & 0 \\ 0 & 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_m \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

3

Dimensional reduction generates a 4-dimensional Higgs potential,

$$V(\Phi) = \frac{g^2}{2} \operatorname{tr}_k \left(\frac{1}{4g^2r^2} \begin{pmatrix} m\mathbf{1}_{k_0} & 0 & \cdots & 0 \\ 0 & (m-2)\mathbf{1}_{k_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -m\mathbf{1}_{k_m} \end{pmatrix} - [\Phi, \Phi^\dagger] \right)^2,$$

where g is the 6-dimensional gauge coupling. The minimisation of the Higgs potential gives a vacuum structure that depends on the monopole charges $p_i = m - 2i$.

2.2.1 Example: $SU(3) \to SU(2) \times U(1) \to U(1)$

As a concrete example, consider the case with $G \cong SU(3)$ and $m = 1$ (fundamental of $SU(2)$), so that $k = 3$ and $k_0 = 2$, $k_1 = 1$. In this case there is one unit charge monopole and one anti-monopole sector in the internal space which give a symmetry breaking pattern

$$SU(3) \xrightarrow{\text{reduction}} SU(2) \times U(1) \xrightarrow{\text{dynamics}} U(1),$$

so $K \cong SU(2) \times U(1)$ is broken dynamically to $U(1)$ (for details, see [10]).

There is only one Higgs multiplet, ϕ, which is a 2-component vector, and the minimum of $V(\phi)$ is at $\phi_0 = \begin{pmatrix} 0 \\ \frac{1}{2gr} \end{pmatrix}$ in a suitable gauge. Perturbing around this vacuum gives $\phi = \begin{pmatrix} 0 \\ \frac{1}{2gr} + h \end{pmatrix}$, with h real, and the Higgs mass works out to be $m_h = \frac{1}{r}$.

The three gauge boson masses are $m_{W^\pm} = \frac{1}{2} m_Z = \frac{1}{\sqrt{2}r}$ while the Weinberg angle evaluates to $\sin^2 \theta_W = \frac{3}{4}$. Clearly this is not a phenomenologically viable model for electroweak interactions, as the gauge boson masses and the Weinberg angle are wrong, but it is nevertheless instructive.

2.2.2 Example: $SU(3k') \to SU(k')$

As a second example take $G \cong SU(k)$. Let $m = 2$ (adjoint of $SU(2)$) and choose $k_0 = k_1 = k_2 = k'$, so that $k = 3k'$. There are now three sectors in the internal space, one charge two monopole, its anti-monopole, and a trivial sector. The symmetry breaking scheme in this case is

$$SU(3k') \xrightarrow{\text{reduction}} SU(k')^3 \times U(1)^2 \xrightarrow{\text{dynamics}} SU(k')_{diag}.$$

There are two Higgs multiplets, ϕ_1 and ϕ_2, both of which are $k \times k$ matrices. The Higgs potential is

$$V(\Phi) = g^2 \operatorname{tr}_k \left((\phi_1{}^\dagger \phi_1)^2 - \phi_1{}^\dagger \phi_1 \phi_2{}^\dagger \phi_2 + (\phi_2{}^\dagger \phi_2)^2 \right) - \frac{1}{2r^2} \operatorname{tr}_k \left(\phi_1{}^\dagger \phi_1 + \phi_2{}^\dagger \phi_2 \right),$$

and we expand ϕ_i around the vacuum as

$$\phi_i = \frac{\sqrt{i(3-i)}}{2gr} \mathbf{1}_k + h_i,$$

with $h_i = h_i^\dagger$, $i = 1, 2$.

Diagonalising the Higgs mass matrix produces two distinct eigenvalues $m_h^2 = \frac{3}{r^2}, \frac{1}{r^2}$. There are $k'^2 - 1$ gauge bosons with mass $m_W^2 = \frac{1}{2r^2}$, $k'^2 - 1$ with $m_{W'}^2 = \frac{3}{2r^2}$, while two Z-bosons acquire masses $m_Z^2 = \frac{1}{4r^2}$ and $m_{Z'}^2 = \frac{9}{4r^2}$.

