A Three-Generation Calabi-Yau Manifold
with Small Hodge Numbers

Volker Braun[1], Philip Candelas[2],

and

Rhys Davies[3]

[1]*Dublin Institute for Advanced Studies*
10 Burlington Road
Dublin 4, Ireland

[2]*Mathematical Institute*
University of Oxford
24-29 St. Giles', Oxford OX1 3LB, UK

[3]*Rudolf Peierls Centre for Theoretical Physics*
University of Oxford
1 Keble Road, Oxford OX1 4NP, UK

arXiv:0910.5464v1 [hep-th] 28 Oct 2009

Abstract

We present a complete intersection Calabi-Yau manifold Y that has Euler number -72 and which admits free actions by two groups of automorphisms of order 12. These are the cyclic group $\mathbb{Z}_{12}$ and the non-Abelian dicyclic group Dic_3. The quotient manifolds have $\chi = -6$ and Hodge numbers $(h^{11}, h^{21}) = (1, 4)$. With the standard embedding of the spin connection in the gauge group, Y gives rise to an E_6 gauge theory with 3 chiral generations of particles. The gauge group may be broken further by means of the Hosotani mechanism combined with continuous deformation of the background gauge field. For the non-Abelian quotient we obtain a model with 3 generations with the gauge group broken to that of the standard model. Moreover there is a limit in which the quotients develop 3 conifold points. These singularities may be resolved simultaneously to give another manifold with $(h^{11}, h^{21}) = (2, 2)$ that lies right at the tip of the distribution of Calabi–Yau manifolds. This strongly suggests that there is a heterotic vacuum for this manifold that derives from the 3 generation model on the quotient of Y. The manifold Y may also be realised as a hypersurface in the toric variety. The symmetry group does not act torically, nevertheless we are able to identify the mirror of the quotient manifold by adapting the construction of Batyrev.

Contents

1. Introduction

Here we report the existence of a Calabi–Yau manifold Y with Euler number -72 that admits freely acting symmetry groups of order 12. Thus the quotients are smooth and have Euler number -6. The two symmetry groups[1] are $\mathbb{Z}_{12}$ and a non-Abelian group $\mathfrak{G}$ which is isomorphic to the dicyclic group[2] Dic_3 and for both these groups the quotient manifold has Hodge numbers $(h^{11}, h^{21}) = (1, 4)$. With the standard embedding of the spin connection in the gauge group, the quotient manifolds correspond to string theory vacua with gauge symmetry E_6 and 3 net chiral generations of particles.

Two avenues are open to break the gauge symmetry further. The first is the Hosotani mechanism [1], whereby one assigns vacuum expectation values to Wilson line operators corresponding to homotopically non-trivial paths on the quotient manifold. The resulting gauge group is then the subgroup of E_6 that commutes with these Wilson line operators. The other avenue is to choose a heterotic vacuum by deforming the structure group of the background gauge field away from the $SU(3)$ that is provided by the standard embedding of the spin connection in the gauge group. Neither mechanism, on its own, is able to break the symmetry down to the group $SU(3){\times}SU(2){\times}U(1)$ of the standard model [2, 3]. However by combining the two mechanisms, for the quotient with non-Abelian fundamental group $\mathfrak{G}$, the gauge group may be reduced to that of the standard model. The way this comes about is that we can start with the standard embedding, with 3 net chiral families of E_6. The Hosotani mechanism can break the gauge group in basically two ways to either $SU(3){\times}SU(2){\times}U(1){\times}U(1)'$ or to $SU(4){\times}SU(2){\times}U(1)$, the reduction in the rank of the gauge group being due to the fact that the fundamental group of the quotient is non-Abelian. To break the remaining unwanted gauge symmetry it is necessary to give a VEV to some standard model singlet field(s), a process which we understand to being equivalent to deforming the background gauge field. This process is tightly constrained precisely because we are dealing with a manifold with few parameters. For the deformation to be possible requires the existence of a vector-like pair of such fields. It transpires that these exist for the case of $SU(4){\times}SU(2){\times}U(1)$, where we can give a VEV to a field transforming in the $(\mathbf{4}, \mathbf{1})_{\frac{1}{2}}$ representation of the gauge group. This leads to a unique choice of Wilson lines, and hence massless spectrum. If, however, we first break to $SU(3){\times}SU(2){\times}U(1){\times}U(1)'$ we find, despite the fact that this group appears more promising at the outset, that the deformations required to give the appropriate VEV's to the exotic standard model singlets do not, in fact, exist.

Heterotic vacua are of great interest in and of themselves. However we find it interesting also that, having started with a vacuum corresponding to the standard embedding of the spin connection in the gauge group, we have been led to deform to a heterotic vacuum. This

---

[1] In the following, we will denote the cyclic group of order n as $\mathbb{Z}_n$.

[2] There are three non-Abelian groups of order 12. These are Dih_6, the group of symmetries of a regular hexagon, the alternating group A_4, which is also the symmetry group of a regular tetrahedron, and the group Dic_3 which is neither of the foregoing. We will describe this group in detail in Subsection 2.1.

point takes perhaps additional significance from the fact that the manifold $Y_{\mathfrak{G}}$ lies almost at the tip of the distribution of Calabi–Yau manifolds and is closely related to a manifold with $(h^{11}, h^{21}) = (2, 2)$ which occupies a remarkable position at the very tip of the distribution, in a way that we pause to explain.

A Calabi–Yau manifold is partially characterised by the Hodge numbers (h^{11}, h^{21}). These are topological numbers which also count the number of parameters that deform the Kähler class and the complex structure of the manifold. For the purposes of the present discussion, let us call the sum $h^{11} + h^{21}$ the *height* of the manifold. The height of $Y_{\mathfrak{G}}$ is 5 and, together with its mirror, these are the only manifolds known with this height. The only manifolds known with smaller height all have height 4; with this height manifolds are known with $(h^{11}, h^{21}) = (1, 3), (2, 2)$ and $(3, 1)$. A notable feature of the model presented here is therefore that three generations are achieved in an economical manner, since the height is the least possible for a '3-generation' manifold with the standard embedding. The 3-generation Yau manifold [4, 5], by contrast, has $(h^{11}, h^{21}) = (6, 9)$ (for a discussion of the phenomenology of this model see [6, 7]), and a manifold presented in [8] has $(h^{11}, h^{21}) = (5, 8)$ so is little better, in this sense. There is no shortage, as such, of Calabi–Yau manifolds with $|\chi| = 6$ since there is also a manifold with Hodge numbers $(10, 13)$ and many manifolds with Hodge numbers $(h, h+3)$ and $(h+3, h)$ for $h \geq 15$ that are provided by constructions of manifolds as hypersurfaces in weighted projective spaces and in toric varieties [9, 10, 11]. Calabi–Yau manifolds admitting freely-acting discrete symmetries seem to be rare, and those cases where the group is large seem to be very rare indeed. The remarkable features of the manifold presented here are the related facts that $|\chi| = 6$ is achieved by Hodge numbers that are so small and that the fundamental group is large, by the standards of these groups.

The manifold $Y_{\mathfrak{G}}$ has $4 \, (= h^{21})$ parameters that correspond to its complex structure. The covering manifold Y has symmetries beyond those of $\mathfrak{G}$, these are not freely acting but are nevertheless symmetries, and some of these descend to the quotient $Y_{\mathfrak{G}}$. For generic values of the complex structure parameters the symmetry is $\mathbb{Z}_2$ but there are loci in the parameter space where the symmetry is enhanced. There is a two parameter subvariety, Γ, of the parameter space for which the symmetry group is maximally enhanced to the dihedral group Dih_6. On Γ the generic variety $Y_{\mathfrak{G}}$ is not smooth but has three conifold points. It is a striking fact that these singularities may be simultaneously resolved to give a new manifold $\widehat{Y}_{\mathfrak{G}}$ with $(h^{11}, h^{21}) = (2, 2)$ which is, as we have noted, currently at the very tip of the distribution. This transition suggests that there is a heterotic model associated with $\widehat{Y}_{\mathfrak{G}}$, with three generations, that derives from the standard embedding of the gauge group in the tangent bundle via transgression [12]. The idea being that $Y_{\mathfrak{G}}$ makes a conifold transition to $\widehat{Y}_{\mathfrak{G}}$ while the bundle makes a 'smooth' transition to a bundle on $\widehat{Y}_{\mathfrak{G}}$ that derives from the tangent bundle on $Y_{\mathfrak{G}}$. Thus we are led from a 3-generation model that derives from the standard embedding on $Y_{\mathfrak{G}}$ to a 3-generation heterotic model on the manifold $\widehat{Y}_{\mathfrak{G}}$.
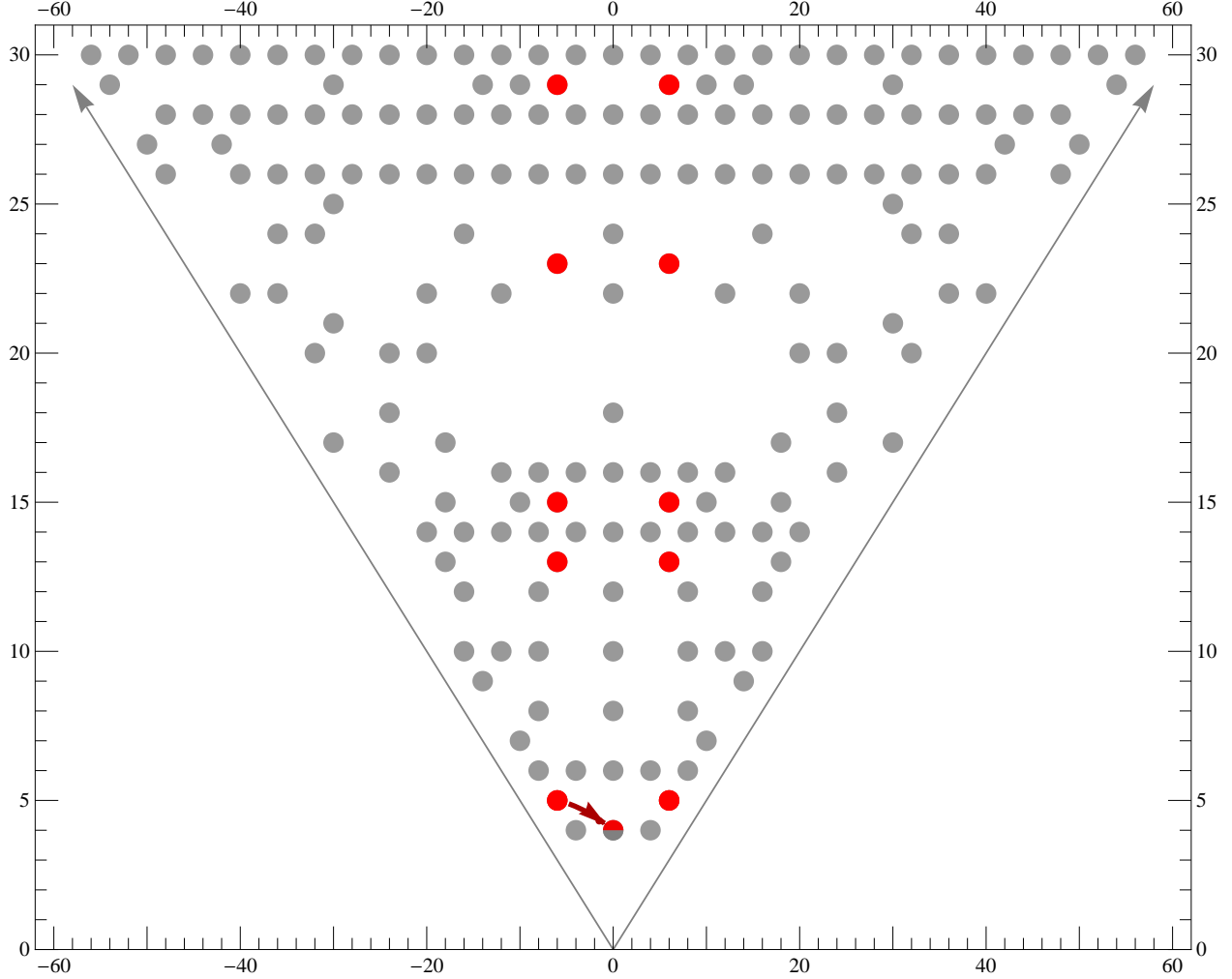
Figure 1: *The very tip of the distribution of Calabi–Yau manifolds showing the manifolds that have $h^{11}+h^{21} \leq 30$. The Euler number $\chi = 2(h^{11}-h^{21})$ is plotted horizontally, the height $h^{11}+h^{21}$ is plotted vertically and the oblique axes bound the region $h^{11} \geq 0$, $h^{21} \geq 0$. The quotients $Y/\mathfrak{G}$ with $|\mathfrak{G}| = 12$, that we discuss here, have $(h^{11}, h^{21}) = (1,4)$ these and their corresponding mirrors are shown. The conifold transition between the manifolds with $(h^{11}, h^{21}) = (1,4)$ and $(h^{11}, h^{21}) = (2,2)$ is indicated by the red arrow. Manifolds with $|\chi| = 6$ are distinguished by red dots and there are many of these for $h^{11}+h^{21} > 30$. The manifolds indicated in this plot with $h^{11}+h^{21} \leq 24$ are identified in [8, Table 9].*

The transgression that we propose is shown in Figure 1 with an arrow. The figure illustrates the fact that the manifolds that we are discussing lie at the very tip of the distribution.

There are many aspects of the phenomenology of these models that we do not discuss here. Notably absent is a discussion of the Yukawa couplings; however it is scarcely possible to discuss these without first achieving some understanding of the parameter space of the model

3

and of the nature of the manifold that is the mirror to $Y_{\mathfrak{G}}$ and this is the aim of the present work. A discussion of the Yukawa couplings and the quantum corrections to the picture presented here is a subject to which we hope to return.

The manifold Y can be represented as a CICY (complete intersection Calabi–Yau manifold) in several ways, and for two of these the group actions can be realised as linear transformations of the embedding space. The corresponding configuration matrices are

$$
\begin{array}{c}
\mathbb{P}^2 \\
\mathbb{P}^2 \\
\mathbb{P}^2 \\
\mathbb{P}^2
\end{array}
\left[
\begin{array}{ccccc}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1}
\end{array}
\right]
\ , \qquad
\begin{array}{c}
\mathbb{P}^1 \\
\mathbb{P}^1 \\
\mathbb{P}^1 \\
\mathbb{P}^1 \\
\mathbb{P}^1 \\
\mathbb{P}^1
\end{array}
\left[
\begin{array}{ccc}
\mathbf{1} & 0 & \mathbf{1} \\
\mathbf{1} & 0 & \mathbf{1} \\
\mathbf{1} & 0 & \mathbf{1} \\
0 & \mathbf{1} & \mathbf{1} \\
0 & \mathbf{1} & \mathbf{1} \\
0 & \mathbf{1} & \mathbf{1}
\end{array}
\right]
\tag{1.1}
$$

The reader unfamiliar with the notation is referred to Section 2 for an explanation. It suffices here to recall that the statement that Y corresponds to the first configuration is the statement that Y can be realised in $(\mathbb{P}^2)^4$ as the complete intersection of 5 polynomials whose multidegrees, in the homogeneous coordinates of the four $\mathbb{P}^2$'s, are given by the columns of the matrix. The same manifold can also be realised in $(\mathbb{P}^1)^6$ as the complete intersection of three polynomials with the stated multidegrees. These configurations give us a very explicit description of the manifold Y in terms of polynomials.

Compactifications on Calabi–Yau manifolds typically lead to a large numbers of massless neutral scalars in the low energy theory that are associated with the deformations of the gauge bundle on the manifold. In this case we begin with the gauge bundle equal to the tangent bundle, and have a heuristic argument based on the fact that the configuration matrices that describe Y are all multilinear that suggests that the tangent bundle deforms with few parameters. We can at least show that there are no deformations of the tangent bundle that arise as deformations of the differentials of the defining polynomials.

If one thinks carefully about the configurations (1.1), another picture of Y emerges. The manifold can be realised as a hypersurface, that is by a single equation, in a space $\mathcal{S}\times\mathcal{S}$ where $\mathcal{S}$ is the twofold defined by two bilinear equations in $\mathbb{P}^2\times\mathbb{P}^2$ or a single trilinear equation in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. The surface specified in this way is the del Pezzo[3] surface dP_6. The manifold Y is then an anti-canonical hypersurface inside the fourfold $\mathcal{S}\times\mathcal{S}$. The surface $\mathcal{S} = \mathrm{dP}_6$ is toric so we find ourselves within the general framework of toric hypersurfaces and reflexive polyhedra [13]. It turns out that the symmetries of Y are also best understood as descending from symmetries of dP_6.

For a review of constructions of Calabi–Yau manifolds see [12, 14], which includes a description of the construction of CICY manifolds. The construction of this class of manifolds

---

[3]Note that we will be following the mathematical tradition and label the del Pezzo surfaces dP_d by their degree d. The blow-up of $\mathbb{P}^2$ at n generic points is a del Pezzo surface of degree $d = 9 - n$.

was inspired by its best known member, which is Yau's three-generation manifold. A list of almost 8,000 manifolds was constructed in this way [15]. Although a rather special class of manifolds (the Euler numbers, for example, all fall in the range $-200 \leq \chi \leq 0$) these provide many examples of Calabi–Yau manifolds that admit freely acting symmetries. The list was searched [16, 17] for manifolds that admit a freely acting symmetry such that the quotient has $\chi = -6$ and in these searches, to the chagrin of one of the present authors, the manifold presented here, which occurs 3 times in the list of [15] (two of these presentations are the configurations displayed above, the third is a hybrid of the two), was wrongly rejected. The manifold has subsequently lain, largely unremarked, in the list for more than 20 years. In [16] one of the presentations of this manifold was recognised as admitting a symmetry group $\mathbb{Z}_3$. More recently [8] it was recognised that the symmetry group could be promoted to $\mathbb{Z}_6$. The groups of order 12 finally came to light in the course of a recent project to classify all the freely acting symmetries for the manifolds of the CICY list. It transpires that, apart from the covering manifold of Yau's three-generation manifold, the manifold presented here is the only one to admit a smooth quotient with $\chi = -6$.

The layout of the paper is as follows. In Section 2 we describe the covering manifold, the free group action by the group Dic_3, and the relation to the del Pezzo surface dP_6. In Section 3 we use this information to examine possible gauge symmetry breaking patterns and the resulting 4D theories. We turn in Section 4 to explore the existence of additional discrete symmetries of the covering manifold which are not freely acting but nevertheless play a role in the low energy theory. Our main interest here is to identify the symmetries that survive as symmetries of the quotient manifold. One conclusion is that the quotient manifolds have a $\mathbb{Z}_2$-symmetry for all values of the complex structure parameters. Additional symmetries arise at various loci in the parameter space. The symmetry group of the quotient is a subgroup of Dih_6 (the group corresponding to the symmetries of a regular hexagon) and is another non-Abelian group of order 12. There is a 2-dimensional subspace of the parameter space where the symmetry group of the quotient is the full group Dih_6. The corresponding quotient varieties are, however, all singular and, generically, have three conifold points. These three-nodal varieties are especially interesting since they offer the possibility of transgression to heterotic models with 3 generations based on manifolds with $(h^{11},\, h^{21}) = (2, 2)$ as discussed above.

We take up in detail the issue of the 3-nodal varieties with symmetry Dih_6 in Section 5. The nodes of the quotient derive from a very beautiful configuration of 36 nodes on the covering space. The singularities of a nodal variety may be resolved locally to give a smooth complex manifold but it is not generally the case that the nodes may be resolved such that the resulting manifold is Kähler. In the present case however there is a configuration of non-Cartier divisors that contain the nodes and by blowing up along these divisors we are able to show that there is a Calabi–Yau resolution in this case. In Section 6 we use the formalism of toric geometry to describe $Y_{\mathfrak{G}}$ and its mirror, and demonstrate that, rather unexpectedly, the mirror also admits an action by the group $\mathfrak{G}$. We are fortunate to be able to do this

5

since, although the covering space Y is a hypersurface in a toric variety the symmetries of $\mathfrak{G}$ do not act torically. Nevertheless, the structure of the reflexive polyhedra associated with Y is such that we are able to identify the mirror manifold of $Y_{\mathfrak{G}}$.

