A Description of Kitaev's Honeycomb model with Toric-Code Stabilizers

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(Dated: March 30, 2009)

We present a description of the Kitaev honeycomb lattice model as a BCS type system. A 2-dimensional fermionization procedure is outlined and the derived eigenstates of the system are shown to be Cooper-paired products of toric-code states. We extend our analysis to a torus, giving particular attention to the ground state of the fully periodic vortex-free sector.

PACS numbers: 05.30.Pr, 75.10.Jm, 03.65.Vf

In the seminal work on his honeycomb lattice model, Kitaev outlined a connection between the non-Abelian phase of the system and chiral p-wave superconductors [1]. This has been made more explicit recently where a number of authors (see for example [2, 3]) have connected the ground state sector of the system with the spinless p-wave Hamiltonian used in [4] to relate the BCS wavefunction [5] and the Moore-Read Pfaffian [6]. One of the drawbacks of the various fermionization techniques is that they tend to obscure the re-interpretation of these states in terms of the the original spin quantum numbers. This has hindered the comparison of the BCS state with the extensive perturbative predictions on the model [1, 7, 8, 9, 10, 11] and in turn clouded the understanding of the topological transition between non-Abelian and Abelian phases of the system, the latter of which, in the leading non-trivial order, is equivalent to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ toric-code [12].

Here we will outline an exact fermionization procedure that is closely related to the perturbative analysis of [7, 8, 9]. The technique can be used to include terms that break time-reversal symmetry and it is thus possible, as in [3], to reduce the system exactly to the form used by Read and Green [4]. Importantly the procedure also allows the eigenstates of the system to be interpreted as BCS products of toric-code states, thereby illuminating the relationship between the Abelian and the non-Abelian topological phases. This interpretation can also be applied without conflict to other studies of the system based on alternative fermionization methods, see for example [1, 2, 3, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. We finish by outlining how to extend the method to a torus and discuss the role that the homologically non-trivial loop symmetries must play in any interpretation of the system as a superconducting fluid. We give particular attention to the fully periodic vortex free sector and show how derive the correct low-energy states from the BCS product.

The system consists of spins on the sites of a hexagonal lattice. The Hamiltonian can be written as

$$H = -\sum_{\alpha \in \{x,y,z\}} \sum_{i,j} J_\alpha K^\alpha_{i,j} \tag{1}$$

where $K^\alpha_{ij} = \sigma^\alpha_i \sigma^\alpha_j$ denotes a directional spin exchange interaction occurring between the sites i, j connected by a α-link see FIG. 1. We define a the basic unit cell of the lat-

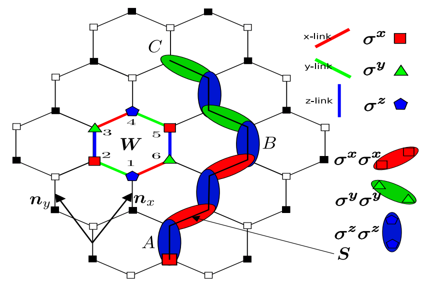

FIG. 1: The plaquette operator $\boldsymbol{W}$ and the fermionic string $\boldsymbol{S}$

tice with the two unit vectors $\boldsymbol{n}_x$ and $\boldsymbol{n}_y$ as shown in FIG. 1. By contracting each z-link to a single point we define the position vector labeling the z-dimers on a square lattice as $\boldsymbol{q} = q_x \boldsymbol{n}_x + q_y \boldsymbol{n}_y$.

