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Chiral fermions and Torsion in the Early Universe

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Torsion arising from fermionic matter in the Einstein-Cartan formulation of general relativity is considered in the context of Robertson-Walker geometries and the early Universe. An ambiguity in the way torsion arising from hot fermionic matter in chiral models should be implemented is highlighted and discussed. In one interpretation, the non-zero torsion present in chiral models gives a negative contribution to the energy density which ameliorates the Big Bang singularity or even, in extreme cases, eliminates it completely giving bounce solutions for early Universe cosmology.

It is often the case that quantum matter acts as a source for a classical field in situations where quantum aspects of the field itself can be ignored. This approximation has proven extremely useful for Einstein's equations where

$$G_{ab} = 8\pi G < T_{ab} >, \tag{1}$$

works well when the matter source is degenerate fermionic matter, where $<>$ is a quantum expectation value, and for thermal radiation, where $<>$ is a thermal average of photons. There are difficulties with this approach however, not least that the singularities inherent in fully fledged quantum field theory for the sources render (1) ambiguous and some criterion for cutting off the integrals must be introduced. For example it is well known that a naïve calculation of the vacuum energy density of the standard model of particle physics leads to far too high a value of the cosmological constant to be compatible with observations [1]. Nevertheless (1) seems to work well in the early Universe when the dynamics is dominated by radiation, as long as temperatures are well below the Planck temperatures. In the radiation dominated Universe Einstein's equations boil down to the Friedmann-Robertson-Walker (FRW) equation, ignoring spatial curvature this is

$$\frac{\dot a^2}{a^2} = \frac{8\pi G}{3} < T_{00} >= N_{eff}\frac{4\pi^3}{45}\frac{T^4}{m_{Pl}^2} \tag{2}$$

where $a(t)$ is the cosmological scale factor and N_{eff} is the effective number of degrees of freedom in the relativistic gas,

$$N_{eff} := N_B + \frac{7}{8}(N_+ + N_-) \tag{3}$$

with N_B the number of bosonic degrees of freedom (2 for photons), N_+ and N_- are the number of positive chirality and negative chirality fermionic degrees of freedom respectively (for a standard model neutrino $N_+ = 2$, $N_- = 0$; for a Dirac fermion $N_+ = N_- = 2$), [2]. The Planck mass, $m_{Pl}^2 = G^{-1}$ (we use units with $\hbar = c = 1$), appears in this formula not because we are considering a theory of quantum gravity but because of the quantum nature of the source for classical gravity.

In the Einstein-Cartan formulation of general relativity fermionic matter is expected to induce torsion (recent bounds on the magnitude of the torsion have been derived from cosmic microwave polarization in [3]). When the connection is varied in the Einstein-Cartan action the torsion two-forms $\tau^a = \frac{1}{2}\tau^a{}_{bc}e^b \wedge e^c$ are determined by a spinor field Ψ via the algebraic equation

$$\tau^a = 2\pi G\epsilon_{abcd}(\overline{\Psi}\gamma^5\gamma^d\Psi)e^b \wedge e^c, \tag{4}$$

[4] ($a, b, c, \ldots$ are orthonormal indices).

In the spirit of (1) the equation of motion (4) would be interpreted as

$$\tau_{a,bc} = 4\pi G\epsilon_{abcd} < \overline{\Psi}\gamma^5\gamma^d\Psi > . \tag{5}$$

We shall examine the effect of torsion arising from relativistic fermions in the early Universe, assuming isotropy and spatial homogeneity of both the geometry and the matter. It will be assumed that the metric of Robertson-Walker type and that the energy-momentum is of the form

$$T_{ab} = \begin{pmatrix} \rho & 0 \\ 0 & p\,\delta_{ij} \end{pmatrix} \tag{6}$$

where the density ρ and pressure p are homogeneous and depend only on time and $i, j = 1, 2, 3$ are space-like indices.