While our main interest is with the case of the quotient by the non-Abelian group we present in Section 7 a brief account of the analysis for the $\mathbb{Z}_{12}$ quotient. Two appendices deal with an alternative presentation of the covering manifold and a demonstration that the tangent bundle of Y does not have any deformations that correspond to deformations of the differentials of the defining polynomials.
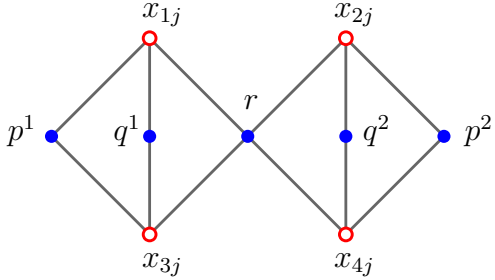
An account of the computer search for free group actions on CICY's and their classification will be presented elsewhere.

2. A Manifold with a $\chi = -6$ Quotient

2.1. The manifold $Y^{8,44}$

The manifold that we shall denote by $Y^{8,44}$ is specified by the configuration

$$Y^{8,44} = \begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \left[\begin{array}{ccccc} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array} \right]^{8,44}_{-72}$$

The manifold is defined by 5 polynomials, that we denote by p^1, p^2, q^1, q^2, and r that act in $(\mathbb{P}^2)^4$. We take coordinates $x_{\alpha j}$ for the four $\mathbb{P}^2$, where the indices α and j take values in $\mathbb{Z}_4$ and $\mathbb{Z}_3$, respectively. The columns of the matrix correspond to the degrees of the polynomials in the coordinates of each space, in the order $\{p^1, q^1, r, q^2, p^2\}$. The diagram on the right encodes the same information and shows how the polynomials, represented by the blue dots, depend on the variables of the four $\mathbb{P}^2$'s, which correspond to the open red dots. The fact that the dots are all connected by single lines in the diagram corresponds to the fact that the polynomials are all multilinear. We begin by seeking equations that are covariant under the cyclic permutation of the four $\mathbb{P}^2$'s,

$$g_4 : \ x_{\alpha j} \ \rightarrow \ x_{\alpha+1,j} \ , \quad p^1 \leftrightarrow p^2 \ , \quad q^1 \leftrightarrow q^2 \ , \quad r \ \rightarrow \ r \ .$$

For w_j and z_k homogeneous coordinates on $\mathbb{P}^2 \times \mathbb{P}^2$ define bilinear polynomials p and q by

$$p(w, z) \ = \ \sum_{jk} A_{jk} \, w_j z_k \ , \qquad q(w, z) \ = \ \sum_{jk} B_{jk} \, w_j z_k \ ,$$

where the coefficient matrices A_{jk} and B_{jk} are symmetric. Define also g_4-invariant polynomials

$$m_{ijk\ell} \ = \ \frac{1}{4} \sum_{\alpha} x_{\alpha,i} \, x_{\alpha+1,j} \, x_{\alpha+2,k} \, x_{\alpha+3,\ell} \ .$$

In terms of these quantities we may take defining polynomials, for Y, of the form

$$p^1 \ = \ p(x_1, x_3) \ , \qquad p^2 \ = \ p(x_2, x_4)$$

$$q^1 \ = \ q(x_1, x_3) \ , \qquad q^2 \ = \ q(x_2, x_4)$$

$$r \ = \ \sum_{ijk\ell} C_{ijk\ell} \, m_{ijk\ell} \ .$$

Note that the quantities $m_{ijk\ell}$ are cyclically symmetric in their indices so, in the definition of r, we take the sum over indices i, j, k, ℓ to run over combinations that are identified up

to cyclic permutation and the coefficients $C_{ijk\ell}$ to be cyclically symmetric. Consider now a second symmetry

$$g_3 : \quad x_{\alpha j} \ \rightarrow \ \zeta^{(-1)^\alpha j} x_{\alpha j}, \quad p^1 \ \rightarrow \ p^1, \quad p^2 \ \rightarrow \ p^2, \quad q^1 \ \rightarrow \ \zeta\, q^1, \quad q^2 \ \rightarrow \ \zeta^2 q^2,$$

where ζ is a nontrivial cube root of unity. Covariance under g_3 restricts the coefficients that can appear in the defining polynomials. We see that

$$A_{jk} \ = \ 0 \ \ \text{unless} \ \ j + k \equiv 0 \mod 3$$
$$B_{jk} \ = \ 0 \ \ \text{unless} \ \ j + k \equiv 2 \mod 3$$
$$C_{ijk\ell} \ = \ 0 \ \ \text{unless} \ \ i + k \equiv j + \ell \mod 3 \ .$$

Thus, removing overall scales, p and q are of the form

$$p(w, z) \ = \ w_0 z_0 + a \left(w_1 z_2 + w_2 z_1 \right) , \quad q(w, z) \ = \ w_1 z_1 + b \left(w_0 z_2 + w_2 z_0 \right),$$

while r is a linear combination of 9 of the $m_{ijk\ell}$

$$
\begin{aligned}
r \ = \ & c_0\, m_{0000} + c_1\, m_{1111} + c_2\, m_{2222} + c_3\, m_{0011} + c_4\, m_{0212} \\
& + \ c_5\, m_{0022} + c_6\, m_{1122} + c_7\, m_{0102} + c_8\, m_{0121}.
\end{aligned}
\tag{2.1}
$$

The freedom to redefine the coordinates $x_{\alpha j}$ is restricted by the action of g_3 and g_4. The remaining freedom allows only a two-parameter redefinition

$$(x_{\alpha 0},\ x_{\alpha 1},\ x_{\alpha 2}) \ \rightarrow \ (x_{\alpha 0},\ \lambda\, x_{\alpha 1},\ \mu\, x_{\alpha 2}) \tag{2.2}$$

which we may use to set the constants a and b that appear in p and q to unity. We may also redefine r by multiples of the other polynomials. Let $\tilde{p}$ and $\tilde{q}$ be generic polynomials with the same degrees and covariance properties as p and q, that is

$$\tilde{p}(w, z) \ = \ a_1 w_0 z_0 + a_2 (w_1 z_2 + w_2 z_1) , \quad \tilde{q}(w, z) \ = \ b_1 w_1 z_1 + b_2 (w_0 z_2 + w_2 z_0) \ .$$

These provide a four-parameter freedom to redefine r:

$$r \ \rightarrow \ r + \Big(\tilde{p}(x_1, x_3)\, p(x_2, x_4) + p(x_1, x_3)\, \tilde{p}(x_2, x_4) \Big) + \Big(\tilde{q}(x_1, x_3)\, q(x_2, x_4) + q(x_1, x_3)\, \tilde{q}(x_2, x_4) \Big) \ .$$

The coefficients c_2, c_3 and c_4 are unaffected by this process but we may use this freedom to eliminate the terms in eq. (2.1) with coefficients c_5, c_6, c_7 and c_8, say. An overall scale is of no consequence so the resulting equation has four parameters. We summarise the present form of the polynomials as

$$
\begin{aligned}
p^1 \ &= \ x_{10}\, x_{30} + x_{11}\, x_{32} + x_{12}\, x_{31} \ , \qquad q^1 \ = \ x_{11}\, x_{31} + x_{10}\, x_{32} + x_{12}\, x_{30} \ , \\[4pt]
p^2 \ &= \ x_{20}\, x_{40} + x_{21}\, x_{42} + x_{22}\, x_{41} \ , \qquad q^2 \ = \ x_{21}\, x_{41} + x_{20}\, x_{42} + x_{22}\, x_{40} \ , \\[4pt]
r \ &= \ c_0\, m_{0000} + c_1\, m_{1111} + c_2\, m_{2222} + c_3\, m_{0011} + c_4\, m_{0212} \ .
\end{aligned}
\tag{2.3}
$$

Consider again the coordinate transformations (2.2). We see that the equations p^1, p^2, q^1 and q^2 are invariant provided $\lambda\mu = 1$ and $\mu^3 = 1$. We take, therefore, $\lambda = \zeta$ and $\mu = \zeta^2$. The polynomial r, however is not invariant since, under the transformation, $m_{ijk\ell} \to \zeta^{i+j+k+\ell} m_{ijk\ell}$. The effect is equivalent to changing the coefficients. In this way we see that there is a $\mathbb{Z}_3$-action on the coefficients, and that we should identify

$$(c_0,\, c_1,\, c_2,\, c_3,\, c_4) \;\simeq\; (c_0,\, \zeta\, c_1,\, \zeta^2 c_2,\, \zeta^2 c_3,\, \zeta^2 c_4)\,. \tag{2.4}$$

Returning to the symmetries g_4 and g_3: these generate a group that we shall denote by $\mathfrak{G}$. Note that

$$g_4\, g_3 = g_3^2\, g_4\,,$$

so the group is non-Abelian. This relation, however, permits the enumeration of the elements of the group as $g_3^m\, g_4^n$, $0 \leq m \leq 2$, $0 \leq n \leq 3$. Thus $\mathfrak{G}$ has order 12 and is isomorphic to the dicyclic group Dic_3. A fact that will be useful shortly is that the element $g_6 = g_3^2 g_4^2$ generates a $\mathbb{Z}_6$ subgroup of $\mathfrak{G}$ and that the elements of $\mathfrak{G}$ may also be enumerated as $g_6^m\, g_4^n$ with $0 \leq m \leq 5$ and $n = 0$ or 1. The commutation relation above may be expressed as $g_4 g_3 g_4^{-1} = g_3^2$, so the group contains $\mathbb{Z}_3$ as a normal subgroup, and can be thought of as a semi-direct product $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$.

$g_3^m\, g_4^n$	1	g_4	g_4^2	g_4^3	g_3	$g_3\, g_4$	$g_3\, g_4^2$	$g_3\, g_4^3$	g_3^2	$g_3^2\, g_4$	$g_3^2\, g_4^2$	$g_3^2\, g_4^3$
$g_6^m\, g_4^n$	1	g_4	g_6^3	$g_6^3 g_4$	g_6^2	$g_6^2 g_4$	g_6^5	$g_6^5 g_4$	g_6^4	$g_6^4 g_4$	g_6	$g_6 g_4$
Order	1	4	2	4	3	4	6	4	3	4	6	4

Table 1: *The elements of the group Dic_3 presented in the form $g_3^m\, g_4^n$, for $0 \leq m \leq 2$, $0 \leq n \leq 3$, and $g_6^m\, g_4^n$, for $0 \leq m \leq 5$, $n = 0, 1$, together with the order of each element.*

In order to check for fixed points of $\mathfrak{G}$ note that if an element h has a fixed point then so has h^m for each $m \geq 1$. The order of an element of $\mathfrak{G}$, that is not the identity, must divide the order of the group so can be 2, 3, 4, or 6. If an element h has order 4 then h^2 has order 2 and if it has order 6 then h^3 has order 2. Hence it is enough to check the elements of order 2 and 3 for fixed points. The only elements of order 3 are g_3 and g_3^2 and if g_3^2 has a fixed point then so has $g_3^4 = g_3$. Thus it suffices to check g_3 and g_4^2, the latter being the unique element of order 2.

A fixed point of g_3 is such that $x_{\alpha j}$ takes one of the values $\{(1,0,0), (0,1,0), (0,0,1)\}$, for each α. Thus there are 3^4 fixed points in the embedding space $(\mathbb{P}^2)^4$. It is an easy check to see that these points do not coincide with simultaneous zeros of the defining polynomials provided none of the coefficients c_2, c_3, and c_4 vanish.

A fixed point of g_4^2 is such that $x_{1j} = x_{3j} = w_j$ and $x_{2j} = x_{4j} = z_j$ for some w_j and z_j that satisfy the equations

$$p(w, w) = 0 , \qquad p(z, z) = 0$$
$$q(w, w) = 0 , \qquad q(z, z) = 0$$
$$r(w, z, w, z) = 0$$

and it is easily checked that these five equations do not have a solution in $\mathbb{P}^2 \times \mathbb{P}^2$ for generic values of the parameters. We pause to do this explicitly since the values of the parameters for which there *are* fixed points will be of interest later. The equations $p(w, w) = q(w, w) = 0$ have four solutions for w. These are $w = (0, 0, 1)$ and $w = (1, \omega, -\omega^2/2)$, for $\omega^3 = 1$, and the solutions for z are the same, giving rise to 16 points in $\mathbb{P}^2 \times \mathbb{P}^2$. The polynomial $r(w, z, w, z)$ does not vanish on any of these points unless at least one of the quantities

$$c_2 , \quad c_2 + 2c_4 , \quad c_0^3 + c_1^3 + d_2^3 - 3\, c_0 c_1 d_2 , \tag{2.5}$$

where $16 d_2 = c_2 + 16 c_3 + 4 c_4$, vanishes.

We have checked that the polynomials (2.3) are transverse following the methods of [8]. We conclude that there exists a smooth quotient Calabi-Yau manifold, which we will denote by

$$Y_{\mathfrak{G}} := Y/\mathfrak{G}.$$

Now we may regard the manifold $Y^{8,44}$ as a hypersurface in $\mathcal{S} \times \mathcal{S}$, where $\mathcal{S}$ is the surface

$$\mathcal{S} = \begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.6}$$

which has Euler number 6 and is the del Pezzo surface dP_6 obtained by blowing up three points of $\mathbb{P}^2$ that are in general position. It is instructive to verify this explicitly and to locate the three blown up points by considering the defining equations (2.3). The polynomials p^1 and q^1, that define the first copy of $\mathcal{S}$, are $p^1 = x_1^T A\, x_3 = 0$ and $q^1 = x_1^T B\, x_3 = 0$ where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Given $x_1 \in \mathbb{P}^2$, consider the corresponding values of x_3 which solve these equations. For generic values, x_3 is determined up to scale, hence uniquely as a point of $\mathbb{P}^2$, as the vector orthogonal to $x_1^T A$ and $x_1^T B$, but at the three points coming from left eigenvectors of AB^{-1} we have $x_1^T A \propto x_1^T B$, and there is a whole $\mathbb{P}^1$ of solutions for x_3. These three points are $x_1 = (1, 1, 1)$, $(1, \zeta, \zeta^2)$ and $(1, \zeta^2, \zeta)$, with ζ again a non-trivial cube root of unity. The corresponding $\mathbb{P}^1$'s are the exceptional curves E_1, E_2, E_3. Also important to us are the three lines L_{ij} in $\mathcal{S}$ that correspond to the lines in the $\mathbb{P}^2$ that join the points that are blown up to E_i and E_j. The three L_{ij} together with the three E_i form a hexagon in $\mathcal{S}$, as sketched in Figure 2. We will see presently that the order 6 symmetry g_6 acts on this hexagon by rotation.
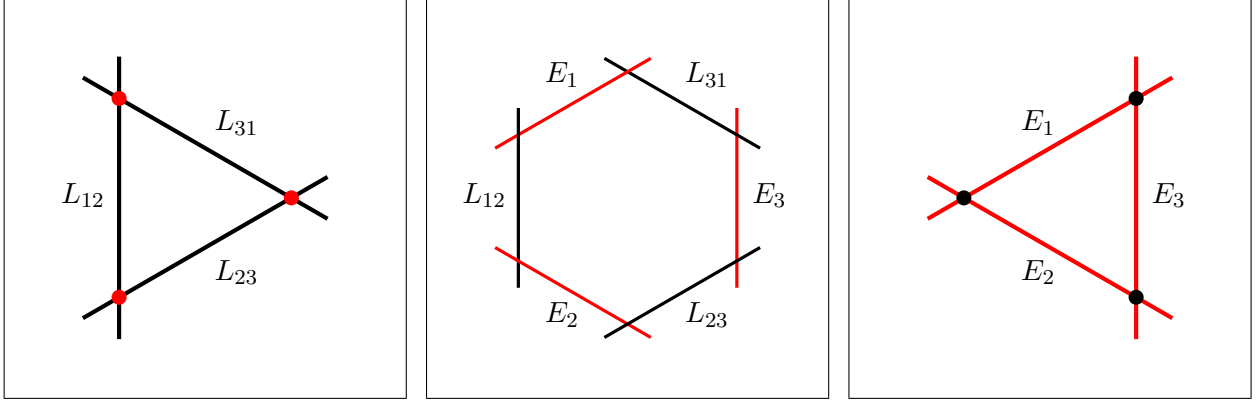
Figure 2: *A sketch of dP$_6$, in the centre, showing the hexagon formed by the six (-1)-lines. This surface may be realised in $\mathbb{P}^2 \times \mathbb{P}^2$ as the locus $p(w,z) = q(w,z) = 0$ defined by two bilinear polynomials in the coordinates w_j and z_j of the two $\mathbb{P}^2$'s. If we project to the first $\mathbb{P}^2$ then the image is as in the sketch on the left, in which the three lines E_i project to points. If, instead, we project to the second $\mathbb{P}^2$ then the image is as in the sketch on the right, in which the three lines L_{ij} have been projected to points.*

2.2. The group representations of Dic$_3$

Before proceeding we pause to describe the representations of the group $\mathfrak{G} \cong$ Dic$_3$. There are four one-dimensional representations of Dic$_3$, in which g_3 acts trivially and g_4 is multiplication by one of the fourth roots of unity. We will denote these, in an obvious notation, by R_1, R_i, R_{-1} and R_{-i}. These are the homomorphisms of Dic$_3$ to its Abelianisation $\mathbb{Z}_4$. There are also two distinct two-dimensional representations, distinguished in a coordinate-invariant way by $\text{Tr}(g_4^2) = \pm 2$. These we will denote by $R_\pm^{(2)}$. For completeness we display the corresponding matrices in a basis for which g_3 is diagonal:

$$R_+^{(2)}: \quad g_3 \rightarrow \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}, \quad g_4 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$R_-^{(2)}: \quad g_3 \rightarrow \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}, \quad g_4 \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The non-obvious tensor products of representations are as follows

$$R_1 \otimes R_\pm^{(2)} \;=\; R_{-1} \otimes R_\pm^{(2)} = R_\pm^{(2)}, \qquad R_i \otimes R_\pm^{(2)} \;=\; R_{-i} \otimes R_\pm^{(2)} = R_\mp^{(2)}$$

$$R_+^{(2)} \otimes R_+^{(2)} \;=\; R_1 \oplus R_{-1} \oplus R_+^{(2)}, \qquad R_-^{(2)} \otimes R_-^{(2)} \;=\; R_1 \oplus R_{-1} \oplus R_+^{(2)}$$

$$R_+^{(2)} \otimes R_-^{(2)} \;=\; R_i \oplus R_{-i} \oplus R_-^{(2)}$$

11

2.3. Group action on homology

The cohomology group $H^2(Y^{8,44})$ descends from that of $H^2(\mathcal{S}\times\mathcal{S}) = H^2(\mathcal{S})+H^2(\mathcal{S})$. For each $\mathcal{S}$ we have $h^{11}(\mathcal{S}) = 4$ and the cohomology group is spanned by (the duals of) the hyperplane class, H, and the three E_i. The intersection numbers of the four classes $\{H, E_1, E_2, E_3\}$ are given by a matrix η

$$H \cdot H \;=\; 1\,, \quad H \cdot E_i \;=\; 0\,, \quad E_i \cdot E_j \;=\; -\delta_{ij} \quad \text{so} \quad \eta \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

By considering the intersection numbers of L_{ij} with H and the E_k one sees that

$$L_{ij} \;=\; H - E_i - E_j$$

and that the L_{ij} are also (-1)-lines, that is $L_{ij} \cdot L_{ij} = -1$. Of course, everything said above also applies on the second copy of $\mathcal{S}$. To distinguish the cohomology classes coming from this copy, we denote them by $\widetilde{H}, \widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3$.