Consider now loops of K operators, $K^{\alpha^{(1)}}_{ij} K^{\alpha^{(2)}}_{jk}, \ldots\ldots, K^{\alpha^{(n)}}_{li}$, where $\alpha^{(m)} \in x, y, z$. Any loop constructed in this way commutes with the Hamiltonian and with all other loops. The shortest loop symmetries are the plaquette operators

$$\boldsymbol{W}_{\boldsymbol{q}} = \sigma^z_1 \sigma^x_2 \sigma^y_3 \sigma^z_4 \sigma^x_5 \sigma^y_6, \tag{2}$$

where the numbers 1 through 6 label lattice sites on single hexagonal plaquette, see FIG. 1. We will use the convention that $\boldsymbol{q}$ denotes the z-dimer at the bottom of the plaquette. The commutation relations imply that we may choose energy eigenvectors $|n\rangle$ such that $W_{\boldsymbol{q}} = \langle n|\boldsymbol{W}_{\boldsymbol{q}}|n\rangle = \pm 1$. If $W_{\boldsymbol{q}} = -1$ then we say that the state $|n\rangle$ carries a vortex at $\boldsymbol{q}$.

On a torus, the plaquette operators are not independent, as they obey $\prod W_{\boldsymbol{q}} = I$. There are also two independent homologically non-trivial loop symmetries. We are free to choose any two closed loop operators that traverse the torus as long as they cannot be deformed into each other (by plaquette multiplication). The other homologically non-trivial loop symmetries can be constructed from the products of these two operators and the $N/2 - 1$ independent plaquette operators, see [10]. Note that when the periodicity of the toroidal configuration is specified by lattice vectors which are integer multiples of the unit vectors i.e. $\boldsymbol{X} = N_x \boldsymbol{n}_x$ and $\boldsymbol{Y} = N_y \boldsymbol{n}_y$, it is natural to use the overlapping products of alternating z- and x-links ($L_x = \prod K^z_{ij} K^x_{jk}$) and alternating z- and y-links

($L_y = \prod K^z_{ij} K^y_{jk}$) as the two independent homologically non-trivial symmetries.

The Hamiltonian (1) is often extended to include perturbing terms that (i) are sums of K operator products (ii) open a gap in the B-phase (iii) break time-reversal symmetry (T-symmetry), see [1, 3, 18] and the general analysis of the link or bond algebras in [23]. The breaking of T-symmetry is essential for relating the model to chiral p-wave superconductors. A generalisation of the type given in [3] is needed to get the precise $(k_x + ik_y)$ pairing symmetry related to the Ising CFT model and the $\nu = 5/2$ quantum Hall state through the Moore-Read Pfaffian [4]. As the procedure we will outline here gives the same physical results as the quoted references for generalised T-symmetry breaking we will keep the discussion as simple as possible here and restrict the explicit calculations to the original T-symmetric terms.

We begin our derivation by first noting that in [7], the Hamiltonian (1) was written in terms hard-core bosons and effective spins of the z-dimers using the mapping:

$$\begin{aligned} |\uparrow_{\blacksquare}\uparrow_{\square}\rangle &= |\Uparrow, 0\rangle, \quad |\downarrow_{\blacksquare}\downarrow_{\square}\rangle = |\Downarrow, 0\rangle, \\ |\uparrow_{\blacksquare}\downarrow_{\square}\rangle &= |\Uparrow, 1\rangle, \quad |\downarrow_{\blacksquare}\uparrow_{\square}\rangle = |\Downarrow, 1\rangle. \end{aligned} \tag{3}$$

The labels on the L.H.S. indicate the states of the z-dimer in the computational basis. The first quantum number of the kets on the R.H.S. represents the effective spin of the square lattice and the second is the bosonic occupation number. The presence of a boson indicates an anti-ferromagnetic configuration of the spins connected by a z-link.