The Riemann tensor involves the square of the connection and the net effect of including the torsion (4) into the gravitational connection is that Einstein's equations are modified to

$$3\left(\frac{\dot a^2}{a^2} - \frac{\tau^2}{4}\right) = 8\pi G\rho \tag{7}$$

$$-\frac{2\ddot a}{a} - \frac{\dot a^2}{a^2} + \frac{\tau^2}{4} = \frac{8\pi G}{3}\rho, \tag{8}$$

where $a(t)$ is the Robertson-Walker scale factor and

$$\tau^2 = -\tau^a\tau_a = -16\pi^2G^2(\overline{\Psi}\gamma^5\gamma^a\Psi)(\overline{\Psi}\gamma^5\gamma_a\Psi) \tag{9}$$

(the metric signature $(-,+,+,+)$). Eliminating τ from (7) and (8) gives

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\rho. \qquad (10)$$

We assume in (8) that the pressure $p = \frac{\rho}{3}$ for relativistic matter, or equivalently that the energy-momentum tensor has zero trace, as befits highly relativistic fermions, and we ignore any possible spatial curvature. The Clifford algebra convention is $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$, with γ^0 hermitian.

Equations (7) and (8) are are not independent and are related by the (first) Bianchi identity. We shall see below that this has important implications for the form of ρ.

Since fermions constitute quantum matter, it seems natural to interpret (4) in the early Universe as meaning a thermal average (5). However there is an ambiguity as to whether (9) should be interpreted using

$$< (\overline{\Psi}\gamma^5\gamma^a\Psi)(\overline{\Psi}\gamma^5\gamma_a\Psi) > \qquad (11)$$

or

$$< \overline{\Psi}\gamma^5\gamma^a\Psi >< \overline{\Psi}\gamma^5\gamma_a\Psi > . \qquad (12)$$

These are different in general. The former can be Fierz rearranged to give

$$< (\overline{\Psi}\gamma^5\gamma^a\Psi)(\overline{\Psi}\gamma^5\gamma_a\Psi) > \; = \; 4 < (\Psi_+^\dagger\Psi_-)(\Psi_-^\dagger\Psi_+) > \qquad (13)$$

where Ψ_+ and Ψ_- are the positive and negative chirality components of Ψ. This is always positive definite for Dirac spinors and vanishes for Weyl spinors [4], hence $\tau^2 \leq 0$ in (9). The cosmological consequences of this formulation in inflationary models are explored in [5]. The same philosophy, applied to spin densities rather than the pseudo-vector $\overline{\Psi}\gamma^5\gamma^a\Psi$, is followed in [6] and [7].

We reach a radically different conclusion if we use (12), which follows from taking the thermal average of (4) before calculating the Riemann tensor. Applying the usual Robertson-Walker assumptions of spatial homogeneity and isotropy to the connection, and hence the torsion, we would conclude that, in the cosmic frame,

$$< \overline{\Psi}\gamma^5\gamma^i\Psi >= 0, \qquad (14)$$

while

$$< \overline{\Psi}\gamma^5\gamma^0\Psi >= n_- - n_+ \qquad (15)$$

where n_+ and n_- are the number density of positive and negative chirality fermions respectively. In this interpretation

$$\tau^2 = 16\pi^2 G^2 (n_+ - n_-)^2 \qquad (16)$$

is positive in any chiral model for matter with $n_+ \neq n_-$. We therefore define

$$\tau = 4\pi G(n_+ - n_-), \qquad (17)$$

in terms of which the non-vanishing components of the torsion are

$$\tau_{i,jk} = \epsilon_{ijk}\tau.$$

This conclusion is not incompatible with the conclusion of [8], where classical solutions of the Weyl equation were analyzed in spherical symmetric space-times with torsion. Thermal averages do not necessarily have the same symmetries as solutions of the equations of motion. The general form of the torsion compatible with Robertson-Walker symmetries was given in [9]. The fact that chiral fermions can have interesting consequences when torsion is taken into consideration has been noticed before, in the context of anomalies for lepton currents in the Standard Model of particle physics [10].