(-1)-line	w	z
E_1	$(1,1,1)$	$z_0 + z_1 + z_2 = 0$
L_{12}	$w_0 + \zeta w_1 + \zeta^2 w_2 = 0$	$(1, \zeta^2, \zeta)$
E_2	$(1, \zeta, \zeta^2)$	$z_0 + \zeta^2 z_1 + \zeta z_2 = 0$
L_{23}	$w_0 + w_1 + w_2 = 0$	$(1,1,1)$
E_3	$(1, \zeta^2, \zeta)$	$z_0 + \zeta z_1 + \zeta^2 z_2 = 0$
L_{31}	$w_0 + \zeta^2 w_1 + \zeta w_2 = 0$	$(1, \zeta, \zeta^2)$

Table 2: *The equations that define the six (-1)-lines in the coordinates $(w_i,\,z_j)$ of $\mathbb{P}^2\times\mathbb{P}^2$.*

We know the action of the group generators on the spaces, and this allows us to calculate the induced action on $H^2(\mathcal{S}\times\mathcal{S})$. Choosing the ordered basis $\{H, E_1, E_2, E_3, \widetilde{H}, \widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3\}$, we can write 8×8 matrices $U(g)$ representing the action of $g \in \mathfrak{G}$. It is clear that g_3 preserves the hyperplane classes and rotates the exceptional curves into each other, so that

$$U(g_3) \;=\; \begin{pmatrix} \mathsf{G}_3 & 0 \\ 0 & \mathsf{G}_3^2 \end{pmatrix} \quad \text{with} \quad \mathsf{G}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The action of g_4 is slightly more complicated to read off. The coordinates for the two copies of $\mathcal{S}$ are (x_{1j}, x_{3k}) and (x_{2j}, x_{4k}) and g_4 acts by mapping

$$(x_1, x_3) \rightarrow (x_2, x_4), \quad \text{while} \quad (x_2, x_4) \rightarrow (x_3, x_1).$$

We can think of this action as being an exchange of the two copies of $\mathcal{S}$ followed by the involution $x_1 \leftrightarrow x_3$ on the first copy. We need to calculate the action of this involution on $H^2(\mathcal{S})$. To this end, choose one of the exceptional curves, say E_1, which lies above $(1,1,1)$ in $\mathbb{P}^2_{x_1}$. Then it is described in $\mathbb{P}^2_{x_3}$ by the line $x_{30} + x_{31} + x_{32} = 0$. On the coordinates, the involution acts as $x_1 \leftrightarrow x_3$, so it maps E_1 to the curve described by $x_{10} + x_{11} + x_{12} = 0$ and $(x_{30}, x_{31}, x_{32}) = (1,1,1)$. Since this line passes through both $(1, \zeta, \zeta^2)$ and $(1, \zeta^2, \zeta)$, which are the other points that are blown up, we see that it is actually the line referred to earlier as L_{23}. Thus the action of the involution is $E_1 \leftrightarrow L_{23}$, or more generally $E_i \leftrightarrow L_{i+1,i+2}$. This is enough information to work out the action of g_4 on $H^2(\mathcal{S} \times \mathcal{S})$, with respect to the basis given above. The result is

$$U(g_4) = \begin{pmatrix} 0 & \mathsf{G}_2 \\ 1 & 0 \end{pmatrix} \quad \text{with} \quad \mathsf{G}_2 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

It is a quick check that $\mathsf{G}_2^2 = \mathsf{G}_3^3 = 1$ and that G_2 and G_3 commute and preserve the intersection matrix η. It is useful also to express the transformations in terms of the (-1)-lines of the two copies of $\mathcal{S}$, eliminating the explicit reference to H. Denote by D_a and $\widetilde{D}_b$, $a, b \in \mathbb{Z}_6$, the six (-1)-lines on the two copies of $\mathcal{S}$

$$D_a = (E_1, L_{12}, E_2, L_{23}, E_3, L_{31}) \quad \text{and} \quad \widetilde{D}_b = (\widetilde{E}_1, \widetilde{L}_{12}, \widetilde{E}_2, \widetilde{L}_{23}, \widetilde{E}_3, \widetilde{L}_{31}). \quad (2.7)$$

In terms of these the action of the generators g_6 and g_4 is

$$g_6 : D_a \times \widetilde{D}_b \rightarrow D_{a+1} \times \widetilde{D}_{b-1} \quad \text{and} \quad g_4 : D_a \times \widetilde{D}_b \rightarrow D_{b+3} \times \widetilde{D}_a. \quad (2.8)$$

If we change our basis for $H^2(\mathcal{S} \times \mathcal{S})$ such that $U(g_3)$ becomes diagonal, we can compare with Subsection 2.2 and see that the eight-dimensional representation decomposes into the sum

$$R_1 \oplus R_{-1} \oplus R_i \oplus R_{-i} \oplus R_+^{(2)} \oplus R_+^{(2)}$$

In particular, there is only a single invariant, implying $b_2 = 1$ for the quotient. This invariant corresponds to the canonical class, as it must, which we can see explicitly as follows. The group element $g_6 = g_3^2 g_4^2$ generates a $\mathbb{Z}_6$ subgroup, as noted previously. We have

$$U(g_6) = \begin{pmatrix} \mathsf{G}_6 & 0 \\ 0 & \mathsf{G}_6^{-1} \end{pmatrix} \quad \text{where} \quad \mathsf{G}_6 = \mathsf{G}_2 \mathsf{G}_3^2 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & -1 \end{pmatrix}$$

13

which acts separately on the two homology bases. The two canonical classes

$$K_{\mathcal{S}} \; = \; 3H - E_1 - E_2 - E_3 \qquad \text{and} \qquad K_{\widetilde{\mathcal{S}}} \; = \; 3\widetilde{H} - \widetilde{E}_1 - \widetilde{E}_2 - \widetilde{E}_3$$

are invariant under g_6 and since the eigenvalues of G_6 are $\{1, -1, \zeta, \zeta^2\}$ we see that these are the only two invariant homology classes. A class that is invariant under g_6 is invariant under $\mathfrak{G}$ if it is also invariant under g_4 and we immediately see that the only such invariant combination is $K_{\mathcal{S}} + \widetilde{K}_{\mathcal{S}}$. The fact that $b_2 = 1$ for the quotient variety implies $h^{11} = 1$ for the hypersurface $Y_{\mathfrak{G}}$. Since the Euler number divides by the order of the group, $\chi = -72/12 = -6$ and it follows that $h^{21} = 4$, in agreement with our count of parameters in the defining polynomials.

3. Rudiments of the Phenomenology

Compactification of $E_8{\times}E_8$ heterotic string theory on the manifold $Y_{\mathfrak{G}}$, with the standard embedding, leads to a 4D effective theory with unbroken gauge group $E_6{\times}E_8$ with four chiral multiplets in the **27** of E_6 and one in the $\overline{\bf 27}$. We believe there are few E_6 singlets, since these would correspond to rank-three deformations of the gauge bundle, and we argue in Appendix B that there are few of these. We wish to break the E_6 gauge symmetry further to achieve, if possible, the gauge group $G_{\rm SM} = SU(3){\times}SU(2){\times}U(1)$ of the standard model. Given that the unbroken gauge group is the commutant in E_8 of the holonomy group of the gauge connection, there are two related mechanisms at our disposal for gauge symmetry breaking. The first is to continuously deform the internal gauge field, which corresponds to the Higgs mechanism in the 4D theory. We may take the vector bundle corresponding to the background gauge field to be a deformation of $\mathcal{T} \oplus \mathcal{O}$ or of $\mathcal{T} \oplus \mathcal{O} \oplus \mathcal{O}$, where $\mathcal{T}$ is the tangent bundle of $Y_{\mathfrak{G}}$ and $\mathcal{O}$ is a trivial line bundle. In this way the structure group of the bundle becomes, in many cases, $SU(4)$, for deformations of $\mathcal{T} \oplus \mathcal{O}$, or $SU(5)$, for the case of $\mathcal{T} \oplus \mathcal{O} \oplus \mathcal{O}$. The commutants of these groups in E_8 are Spin(10) and $SU(5)$, respectively, which are attractive groups for phenomenology. In a specific model, however, it requires checking that suitably deformations of the bundles exist in order to break the symmetry as desired. We will see in Subsection 3.1 that on $Y_{\mathfrak{G}}$ it is possible to obtain Spin(10) but not $SU(5)$ in this way.

The second possibility is to invoke the Hosotani mechanism, whereby we avail ourselves of the fact that $Y_{\mathfrak{G}}$ is multiply connected to give non-zero values to Wilson lines around homotopically nontrivial paths in $Y_{\mathfrak{G}}$. This amounts to taking the gauge bundle to be $\mathcal{T} \oplus \mathcal{W}$, where $\mathcal{W}$ is a non-trivial flat bundle. $\mathcal{W}$ is specified by choosing a group homomorphism $\Phi : \pi_1\left(Y_{\mathfrak{G}}\right) \cong \mathfrak{G} \to E_6$ and letting the holonomy of the connection on $\mathcal{W}$ around a path γ be given by $\Phi([\gamma])$. The holonomy group therefore acquires an extra discrete factor $\Phi(\mathfrak{G})$. The smallest resulting unbroken gauge group containing $G_{\rm SM}$ is $SU(3){\times}SU(2){\times}U(1){\times}U(1)$. In principle the extraneous $U(1)$ can be broken by a vacuum expectation value for a standard model singlet, corresponding to a continuous deformation of the gauge bundle away from $\mathcal{T} \oplus \mathcal{W}$. In fact in this model all such fields have a D-term potential forcing their VEV to be zero. We therefore consider a less minimal choice of Wilson lines, for which the larger group $SU(4){\times}SU(2){\times}U(1)$ is unbroken. In this case there are flat directions in the low energy theory along which the group is Higgsed to exactly $G_{\rm SM}$.

In the following we make extensive use of standard group theory, for which the comprehensive reference is the review by Slansky [18].

3.1. Deforming the gauge bundle

We obtain unbroken E_6 gauge symmetry by choosing the non-trivial part of the gauge bundle to be equal to the tangent bundle $\mathcal{T}$ of the manifold, with structure group $SU(3)$. We will now consider, as discussed above, taking instead a non-trivial deformation of $\mathcal{T} \oplus \mathcal{O}$ or $\mathcal{T} \oplus \mathcal{O} \oplus \mathcal{O}$, with structure group $SU(4)$ or $SU(5)$, respectively. Since deformation is a continuous process, it must correspond to the Higgs mechanism in the low energy theory, whereby the vector multiplets lying outside the unbroken sub-algebra gain mass by eating chiral multiplets with the same charges. The gauge bosons transform in the adjoint representation of E_6, while the families and antifamilies transform in the $\mathbf{27}$ and $\overline{\mathbf{27}}$, respectively. These representations decompose under the $\mathrm{Spin}(10)$ subgroup as:

$$\mathbf{78} \; = \; \mathbf{45} \oplus \mathbf{16} \oplus \overline{\mathbf{16}} \oplus \mathbf{1} \;, \qquad \mathbf{27} \; = \; \mathbf{16} \oplus \mathbf{10} \oplus \mathbf{1} \;, \qquad \overline{\mathbf{27}} \; = \; \overline{\mathbf{16}} \oplus \mathbf{10} \oplus \mathbf{1} \;.$$

So if E_6 is Higgsed to $\mathrm{Spin}(10)$, the vector multiplets in $\mathbf{16} \oplus \overline{\mathbf{16}} \oplus \mathbf{1}$ will eat corresponding chiral multiplets, leaving the following representation

$$(3 \times \mathbf{16}) \oplus (5 \times \mathbf{10}) \oplus (4 \times \mathbf{1})$$

Notice that all anti-generations are gone. Although $\mathbf{10}$'s and singlets can obtain mass through the Yukawa couplings when E_6 is broken, one would hope that at least one of the $\mathbf{10}$'s remains massless, since the standard model Higgs lives in this multiplet. Now suppose we want to go further, and deform to an $SU(5)$ bundle, in order to break $\mathrm{Spin}(10)$ to $SU(5)$. Once again, this would be a continuous process, and so correspond to the Higgs mechanism in the low energy theory, so we can repeat the analysis above. The relevant representations decompose as follows:

$$\mathbf{45} \; = \; \mathbf{24} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{1} \;, \qquad \mathbf{16} \; = \; \mathbf{10} \oplus \overline{\mathbf{5}} \oplus \mathbf{1} \;, \qquad \mathbf{10} \; = \; \mathbf{5} \oplus \overline{\mathbf{5}} \;.$$

So if such a Higgsing were possible, there would have to be chiral multiplets transforming as $\mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{1}$ to be eaten by the corresponding vector fields. But there are no chiral multiplets transforming as $\overline{\mathbf{10}}$, so we conclude that this cannot happen. Notice that this is a consequence of having only one anti-generation of E_6 to begin with. The simple analysis above relies on general features of supersymmetric physics in 4D, but the complete mathematical answer to whether one can deform $\mathcal{T} \oplus \mathcal{O}^{\oplus r}$ to a slope-stable rank $3 + r$ bundle on any Calabi-Yau threefold X was given in [19]. Roughly, this involves three ingredients:

1. A choice of r linearly independent classes $\alpha_i \in H^1(X, \mathcal{T}^\vee)$ corresponding to an extension

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow 0.$$

   This amounts to the choice of r $\overline{\mathbf{27}}$ that we want to use in the Higgsing of E_6.

2. A choice of r linearly independent classes $\beta_i \in H^1(X, \mathcal{T})$ corresponding to the opposite extension

$$0 \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{T} \longrightarrow 0.$$

   This amounts to the choice of the corresponding $\mathbf{27}$s.

3. A family of vector bundles interpolating between $\mathcal{F}_1$ and $\mathcal{F}_2$. This ensures that, to all orders in perturbation theory, there is no superpotential term that forbids the VEVs.

We again notice that there is no deformation at $r = 2$ because one would need linearly independent α_1, $\alpha_2 \in H^1(Y_\mathfrak{G}, \mathcal{T})$. But $\dim H^1(Y_\mathfrak{G}, \mathcal{T}) = h^{11}(Y_\mathfrak{G}) = 1$, so there are no two linearly independent cohomology classes. This is a simple example of the beautiful interplay between mathematical features of the compactification and low energy supersymmetric field theory; for a recent analysis of vector bundle stability in this spirit see [20, 21].

In summary, deforming the tangent bundle can at most break the gauge group to $\mathrm{Spin}(10)$, but nothing smaller. We might then consider further breaking via the Hosotani mechanism, but as explained in [3], it is not possible to break $\mathrm{Spin}(10)$ directly to G_{SM} this way. It can however be broken to $G_{\mathrm{SM}} \times U(1)$, as in [22], with the extra $U(1)$ corresponding to baryon number minus lepton number. This is an option for the Abelian quotient of $Y^{8,44}$, with fundamental group $\mathbb{Z}_{12}$, but not for $Y_\mathfrak{G}$.

3.2. Breaking E_6 by the Hosotani mechanism

In the previous section we separated Wilson lines and gauge bundle deformations into commuting subgroups of E_6, and found we could not get all the way to the standard model gauge group G_{SM}. We now consider relaxing this restriction. First we will choose values for the Wilson lines, to partially break E_6, and then determine whether we can deform the resulting bundle to obtain just G_{SM}. The maximal symmetry breaking achievable with $\mathfrak{G}$-valued Wilson lines is $E_6 \to G_{\mathrm{SM}} \times U(1)'$, which we describe below. Unfortunately, in this case it is impossible to remove the extraneous $U(1)'$ by a deformation. In the following subsection we therefore describe a model in which the Wilson lines leave $SU(4) \times SU(2) \times U(1)$ unbroken, and a bundle deformation can break this to precisely G_{SM}.

3.2.1. Model 1

First we follow the original approach of [2] and take the image of $\Phi : \pi_1(Y_{\mathfrak{G}}) \to E_6$ to lie in the maximal subgroup $SU(3)_C \times SU(3)_L \times SU(3)_R$. There is in fact a unique choice of such an embedding that achieves the maximal breaking of E_6 to $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)'$, namely

$$\Phi(g_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix},$$

$$\Phi(g_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.1}$$

In particular, the rank of the unbroken gauge group is reduced from 6 down to 5, which would not be possible with Abelian Wilson lines. The next step is to compute the massless particle spectrum. Note that the $\mathbf{27}$ of E_6 decomposes as

$$\mathbf{27} = (\mathbf{3}, \mathbf{3}, \mathbf{1}) \oplus (\overline{\mathbf{3}}, \mathbf{1}, \overline{\mathbf{3}}) \oplus (\mathbf{1}, \overline{\mathbf{3}}, \mathbf{3})$$

and, therefore, under the unbroken gauge group as

$$\mathbf{27} = \underbrace{(\mathbf{3}, \mathbf{2})_{\frac{1}{3}, 2} \oplus (\overline{\mathbf{3}}, \mathbf{1})_{-\frac{4}{3}, 2} \oplus (\mathbf{1}, \mathbf{1})_{2, 2}}_{\mathbf{10}} \oplus \underbrace{(\overline{\mathbf{3}}, \mathbf{1})_{\frac{2}{3}, -1} \oplus (\mathbf{1}, \mathbf{2})_{-1, -1}}_{\overline{\mathbf{5}}}$$

$$\oplus \underbrace{(\mathbf{3}, \mathbf{1})_{-\frac{2}{3}, -4} \oplus (\mathbf{1}, \mathbf{2})_{1, -4}}_{\mathbf{5}} \oplus \underbrace{(\overline{\mathbf{3}}, \mathbf{1})_{\frac{2}{3}, -1} \oplus (\mathbf{1}, \mathbf{2})_{-1, -1}}_{\overline{\mathbf{5}}} \oplus \underbrace{(\mathbf{1}, \mathbf{1})_{0, 5}}_{\mathbf{1}} \oplus \underbrace{(\mathbf{1}, \mathbf{1})_{0, 5}}_{\mathbf{1}}.$$

Here, the subscripts represent the charges under $U(1)_Y$ and $U(1)'$, respectively and the underbraces gather together the $SU(5)$ representations. The first line is the field content of a single standard model generation, and the rest consists of exotics. There are two potentially important things to notice. First, two of the exotics are standard model singlets, but charged under $U(1)'$, so if some combination of these develops a VEV it will break the symmetry to exactly G_{SM}. Second, if $U(1)'$ does get broken, there is nothing forbidding mass terms for all of the exotics, since they can pair up to give standard model singlets. Unfortunately, the unique choice eq. (3.1) of $\mathfrak{G}$ Wilson lines also projects out the one vector-like pair of standard model singlets (the calculation proceeds as described below for the second choice of embedding). Therefore, it is impossible to Higgs the remaining $U(1)'$.