2.2.3 Quiver diagrams

This construction generates quiver gauge theories on $\mathcal{M}_4$. Writing the Lie algebra of $SU(2)$ in the form $[J_3, J_\pm] = \pm 2 J_\pm$, the Higgs fields give rise to a chain of bundle maps Φ_i:

$$0 \;\longrightarrow\; B_0 \;\xrightarrow{\Phi_1}\; B_1 \;\xrightarrow{\Phi_2}\; \cdots \;\xrightarrow{\Phi_{m-1}}\; B_{m-1} \;\xrightarrow{\Phi_m}\; B_m \;\longrightarrow\; 0.$$

The isometry group $SU(2)$ is rather special in that there is only one raising and one lowering operator, so the quiver diagram is always a chain. Higher rank isometry and holonomy groups generate more complicated quiver diagrams in general.

2.3 A more general example: Complex projective plane

As a more general example consider $\mathbf{CP}^2 \simeq SU(3)/U(2)$ (for details see [9] and [11]). Label the irreducible representations of $SU(3)$ by $\{l, \bar{l}\}$, corresponding to the Young tableau
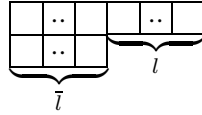
Denote irreducible representations of $SU(2) \times U(1)$ by (n, m), with $n = 2I$ (isospin) and $m = 3Y$ (hypercharge). Then under the embedding $U(2) \hookrightarrow SU(3)$, the irreducible representations decompose as $\{l, \bar{l}\} \to \oplus (n, m) := W_{l,\bar{l}}$, where $W_{l,\bar{l}}$ represents the set of all $SU(2) \times U(1)$ irreducible representations in $\{l, \bar{l}\}$. For example, $W_{1,0}$ has two elements: $\mathbf{3} \to \mathbf{2}_1 \oplus \mathbf{1}_{-2}$.

The root diagram for $SU(3)$ is
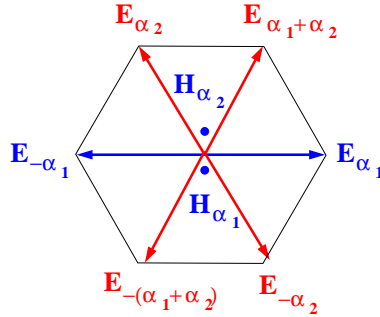
For any given irreducible representation $\{l, \bar{l}\}$, E_{α_2} and $E_{\alpha_1 + \alpha_2}$ map between elements of $W_{l,\bar{l}}$ with different isospin and can be decomposed into components that increase the isospin and components that decrease it:

$$E_{\alpha_2} = E_{\alpha_2}^+ + E_{\alpha_2}^-, \qquad E_{\alpha_1 + \alpha_2} = E_{\alpha_1 + \alpha_2}^+ + E_{\alpha_1 + \alpha_2}^-,$$

with

$$E_{\alpha_2}^\pm : (n, m) \longrightarrow (n \pm 1, m + 3), \qquad E_{\alpha_1 + \alpha_2}^\pm : (n, m) \longrightarrow (n \pm 1, m + 3).$$

Choosing a basis of orthonormal 1-forms for $\mathbf{CP}^2$ which is compatible with the complex structure, β^1, β^2, $\bar{\beta}^1$, $\bar{\beta}^2$, define the Lie-algebra valued 1-forms $\beta_{n,m}^\pm$, together with their complex conjugates, via the relations

$$\beta^\pm := \beta^1 E_{\alpha_1 + \alpha_2}^\pm + \beta^2 E_{\alpha_2}^\pm = \sum_{(n,m) \in W_{l,\bar{l}}} \beta_{n,m}^\pm.$$

There is then a Higgs fields, $\phi_{n,m}^\pm$, associated with each $\beta_{n,m}^\pm$.