Both (11) and (12) have interesting, though very different, cosmological consequences. The form (11), being the square of a vector, has a dual description as the square of a 3-form and as such is in the class of models described in [11]. Indeed a term of this form is present in the Landau-Ginsparg models discussed in [11], though the stabilising quartic term is absent and there is no kinetic term here. A kinetic term would require time derivatives of the torsion and so would go beyond Einstein-Cartan theory — such terms would be expected to appear in an effective action description of gravity involving higher derivatives and powers of the Riemann tensor but we shall focus here on (11). The form (12) was explored in a cosmological context in [5].

So which should one use (11) or (12)? Weinberg [12] takes the point of view that there is nothing special about torsion: it is just another tensor and one can always move it to the right hand side of Einstein's equations and consider it to be part of the matter rather than part of the geometry. We see here that, in the context of (1), there is an ambiguity. If the torsion terms are absorbed into the energy momentum tensor before expectation values are taken then it would seem that (11) is appropriate. In the Einstein-Cartan formulation however the torsion is determined by the equation of motion (4), in which the square of the torsion does not appear. If the gravitational field itself is not quantised, it is hard to see any interpretation of (4) other than (11). When the Riemann tensor is calculated it is then (12) that arises and not (11). Much of the literature has focused on (11) however so in this letter the consequences of (5) and (12) will be explored and developed.

In a thermal state the number density depends on temperature, when the temperature is large enough all fermions are relativistic and

$$n_\pm(T) = \frac{3}{4}\frac{\zeta(3)}{\pi^2}N_\pm T^3, \qquad (18)$$

where $\zeta(3) = \sum_{p=1}^\infty \frac{1}{p^3} \approx 1.202$ is the Riemann ζ-function. For a model with N_+ positive chirality degrees of freedom and N_- negative chirality degrees of freedom

$$\tau = \frac{3\zeta(3)}{\pi}(N_+ - N_-)\frac{T^3}{m_{Pl}^2} := A\frac{T^3}{m_{Pl}^2}, \qquad (19)$$

where $A = \frac{3\zeta(3)}{\pi}(N_+ - N_-)$.

When there is torsion the Bianchi identity does not require that G_{ab} be co-variantly constant, in general one has

$$\nabla_b G^{ba} = -\tau^c{}_{bc} G^{ba} + \frac{1}{2}\widetilde{R}^{adbc}\tau_{d,bc}, \qquad (20)$$

where $\widetilde{R}^{adbc} := \frac{1}{4}\epsilon^{ada'd'} R_{a'd'b'c'}\epsilon^{bcb'c'}$. In the case of Friedmann-Robertson-Walker (FRW) Universes under study here only the second term on the right hand side contributes giving

$$\nabla_b G^{b0} = -\frac{3}{2}\frac{\tau}{a}\frac{d}{dt}(\tau a), \qquad \nabla_b G^{bi} = 0. \qquad (21)$$

One strategy is to demand $\nabla_b G^{ba} = 0$ and use this to determine the torsion, implying that $\tau \propto 1/a$ [13], but this is too restrictive for our purposes. Instead we take thermodynamic averages as above and use (19) for the form of the torsion.

Assuming adiabatic expansion of the Universe requires that T is inversely proportional to the cosmological scale factor a, $T \propto 1/a$, [2]. The scale factor a can then be eliminated from Einstein's equations in favour of T to give

$$\frac{\dot{T}^2}{T^2} = \frac{8\pi}{3}\frac{\rho}{m_{Pl}^2} + \frac{A^2}{4}\frac{T^6}{m_{Pl}^4} \qquad (22)$$

$$\frac{\ddot{T}}{T} = 8\pi\frac{\rho}{m_{Pl}^2} + \frac{A^2}{2}\frac{T^6}{m_{Pl}^4}. \qquad (23)$$

The behaviour $\tau^2 \sim 1/a^6$ for torsion arising from spin in the early Universe has been noted before, [14, 15] — what is new in the discussion here is that the torsion arising from thermal fermions is only non-zero in chiral theories.