3.2.2. Model 2

We must therefore turn to an embedding of $\mathfrak{G}$ in E_6 which leaves a larger unbroken gauge symmetry. This time we will choose a different maximal subgroup in which to embed $\mathfrak{G}$, which will turn out to be equivalent to using $SU(3)^3$, but is clearer in this case. Therefore consider the maximal subgroup $SU(6) \times SU(2)$ of E_6. We can define the map Φ by giving the images of g_3, g_4 in this maximal subgroup:

$$\Phi(g_3) = \mathbf{1}_6 \times \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}$$

$$\Phi(g_4) = \mathrm{diag}(\,\mathrm{i},\ \mathrm{i},\ \mathrm{i},\ \mathrm{i}, -1, -1) \times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The generators of E_6 which do not lie in $SU(6) \times SU(2)$ transform as $(\mathbf{20}, \mathbf{2})$, so none of these are invariant under $\Phi(\mathfrak{G})$, and the unbroken gauge group is the commutant of $\Phi(\mathfrak{G})$ in $SU(6) \times SU(2)$. It is easy to check that this is $SU(4) \times SU(2) \times U(1) \subset SU(6)$. We now work out the representations of the unbroken group into which the matter fields fall. The $\mathbf{27}$ of E_6 decomposes under $SU(6) \times SU(2)$ as

$$\mathbf{27} = (\mathbf{15}, \mathbf{1}) \oplus (\overline{\mathbf{6}}, \mathbf{2})$$

and therefore under the unbroken gauge group as follows:

$$\mathbf{27} = (\mathbf{4}, \mathbf{2})_{\frac{1}{2}} \oplus (\mathbf{6}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{1})_2 \oplus \left[2 \times (\overline{\mathbf{4}}, \mathbf{1})_{\frac{1}{2}}\right] \oplus \left[2 \times (\mathbf{1}, \mathbf{2})_{-1}\right]$$

where the subscripts denote the $U(1)$ charges with a convenient normalisation. We pause momentarily to describe how the structure of the standard model appears in this setup. $SU(4)$ has $SU(3) \times U(1)$ as a maximal subgroup, and the $SU(3)$ is that of colour, while a combination of this $U(1)$ with the explicit one gives the hypercharge group $U(1)_Y$. The explicit $SU(2)$ is weak $SU(2)$. We can also describe the representations in terms of the quantum numbers of left-handed standard model fields. The $(\mathbf{1}, \mathbf{2})_{-1}$ and $(\mathbf{1}, \mathbf{1})_2$ transform like the lepton doublet l (or down-type Higgs H_d) and the anti-electron e^c respectively. The $(\mathbf{4}, \mathbf{2})_{\frac{1}{2}}$ contains the quark doublet Q and an up-type Higgs H_u. The $(\overline{\mathbf{4}}, \mathbf{1})_{\frac{1}{2}}$ contains the anti-down quark d^c and a standard model singlet (or 'right-handed neutrino') N. Finally, the $(\mathbf{6}, \mathbf{1})_{-1}$ contains the anti-up quark u^c and an exotic colour triplet. It is clear from the above that the extended symmetry can be broken to that of the standard model by giving a VEV to a field in the $(\overline{\mathbf{4}}, \mathbf{1})_{\frac{1}{2}}$ representation, as this is the only one containing a standard model singlet. To decide whether or not this is possible, we must first determine which fields are massless. In the presence of discrete holonomy (VEVs for Wilson lines), it is no longer necessarily true that the light charged multiplets will be the content of h^{11} $\overline{\mathbf{27}}$'s and h^{21} $\mathbf{27}$'s. This is so because the fields no longer transform simply as cohomology classes of the manifold; this representation of $\mathfrak{G}$ is tensored with the representation of $\Phi(\mathfrak{G}) \subset E_6$. Calculating the light spectrum in this case proceeds as follows: In the absence of flux lines

$SU(4){\times}SU(2){\times}U(1)$ representation	$\Phi(\mathfrak{G})$ rep.	No. of light copies	No. of light conjugates
$(\mathbf{4},\mathbf{2})_{\frac{1}{2}}$	$R_{-\mathrm{i}}$	4	1
$(\mathbf{6},\mathbf{1})_{-1}$	R_{-1}	4	1
$(\mathbf{1},\mathbf{1})_{2}$	R_{1}	4	1
$(\overline{\mathbf{4}},\mathbf{1})_{\frac{1}{2}}$	$R_{+}^{(2)}$	8	2
$(\mathbf{1},\mathbf{2})_{-1}$	$R_{-}^{(2)}$	6	0

Table 3: *The transformation properties of the $\mathbf{27}$ fields under the unbroken gauge group and $\Phi(\mathfrak{G})$, the discrete part of the holonomy group. The last two columns give the number of zero modes transforming in each representation and its conjugate.*

the structure group of the gauge bundle is SU(3), and an index theorem tells us that the difference between the number of zero modes for a $\mathbf{3}$ and a $\overline{\mathbf{3}}$ is half the Euler number of $Y_{\mathfrak{G}}$. This corresponds to the net number of generations, since $\mathbf{3}$ and $\overline{\mathbf{3}}$ come paired with $\mathbf{27}$ and $\overline{\mathbf{27}}$ of E_6. Discrete holonomy valued in E_6 does not change the curvature of the connection, so because the index is calculated via a curvature integral, it does not change the net number of generations, and we still have 3 chiral $\mathbf{27}$s. However, the vector-like pairs no longer need to form complete E_6 multiplets, since E_6 is broken. If we can explicitly calculate the zero modes coming from the $\overline{\mathbf{27}}$, we know that these must be paired with their conjugates from a $\mathbf{27}$ so as not to violate the index theorem. We have already calculated the representation of $\mathfrak{G}$ on $(1,1)$-forms in Subsection 2.3, so we just need to work out the action of $\Phi(\mathfrak{G})$ on the $\overline{\mathbf{27}}$, tensor these together and pick out the invariants. It will be simpler to calculate how $\Phi(\mathfrak{G})$ acts on the $\mathbf{27}$ and then conjugate this, since we have written everything so far in terms of the $\mathbf{27}$. We know the transformation properties of the $\mathbf{27}$ under $SU(6){\times}SU(2)$, and thus under $\Phi(\mathfrak{G})$. The results are shown in Table 3. The most important outcome is the appearance of two vector-like pairs of the $(\overline{\mathbf{4}},\mathbf{1})_{\frac{1}{2}}$. On general grounds, these fields correspond to classical D-flat directions breaking the gauge symmetry to that of the standard model. Furthermore, the (renormalisable) superpotential contains only terms linear in the singlet components, meaning that at this level the F-flat conditions $W = dW = 0$ are also still satisfied by giving this component an expectation value.

Suppose then that we give a supersymmetric VEV to some linear combination of the $(\overline{\mathbf{4}},\mathbf{1})_{\frac{1}{2}}$ fields (and those transforming in the conjugate representation). The discussion is then familiar: by a gauge transformation, we can take the VEV to be $\langle\phi\rangle = v\,(0,0,0,1)$, which is clearly invariant under $SU(3) \subset SU(4)$. It is also invariant under a particular $U(1)$

subgroup of $SU(4){\times}U(1)$, which commutes with $SU(3)$ and is given by

$$e^{i\theta} \;\to\; \left(\begin{array}{cccc|c} e^{-i\theta} & & & & \\ & e^{-i\theta} & & & \\ & & e^{-i\theta} & & \\ & & & e^{3i\theta} & \\ \hline & & & & e^{3i\theta} \end{array}\right)$$

The unbroken gauge group will therefore be $SU(3){\times}SU(2){\times}U(1)$, and with an appropriate normalisation of the $U(1)$ generator the representations break down as follows:

$$(\mathbf{4},\mathbf{2})_{\frac{1}{2}} = (\mathbf{3},\mathbf{2})_{\frac{1}{3}} \oplus (\mathbf{1},\mathbf{2})_1 \;,\quad (\overline{\mathbf{4}},\mathbf{1})_{\frac{1}{2}} = (\overline{\mathbf{3}},\mathbf{1})_{\frac{2}{3}} \oplus (\mathbf{1},\mathbf{1})_0$$

$$(\mathbf{6},\mathbf{1})_{-1} = (\mathbf{3},\mathbf{1})_{-\frac{2}{3}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-\frac{4}{3}} \;,\quad (\mathbf{1},\mathbf{1})_2 = (\mathbf{1},\mathbf{1})_2 \;,\quad (\mathbf{1},\mathbf{2})_{-1} = (\mathbf{1},\mathbf{2})_{-1}$$

We therefore see explicitly that the unbroken $U(1)$ is hypercharge. We can now ask what the light matter content of the theory will be after this Higgsing of the extra symmetry. Certainly chiral multiplets in the representation $(\mathbf{3},\mathbf{1})_{-\frac{2}{3}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{\frac{2}{3}} \oplus (\mathbf{1},\mathbf{1})_0$ will be absorbed into massive vector multiplets, but deciding which other fields gain masses will require detailed knowledge of the superpotential. Above we have embedded the Wilson lines in the $SU(6){\times}SU(2)$ maximal subgroup of E_6. In fact they also lie in the $SU(3){\times}SU(3){\times}SU(3)$ maximal subgroup, and our choice corresponds to one of the cases described in [2]. In this case the preceding discussion is more transparent than that if one proceeds via $SU(3)^3$, but it is easy to check that the results are the same.

4. Symmetries of $Y^{8,44}$ and $Y^{8,44}/\mathfrak{G}$

4.1. Symmetries of $dP_6 \times dP_6$

The study of possible symmetries of the quotient manifold is necessarily related to a study of the parameter space of the manifold since such symmetries will exist only for special values of the parameters. For the manifold $Y^{8,44}$ and its quotient the symmetries originate in the symmetries of $\mathcal{S} \cong dP_6$. We summarise these following [23, pp38-39]. The dP_6 surface contains, as remarked previously, six (-1)-curves that intersect in a hexagon. It turns out that the entire symmetry group of the hexagon, that is, the dihedral group

$$\mathrm{Dih}_6 = \mathbb{Z}'_2 \ltimes \left(\mathbb{Z}_2 \times \mathbb{Z}_3 \right)$$

acts on the dP_6. In addition, the dP_6 surface is toric, corresponding to the fan over the polygon shown in Figure 5, and so is acted on by the torus $(\mathbb{C}^*)^2$.

As previously, we realise the del Pezzo surface as the complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$ of the polynomials

$$p \;=\; w_0 z_0 + w_1 z_2 + w_2 z_1 \;, \quad q \;=\; w_1 z_1 + w_0 z_2 + w_2 z_0 \;.$$

In these coordinates, the symmetry group acts as follows:

- $\mathbb{Z}_2$ acts via the coordinate exchange

$$(w_0, w_1, w_2) \longleftrightarrow (z_0, z_1, z_2) \;. \tag{4.1}$$

- $\mathbb{Z}_3$ acts as the phase rotation

$$\Big((w_0, w_1, w_2), (z_0, z_1, z_2) \Big) \longmapsto \Big((w_0, \zeta w_1, \zeta^2 w_2), (z_0, \zeta z_1, \zeta^2 z_2) \Big) \;. \tag{4.2}$$

- $\mathbb{Z}'_2$ acts via the coordinate exchange

$$\Big((w_0, w_1, w_2), (z_0, z_1, z_2) \Big) \longmapsto \Big((w_1, w_0, w_2), (z_1, z_0, z_2) \Big) \;. \tag{4.3}$$

- The toric action will be described in Section 6, where we discuss the toric point of view in detail.

4.2. Symmetries of the quotient

If we refer to the group above as $\mathrm{Aut}(\mathcal{S})$ then the symmetries of $\mathcal{S} \times \mathcal{S}$ are

$$\mathrm{Aut}(\mathcal{S} \times \mathcal{S}) \;=\; \mathbb{Z}_2 \ltimes \Big(\mathrm{Aut}(\mathcal{S}) \times \mathrm{Aut}(\mathcal{S}) \Big) \;.$$

Not all of these symmetries descend to symmetries of the quotient $Y_\mathfrak{G}$ since an element of $\text{Aut}(\mathcal{S} \times \mathcal{S})$ can (i) fail to be a symmetry of the covering manifold Y owing to the fact that it does not preserve the hypersurface $r = 0$, or (ii) it may fail to commute appropriately with $\mathfrak{G}$. We shall see presently that the symmetry group of the quotient $Y_\mathfrak{G}$ is reduced to a subgroup of (a single copy of) the dihedral group Dih_6. This subgroup is $\mathbb{Z}_2$ for generic values of the parameters, $\mathbb{Z}_2 \times \mathbb{Z}_2$ on a certain 3 parameter family of manifolds and the full group Dih_6 for an interesting 2 parameter family of singular varieties. The condition that, to be a symmetry of the quotient, the symmetry must commute appropriately with $\mathfrak{G}$ is very restrictive so we postpone testing for the preservation of the hypersurface $r = 0$ and begin with a discussion parallel to that of [2]. Let π denote the operation of taking the quotient by $\mathfrak{G}$ then the condition that a linear symmetry h be a symmetry of the quotient is that $h\,\pi(x) = \pi(hx)$, which is the condition that for each $g \in \mathfrak{G}$ there is an element $g' \in \mathfrak{G}$ such that $gh = hg'$. Since our group $\mathfrak{G}$ is generated by g_3 and g_4 this is the condition that, for each g and h

$$hgh^{-1} = g_3^m\, g_4^n$$

for some integers $0 \le m \le 2$, $0 \le n \le 3$. It suffices to apply this condition to the generators g_3 and g_4 themselves. The form of the right hand side of this relation is restricted by the fact that hg_3h^{-1} must be an element of $\mathfrak{G}$ that is of order 3 and hg_4h^{-1} must be of order 4. There are just two elements of order 3, which are g_3 and g_3^2, and six of order 4, which are the elements $g_3^m\, g_4$ and $g_3^m\, g_4^3$ for $0 \le m \le 2$. Thus we may write

$$hg_3h^{-1} = g_3^{1+k} \ , \quad hg_4h^{-1} = g_3^m\, g_4^{1+2n} \ ; \quad k = 0,\,1\ , \quad 0 \le m \le 2\ , \quad n = 0,\,1\ . \quad (4.4)$$

Consider first the case that k, m and n all vanish; these are the symmetries h that commute with $\mathfrak{G}$. A symmetry that commutes with both g_3 and g_4 must be of the form h_3^ℓ or $g_4^2\, h_3^\ell$, $0 \le \ell \le 2$ where

$$h_3: \ x_{\alpha j} \ \to \ \zeta^j\, x_{\alpha j} \ .$$

This is the $\mathbb{Z}_3$ symmetry of (4.2), understood as applying symmetrically to the two copies of $\mathcal{S}$, and also a symmetry operation that we first saw in Subsection 2.1. Consider now the symmetry of (4.1), which we take to act on the first copy of $\mathcal{S}$ only since acting equally on both copies is equivalent to the operation g_4^2,

$$h_2: \ x_{1,j} \ \leftrightarrow \ x_{3,j}\ , \quad x_{2,j} \ \to \ x_{2,j}\ , \quad x_{4,j} \ \to \ x_{4,j} \ .$$

It is an easy check that h_2 satisfies (4.4) with $(k, m, n) = (0, 0, 1)$. Next we take a symmetry h_2' corresponding to (4.3), which we take to act symmetrically on the two copies of $\mathcal{S}$

$$h_2': \ x_{\alpha, j} \ \to \ x_{\alpha, 1-j}$$

This satisfies (4.4) with $(k, m, n) = (1, 0, 0)$. We note also that $h = g_3^2$ satisfies (4.4) with $(k, m, n) = (0, 1, 0)$. In terms of these symmetries we can give a transformation

$$h = (h_2')^n\, g_3^{2m}\, h_2^k$$

23

that satisfies (4.4) for general values of the integers k, m and n. Furthermore this solution to the conditions is uniquely determined modulo h_3 and g_4^2 since if h and $\tilde{h}$ both satisfy (4.4), for given k, m and n, then $\tilde{h}h^{-1}$ commutes with $\mathfrak{G}$. We have shown that the symmetry group of $Y_{\mathfrak{G}}$ is a subgroup of $\mathfrak{H} = \langle h_2, h_2', h_3 \rangle$. One sees that

$$h_2 h_3 = h_3 h_2 , \quad h_2 h_2' = h_2' h_2 \quad \text{and} \quad h_2' h_3 = h_3^2 h_2' .$$

We set $h_6 = h_2 h_3^2$ so that $h_2 = h_6^3$ and $h_3 = h_6^2$. Thus $\mathfrak{H}$ is generated by h_6 and h_2' and we note that

$$h_2' h_6 = h_6^5 h_2' .$$

We see that $\mathfrak{H} \cong \mathrm{Dih}_6$ and we recover, in this way, the dihedral group.

Symmetry	Relation	$x_{\alpha j}$	D_a	$\widetilde{D}_b$
g_3	g_6^2	$\zeta^{(-1)^\alpha j} x_{\alpha j}$	D_{a+2}	$\widetilde{D}_{b-2}$
g_4		$x_{\alpha+1, j}$	$\widetilde{D}_a$	D_{b+3}
g_6	$g_3^2 g_4^2$	$\zeta^{(-1)^{\alpha+1} j} x_{\alpha+2, j}$	D_{a+1}	$\widetilde{D}_{b-1}$
h_2	h_6^3	$x_{1j} \leftrightarrow x_{3j}, \ x_{2j} \to x_{2j}, \ x_{4j} \to x_{4j}$	D_{a+3}	$\widetilde{D}_b$
h_2'		$x_{\alpha, 1-j}$	D_{6-a}	$\widetilde{D}_{6-b}$
h_3	h_6^2	$\zeta^j x_{\alpha j}$	D_{a+2}	$\widetilde{D}_{b+2}$
h_6	$h_2 h_3^2$	$x_{1j} \to \zeta^2 x_{3j}, \ x_{2j} \to \zeta^2 x_{2j},$ $x_{3j} \to \zeta^2 x_{1j}, \ x_{4j} \to x_{4j}$	D_{a+1}	$\widetilde{D}_{b-2}$

Table 4: *A table of the various symmetry operations that we have met with their actions on the coordinates and the (-1)-lines D_a and $\widetilde{D}_b$.*

We have met a number of symmetry operations in the course of this discussion and we gather these together in Table 4 for reference. We now examine which of the symmetries of $\mathfrak{H}$ preserve the hypersurface $r = 0$. The transformation h_2 affects only the coordinates x_{1j} and x_{3k}. The effect on the polynomials m_{ijkl} is $m_{ijkl} \to m_{kjil}$ and this clearly preserves the polynomials m_{iiii} and it also preserves m_{0011} and m_{0212} since these are transformed to polynomials m_{ijkl} whose indices are related to the original polynomials by cyclic permutation. Thus $Y_{\mathfrak{G}}$ is invariant under h_2 for all values of the parameters. The transformation h_2' transforms $m_{ijkl} \to m_{1-i, 1-j, 1-k, 1-l}$. The effect is to interchange the polynomials m_{0000} and m_{1111},

the other terms in r being invariant since the indices transform by a cyclic permutation. Thus r is invariant if $c_0 = c_1$. For generic coefficients satisfying this condition, the quotient variety is smooth and is invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action generated by h'_2 and h_2. The transformation h_3 has been discussed in Subsection 2.1 and can be understood as inducing the transformation $(c_0,\, c_1,\, c_2,\, c_3,\, c_4) \to (c_0,\, \zeta c_1,\, \zeta^2 c_2,\, \zeta^2 c_3,\, \zeta^2 c_4)$ on the parameters. If we make the choice $c_0 = c_1 = 0$ then r also transforms homogeneously, $r \to \zeta^2 r$. Thus there is a $\mathbb{P}^2$ within the parameter space corresponding to parameters $(0, 0, c_2, c_3, c_4)$ for which each quotient $Y_{\mathfrak{G}}$ has a group Dih_6 of automorphisms. These varieties are, however, all singular and, for generic values of c_2, c_3 and c_4, have 3 nodes. These nodal varieties will be studied in detail in Section 5, where we demonstrate that the nodes can be resolved to obtain a new family of Calabi-Yau manifolds. There are also two isolated points $c_j = (1, 0, 0, 0, 0)$ and $c_j = (0, 1, 0, 0, 0)$ that correspond to Dih_6-invariant varieties that are very singular.

Although we have not yet described the torus action, we note here that for the special parameter choice $c_j = (0, 0, 1, 4, 4)$, the hypersurface $r = 0$ is invariant under the complete $(\mathbb{C}^*)^4$ action. This is a point corresponding to a very singular variety.