2.3.1 Example: Adjoint representation

For example, the adjoint representation $l = \bar{l} = 1$ of $SU(3)$ decomposes as

$$W_{1,1} = \{ \overset{K}{(1,3)} \oplus \overset{\overline{K}}{(1,-3)} \oplus \overset{\pi}{(2,0)} \oplus \overset{\eta}{(0,0)} \},$$

where the different $SU(2) \times U(1)$ representations are also indicated by their usual particle physics notation. Choosing the gauge group to be

$$G = U(k) \longrightarrow U(k_{1,3}) \times U(k_{1,-3}) \times U(k_{2,0}) \times U(k_{0,0}),$$

with $k = 2k_{1,3} + 2k_{1,-3} + 3k_{2,0} + k_{0,0}$, there are four Higgs fields mapping between the $SU(2) \times U(1)$ representations and the quiver diagram assumes the form
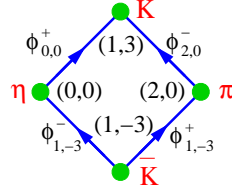
For illustrative purposes, we further specialise to the case $k_{1,3} = k_{1,-3} = k_{2,0} = k_{0,0} = k'$. Then dimensional reduction gives $K \cong U(k')^4$,

$$U(8k') \longrightarrow U(k')^4,$$

and $\phi_{n,m}^{\pm}$ are $k' \times k'$ complex matrices acted on by some $SU(k')_L \times SU(k')_R$ subgroup. The symmetry is further reduced by dynamical symmetry breaking

$$SU(8k') \longrightarrow SU(k')^4 \times U(1)^3 \longrightarrow SU(k')_{diag}$$
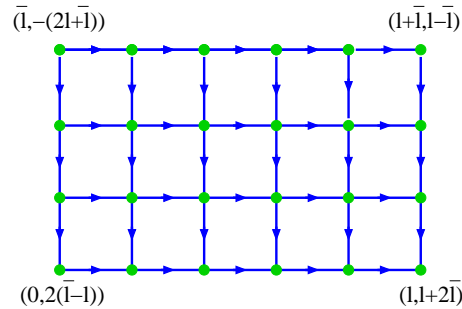
and the Higgs potential minimised by

$$\phi_{n,m}^{\pm}{}^0 = \frac{\sqrt{3}}{2gr} U_{n,m}^{\pm},$$

where $U_{0,0}^+, U_{1,-3}^+, U_{2,0}^-, U_{1,-3}^-$ are four unitary matrices satisfying one extra condition

$$U_{2,0}^- U_{1,3}^+ = U_{0,0}^+ U_{1,-3}^-. \tag{1}$$

2.3.2 Quiver diagrams

For a general $SU(3)$ irreducible representation, $\{l, \bar{l}\}$, the quiver diagram is

The total number of Higgs matrices (blue links) is $2l\bar{l} + l + \bar{l}$, while the number of gauge groups (green dots) is $(l+1)(\bar{l}+1)$. If $k_{n,m} = k'$ are all equal, then all Higgs fields are $k' \times k'$ matrices and $V(\Phi)$ is minimised by those Higgs fields all proportional to unitary matrices, with constraints of the form (1) on the unitary matrices around any plaquette. Interpreting the Higgs fields as a $SU(k')$ lattice gauge field on the quiver lattice, the constraints are satisfied by demanding the trivial gauge configuration on the quiver lattice.

3 Fermions and Yukawa couplings

3.1 Twisted Dirac operators on S^2

To study how dimensionally reduced fermions and Yukawa couplings emerge in these models, we first consider the simplest non-trivial example of S^2. Represent the Dirac operator for a fermion with unit charge in the presence of a magnetic monopole on S^2 of charge p by $D\!\!\!\!/\,_{S^2}^{(p)}$. Mathematically, this is the Dirac operator twisted with the p-th tensor power of the tautological line bundle L [12].