The operations of the original spin Hamiltonian then become (see [7, 8, 9])

$$\begin{array}{ll} \sigma^x_{\boldsymbol{q},\blacksquare} = \tau^x_{\boldsymbol{q}}(b^\dagger_{\boldsymbol{q}} + b_{\boldsymbol{q}}) & , \ \sigma^x_{\boldsymbol{q},\square} = b^\dagger_{\boldsymbol{q}} + b_{\boldsymbol{q}}, \\ \sigma^y_{\boldsymbol{q},\blacksquare} = \tau^y_{\boldsymbol{q}}(b^\dagger_{\boldsymbol{q}} + b_{\boldsymbol{q}}) & , \ \sigma^y_{\boldsymbol{q},\square} = i\,\tau^z_{\boldsymbol{q}}(b^\dagger_{\boldsymbol{q}} - b_{\boldsymbol{q}}), \\ \sigma^z_{\boldsymbol{q},\blacksquare} = \tau^z_{\boldsymbol{q}} & , \ \sigma^z_{\boldsymbol{q},\square} = \tau^z_{\boldsymbol{q}}(I - 2b^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}}), \end{array} \tag{4}$$

where $\tau^a_{\boldsymbol{q}}$ is the Pauli operator acting on the effective spin at position $\boldsymbol{q}$ and $b^\dagger(b\ )$ are the canonical creation(annihilation) operators for the hard-core bosons. In this notation the Hamiltonian itself becomes

$$\begin{aligned} H &= -J_x \sum_{\boldsymbol{q}} (b^\dagger_{\boldsymbol{q}} + b_{\boldsymbol{q}}) \tau^x_{\boldsymbol{q}+\boldsymbol{n}_x} (b^\dagger_{\boldsymbol{q}+\boldsymbol{n}_x} + b_{\boldsymbol{q}+\boldsymbol{n}_x}) \\ &- J_y \sum_{\boldsymbol{q}} i\tau^z_{\boldsymbol{q}} (b^\dagger_{\boldsymbol{q}} - b_{\boldsymbol{q}}) \tau^y_{\boldsymbol{q}+\boldsymbol{n}_y} (b^\dagger_{\boldsymbol{q}+\boldsymbol{n}_y} + b_{\boldsymbol{q}+\boldsymbol{n}_y}) \\ &- J_z \sum_{\boldsymbol{q}} (I - 2b^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}}). \end{aligned} \tag{5}$$

In this representation the plaquette operators of the original Hamiltonian are

$$\boldsymbol{W}_{\boldsymbol{q}} = (I - 2\boldsymbol{N}_{\boldsymbol{q}})(I - 2\boldsymbol{N}_{\boldsymbol{q}+\boldsymbol{n}_y})\boldsymbol{Q}_{\boldsymbol{q}} \tag{6}$$

where $\boldsymbol{N}_{\boldsymbol{q}} = b^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}}$ and $\boldsymbol{Q}_{\boldsymbol{q}} = \tau^z_{\boldsymbol{q}}\ \tau^y_{\boldsymbol{q}+\boldsymbol{n}_x} \tau^y_{\boldsymbol{q}+\boldsymbol{n}_y} \tau^z_{\boldsymbol{q}+\boldsymbol{n}}$. We can generalise the expression (6) to include products of plaquette operators. Of particular importance later, because of the conventions used, will be the products arranged vertically on the effective lattice. We have in this case

$$\begin{aligned} \boldsymbol{F}_{q_x,q_y} &\equiv \prod_{q'_y=0}^{q_y} \boldsymbol{W}_{q_x,q'_y} \\ &= (I - 2\boldsymbol{N}_{q_x,0})(I - 2\boldsymbol{N}_{q_x,q_y}) \prod_{q'_y=0}^{q_y} \boldsymbol{Q}_{q_x,q'_y} \end{aligned} \tag{7}$$

and we see that only the bosons at the upper and lower left corners of the plaquette product need to be taken into account.