5. Conifold Transition to a Manifold with $(h^{11}, h^{21}) = (2, 2)$

We have noted that when $c_0 = c_1 = 0$ there is a two-parameter family of Dih_6 invariant varieties $Y_{\mathfrak{G}}$ and that these are all singular, the generic member having 3 nodes. These arise as 36 nodes on the covering manifold $Y^{8,44}$ which form three $\mathfrak{G}$-orbits, or a single $\mathbb{Z}_3 \times \mathfrak{G}$ orbit where the $\mathbb{Z}_3$ is generated by h_3. The fact that there are 36 singularities for generic (c_2, c_3, c_4) is best checked by a Gröbner basis calculation. The location of the singularities will be given presently and once it is known that these are the only singularities then it is easy to check that these are nodes by expanding the equations in a neighborhood of a singular point. Owing to the $\mathfrak{H} \ltimes \mathfrak{G}$ action it is sufficient to examine any one of the singularities locally. We describe the resolution of the three nodes on $Y_{\mathfrak{G}}$ in two steps. First we demonstrate that the nodal varieties Y admit Kähler small resolutions, by identifying smooth divisors which intersect the nodes in an appropriate way, and blowing up along these divisors. We then show that such a resolution is $\mathfrak{G}$-equivariant, and therefore yields a resolution of $Y_{\mathfrak{G}}$.

5.1. The parameter space of 3-nodal quotients $Y_{\mathfrak{G}}$

Before proceeding we pause to describe the parameter space Γ of 3-nodal, Dih_6-invariant quotients $Y_{\mathfrak{G}}$. The generic quotient with $c_0 = c_1 = 0$ has three nodes however on a certain locus within this space there are more severe degenerations. We find this locus to consist of the components listed in Table 5.

These loci arise in the Groebner basis calculation that finds the nodes. We have seen most of these conditions at some point in this paper. For example several of the linear conditions are those for elements of $\mathfrak{G}$ to have fixed points on $Y^{8,44}$. When this occurs, the covering space $Y^{8,44}$ necessarily has a node at the fixed point, so $Y_{\mathfrak{G}}$ develops an isolated singularity which is a quotient of a node [24]. The remaining linear conditions lead to varieties with additional nodes and/or orbifolds of nodes. We shall see the significance of the initial, quadratic, condition shortly. A sketch indicating the intriguing manner in which these components intersect is given in Figures 3.

The curves listed in Table 5 show an intriguing symmetry under the $\mathbb{Z}_2$-automorphism

$$c_2 \ \rightarrow \ 2\,c_3 \ , \quad c_3 \ \rightarrow \ \frac{1}{2}c_2 \ , \quad c_4 \ \rightarrow \ -\,c_4 \ .$$

This operation fixes the curves $\Gamma^{(\mathrm{o})}$, $\Gamma^{(\mathrm{v})}$ and $\Gamma^{(\mathrm{x})}$ and interchanges the curves

$$\Gamma^{(\mathrm{i})} \ \leftrightarrow \ \Gamma^{(\mathrm{ii})} \ , \quad \Gamma^{(\mathrm{iii})} \ \leftrightarrow \ \Gamma^{(\mathrm{iv})} \ , \quad \Gamma^{(\mathrm{vi})} \ \leftrightarrow \ \Gamma^{(\mathrm{vii})} \ , \quad \Gamma^{(\mathrm{viii})} \ \leftrightarrow \ \Gamma^{(\mathrm{ix})} \ .$$

We are unclear whether this is a genuine symmetry of the geometry of the parameter space. It is not a symmetry at the same level as the $\mathbb{Z}_3$-symmetry (2.4) which arises from a coordinate transformation that preserves the form of the defining polynomials of Y. We can see this by

Component	Equation	Type of Singularity
$\Gamma^{(\mathrm{o})}$	$4\,c_2 c_3 - c_4^2 \;=\; 0$	3 higher order singularities
$\Gamma^{(\mathrm{i})}$	$4\,c_2 + c_3 - 2\,c_4 \;=\; 0$	6 nodes
$\Gamma^{(\mathrm{ii})}$	$c_2 + 16\,c_3 + 4\,c_4 \;=\; 0$	3 nodes, 1 g_4^2-node, 1 g_4-node
$\Gamma^{(\mathrm{iii})}$	$c_2 \;=\; 0$	3 nodes, 1 $\mathfrak{G}$-node
$\Gamma^{(\mathrm{iv})}$	$c_3 \;=\; 0$	3 nodes, 1 g_3-node
$\Gamma^{(\mathrm{v})}$	$c_4 \;=\; 0$	3 nodes, 1 g_3-node
$\Gamma^{(\mathrm{vi})}$	$8\,c_3 + c_4 \;=\; 0$	4 nodes
$\Gamma^{(\mathrm{vii})}$	$4\,c_2 - c_4 \;=\; 0$	4 nodes
$\Gamma^{(\mathrm{viii})}$	$c_3 - c_4 \;=\; 0$	5 nodes
$\Gamma^{(\mathrm{ix})}$	$c_2 + 2\,c_4 \;=\; 0$	3 nodes, 1 g_4^2-node
$\Gamma^{(\mathrm{x})}$	$c_2 + 2\,c_3 \;=\; 0$	3 nodes

Table 5: *The generic member of* Γ *is a variety with 3 nodes but along these curves the varieties develop more severe singularities. Some of these extra singularities are orbifolds of nodes as indicated. A* g_3-*node, for example, is a node fixed by the symmetry* g_3.

noting that the transformation interchanges, for example, varieties on curve $\Gamma^{(\mathrm{i})}$, which have 6 nodes, with varieties on curve $\Gamma^{(\mathrm{ii})}$ which have singularities of a different type. The curve $\Gamma^{(\mathrm{x})}$ is not a curve of varieties that have more severe singularities since the number of nodes is, generically, three. We include it in the discriminant as a curve that, being fixed under the automorphism, may require special consideration.

5.2. The nodes and their resolution

To describe the location of the singularities we refer to the (-1)-lines on the two copies of $\mathcal{S}$ labelled and ordered as in (2.7). On each D_a we specify a point, $\mathrm{pt}(D_a)$ by the condition $x_{12}\,x_{32} = 0$. Thus $\mathrm{pt}(E_1)$, for example, is specified by the coordinates $x_{ij} = (1,1,1)$ and $x_{3j} = (1,-1,0)$ and $\mathrm{pt}(L_{12})$ has, as we see from Table 2, $x_{1j} = (1,-\zeta^2,0)$ and $x_{3j} = (1,\zeta^2,\zeta)$.

In an analogous way we define on each $\widetilde{D}_a$ a point, $\text{pt}(\widetilde{D}_a)$, by the condition $x_{22}\,x_{42} = 0$. We often abbreviate $\text{pt}(D_a) = \text{pt}_a$ and $\text{pt}(\widetilde{D}_a) = \widetilde{\text{pt}}_a$ respectively. The 36 nodes are the points

$$\text{pt}(D_a) \times \text{pt}(\widetilde{D}_b) \; ; \quad a, b \in \mathbb{Z}_6 \ .$$

The action of the symmetries $\mathfrak{H} \ltimes \mathfrak{G}$ on the nodes follows from the action of the symmetries on the (-1)-lines. For g_6 and g_4 these are given by (2.8) and the action of h_3 is easily read off from Table 2. We have

$$g_6 : \text{pt}_a \times \widetilde{\text{pt}}_b \;\rightarrow\; \text{pt}_{a+1} \times \widetilde{\text{pt}}_{b-1} \ , \quad g_4 : \text{pt}_a \times \text{pt}_b \;\rightarrow\; \text{pt}_{b+3} \times \widetilde{\text{pt}}_a$$

$$h_6 : \text{pt}_a \times \widetilde{\text{pt}}_b \;\rightarrow\; \text{pt}_{a+1} \times \widetilde{\text{pt}}_{b-2} \ , \quad h_2' : \text{pt}_a \times \widetilde{\text{pt}}_b \;\rightarrow\; \text{pt}_{6-a} \times \widetilde{\text{pt}}_{6-b} \ . \tag{5.1}$$

 Notice that the generators of $\mathfrak{G}$ preserve the sum $a + b$ mod 3 so this sum distinguishes the $\mathfrak{G}$-orbits. The transformation h_3 permutes these orbits so the 36 nodes form a single orbit under $\mathfrak{H} \ltimes \mathfrak{G}$. The points $\text{pt}_a \in \mathcal{S}$ have the property that if the coordinates x_{1j} and x_{3j} are restricted to these values then the equation $r = 0$ is identically satisfied for all $(x_{2\,j}, x_{4\,k}) \in \widetilde{\mathcal{S}}$. Thus $\mathcal{D}_a = \text{pt}_a \times \widetilde{\mathcal{S}}$ is a Weil divisor in Y, as is $\widetilde{\mathcal{D}}_b = \mathcal{S} \times \widetilde{\text{pt}}_b$. The six divisors $\mathcal{D}_a$ are mutually disjoint, and each contains six nodes. The same applies to the six divisors $\widetilde{\mathcal{D}}_b$, and the two collections intersect precisely in the 36 nodes $\text{pt}_a \times \widetilde{\text{pt}}_b$. The configuration of the divisors and nodes is sketched in Figure 4. As we will see below, the given divisors are non-Cartier in a neighbourhood of each node, and we can blow up Y along such a divisor to obtain a small resolution of each node it contains. We may therefore resolve all 36 nodes by blowing up each of the 'horizontal' divisors $\mathcal{S} \times \widetilde{\text{pt}}_b$. In this way we obtain a Kähler manifold $\widehat{Y}$ that has vanishing first Chern class and $\chi = 0$. Alternatively, we can blow up each of the 'vertical' divisors $\text{pt}_a \times \widetilde{\mathcal{S}}$ but as we will see, this gives the same variety.

We may examine the singularities and their resolutions locally since, as remarked previously, all the singularities are related by the group $\mathfrak{H} \ltimes \mathfrak{G}$. We expand about a singularity by writing $x_{\alpha 0} = 1$ and $x_{\alpha j} = x_{\alpha j}^{\sharp} + \epsilon_{\alpha j}$ for $j = 1, 2$, where the $x_{\alpha j}^{\sharp}$ are the coordinates of a singularity. We have 8 coordinates $\epsilon_{\alpha j}$ and we may solve the four equations $p^1 = p^2 = q^1 = q^2 = 0$ for the $\epsilon_{\alpha 2}$ as functions of the $\epsilon_{\alpha 1}$. We are left with the constraint $r = 0$ which, for the point $x^{\sharp} = (1, 1, 1) \times (1, -1, 0) \times (1, 1, 1) \times (1, -1, 0)$, becomes

$$\epsilon_1(A\,\epsilon_2 - B\,\epsilon_4) - \epsilon_3(B\,\epsilon_2 - C\,\epsilon_4) \;=\; 0 \tag{5.2}$$

where

$$A \;=\; 2(c_2 + c_3 + c_4) \ , \quad B \;=\; 4c_2 - 2c_3 + c_4 \ , \quad C \;=\; 2(4c_2 + c_3 - 2c_4) \ .$$

We see that the singularity is indeed a node provided that the determinant of the matrix associated to the quadratic form does not vanish. This determinant is proportional to $(AC - B^2)^2 = 3^4(c_4 - 4c_2 c_3)^2$, so this requirement provides an understanding of the quadratic
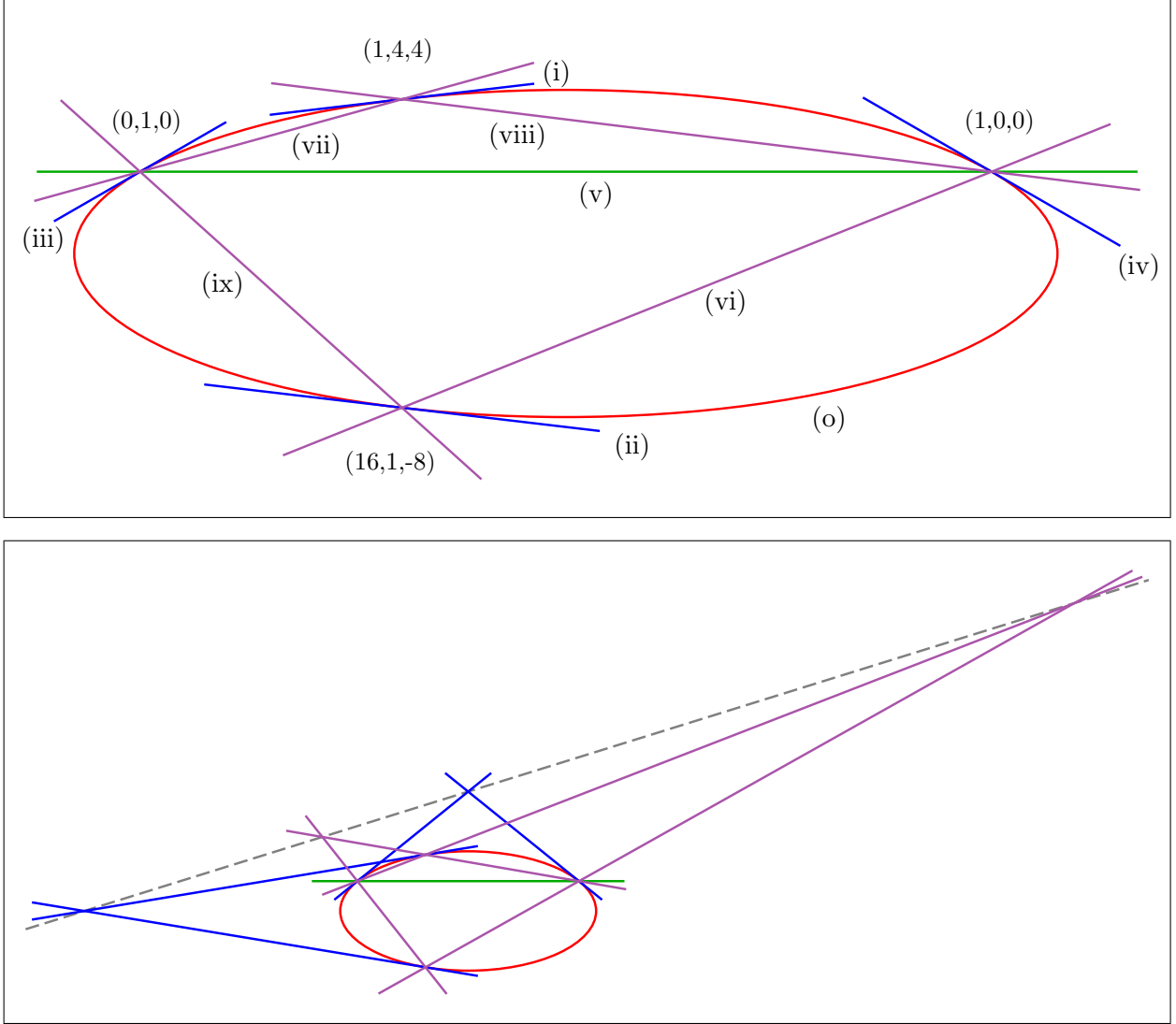
28

Figure 3: *Two sketches of the surface Γ, the locus of Dih$_6$-invariant varieties showing the discriminant of the space of 3-nodal varieties. The components of the discriminant locus are labeled according to Table 5. For the resolved manifold with Hodge numbers $(h^{11}, h^{21}) = (2, 2)$ this is the space of complex structures. The second sketch zooms out to show how the components intersect. The four intersections of the pairs of blue and purple lines lie on the dashed line $\Gamma^{(x)}$.*

condition in Table 5. We now see that the 'vertical' divisor $\mathcal{D}_0 = \mathrm{pt}_0 \times \widetilde{\mathcal{S}}$ is non-Cartier in a neighbourhood of the node, being given by the two local equations $\epsilon_1 = \epsilon_3 = 0$ (by definition, a Cartier divisor is given by a single polynomial in each affine patch). We may blow up Y along this divisor by introducing a $\mathbb{P}^1$ with coordinates (s_1, s_2), and considering the following equation in $Y \times \mathbb{P}^1$:

$$\epsilon_1 \, s_1 + \epsilon_3 \, s_2 \;=\; 0 \tag{5.3}$$

29

If $\pi : Y \times \mathbb{P}^1 \to Y$ is projection onto the first factor, then the blow up of Y is given by

$$\widehat{Y} = \overline{\pi^{-1}(Y \setminus \mathcal{D}_0)} \ ,$$

that is, $\widehat{Y}$ is the closure of the preimage. One can check that $\widehat{Y}$ defined in this way is indeed just Y with each node on $\mathcal{D}_0$ replaced by a $\mathbb{P}^1$. Alternatively we may blow up along the 'horizontal' divisor $\widetilde{\mathcal{D}}_0 = \mathcal{S} \times \widetilde{\mathrm{pt}}_0$; the discussion is the same, but instead of (5.3) we take

$$(B\,\epsilon_2 - C\,\epsilon_4)\,s_1 + (A\,\epsilon_2 - B\,\epsilon_4)\,s_2 \;=\; 0 \ . \tag{5.4}$$

The vanishing of the combinations $B\,\epsilon_2 - C\,\epsilon_4$ and $A\,\epsilon_2 - B\,\epsilon_4$ is equivalent to the vanishing of ϵ_2 and ϵ_4, provided $AC - B^2 \neq 0$. We can see that the two resolutions are identical by observing that they are each given by the following matrix equation in $\mathbb{C}^4 \times \mathbb{P}^1$, where $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \mathbb{C}^4$:

$$\begin{pmatrix} \epsilon_1 & \epsilon_3 \\ B\,\epsilon_2 - C\,\epsilon_4 & A\,\epsilon_2 - B\,\epsilon_4 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \;=\; 0 \ .$$

We can now argue that our resolution of the 36 nodes on Y is $\mathfrak{G}$-equivariant, and therefore gives a resolution of the 3 nodes on $Y_{\mathfrak{G}}$. Suppose we blow up the six divisors $\mathcal{D}_a$. These are just permuted by g_6, so the resolution is manifestly $\mathbb{Z}_6$ invariant. The element g_4 on the other hand interchanges the six $\mathcal{D}_a$ for the six $\widetilde{\mathcal{D}}_a$, but we have shown above that at each node $\mathrm{pt}_a \times \widetilde{\mathrm{pt}}_b$ we obtain the same resolution whether we blow up $\mathcal{D}_a$ or $\widetilde{\mathcal{D}}_b$. The resolution is therefore also equivariant under the g_4 action, and thus under the action of the whole group $\mathfrak{G}$. Finally we can ask about the Hodge numbers of $\widehat{Y}_{\mathfrak{G}}$. The space of complex
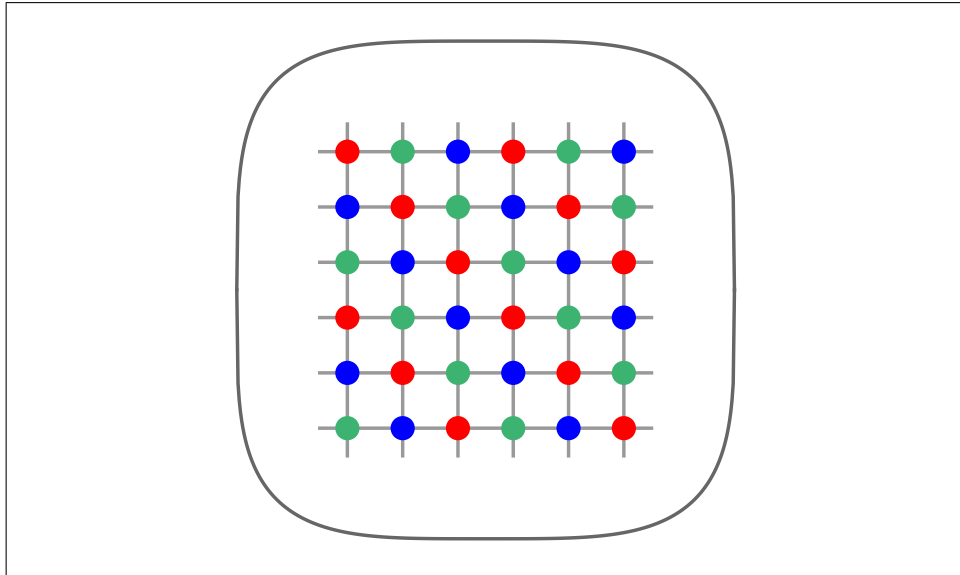

Figure 4: *The divisors $\mathrm{pt}_a \times \widetilde{\mathcal{S}}$ and $\mathcal{S} \times \widetilde{\mathrm{pt}}_b$. These intersect in the 36 nodes which form three $\mathfrak{G}$-orbits that are distinguished by colour.*

structures is two-dimensional, so $h^{21}(\widehat{Y}_{\mathfrak{G}}) = 2$. To obtain h^{11} note that we blow up a single divisor on $Y/\mathbb{Z}_6$, and this resolution happens also to be g_4-covariant, so we simply have $h^{11}(\widehat{Y}_{\mathfrak{G}}) = h^{11}(Y_{\mathfrak{G}}) + 1 = 2$. This is consistent, because we obtain $\widehat{Y}_{\mathfrak{G}}$ by resolving 3 nodes, which gives $\chi(\widehat{Y}_{\mathfrak{G}}) = \chi(Y_{\mathfrak{G}}) + 6 = 0$.