For a given p, the eigenspinors will be denoted by $\chi_{j,p;l}$ and have eigenvalues

$$\mu_{j,p} = \pm\frac{1}{r}\sqrt{\left(j + \frac{1+p}{2}\right)\left(j + \frac{1-p}{2}\right)}$$

so that

$$D\!\!\!\!/\,_{S^2}^{(p)}\chi_{j,p;l}(y) = \mu_{j,p}\chi_{j,p;l}(y).$$

For p even the quantum number j is half-integral while for odd p it is integral: in both cases $j \geq \frac{|p|+1}{2}$ and the degeneracy is $2j + 1$, labelled by $l = 0, 1, \ldots, 2j$. The eigenspinors can be decomposed into their positive and negative chirality components

$$\chi_{j,p;l} = \begin{pmatrix} \chi_{j,p;l}^+ \\ \pm\chi_{j,p;l}^- \end{pmatrix},$$

where the sign corresponds to the sign of the eigenvalue.

In addition, for the special value $j = \frac{|p|-1}{2}$ when $p \neq 0$, there are $|p|$ zero modes: for $p \geq 1$ there are p negative chirality modes, which we denote by

$$\chi_{p;r}^-, \qquad r = 0, 1, \ldots, p - 1,$$

while for $p \leq -1$ there are $|p|$ positive chirality modes,

$$\chi_{p;r}^+, \qquad r = 0, 1, \ldots, |p| - 1.$$

For a given monopole charge, the index of the Dirac operator is

$$\text{Index}\big(D\!\!\!\!/\,_{S^2}^{(p)}\big) = -p.$$

The Dirac operator on $\mathcal{M}_6$ splits up into the direct sum of 4-dimensional and 2-dimensional Dirac operators

$$D\!\!\!\!/\,_{(6)} = D\!\!\!\!/\,_{(4)} \otimes \mathbf{1}_2 + \gamma_5 \otimes D\!\!\!\!/\,_{S^2}.$$

At first sight zero modes of the Dirac operator on S^2 might be expected to manifest themselves as massless fermions for the Dirac operator on $\mathcal{M}_4$, but we shall see below that this is not always the case.

After dimensional reduction a fermion on $\mathcal{M}_6$, e.g. in the fundamental of $U(k)$, will decompose as

$$\Psi(x, y) = \begin{pmatrix} \Psi^+(x, y) \\ \Psi^-(x, y) \end{pmatrix}$$

where the $\pm$ signs refer to the S^2 chirality, not 4-dimensional or 6-dimensional chirality. Indeed Ψ itself could be either Dirac or Weyl in 6-dimensions. In the equivariant dimensional reduction framework only zero modes on S^2 are compatible with $SU(2)$ symmetry: $j > \frac{|p|-1}{2}$ correspond to higher harmonics which

do not have this symmetry and correspond to 4-dimensional fermions with masses of order $\frac{1}{r}$. Focusing on zero modes, the 6-dimensional fermions $\Psi^{\mp}$ decompose as

$$
\begin{aligned}
\Psi^-(x,y) &= \oplus_{r=0}^{p_i-1} \widetilde{\psi}_{p_i;r}(x)\chi^-_{p_i;r}(y), \ \Psi^+ = 0 \quad (p_i \geq 1) \\
\Psi^+(x,y) &= \oplus_{r=0}^{|p_i|-1} \psi_{p_i;r}(x)\chi^+_{p_i;r}(y), \ \Psi^- = 0 \quad (p_i \leq -1),
\end{aligned}
$$

where $\widetilde{\psi}_{p_i;r}(x)$ and $\psi_{p_i;r}(x)$ are either Dirac spinors in 4-dimensions, if Ψ is Dirac in 6-dimensions, or Weyl spinors of opposite chirality, if Ψ is Weyl in 6-dimensions.