The relation (6) allows one to write down an orthonormal basis for the full honeycomb(brick-wall) system [9]. Explicitly we can write $|\{Q_{\boldsymbol{q}}\}, \{\boldsymbol{q}\}\rangle$ where the quantum numbers are the eigenvalues $Q_{\boldsymbol{q}}$ of the operator $\boldsymbol{Q}_{\boldsymbol{q}}$ and the bosonic position vectors $\boldsymbol{q}$. Any state with a given $\{\boldsymbol{q}\}$ is determined, up to a phase, by the stabilizers $\boldsymbol{Q}_{\boldsymbol{q}}|\psi\rangle = \pm|\psi\rangle$, that reflect the underlying vorticity and bosonic position vectors, through the mapping (4). The state $|\{Q_{\boldsymbol{q}}\}, \{\emptyset\}\rangle\ \forall Q_{\boldsymbol{q}} = 1$ is unitarily equivalent to the toric-code ground state in the square lattice effective spin representation [1, 12] and from the perturbation theory this state is known to be the leading contribution to the actual ground state of the full hexagonal system [8, 10]. To fully specify a state on a torus one must also specify two additional quantum numbers associated with the homologically non-trivial loop symmetries. In our case we choose these to be the eigenvalues l_x and l_y of the operators L_x and L_y described above.

We now define a particular string operator using overlapping products of the K^α_{ij} terms of the original Hamiltonian. The primary function of the string will be to break/fix z-dimers at a particular location $\boldsymbol{q}$ of the lattice. Our convention will be to first apply a single σ^x term to a black site which we set to be the origin. The rest of the string is made by applying first alternating K^z_{ij} and K^x_{jk} until we reach a required length and then apply alternating K^z_{lm} and K^y_{mn} terms ending on the black site at $\boldsymbol{q}$, see FIG. 1. Explicitly we write

$$\begin{aligned} S_{\boldsymbol{q}} \ \equiv\ & \sigma^y_{(q_x,q_y),\blacksquare}\sigma^y_{(q_x,q_y-1),\square}\sigma^z_{(q_x,q_y-1)\square} \\ & ...\sigma^y_{(q_x,1),\blacksquare}\sigma^y_{(q_x,0),\square}\sigma^z_{(q_x,0),\square}\sigma^z_{(q_x,0)\blacksquare}\sigma^x_{(q_x,0)\blacksquare} \\ & ...\sigma^x_{(1,0)\blacksquare}\sigma^x_{(0,0)\square}\sigma^z_{(0,0),\square}\sigma^z_{(0,0),\blacksquare}\sigma^x_{(0,0),\blacksquare} \end{aligned} \tag{8}$$

Using the representation of [7, 8, 9] we can decompose (8) into the effective spin and bosonic subspaces, i.e. $S = S_e \otimes S_b$. In this decomposition there are four different types of structures to observe on the effective lattice: (1) the line including the starting point A up to, but not including, the turning point B, (2) the turning point $B = (q_x, 0)$, (3) the exclusive interval BC, and (4) the end point $C = (q_x, q_y)$, see FIG 2 and TABLE I.

The operator $S_{\boldsymbol{q}}$ squares to unity while different operators $S_{\boldsymbol{q}}, S_{\boldsymbol{q}'}$ anti-commute with each other. This lead us to identify the string $S_{\boldsymbol{q}}$ with the following sum of fermionic creation and annihilation operators: $S_{\boldsymbol{q}} = c^\dagger_{\boldsymbol{q}} + c_{\boldsymbol{q}} = (b^\dagger_{\boldsymbol{q}} + b_{\boldsymbol{q}})S'_{\boldsymbol{q}}$ where $S'_{\boldsymbol{q}}$ can be determined from TABLE I. Individually our fermionic canonical creation and annihilation operators are

$$c^\dagger_{\boldsymbol{q}} = b^\dagger_{\boldsymbol{q}} S'_{\boldsymbol{q}}, \quad c_{\boldsymbol{q}} = b_{\boldsymbol{q}} S'_{\boldsymbol{q}} \tag{9}$$

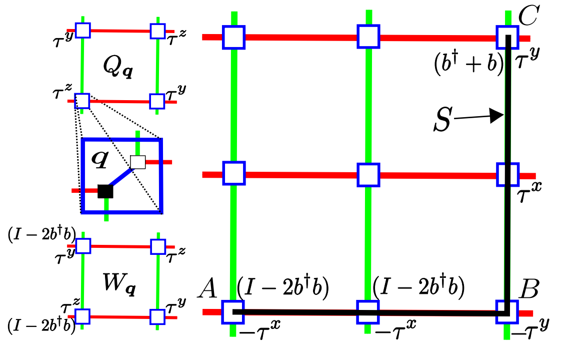

FIG. 2: Bosonic and effective spin decomposition of the operator string S.