The considerations above suggest, along the lines of [12], that there is a 3-generation heterotic model on $\widehat{Y}_{\mathfrak{G}}$ that derives from the model that we have presented on $Y_{\mathfrak{G}}$. This is an intriguing possibility, not least because the automorphism group of $Y_{\mathfrak{G}}$ is at least Dih_6 at all points in its moduli space, so any such theory will feature this quite large discrete symmetry group.

6. Toric Considerations and the Mirror Manifold

6.1. Toric generalities

Let us briefly review the construction of a toric variety in terms of homogeneous coordinates, as described by Cox [25] (there are several reviews of toric geometry to be found in the physics literature [26, 27, 28], see also the text by Fulton [29]). Abstractly, an n-dimensional toric variety is an algebraic variety X containing the algebraic torus $\mathbf{T}^n \cong (\mathbb{C}^*)^n$ as a dense open subset, such that the group action of the torus on itself extends to an action on X. Each such X is given by a 'fan', which consists of a collection of strictly convex, rational, polyhedral cones in an n-dimensional lattice $N \cong \mathbb{Z}^n$. Each face of a cone is also considered to be a cone and forms part of the fan. Furthermore two cones can intersect only in a cone which is a face of each. Let $\{v_\rho\}_{\rho=1...d}$ be the set of lattice vectors generating the one-dimensional cones, and denote by ∇ the polyhedron given by the convex hull of $\{v_\rho\}$. These vectors will generally satisfy $d - n$ relations

$$\sum_{\rho=1}^{d} Q_{r\rho}\, v_\rho = 0 \quad r = 1, \ldots, d-n \tag{6.1}$$

We associate a complex variable x_ρ with each one-dimensional cone; these will be our homogeneous coordinates. Our first step is to delete a subset of the space $\mathbb{C}^d$ spanned by these variables: if some set $v_{\rho_1}, \ldots, v_{\rho_k}$ does *not* span a cone in the fan, we remove the set $\{x_{\rho_1} = \ldots = x_{\rho_k} = 0\}$. The toric variety X is then a quotient of this space[4] by $(\mathbb{C}^*)^{d-n}$, where for each relation in (6.1) we impose

$$(x_1, \ldots, x_d) \sim (\lambda^{Q_{r1}} x_1, \ldots, \lambda^{Q_{rd}} x_d) \quad \lambda \in \mathbb{C}^*$$

Batyrev described a general method for constructing mirror pairs of Calabi-Yau manifolds as hypersurfaces in toric varieties [13]. Let M be the dual lattice to N, and define the polyhedron $\Delta \subset M$ dual to ∇ as follows

$$\Delta = \{u \in M \mid u \cdot v \geq -1 \ \ \forall\, v \in \nabla\} \tag{6.2}$$

Points in the dual lattice M give rise rise to holomorphic functions on $\mathbf{T}^n \subset X$ via

$$u \mapsto \prod_i t_i^{u_i} \tag{6.3}$$

where $\{u_i\}$ are the components of u relative to the standard basis for N. When the polyhedra are reflexive, the integral points of the dual polyhedron Δ correspond in this way to sections of the anticanonical bundle of X, so the closure of the vanishing locus of a linear combination of these monomials is a Calabi-Yau variety. We can also dualise this construction: taking cones over the faces of Δ gives the fan of another toric variety X^*, and points in ∇ correspond to sections of its anti-canonical bundle. In this way we obtain the mirror family.

---

[4]We ignore possible discrete factors in the quotient group, which will not be relevant here.

6.2. The Newton polyhedron and its dual

It is a felicitous fact that the del-Pezzo surface dP_6 is toric and has a fan with six one-dimensional cones $\{v_a\}$ as shown in Figure 5. These correspond to toric divisors and so are in correspondence with the D_a. Let Σ denote this collection of one-dimensional cones

$$\Sigma = \{v_a\} = \left\{ (0,1), (-1,1), (-1,0), (0,-1), (1,-1), (0,1) \right\} .$$

We can also think of Σ as the polygon over the fan, which in the present case is a hexagon. The dual polygon is naïvely Σ rotated by 90°, but by appropriate choice of coordinates in M we can simply identify the two. We have previously been specifying $\mathcal{S} \cong \mathrm{dP}_6$ as a complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$, as described in Section 2. In order to describe the toric action it is convenient to first make a change of coordinates to

$$y_{\alpha j} = \sum_k M_{jk}\, x_{\alpha k} \quad \text{where} \quad M = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{pmatrix} \tag{6.4}$$

This brings the polynomials p^1 and q^1 to the form

$$p^1 = y_{10}\, y_{30} + y_{11}\, y_{31} + y_{12}\, y_{32} , \quad q^1 = y_{10}\, y_{30} + \zeta\, y_{11}\, y_{31} + \zeta^2\, y_{12}\, y_{32}$$

In these coordinates the blown-up points in the y_1 plane are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. It is easy to see that for any $(\mu, \lambda) \in (\mathbb{C}^*)^2$, the polynomials are invariant under

$$(y_{10}, y_{11}, y_{12}) \times (y_{30}, y_{31}, y_{32}) \rightarrow (y_{10}, \lambda\, y_{11}, \mu\, y_{12}) \times (y_{30}, \lambda^{-1}\, y_{31}, \mu^{-1}\, y_{32})$$
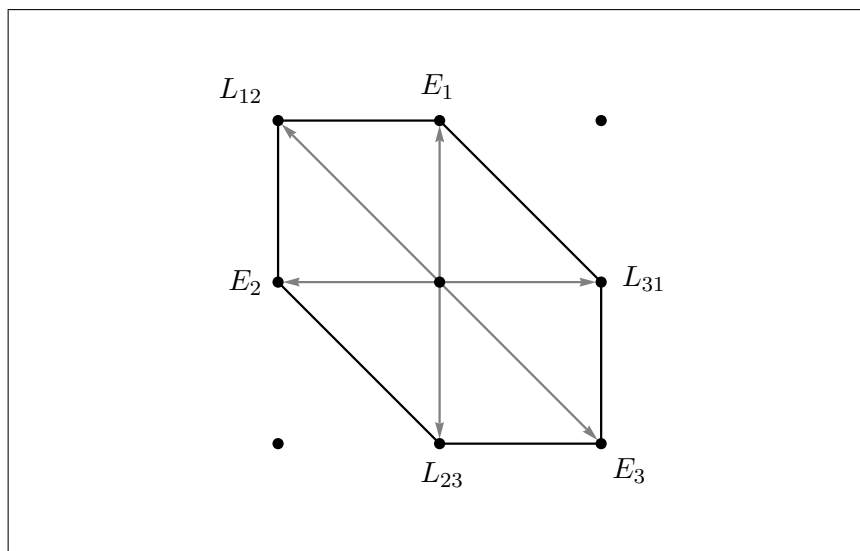

Figure 5: *The fan and polygon for* dP_6. *The one-dimensional cones can be taken to correspond to the divisors as indicated.*

The torus $(\mathbb{C}^*)^2$ can then be explicitly embedded in dP_6 follows:

$$(t_1, t_2) \;\rightarrow\; (1, t_1\,\zeta^2, t_2\,\zeta) \times (1, t_1^{-1}\,\zeta^2, t_2^{-1}\,\zeta) \subset \mathbb{P}^2 \times \mathbb{P}^2. \tag{6.5}$$

We can do the same for the second copy $\widetilde{\mathcal{S}}$, and call the extra toric variables t_3, t_4. The polyhedron, ∇, over the fan for $\mathcal{S} \times \widetilde{\mathcal{S}}$ is obtained as the convex hull of the union of Σ with a second orthogonal copy $\widetilde{\Sigma}$, corresponding to $\widetilde{\mathcal{S}}$

$$\nabla \;=\; \mathrm{Conv}(\Sigma,\, \widetilde{\Sigma}) \;=\; \big\{\, (v_a,\, 0) \,\big|\, v_a \in \Sigma \big\} \cup \big\{\, (0,\, v_b) \,\big|\, v_b \in \widetilde{\Sigma} \big\} \tag{6.6}$$

The Newton polyhedron, Δ, is dual to ∇, in the sense of Subsection 6.2, and is given by the Minkowski sum of Σ and $\widetilde{\Sigma}$,

$$\Delta \;=\; \mathrm{Mink}(\Sigma,\, \widetilde{\Sigma}) \;=\; \big\{\, (v_a,\, v_b) \,\big|\, v_a \in \Sigma,\, v_b \in \widetilde{\Sigma} \big\}\,. \tag{6.7}$$

The polyhedron ∇ has 12 vertices, which are the points given explicitly by (6.6), and no other integral points apart from the origin. It has 36 three-faces that are tetrahedra. The polyhedron Δ has 36 vertices and 12 three-faces that are all hexagonal prisms, these each have two hexagonal two-faces and six rectangular two-faces. The 36 points given explicitly by (6.7) are the vertices. The polyhedron also contains the origin and 12 additional integral points, one interior to each of the hexagonal two-faces. These 12 additional points are in fact the vertices of ∇, so we have $\nabla \subset \Delta$. These facts are most quickly established by having recourse to a programme such as POLYHEDRON or PALP [30] that analyse reflexive polyhedra. We can, with benefit of hindsight, get useful insight into this structure and understand Δ and ∇ rather simply in terms of the divisors D_a and $\widetilde{D}_b$. We will not rederive the structure of the polyhedra but will content ourselves with giving a description using only what we know about these divisors. It is easy to see from the relation (6.7) that there is a one-to-one correspondence between vertices of Δ and pairs of divisors

$$\nu_{a\,b} \;\leftrightarrow\; (D_a, \widetilde{D}_b)\,, \quad a, b \in \mathbb{Z}_6\,.$$

The integral points $\tilde{\imath}_b$ that are interior to the six blue hexagonal two-faces each correspond to a divisor $\widetilde{D}_b$ and the vertices of this hexagonal face are the $\nu_{a\,b}$ as a varies. The same vertices arise in both the blue and pink three-faces and this gives the correspondence between the blue and pink two-faces in Figure 6. The rectangular two-faces contain the vertices $\{\nu_{a,b},\, \nu_{a+1,b},\, \nu_{a+1,b+1},\, \nu_{a,b+1}\}$.

Having associated the points of Δ with the divisors D_a and $\widetilde{D}_b$ we see that there is a natural $\mathfrak{H} \ltimes \mathfrak{G}$ action on the points. Let ρ and σ denote the matrices

$$\rho \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \sigma \;=\; \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

and note[5] that $\rho^6 = \mathbf{1}$ and $\rho^3 = -\mathbf{1}$. It is an immediate check that $\rho\, v_a = v_{a+1}$ and that $\sigma\, v_a = v_{6-a}$. In virtue of the polyhedron of Figure 5 it comes as no surprise that ρ and σ

---

[5]As a $PSL(2, \mathbb{Z})$ matrix ρ is often understood to have order 3 however in the present context an overall sign is significant since $-v_a = v_{a+3}$.

furnish a representation of Dih_6. We know the action of $\mathfrak{H} \ltimes \mathfrak{G}$ on the D_a and $\widetilde{D}_b$ from (5.1) and in this way we see that the action of the generators on the points of Δ is given by a linear action on the lattice M, with matrices

$$
g_6 \;=\; \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}, \qquad g_4 \;=\; \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}
$$

$$
h_6 \;=\; \begin{pmatrix} \rho & 0 \\ 0 & -\rho \end{pmatrix}, \qquad h_2' \;=\; \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.
$$

The method we have used relies on the fact that the polygon Σ is self-dual; a more general approach would be to use (6.5) to find the action of $\mathfrak{H} \ltimes \mathfrak{G}$ on the toric coordinates, and translate this to an action on M by utilising the relationship in (6.3).

So far we have described Δ. The polyhedron ∇ is simpler. The 12 vertices of ∇ are the points ι_a and $\tilde{\iota}_b$. Thus ∇ is contained in Δ and the vertices of ∇ are the points interior to the two-faces of Δ. The three-faces are the 36 tetrahedra with vertices $\{\iota_a, \iota_{a+1}, \tilde{\iota}_b, \tilde{\iota}_{b+1}\}$. Owing to the fact that ∇ is contained in Δ, the group acts on the points of ∇ in the same representation as the action on Δ.
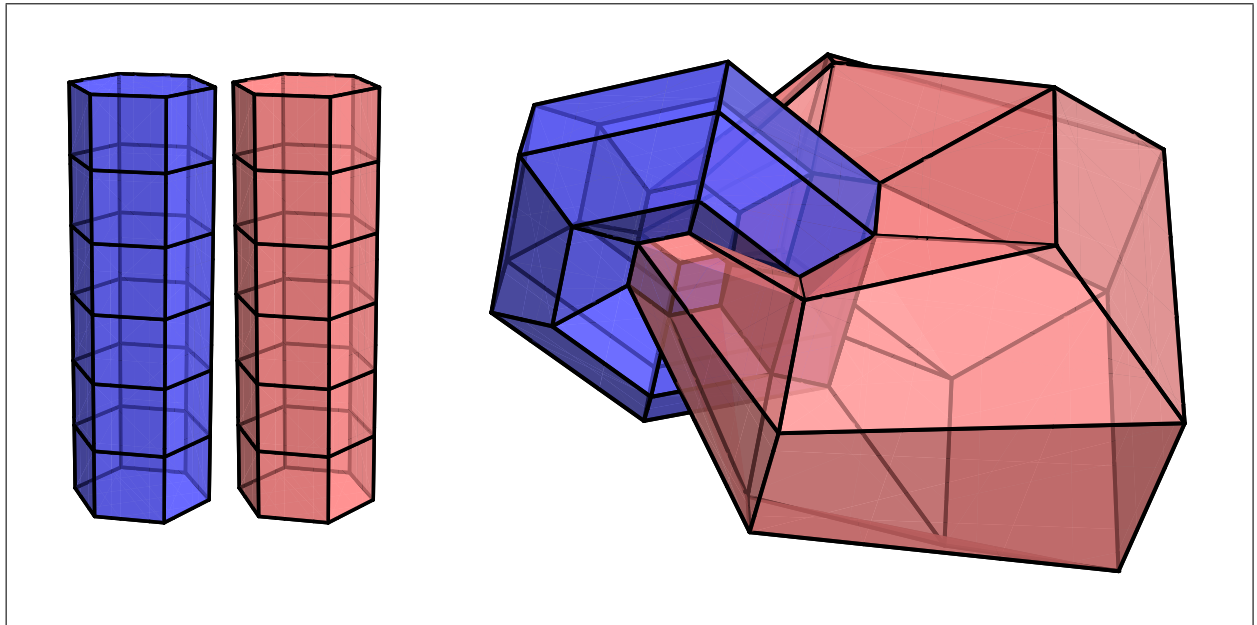

Figure 6: *The three-faces of the polyhedron Δ. Six hexagonal three-faces are stacked to make the blue prism on the left and the remaining six are stacked to make the pink prism. The top and bottom faces of these prisms are identified to make the two intersecting rings shown on the right. The faces that are depicted as exposed are identified between the two rings with the result that there is in fact no boundary.*
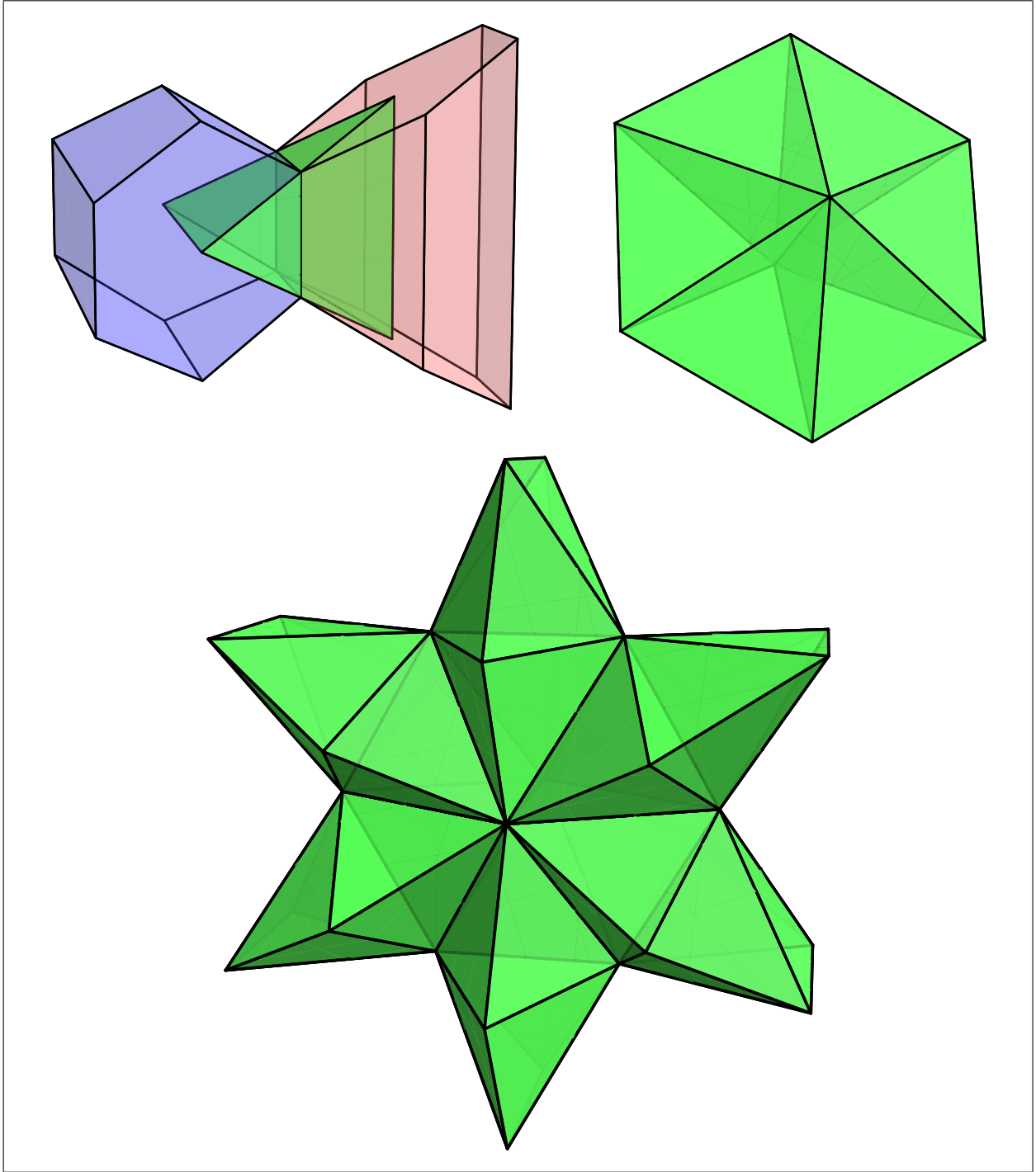
Figure 7: *The vertices of the three-faces of the polyhedron* ∇ *are the interior points to the two-faces of* Δ. *The three-faces are tetrahedra, as shown in the first figure. Note however a hazard of projecting from four dimensions to three. Four of the vertices of* Δ *project onto the faces of the tetrahedron but they do not lie on the tetrahedron as they appear in the figure. Six of these tetrahedra fit together to form the polyhedron on the right. Six of these polyhedra, in turn, fit together to form the star-shaped polyhedron, with the exposed faces identified in pairs.*

6.3. Triangulations

The mirror of $Y^{8,44}$ is realised as the resolution of a hypersurface in the toric variety defined by the fan over the faces of Δ. The toric variety defined by Δ is singular, and since the singularities will intersect a hypersurface, so is the hypersurface. The singularities of the hypersurface are resolved by resolving the singularities of the embedding space. This is done by subdividing the cones to refine the fan. The cones are subdivided by subdividing the three-faces of Δ into smaller polyhedra and the ambient variety becomes smooth if the faces of Δ are divided into polyhedra of minimal volume (which must then be tetrahedra of minimal volume). This process of subdividing the top-dimensional faces is known as triangulation.