Not all of the 4-dimensional fermions $\widetilde{\psi}_{p_i;r}(x)$ and $\psi_{p_i;r}(x)$ are massless however [10]. The 6-dimensional Dirac operator involves the 6-dimensional gauge field, which includes the Higgs field after dimensional reduction, and these induce 4-dimensional Yukawa couplings, allowing for the possibility of generating mass terms for 4-dimensional fermions through dynamical symmetry breaking. If, and only if, m is odd there is a 4-dimensional Yukawa coupling linking $\widetilde{\psi}_1$ to ψ_{-1} through

$$
\frac{g}{2} \int_{\mathcal{M}_4} \sqrt{-g_{(4)}} \, \phi^\dagger_{\frac{m+1}{2}} \overline{\psi}_{-1} \, \gamma_5 \, \widetilde{\psi}_1 \, d^4x + h.c.
$$

For the example in §2.2.1, $SU(3) \to SU(2) \times U(1) \to U(1)$, we had $k_0 = 2$, $k_1 = 1$, and $m = 1$. In this case $\widetilde{\psi}_1$ transforms as $\mathbf{2}_1$ under $SU(2) \times U(1)$, ψ_{-1} as $\mathbf{1}_{-2}$, and $\phi = \phi_1$ as $\mathbf{2}_1$. These 4-dimensional fermions pick up a mass $\frac{1}{2r}$ via the Higgs vacuum expectation value, which is of the same order as the masses of the higher harmonic fermions arising from non-zero eigenvalues of the Dirac operator on S^2 and therefore should be removed from consideration if we are assuming higher harmonics are too heavy to be relevant to the physics at low energies.

3.2 Spin^c structures on $\mathbf{CP}^2$

The issue of fermions on $\mathbf{CP}^2$ is complicated because there is a topological obstruction to the existence of a spin structure: due to the fact that the second Stieffel-Whitney class is non-vanishing [13] there is a global obstruction to defining spinors on $\mathbf{CP}^n$ for even n.

Nevertheless fermions can be defined by coupling them to monopoles and/or instantons (spin^c structures). The full spectrum of the twisted Dirac operator is complicated but for equivariant dimensional reduction we only need the zero modes. For fermions coupling to an equivariant monopole of magnetic charge m and an equivariant instanton of topological charge n, the index of the Dirac operator on $\mathbf{CP}^2$ is [11]

$$
\text{Index}\left(\mathbb{D}^{(n,m)}\right) = \frac{1}{8}(n+1)\left(m^2 - (n+1)^2\right).
$$

The fact that this is not an integer if n and m have the same parity, i.e. they are either both even or both odd (e.g. $n = m = 0$), is related to the lack of spin structure on $\mathbf{CP}^2$. Under the embedding $SU(2) \times U(1) \hookrightarrow SU(3)$, $\{l,\overline{l}\} \to \oplus(n,m) =: W_{l,\overline{l}}$, n and m always have the same parity, so any equivariant monopole/instanton background arising from the embedding will not admit global spinors. We therefore allow for a further twist with a monopole of charge $q \in \mathbf{Z} + \frac{1}{2}$ ($2q$ odd) and the index for this twisted gauge field configuration is

$$
\text{Index}\left(\mathbb{D}_q^{(n,m)}\right) = \frac{1}{8}(n+1)\left((m+2q)^2 - (n+1)^2\right).
$$

We shall denote the positive and negative chirality zero modes of this operator, with a given fixed q, by $\chi^+_{n,m,q}$ and $\chi^-_{n,m,q}$ respectively (for notational clarity the degeneracy is not indicated).

3.2.1 Fundamental representation

For $\{l,\overline{l}\} = \{1,0\}$ we have $\{1,0\} \to (1,1) \oplus (0,-2)$, and choosing for example $q = -\frac{1}{2}$ results in

$$
\text{Index}\left(\mathbb{D}_{-1/2}^{(1,1)}\right) = -1, \qquad \text{Index}\left(\mathbb{D}_{-1/2}^{(0,-2)}\right) = 1.
$$

For example, the case $k = 3k'$ with $k_{1,1} = k_{0,-2} = k'$ gives a single $k' \times k'$ Higgs matrix and the symmetry reduction scheme

$$SU(3k') \longrightarrow SU(k') \times SU(k') \times U(1) \longrightarrow SU(k').$$