	S	$S_e \otimes S_b$
[A,B)	$\sigma^x_\square\sigma^z_\square\sigma^z_\blacksquare\sigma^x_\blacksquare$	$-\tau^x \otimes I-2b^\dagger b$
B	$\sigma^y_\square\sigma^z_\square\sigma^z_\blacksquare\sigma^x_\blacksquare$	$-\tau^y \otimes I$
(B,C)	$\sigma^y_\square\sigma^z_\square\sigma^z_\blacksquare\sigma^y_\blacksquare$	$\tau^x \otimes I$
C	$\sigma^y_\blacksquare$	$\tau^y \otimes b^\dagger + b$

TABLE I: The string S as four unique segments. While bosons are only created/destroyed at the endpoint C of the string, the sites in the $[A, B)$ interval also have non-trivial bosonic dependence.

where the strings now insure that the operators $c^\dagger$ and c obey the canonical fermionic anti-commutator relations. It is important to note that, because of the identification of the sum $c^\dagger_{\boldsymbol{q}} + c_{\boldsymbol{q}}$ with the string $S_{\boldsymbol{q}}$, that operators $c^\dagger_{\boldsymbol{q}}$ and $c_{\boldsymbol{q}}$ must both create/annihilate vortices at $\boldsymbol{q} = (-1, 0)$ and $\boldsymbol{q} = (-1, -1)$. However, quadratic terms of fermionic operators will always preserve the underlying vorticity. This has interesting consequences later when we examine the system on a torus.

Similar to [1, 14], we now introduce the generic quadratic Hamiltonian

$$H = \left[\begin{array}{cc} c^\dagger_{\boldsymbol{q}} & c_{\boldsymbol{q}} \end{array}\right] \left[\begin{array}{cc} \xi_{\boldsymbol{q}\boldsymbol{q}'} & \Delta_{\boldsymbol{q}\boldsymbol{q}'} \\ \Delta^\dagger_{\boldsymbol{q}\boldsymbol{q}'} & -\xi^T_{\boldsymbol{q}\boldsymbol{q}'} \end{array}\right] \left[\begin{array}{c} c_{\boldsymbol{q}'} \\ c^\dagger_{\boldsymbol{q}'} \end{array}\right] + C \qquad (10)$$

If we invert (9) and substitute the relevant expressions into the Hamiltonian (5) we get the form (10) with

$$\begin{aligned} \xi_{\boldsymbol{q}\boldsymbol{q}'} &= 2J_z\delta_{\boldsymbol{q},\boldsymbol{q}'} + J_x \boldsymbol{F}_{\boldsymbol{q}-\boldsymbol{n}_y}(\delta_{\boldsymbol{q},\boldsymbol{q}'+\boldsymbol{n}_x} + \delta_{\boldsymbol{q}+\boldsymbol{n}_x,\boldsymbol{q}'}) \\ &+ J_y(\delta_{\boldsymbol{q},\boldsymbol{q}'+\boldsymbol{n}_y} + \delta_{\boldsymbol{q}+\boldsymbol{n}_y,\boldsymbol{q}'}) \\ \Delta_{\boldsymbol{q}\boldsymbol{q}'} &= J_x \boldsymbol{F}_{\boldsymbol{q}-\boldsymbol{n}_y}(\delta_{\boldsymbol{q},\boldsymbol{q}'+\boldsymbol{n}_x} - \delta_{\boldsymbol{q}+\boldsymbol{n}_x,\boldsymbol{q}'}) \\ &+ J_y(\delta_{\boldsymbol{q},\boldsymbol{q}'+\boldsymbol{n}_y} - \delta_{\boldsymbol{q}+\boldsymbol{n}_y,\boldsymbol{q}'}) \end{aligned} \qquad (11)$$