We start with an $\mathfrak{H} \ltimes \mathfrak{G}$-invariant triangulation of Δ by dividing the 3-faces into wedges as shown in Figure 7. We may denote the blue and pink wedges that contain the two-face $\{\nu_{a,b},\, \nu_{a+1,b},\, \nu_{a+1,b+1},\, \nu_{a,b+1}\}$ by W_{ab} and $\widetilde{W}_{ab}$ respectively,

$$W_{ab} = \left\{\nu_{a,b},\, \nu_{a+1,b},\, \nu_{a+1,b+1},\, \nu_{a,b+1},\, \tilde{\iota}_b,\, \tilde{\iota}_{b+1}\right\}$$

$$\widetilde{W}_{ab} = \left\{\nu_{a,b},\, \nu_{a+1,b},\, \nu_{a+1,b+1},\, \nu_{a,b+1},\, \iota_a,\, \iota_{a+1}\right\}.$$

The group $\mathfrak{H} \ltimes \mathfrak{G}$ acts on the wedges in the expected way $g_6 W_{ab} = W_{a+1,b-1}$ and $g_4 W_{ab} = \widetilde{W}_{b+3,a}$, and so on. The triangulation of Δ into the wedges yields a toric variety with singularities along curves, and therefore a hypersurface with point singularities. Each wedge can be cut into 3 tetrahedra of minimal lattice volume. This further subdivision will yield a smooth toric variety. For reasons that will become clear shortly, we only enforce the $\mathfrak{G}$-symmetry at this point. Therefore, a fundamental region for the group action is a three-face of Δ (one-sixth of the blue or pink prism in Figure 6), consisting of 6 wedges. Each wedge can be triangulated in 6 different ways. These 6 possibilities can be distinguished by how they bisect the three rectangular two-faces of the wedge[6], as shown in Figure 8. We will label these choices as (uUu), ..., (dDu) corresponding to whether the three lines go **u**p or **d**own. When assembling the triangulated wedges into the hexagonal fundamental region, we must ensure that the triangulations match along the two-faces of the wedges. Therefore, the whole triangulation can be written as a cyclic string of six wedge-triangulations such that only two u's or two d's meet, that is, $\boxed{\cdots u)(u\cdots}$ or $\boxed{\cdots d)(d\cdots}$. This ensures a consistent triangulation in the interior of the fundamental group. The boundary of the fundamental region will intersect the boundaries of some of its $\mathfrak{G}$-images. compatibility of the triangulations along the outward-facing rectangular two-faces requires that the i-th and the $(i+3)$-rd wedge have the rectangular two-face cut in the same way, both 'U' or both 'D'. Up to symmetries[7] of the fundamental region, there are 6 distinct triangulations of a three-face.

---

[6]A wedge has three rectangular faces and each of these can be bisected in two ways so there are eight ways to bisect the faces. Two of these ways, however, do not correspond to triangulations.

[7]The symmetries of the fundamental region are Dih_6 transformations together with a reflection in a horizontal plane.
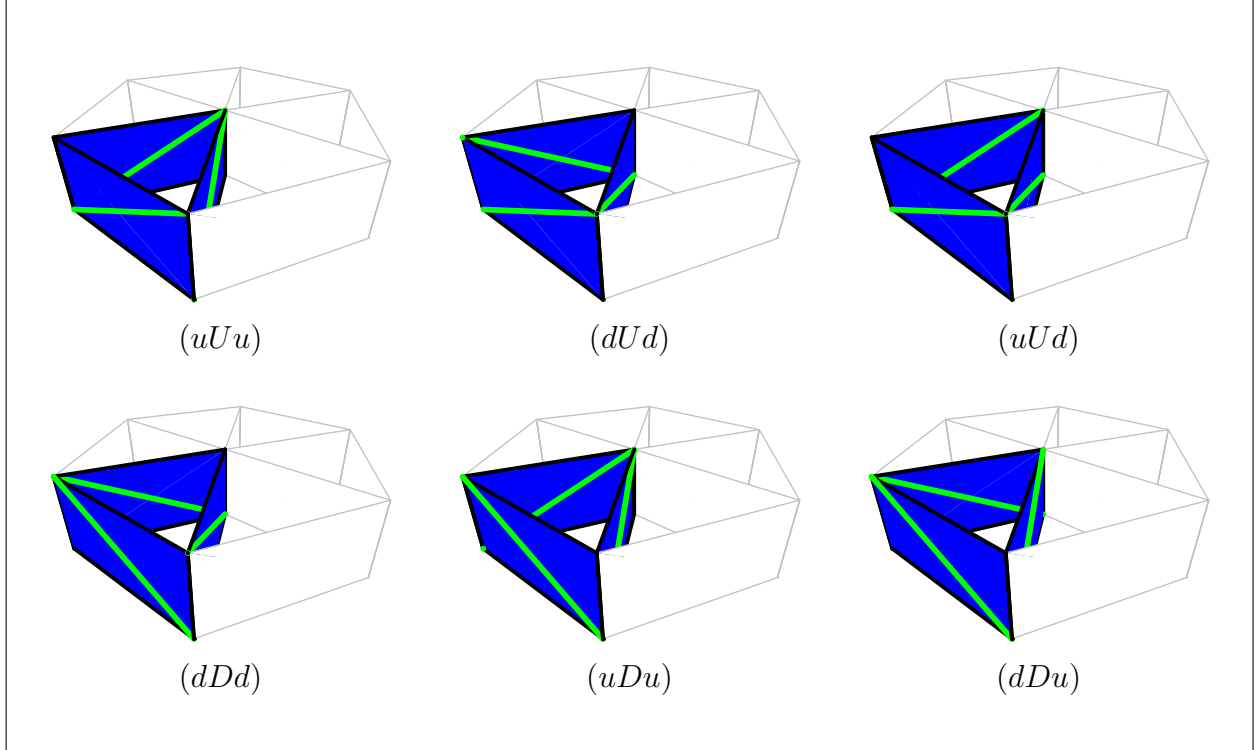
Figure 8: *The* 6 *different triangulations of a wedge inside a three-face of* Δ, *which is a fundamental region for the triangulation.*

These are shown in Table 6. It is interesting that precisely one of these yields a *regular* $\mathfrak{G}$-invariant triangulation of Δ. Recall that a regular triangulation is one that is induced by the 'crease lines' of the graph of a convex support function, a condition that is equivalent to the toric variety being Kähler. We will always use the regular triangulation in the following. It is interesting also that the regular triangulation is not the $\mathfrak{H}$-invariant triangulation $(dDd)(dDd)(dDd)\dots(dDd)$. Thus we learn that the mirror manifold is not Dih$_6$-invariant. Note, however, that the the regular triangulation $(dDu)(uUd)(dDd)(dDu)(uUd)(dDd)$ repeats with period 3. The upper-case characters that correspond to the triangulations of the two-faces of Δ are required to repeat with period three, in virtue of our observations above, but the lower-case characters are not constrained by this requirement. We learn that the mirror remains invariant under the symmetry h_2.

Note that Δ has 49 points, none of which lie interior to a three-face. These yield $h^{11} = 44$ divisor classes after deleting the origin and modding out the 4 linear relations between the points. We have seen that there is a $\mathfrak{G}$-action on Δ and there is a corresponding $\mathfrak{G}$-action on the divisors. Since the group action on the toric hypersurface is free, the coinvariant divisors form a basis for the divisors on the quotient. In other words, one has to identify the divisors on the covering space with their $\mathfrak{G}$-images. There are 4 orbits, and, therefore, $4 \, (= h^{11}(Y_\mathfrak{G}^*))$ linearly independent divisors on the quotient.

Triangulation	Regular
$\{(dDu),\ (uUd),\ (dDu),\ (uDu),\ (uUd),\ (dDd)\}$	No
$\{(dDu),\ (uUd),\ (dDd),\ (dDu),\ (uUd),\ (dDd)\}$	Yes
$\{(dDu),\ (uUd),\ (dDd),\ (dDd),\ (uUu),\ (dDd)\}$	No
$\{(dDu),\ (uDu),\ (uUd),\ (dDd),\ (dDd),\ (uUu)\}$	No
$\{(dDd),\ (dDd),\ (dDd),\ (dDd),\ (dDd),\ (dDd)\}$	No
$\{(dDd),\ (dDd),\ (uUu),\ (dDd),\ (dDd),\ (uUu)\}$	No

Table 6: *Distinct choices for the triangulation of one hexagonal three-face, the fundamental region for the $\mathfrak{G}$-action. The last column states whether the ensuing $\mathfrak{G}$-invariant triangulation of Δ is regular.*

Consider further the $\mathfrak{G}$-action on the points of Δ. These fall into five orbits which we denote by Δ_i, $0 \le i \le 4$. The origin of Δ, which forms an orbit of length one, we denote by Δ_0. There remain the four orbits mentioned above. One of these, which we choose to be Δ_1, consists of the 12 points of Δ that are internal to two-faces. These are the points $\iota_a = (v_a,\ 0)$ and $\tilde{\iota}_b = (0,\ v_b)$. The remaining three orbits consist of the vertices $\nu_{ab} = (v_a,\ v_b)$ which fall into three orbits according to the value of $a + b \bmod 3$. We take Δ_2, Δ_3 and Δ_4 to consist of the vertices ν_{ab} such that $a + b \bmod 3$ takes the values 0, 1 and 2, respectively. We abuse notation by identifying divisors on the quotient with the corresponding orbits of vertices.

As divisors we have a relation

$$\Delta_0 \;=\; -\sum_{i=1}^{4} \Delta_i \ .$$

The calculation that finds the convex piecewise linear function that determines the regularity of the triangulation yields also the generators of the the Mori cone (the dual to the Kähler cone). These are given by

$$
\begin{array}{r c c c c c c}
 & \Delta_0 & \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 \\
\hline
\ell_1 = \big(& -1, & 0, & 0, & -1, & 2 & \big) \\
\ell_2 = \big(& 0, & -1, & 0, & 1, & 0 & \big) \\
\ell_3 = \big(& 0, & 2, & -1, & -1, & 0 & \big) \\
\ell_4 = \big(& 0, & -1, & 2, & 0, & -1 & \big) .
\end{array}
$$

6.4. The mirror manifold

We have previously observed that ∇ is contained in Δ. As a result, the mirror family is defined by a polynomial which is a specialisation of r, given by specialising the coefficients. We will denote this specialisation by r^*. The equation $r^* = 0$ defines a singular variety which, generically, has 72 nodes, and the mirror Y^* is obtained as its resolution. In Section 5, in order to construct the $(2,2)$ manifold, we resolved the 36-nodal varieties by finding a suitable set of divisors and blowing up along these. In this way we demonstrated the existence of a $\mathfrak{G}$-invariant, Calabi-Yau resolution. In the present case we do not know of suitable divisors, however we may now avail ourselves of the techniques of toric geometry. We have just seen that there exists a maximal triangulation of Δ which is regular and $\mathfrak{G}$-invariant, and this provides the $\mathfrak{G}$-invariant, Calabi-Yau resolution of the 72-nodal varieties corresponding to $r^* = 0$.

Let us therefore consider the form of the polynomial r. The integral points of Δ correspond to monomials on the embedded torus, and r, restricted to the torus, is a linear combination of these. A four-parameter family of $\mathfrak{G}$-invariant Laurent polynomials is obtained by writing

$$f = \sum_{i=0}^{4} \gamma_i f_i \quad \text{where} \quad f_i = \sum_{u \in \Delta_i} t^u .$$

Being invariant, this family of Laurent polynomials must be equivalent to the family from (2.3) and the γ_i must be another system of coordinates on the parameter space and so expressible in terms of the c_i. To determine the relations[8], we set $r = f$ on the embedded torus given in (6.5). In this way we find the following correspondence

$$\gamma_0 = c_2 + c_3 + c_4 \qquad\qquad c_0 = 3\big(\zeta\gamma_2 + \zeta^2\gamma_3 + \gamma_4\big)$$

$$\gamma_1 = \frac{1}{12}\big(4c_2 - 2c_3 + c_4\big) \qquad\qquad c_1 = 3\big(\zeta^2\gamma_2 + \zeta\gamma_3 + \gamma_4\big)$$

$$\gamma_2 = \frac{1}{36}\big(4\zeta^2 c_0 + 4\zeta c_1 + (4c_2 + c_3 - 2c_4)\big) \qquad c_2 = \frac{1}{9}\big(\gamma_0 + 12\gamma_1 + 12(\gamma_2 + \gamma_3 + \gamma_4)\big) \quad (6.8)$$

$$\gamma_3 = \frac{1}{36}\big(4\zeta c_0 + 4\zeta^2 c_1 + (4c_2 + c_3 - 2c_4)\big) \qquad c_3 = \frac{4}{9}\big(\gamma_0 - 6\gamma_1 + 3(\gamma_2 + \gamma_3 + \gamma_4)\big)$$

$$\gamma_4 = \frac{1}{36}\big(4c_0 + 4c_1 + (4c_2 + c_3 - 2c_4)\big) \qquad c_4 = \frac{4}{9}\big(\gamma_0 + 3\gamma_1 - 6(\gamma_2 + \gamma_3 + \gamma_4)\big) .$$

Now ∇ is obtained by deleting the vertices of Δ. Thus $\nabla = \Delta_0 \cup \Delta_1$ and the polynomial r^* corresponds to setting $\gamma_2 = \gamma_3 = \gamma_4 = 0$. In virtue of the relations above we see that this is equivalent to the conditions

$$c_0 = c_1 = 0 \quad \text{and} \quad 4c_2 + c_3 - 2c_4 = 0 . \tag{6.9}$$

---

[8]Of course there is a scaling ambiguity in r, so in fact the relations between the γ_i and c_i are only determined up to scale

We learn that the parameter space of the mirror is contained as a curve in the parameter space of $Y_{\mathfrak{G}}$ and, moreover, that this curve lies in Γ and is the curve $\Gamma^{(i)}$ of Table 5.

It is worth pursuing the forms of f and r^* a little further. From the relation

$$f \;=\; \gamma_0 + \gamma_1 f_1 \quad\text{with}\quad f_1 \;=\; t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + t_3 + \frac{1}{t_3} + t_4 + \frac{1}{t_4} + \frac{t_1}{t_2} + \frac{t_2}{t_1} + \frac{t_3}{t_4} + \frac{t_4}{t_3}$$

it is compelling that the point corresponding to the large complex structure limit is the point $\gamma_1 = 0$, which in terms of the c_j is $c_j = (0, 0, 1, 4, 4)$. As may be seen from Figure 3, this is a point where the components of the discriminant locus have a high order contact so it may well be necessary to blow up this point, as in Figure 9, in order to discuss the monodromies about the large complex structure limit adequately. In any event, we have come rather rapidly to an identification of this limit. Returning to the polynomial r^*, we have, on imposing the conditions (6.9)

$$r^* \;=\; \frac{1}{9}\gamma_0 \left(m_{2222} + 4m_{0011} + 4m_{0212}\right) + \frac{4}{3}\gamma_1 \left(m_{2222} - 2m_{0011} + m_{0212}\right) \tag{6.10}$$

The polynomial varying with γ_0 factorises

$$m_{2222} + 4m_{0011} + 4m_{0212} \;=\; s(x_1, x_3)\, s(x_2, x_4) \quad\text{with}\quad s(w, z) \;=\; w_0 z_1 + w_1 z_0 + w_2 z_2 \ .$$

The part of r that varies with γ_1 also simplifies

$$m_{2222} - 2m_{0011} + m_{0212} \;=\; -\frac{1}{2}\, s(x_1, x_3)\, s(x_2, x_4) + \frac{3}{4}\left(s(x_1, x_3)\, x_{22}\, x_{42} + x_{12}\, x_{32}\, s(x_2, x_4)\right) \ .$$

The polynomial s is a natural analogue of p and q. The polynomial p is a sum of monomials such that the coordinate indices sum to 0 mod 3 and the coordinate indices for the monomials of q sum to 2 mod 3. For s the coordinate indices sum to 1 mod 3. Consider now the locus $s = 0$ in dP_6 which corresponds to the locus $p = q = s = 0$ in $\mathbb{P}^2 \times \mathbb{P}^2$. At first sight one might be tempted to identify this as a CICY given by three bilinear equations in $\mathbb{P}^2 \times \mathbb{P}^2$, which is
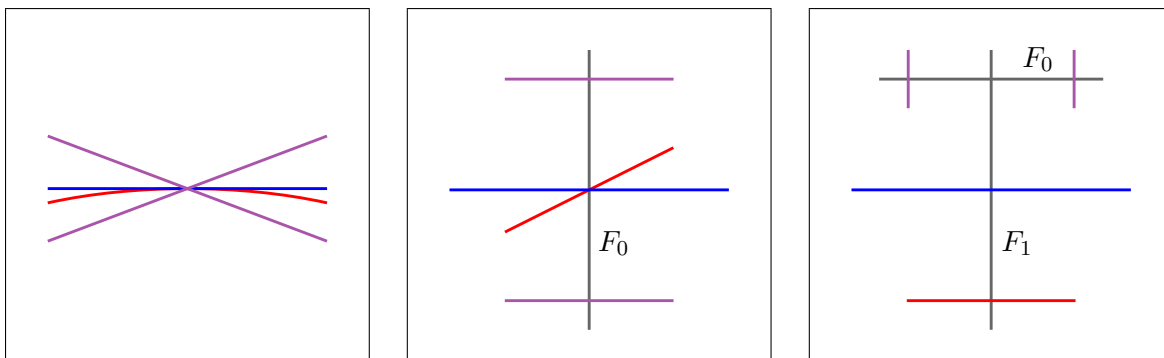

Figure 9: *The resolution of the point $(1, 4, 4)$ of Figure 3 requires a sequence of two blow ups which introduce the two exceptional divisors F_0 and F_1.*

an elliptic curve. This however is a false conclusion owing to the fact that the intersection $p = q = s = 0$ is not transverse. A little thought, and consultation with Table 2, reveals that all six divisors D_a lie in the locus $s = 0$ and that this locus is precisely the hexagon formed by the D_a. We may think of this as a degenerate elliptic curve which has become a chain of six $\mathbb{P}^1$'s. The hexagon less its vertices consists of the six one-dimensional orbits of the torus action and the vertices are the zero-dimensional orbits. Therefore when $\gamma_1 = 0$, so that $r^* = \frac{1}{9}s(x_1, x_3)\, s(x_2, x_4)$, we obtain a very singular variety which is invariant under the whole torus action.