With $2q = -1$, $\chi^+_{0,-2,-\frac{1}{2}}(y)$ and $\chi^-_{1,1,-\frac{1}{2}}(y)$ are the only zero modes giving the equivariant decomposition

$$\Psi = \begin{pmatrix} \psi_{0,-2}(x)\chi^+_{0,-2,-\frac{1}{2}}(y) \\ \widetilde{\psi}_{1,1}(x)\chi^-_{1,1,-\frac{1}{2}}(y) \end{pmatrix},$$

where $\psi_{0,-2}(x)$ and $\widetilde{\psi}_{1,1}(x)$ are either 4-dimensional Dirac spinors on $\mathcal{M}_4$, if Ψ is Dirac in 8-dimensions, or chiral spinors of opposite chirality in 4-dimensions, if Ψ is chiral in 8-dimensions. The induced 4-dimensional Yukawa couplings generate a mass term for these spinors given by

$$\frac{\sqrt{2}}{r}(\psi^\dagger_{0,-2}\gamma_5\,\widetilde{\psi}_{1,1} + \widetilde{\psi}^\dagger_{1,1}\gamma_5\,\psi_{0,-2}).$$

A different choice of q leads to a different conclusion. Taking $2q = 3$ results in

$$\mathrm{Index}\left(D\!\!\!\!/\,^{(1,1)}_{3/2}\right) = 3, \qquad \mathrm{Index}\left(D\!\!\!\!/\,^{(0,2)}_{3/2}\right) = 0.$$

There is no analogue of $\psi_{0,-2}(x)$ in this case and Yukawa couplings cannot generate a mass term in 4-dimensions.

3.2.2 Adjoint representation

Starting from the adjoint representation

$$\{l, \bar{l}\} = \{1, \overline{1}\} \longrightarrow (2,0) \oplus (1,3) \oplus (1,-3) \oplus (0,0),$$

consider the symmetry breaking scheme

$$SU(8k') \longrightarrow SU(k')^4 \times U(1)^3 \longrightarrow SU(k').$$

Choosing, for example, $q = -\frac{3}{2}$ gives

$$\begin{aligned}
\mathrm{Index}\left(D\!\!\!\!/\,^{(2,0)}_{-3/2}\right) &= 0, & \mathrm{Index}\left(D\!\!\!\!/\,^{(1,3)}_{-3/2}\right) &= -1, \\
\mathrm{Index}\left(D\!\!\!\!/\,^{(1,-3)}_{-3/2}\right) &= 8, & \mathrm{Index}\left(D\!\!\!\!/\,^{(0,0)}_{-3/2}\right) &= 1.
\end{aligned}$$

In this case Yukawa couplings generate a mass coupling the 4-dimensional spinors $\widetilde{\psi}_{1,3}(x)$ and $\psi_{0,0}(x)$, but the 8 flavours $\psi_{1,-3}(x)$ remain massless.

4 Conclusions

We have shown that equivariant dimensional reduction with a simple gauge group G gives the following:

- Gauge symmetry reduction $G \to K$ with only one gauge coupling in 4-dimensions, even if K is semi-simple.

- Further dynamical symmetry breaking $K \to K'$ where the vacuum and symmetry breaking patterns, including Higgs and gauge boson masses and Weinberg angles, can be deduced uniquely from group theory and induced representation theory.

- In certain cases the vacuum configuration is related to gauge dynamics on the quiver lattice: the Higgs vacuum corresponds to zero flux on the quiver lattice.

- When fermions are included, chiral theories with families emerge naturally from non-trivial fluxes on S/R.

- Chiral fermions on $\mathcal{M}_d$ do not allow direct mass terms, but Yukawa couplings can give 4-dimensional masses to some of the resulting fermions on $\mathcal{M}_4$. Yukawa couplings can even give masses to some, but not all, zero modes.

The gauge and fermion structure of equivariant dimensionally reduced field theories is clearly very rich. Standard model type Yukawa couplings, with different chiralities belonging to different irreducible representations of the gauge group, arise quite naturally in the models presented here, but an exhaustive analysis of all possibilities would be an ambitious programme and remains to be tackled.

Acknowledgments

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