and $C = -MJ_z$ where M the number of effective spins and $\boldsymbol{F}_{\boldsymbol{q}}$ is defined in (7). We restrict the Hilbert space to the relavent vortex-configuration by replacing $\boldsymbol{F}_{\boldsymbol{q}}$ by the eigenvalues $F_{\boldsymbol{q}}$ of that configuration. In the simplest case of the vortex free sector we have $F_{\boldsymbol{q}} = 1 \quad \forall \boldsymbol{q}$. This sector, because of the theorem by Lieb [25], is known to contain the system ground state and can be solved exactly in the thermodynamic limit by moving to the momentum representation with the Fourier transform $c_{\boldsymbol{q}} = M^{-1/2}\sum c_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{q}}$. After substitution into (11) and anti-symmetrization we have

$$H = \sum_{\boldsymbol{k}} \left[\xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} c_{\boldsymbol{k}} + \frac{1}{2}(\Delta c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}} + \Delta^* c_{-\boldsymbol{k}} c_{\boldsymbol{k}})\right] + C \qquad (12)$$

where $\xi_{\boldsymbol{k}} = \varepsilon_{\boldsymbol{k}} - \mu$ with $\varepsilon_{\boldsymbol{k}} = 2J_x\cos(k_x) + 2J_y\cos(k_y)$ and $\mu = -2J_z$. The gap function is $\Delta_{\boldsymbol{k}} = \alpha_{\boldsymbol{k}} + i\beta_{\boldsymbol{k}}$ with $\alpha_{\boldsymbol{k}} = 0$ and $\beta_{\boldsymbol{k}} = 2J_x\sin(k_x) + 2J_y\sin(k_y)$. If the Hamiltonian is extended, as in Kitaev's original analysis, to include the single products of adjacent K-operators (formula (46) of [1]) one gets $\alpha_{\boldsymbol{k}} = 4\kappa(\sin(k_x) - \sin(k_y) - \sin(k_x - k_y))$ and the form E_k is in exact agreement with the dispersion relation derived there. The procedure also gives agreement with the other fermionization techniques to analyse the extended model [2, 13, 14, 16, 18]. We note in particular that the technique can be used to replicate the dispersion relations of [3] where the p-wave pairing can be tuned to have $k_x + ik_y$ chiral symmetry thus allowing a direct link with the work of Read and Green [4] and subsequent analysis [26, 27, 28, 29], relating the Pfaffian Quantum Hall states, p-wave superconductors and Ising topological model.

The Hamiltonian (12) is diagonalized by Bogoliubov transformation $\gamma_{\boldsymbol{k}} = u_{\boldsymbol{k}} c_{\boldsymbol{k}} - v_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}}$, where $u_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}}$ satisfy $|u_{\boldsymbol{k}}|^2 + |v_{\boldsymbol{k}}|^2 = 1$. We have $H = \sum E_{\boldsymbol{k}}(\gamma^\dagger_{\boldsymbol{k}}\gamma_{\boldsymbol{k}} - 1/2)$, with $E_{\boldsymbol{k}} = \sqrt{\xi^2_{\boldsymbol{k}} + |\Delta_{\boldsymbol{k}}|^2}$, $u_{\boldsymbol{k}} = \sqrt{1/2(1 + \xi_{\boldsymbol{k}}/E_{\boldsymbol{k}})}$, and $v_{\boldsymbol{k}} = i\sqrt{1/2(1 - \xi_{\boldsymbol{k}}/E_{\boldsymbol{k}})}\Theta_{\boldsymbol{k}}$ with $\Theta_{\boldsymbol{k}} = \mathrm{sgn}(-\mathrm{Im}(\Delta_{\boldsymbol{k}})/\xi_{\boldsymbol{k}})$