The reader may be worried about an apparent contradiction between the fact that f_0 appears to be equal to unity, and (6.10), where $f_0 = \frac{1}{9}s(x_1, x_3)\, s(x_2, x_4)$. This is resolved by noticing that in writing $r = f$ only on the torus (6.5), it is implicit that $y_{\alpha 0} = 1$. If we write s in terms of the torus coordinates, we find $s(x_1, x_3) = s(x_2, x_4) = 3$. This is a result of the normalisation $y_{\alpha 0} = 1$. It does not contradict the fact that s vanishes on the hexagon since no point of the hexagon lies on the torus. With $s = 3$, we get $f_0 = \frac{1}{9}s(x_1, x_3)\, s(x_2, x_4) = 1$, so there is, in fact, no contradiction.

For generic γ_1/γ_0, the variety described by this family of polynomials r^* has 72 nodes which form six $\mathfrak{G}$-orbits. These comprise the 36 nodes that we have met previously and 36 nodes that are new and that are located at the points $\mathrm{pt}'_a \times \widetilde{\mathrm{pt}}'_b$, where pt'_a denotes the point on the hexagon corresponding to the intersection of the divisors D_a and D_{a+1}, with $\widetilde{\mathrm{pt}}'_b$ understood analogously. The polyhedron Δ, with its $\mathfrak{G}$-invariant triangulation provides a Calabi–Yau resolution of these nodes. The resolution of 6 nodes on the quotient $Y_\mathfrak{G}$ gives a manifold with $\chi = +6$ and Hodge numbers $(h^{11}, h^{21}) = (4, 1)$, which we identify with the mirror, $Y_\mathfrak{G}^*$, of $Y_\mathfrak{G}$.

In Table 7 we list the values of the parameter for which the variety, whose resolution is $Y_\mathfrak{G}^*$, develops extra singularities. This occurs at the values of γ_0/γ_1 for which $\Gamma^{(i)}$ intersects the other components of the discriminant. In each case where $\Gamma^{(i)}$ intersects another component Γ', for which the generic singularity consists of the 3 nodes that Γ' has in common with $\Gamma^{(i)}$ together with additional singularities, the singularities associated with the intersection are the 3 nodes, together with the 3 extra nodes of $\Gamma^{(i)}$ and the extra singularities of Γ'.

Curve	(o)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)
γ_0/γ_1	∞	4	-12	6	-3	5	∞	∞	-4	4
Sing.		$6, 1\, g_4, 1\, g_4^2$	$6, 1\, \mathfrak{G}$	$6, 1\, g_3$	$6, 1\, g_3$	7			$6, 1\, g_4^2$	6

Table 7: *The values of the parameter γ_0/γ_1 for which the mirror manifold is singular together with the type of singularity. The values given for γ_0/γ_1 are those corresponding to the intersection of $\Gamma^{(i)}$ with the other components of the discriminant locus listed in Table 5. The entry $(6, 1\, g_4, 1\, g_4^2)$, for example means 6 nodes, 1 g_4-node and 1 g_4^2-node. Where the intersection is at the large complex structure limit, for which $\gamma_0/\gamma_1 = \infty$, the singularity is not listed.*

7.   The Abelian Quotient

The manifold $Y^{8,44}$ also admits a free quotient by the Abelian group $\mathbb{Z}_{12}$. With the same conventions as used in Section 2, the action of the group generator is

$$g_{12}: \; x_{\alpha j} \; \rightarrow \; \zeta^j x_{\alpha+1,j} \;, \quad p^1 \rightarrow \zeta^2 p^1 \;, \quad p^2 \rightarrow \zeta^2 p^2 \;, \quad q^1 \leftrightarrow \zeta q^1 \;, \quad q^2 \leftrightarrow \zeta q^2 \;, \quad r \; \rightarrow \; r \;.$$

The covariant polynomials are given by

$$p^1 \;=\; x_{10}\,x_{32} + x_{12}\,x_{30} + x_{11}\,x_{31} \;, \qquad q^1 \;=\; x_{10}\,x_{31} + x_{11}\,x_{30} + x_{12}\,x_{32} \;,$$

$$p^2 \;=\; x_{20}\,x_{42} + x_{22}\,x_{40} + x_{21}\,x_{41} \;, \qquad q^2 \;=\; x_{20}\,x_{41} + x_{21}\,x_{40} + x_{22}\,x_{42} \;,$$

$$r \;=\; C_0\,m_{0000} + C_1\,m_{2211} + C_2\,m_{2121} + C_3\,m_{2010} + C_4\,m_{1110} \;.$$

It is straightforward to check that the corresponding variety is smooth, and that the induced action of $\mathbb{Z}_{12}$ is fixed-point-free. So we obtain another smooth quotient, this time with fundamental group $\mathbb{Z}_{12}$.

7.1.   Group action on homology

The representation theory of $\mathbb{Z}_{12}$ is very straightforward: there are exactly 12 distinct one-dimensional representations, in which the generator of $\mathbb{Z}_{12}$ corresponds to multiplication by one of the twelfth roots of unity. We will denote by R_k the representation in which the generator acts as multiplication by $\exp(2\pi i k/12)$. Then, repeating the type of argument used in Subsection 2.3, we find that $\mathbb{Z}_{12}$ acts on $H^2(Y^{8,44})$ through the representation

$$R_0 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_6 \oplus R_8 \oplus R_9 \oplus R_{10}$$

There is again a single invariant, corresponding to the canonical class, so the Hodge numbers of the quotient are once more $(h^{11}, h^{21}) = (1, 4)$.

7.2. $\mathbb{Z}_{12}$ flux lines

Notice that the argument of Subsection 3.1, that it is impossible to deform $\mathcal{T} \oplus \mathcal{O} \oplus \mathcal{O}$ to a non-split $SU(5)$ bundle, depends only on the fact that $h^{11} = 1$ and, therefore, applies to this case as well. Therefore we must again turn to the Hosotani mechanism. Since $\mathbb{Z}_{12}$ is Abelian, the best the Hosotani mechanism can do in this case is reduce the E_6 gauge symmetry to $SU(3) \otimes SU(2) \otimes U(1)^3$. There are a very large number of choices which can achieve this, and these will generally give different light spectra and interactions. While there may be promising models here, their identification would require a detailed study which is beyond the scope of this paper.
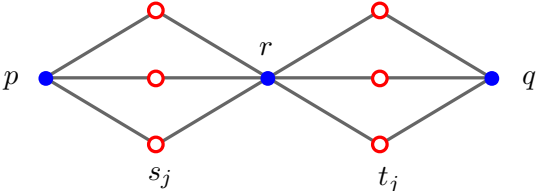
Acknowledgements

A.   Alternative Representations

We have discussed the fact that the manifold $Y^{8,44}$ can be viewed as a hypersurface in $\mathcal{S} \times \mathcal{S}$ embedded via a section of the anticanonical bundle. The same manifold can be represented as a CICY in different ways owing to the fact that there are alternative ways of representing $\mathcal{S} \cong \mathrm{dP}_6$. In addition to the representation (2.6) the following representation is useful

$$\mathcal{S} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array}\!\!\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] . \tag{A.1}$$

The identification follows from the fact that the configuration on the right has Euler number 6 and positive canonical class. In this way we see that the following configuration is an alternative way of representing the manifold $Y^{8,44}$, namely

$$X^{8,44} \;=\; \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array}\!\!\left[\begin{array}{ccc} \mathbf{1} & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} \end{array}\right]^{8,44}$$

There is also a "hybrid" representation, $Z^{8,44}$, that is formed by taking one copy of $\mathcal{S}$ to be represented as in (2.6) and the other as in (A.1). This representation is however less useful owing to the fact that the groups $\mathfrak{G}$ are not represented linearly on the coordinates. The three representations $X^{8,44}$, $Y^{8,44}$ and $Z^{8,44}$ are the three occurrences of the Hodge numbers $(h^{11}, h^{21}) = (8, 44)$ in the CICY list.

The configuration $X^{8,44}$ is described, together with the common $\mathbb{Z}_6$ subgroup of Dic_3 and $\mathbb{Z}_{12}$, in [8, §3.5.2]. We can therefore give a brief account here that concentrates on the action of the enlarged groups. We take coordinates s_{ja} on the first three $\mathbb{P}^1$ and t_{ja} on the last three, where $j = 0, 1, 2$, $a = 0, 1$, and polynomials p, q, and r, as indicated by the diagram.

We can then take the action of the two generators g_3, g_4 to be given by

$$g_3 : s_{j,a} \rightarrow s_{j+1,a} \ , \qquad t_{j,a} \rightarrow t_{j+1,a} \ , \quad p \rightarrow p \ , \quad q \rightarrow q \ , \quad r \rightarrow r \ ,$$

$$g_4 : s_{j,a} \rightarrow (-1)^{a+1} t_{-j,a} \ , \quad t_{j,a} \rightarrow s_{-j,a} \ , \quad p \rightarrow -q \ , \quad q \rightarrow p \ , \quad r \rightarrow r \ .$$

Note that the two generators S and U of [8, §3.5.2] correspond to the symmetries g_3 and g_4^2. We construct the polynomials from the quantities

$$m_{abc} = \sum_{i=0}^{2} s_{i,a} \, s_{i+1,b} \, s_{i+2,c} \ , \qquad n_{abc} = \sum_{i=0}^{2} t_{i,a} \, t_{i+1,b} \, t_{i+2,c}$$

$$l_{abcdef} = \sum_{i=0}^{2} s_{i,a} \, s_{i+1,b} \, s_{i+2,c} \, t_{i,d} \, t_{i+1,e} \, t_{i+2,f}$$

Then, with an appropriate choice of coordinates, the most general polynomials transforming as above are

$$p \;=\; \frac{1}{3} m_{000} + m_{110} \ , \qquad q \;=\; \frac{1}{3} n_{000} + n_{110}$$

$$r \;=\; \tilde{c}_0 \, l_{111111} + \tilde{c}_1 \, l_{001001} + \tilde{c}_2 \, l_{001010} + \tilde{c}_3 \, l_{001100} + \tilde{c}_4 \, (l_{111001} + l_{001111})$$

Note that a further invariant term $\ell_{000011} - \ell_{011000}$ is excluded since

$$\ell_{000011} - \ell_{011000} \;=\; m_{000}\, n_{011} - m_{011}\, n_{000} \;=\; p\, n_{011} - m_{011}\, q$$

and so corresponds merely to a redefinition of r by multiples of the polynomials p and q. This being so, we see that there is a 4 parameter family of polynomials r. It is straightforward to check that for generic values of the four undetermined coefficients, the resulting variety is smooth, and the group action is free.

B.   Deformation of the Tangent Bundle

We will show here that the tangent bundle of the manifold $X^{8,44}$ is rigid, that is, it has no infinitesimal deformations. This is a consequence of the fact that all the defining polynomials are multilinear in their arguments. The rigidity holds already at the level of the covering manifold. We do not have to invoke the group $\mathfrak{G}$ to establish the result. Let a CICY manifold X be defined by polynomials f^σ, $\sigma = 1, 2, \ldots$ and let the embedding spaces have coordinates $x_{\alpha j}$ where α labels the projective space and j runs over the coordinates within each projective space. A vector is a quantity

$$V \;=\; \sum_{\alpha, j} V_{\alpha j} \, \frac{\partial}{\partial x_{\alpha j}}$$

45

and is tangent to X if

$$V(f^\sigma) = \sum_{\alpha,\,j} V_{\alpha j} \frac{\partial f^\sigma}{\partial x_{\alpha j}} \tag{B.1}$$

vanishes on X for each σ. In virtue of the Euler relation

$$\sum_j x_{\alpha j} \frac{\partial f^\sigma}{\partial x_{\alpha j}} = \deg_\alpha(\sigma)\, f^\sigma$$

where $\deg_\alpha(\sigma)$ denotes the homogeneity degree of f^σ as a function of the $x_{\alpha j}$, it is consistent to identify

$$V_{\alpha j} \simeq V_{\alpha j} + \lambda_\alpha\, x_{\alpha j} \tag{B.2}$$

for each α and all λ_α. The condition for tangency (B.1) is deformed by imposing instead

$$\sum_{\alpha,\,j} V_{\alpha j} F^\sigma_{\alpha j} = 0 \quad \text{with} \quad F^\sigma_{\alpha j} = \frac{\partial f^\sigma}{\partial x_{\alpha j}} + \tilde{f}^\sigma_{\alpha j} \tag{B.3}$$

a set of polynomials of the same multidegrees as $\frac{\partial f^\sigma}{\partial x_{\alpha j}}$. In order to maintain consistency with (B.2) we must require that $\sum_j x_{\alpha j} \tilde{f}^\sigma_{\alpha j}$ vanish on X, for each α. It is worth examining the meaning of requiring this quantity to vanish on X more closely. In order for this quantity to vanish on X we must have

$$\sum_j x_{\alpha j}\, \tilde{f}^\sigma_{\alpha j} = \deg_\alpha(\sigma) \sum_\tau m^\sigma{}_\tau\, f^\tau$$

for some matrix of polynomials $m^\sigma{}_\tau$ of appropriate degrees. Now let us write

$$\tilde{f}^\sigma_{\alpha j} = \hat{f}^\sigma_{\alpha j} + \frac{\partial}{\partial x_{\alpha j}} \sum_\tau m^\sigma{}_\tau\, f^\tau$$

so that $\sum_j x_{\alpha j} \hat{f}^\sigma_{\alpha j} = 0$ identically, that is these quantities vanish identically as polynomials. On substituting the decomposition above into (B.3) we find

$$F^\sigma_{\alpha j} = \frac{\partial}{\partial x_{\alpha j}} \left(f^\sigma + \sum_\tau m^\sigma{}_\tau\, f^\tau \right) + \hat{f}^\sigma_{\alpha j}$$

and the quantity in parentheses corresponds to a redefinition of the f^σ. The burden of these comments is that in writing the deformation in the form (B.3) we may, without loss of generality, demand that $\sum_j x_{\alpha j} \tilde{f}^\sigma_{\alpha j}$ vanish identically. We now apply these considerations to the case that the polynomials f^σ are all multilinear. When this is so each deformation $\tilde{f}^\sigma_{\alpha j}$, having the same multidegrees as the derivatives $\partial f^\sigma / \partial x_{\alpha j}$, is independent of the coordinates $x_{\alpha k}$. Thus the condition

$$\sum_j x_{\alpha j} \tilde{f}^\sigma_{\alpha j} = 0$$

forces the $\tilde{f}^\sigma_{\alpha j}$ to vanish identically. Since this fact follows simply from the multilinearity of the configuration the same conclusion holds for the alternative representations $X^{8,44}$ and $Z^{8,44}$ as well as for the various extended representations.

References

[1] Yutaka Hosotani. Dynamical Mass Generation by Compact Extra Dimensions. *Phys. Lett.*, B126:309, 1983.

[2] Edward Witten. Symmetry Breaking Patterns in Superstring Models. *Nucl. Phys.*, B258:75, 1985.

[3] Brett McInnes. Group Theoretic Aspects Of The Hosotani Mechanism. *J. Phys.*, A22:2309–2328, 1989.

[4] Shing-Tung Yau. Compact three-dimensional Kähler manifolds with zero Ricci curvature. In *Symposium on anomalies, geometry, topology (Chicago, Ill., 1985)*, pages 395–406. World Sci. Publishing, Singapore, 1985.

[5] G. Tian and S.-T. Yau. Three-dimensional algebraic manifolds with $C_1 = 0$ and $\chi = -6$. In *Mathematical aspects of string theory (San Diego, Calif., 1986)*, volume 1 of *Adv. Ser. Math. Phys.*, pages 543–559. World Sci. Publishing, Singapore, 1987.

[6] Brian R. Greene, Kelley H. Kirklin, Paul J. Miron, and Graham G. Ross. A Three Generation Superstring Model. 1. Compactification and Discrete Symmetries. *Nucl. Phys.*, B278:667, 1986.

[7] Brian R. Greene, Kelley H. Kirklin, Paul J. Miron, and Graham G. Ross. A Three Generation Superstring Model. 2. Symmetry Breaking and the Low-Energy Theory. *Nucl. Phys.*, B292:606, 1987.

[8] Philip Candelas and Rhys Davies. New Calabi-Yau Manifolds with Small Hodge Numbers. 2008, arXiv:0809.4681.

[9] P. Candelas, M. Lynker, and R. Schimmrigk. Calabi-Yau Manifolds in Weighted P(4). *Nucl. Phys.*, B341:383–402, 1990.

[10] Albrecht Klemm and Rolf Schimmrigk. Landau-Ginzburg string vacua. *Nucl. Phys.*, B411:559–583, 1994, hep-th/9204060.

[11] Maximilian Kreuzer and Harald Skarke. Complete classification of reflexive polyhedra in four dimensions. *Adv. Theor. Math. Phys.*, 4:1209–1230, 2002, hep-th/0002240.

[12] Philip Candelas, Xenia de la Ossa, Yang-Hui He, and Balazs Szendroi. Triadophilia: A Special Corner in the Landscape. *Adv. Theor. Math. Phys.*, 12:2, 2008, arXiv:0706.3134.

[13] V. V. Batyrev. Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. *J. Alg. Geom.*, 3:493–545, 1994.

[14] Tristan Hubsch. *Calabi-Yau manifolds: A Bestiary for physicists*. World Sci. Publishing, Singapore, 1992.

[15] P. Candelas, A. M. Dale, C. A. Lutken, and R. Schimmrigk. Complete Intersection Calabi-Yau Manifolds. *Nucl. Phys.*, B298:493, 1988.

[16] P. Candelas, C. A. Lutken, and R. Schimmrigk. Complete Intersection Calabi-Yau Manifolds. 2. Three Generation Manifolds. *Nucl. Phys.*, B306:113, 1988.

[17] Paul S. Aspinwall, Brian R. Greene, K. H. Kirklin, and P. J. Miron. Searching For Three Generation Calabi-Yau Manifolds. *Nucl. Phys.*, B294:193, 1987.

[18] R. Slansky. Group Theory for Unified Model Building. *Phys. Rept.*, 79:1–128, 1981.

[19] Jun Li and Shing-Tung Yau. The existence of supersymmetric string theory with torsion. 2004, hep-th/0411136.

[20] Lara B. Anderson, James Gray, Andre Lukas, and Burt Ovrut. The Edge Of Supersymmetry: Stability Walls in Heterotic Theory. *Phys. Lett.*, B677:190–194, 2009, 0903.5088.

[21] Lara B. Anderson, James Gray, Andre Lukas, and Burt Ovrut. Stability Walls in Heterotic Theories. *JHEP*, 09:026, 2009, 0905.1748.

[22] Volker Braun, Burt A. Ovrut, Tony Pantev, and Rene Reinbacher. Elliptic Calabi-Yau threefolds with Z(3) x Z(3) Wilson lines. *JHEP*, 12:062, 2004, hep-th/0410055.

[23] Igor V. Dolgachev and Vasily A. Iskovskikh. Finite subgroups of the plane cremona group, 2006.

[24] Rhys Davies. Quotients of the Conifold in Compact Calabi-Yau Threefolds, and New Topological Transitions. In preparation.

[25] David A. Cox. The Homogeneous Coordinate Ring of a Toric Variety, Revised Version. 1993, alg-geom/9210008.

[26] Paul S. Aspinwall, Brian R. Greene, and David R. Morrison. Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory. *Nucl. Phys.*, B416:414–480, 1994, hep-th/9309097.

[27] Vincent Bouchard. Lectures on complex geometry, Calabi-Yau manifolds and toric geometry. 2007, hep-th/0702063.

[28] K. Hori et al. Mirror symmetry. Providence, USA: AMS (2003) 929 p.

[29] W. Fulton. *Introduction to Toric Varieties*. Princeton University Press, 1993.

[30] Maximilian Kreuzer and Harald Skarke. PALP: A Package for analyzing lattice polytopes with applications to toric geometry. *Comput. Phys. Commun.*, 157:87–106, 2004, math/0204356.