The final ingredient that we need to derive the FRW equation is the equation of state relating ρ and T. For a relativistic gas consisting of different particle species the thermal energy-density is

$$\rho_0 = \frac{\pi^2}{30} N_{eff} T^4. \qquad (24)$$

However (24) is incompatible with (22) and (23): (24) is inconsistent with the Bianchi identity and the assumption of adiabaticity. We can keep the assumption of adiabaticity by modifying ρ to

$$\rho = \frac{1}{8\pi}\left(BT^4 + C\frac{T^6}{m_{Pl}^2}\right), \qquad (25)$$

with $B = \frac{4\pi^3}{15} N_{eff}$. Then (22) and (23) are consistent with (25) if and only if

$$C = -\frac{3}{2}A^2. \qquad (26)$$

Using this in (22) we derive the FRW equation with torsion

$$\frac{\dot{T}^2}{T^2} = N_{eff}\frac{4\pi^3}{45}\frac{T^4}{m_{Pl}^2} - \frac{9\zeta(3)^2}{4\pi^2}(N_+ - N_-)^2\frac{T^6}{m_{Pl}^4} \qquad (27)$$

while ρ in (25) must be

$$\rho = \frac{\pi^2}{30} N_{eff} T^4 - \frac{27\zeta(3)^2}{16\pi^3}(N_+ - N_-)^2\frac{T^6}{m_{Pl}^2}. \qquad (28)$$

At temperatures much less than the Planck temperature $T_{Pl} \approx 10^{32}\, K$ the torsion term can only be physically relevant if $(N_+ - N_-)^2 >> N_{eff}$. When the temperature approaches the Planck temperature we expect quantum gravity effects to become important and the classical FRW-equation will break down, so we must restrict our analysis of (27) to $T << T_{Pl}$. For the standard model of particle physics, $N_{eff} = 106.75$ while $(N_+ - N_-)^2 = (3\times 2)^2 = 36$, so, even at $T = m_{Pl}$ the second term on the right hand side of (27) only contributes 4%: the torsion is never relevant in any regime where (27) can be trusted. The effect of torsion is even weaker if right-handed neutrinos exist, as implied by neutrino experiments [16].

However little is known about the fundamental particle content of dark matter — if dark matter is highly chiral, with $(N_+ - N_-)^2 >> N_{eff}$, then the T^6 term in (27) could become important while the dynamics of the early Universe is still governed by classical gravity.

In the regime where (27) can be trusted the last term in (28) makes a negative contribution to the energy density which ameliorates the singularity of the Big Bang and can avoid it completely. Equation (27) implies a maximum temperature given by

$$T_{max}^2 = \frac{16\pi^5}{405\zeta(3)^2}\frac{N_{eff}}{(N_+ - N_-)^2} m_{Pl}^2, \qquad (29)$$

at which $\dot{T} = 0$. If the co-efficient is such that $T_{max} << m_{Pl}$ the classical FRW equation can be used and the Universe could have started expanding from finite temperature at $t = 0$ with $\dot{a} = 0$ and $\ddot{a} > 0$, since at this temperature ρ in (10) is negative. Such a boundary condition is most natural in the context of bouncing cosmologies [14, 17].

It is unlikely that the effects above would have any relevance at temperatures corresponding to grand unified energies, unless there is a huge imbalance between N_{eff} and $(N_+ - N_-)^2$. If there was a period of inflation when the temperature was around $10^{16}\, GeV$ then the torsion discussed here would have a negligible contribution unless $(N_+ - N + -) \approx 10^7 N_{eff}$ which seems unlikely. The above discussion is likely to be at most only relevant to the period before inflation.

At first sight if might seem disconcerting that energy-momentum does not appear to be conserved in this formalism — because of (21) and the Einstein equations T_{ab} cannot be co-variantly constant unless $a\tau$ is constant. However an "improved" energy-momentum tensor, which is conserved, can be defined. We make the co-variant decomposition of the Einstein tensor

$$G_{ab} = \overset{0}{G}_{ab} + \Delta G_{ab} \qquad (30)$$

where $\overset{0}{G}_{ab}$ is the Einstein tensor constructed from the torsion-free connection. We similarly decompose the connection oneforms as

$$\omega^a{}_b = \overset{0}{\omega}{}^a{}_b + \Delta\omega^a{}_b \tag{31}$$

with $\overset{0}{\omega}{}^a{}_b$ the torsion-free connection. Expanding $\Delta\omega^a{}_b = \Delta\omega^a{}_{b,c}\, e^c$ the components $\Delta\omega^a{}_{b,c}$, being the difference of two connections, constitute a tensor field so (31) is again a co-variant decomposition. $\overset{0}{G}_{ab}$ is the zero torsion Einstein tensor for which the first Bianchi identity implies

$$\overset{0}{\nabla}_b \overset{0}{G}{}^{ba} = 0 \tag{32}$$

where $\overset{0}{\nabla}_b$ is the co-variant derivative using $\overset{0}{\omega}{}^a{}_b$. From this follows

$$\nabla_b G^{ba} = \overset{0}{\nabla}_b \left(\Delta G^{ba}\right) + \Delta\omega^b{}_{c,b} G^{ca} + \Delta\omega^a{}_{c,b} G^{bc}. \tag{33}$$

We also have, by definition,

$$\nabla_b T^{ba} = \overset{0}{\nabla}_b\, T^{ba} + \Delta\omega^b{}_{c,b} T^{ca} + \Delta\omega^a{}_{c,b} T^{bc} \tag{34}$$

for T_{ab}. Einstein equations, $G^{ab} = 8\pi G T^{ab}$, now imply

$$\overset{0}{\nabla}_b \left(\Delta G^{ba}\right) = 8\pi G\, \overset{0}{\nabla}_b\, T^{ba}. \tag{35}$$

An "improved" energy-momentum tensor can be defined

$$\mathcal{T}^{ab} := T^{ab} - \frac{1}{8\pi G}\Delta G^{ab} \tag{36}$$

which is conserved using the torsion free connection,

$$\overset{0}{\nabla}_b\, \mathcal{T}^{ba} = 0. \tag{37}$$

In terms of temperature the improved energy-momentum tensor for FRW space-time with torsion is

$$\mathcal{T}_{ab} = \begin{pmatrix} \rho_0 & 0 \\ 0 & \frac{\rho_0}{3}\delta_{ij} \end{pmatrix} - \frac{27\zeta(3)^2}{32\pi^3}(N_+ - N_-)^2 \frac{T^6}{m_{Pl}^2}\delta_{ab} \tag{38}$$

with ρ_0 given in (24). In fact both terms in (38) are separately conserved with the torsion-free connection.

Finally we observe that the geometrical significance of non-zero τ follows from the anti-symmetrised action of two co-variant derivatives on an arbitrary vector field with components U^a,

$$[\nabla_a, \nabla_b]U^c = -\tau^d{}_{ab}\nabla_d U^c + R^c{}_{dab}U^d. \tag{39}$$

In addition to the algebraic (rotation) term involving the Riemann tensor there is a derivative term involving the torsion — a deficit displacement implying that parallelograms generated by parallel transport do not close. The deficit displacement in Robertson-Walker space-time described here is compatible with 3-dimensional rotational symmetry — a vector field with Robertson-Walker symmetries must have $U^i = 0$ and U^0 independent of position, in which case

$$[\nabla_i, \nabla_j]U^0 \;=\; -\tau\, \epsilon_{ij}{}^k \nabla_k U^0. \tag{40}$$

Space-like parallelograms do not close in FRW space with torsion.

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