The ground state, annihilated by all $\gamma_{\boldsymbol{k}}$, and of energy $E_{\text{gs}} = -\frac{1}{2}\int E_{\boldsymbol{k}} d\boldsymbol{k}$, can by inspection be seen to be the BCS type state

$$|\,\text{gs}\rangle = \prod_{\boldsymbol{k}}(u_{\boldsymbol{k}} + v_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}})|\,\text{vac}\rangle. \qquad (13)$$

This expression is similar to the one obtained in Ref. [2], but we note that our fermionization procedure has been designed such that the vacuum state is the toric-code ground state $|\{\boldsymbol{Q}_{\boldsymbol{q}}\}, \{\emptyset\}\rangle$ defined on the effective lattice, while the operators $c^\dagger_{\boldsymbol{l}}$ are, by definition, the Fourier superpositions of the states $|\{\boldsymbol{Q}_{\boldsymbol{q}}\}, \{\boldsymbol{q}\}\rangle$. Note that in the corner of the A phase $(J_z = 1, J_x, J_y \to 0)$ we have $u_{\boldsymbol{k}} \to 1$ and $v_{\boldsymbol{k}} \to 0$ and thus the ground state of the full system $|\,\text{gs}\rangle \to |\{\boldsymbol{Q}_{\boldsymbol{q}}\}, \{\emptyset\}\rangle$ as expected. The expression (13) is, to the best of our knowledge, the first closed form expression for this state that does not require additional spectral projection. It is also noteworthy because it combines two powerful wavefunction descriptors i.e. Cooper pairing and the Stabilizer formalism.

The Hamiltonian (10) may be diagonalised for arbitrary vortex configurations on a torus using the multimode Bogoliubov transformation $\gamma_i = \sum_l (u_{il}c_l - v_{il}c^\dagger_l)$. The quasiparticle excitation energy E_l and the vectors u_{il} and v_{il} are obtained by solving the Bogoliubov-de-Gennes eigenvalue problem

$$\left[\begin{array}{cc} \xi & \Delta \\ \Delta^\dagger & -\xi^T \end{array}\right] \left[\begin{array}{c} u \\ v \end{array}\right] = E \left[\begin{array}{c} u \\ v \end{array}\right], \qquad (14)$$

where the ξ and Δ given in (11) are modified to include the terms that connect both sides of the torus, i.e. the terms that connect the sites $(0, q_y)$ to (N_x, q_y) and $(q_x, 0)$ to (q_x, N_y). The values of these terms are determined from the arrangement of vortices and the quantum numbers of the two independent homologically non-trivial loop symmetries l_x and l_y.

To construct the actual eigenstates using the multimode γ_i it is easiest to use the Hartree-Fock-Bogoliubov (HFB) projection $|\,\mathrm{gs}\rangle = \prod_i \gamma_i |\,\mathrm{vac}\rangle$, where the vacuum state is the appropriate toric-code state on the full hexagonal lattice. This state can be brought into BCS form (13) by making use of the Bloch-Messiah theorem [30]. For the vortex free sector, with the important exception of the fully periodic state with $(l_x, l_y) = (-1, -1)$ in the B-phase, one observes that the eigenstate energy can be simply calculated as a discrete sum over the energy function $E_{\boldsymbol{k}}$ calculated above i.e. $E_{\mathrm{gs}} = -\frac{1}{2}\sum_{k_x,k_y} E_{\boldsymbol{k}}$, where the k_α run over $\theta_\alpha + 2\pi\frac{n_\alpha}{N_\alpha}$ with the integer n_α running from 0 to $N_\alpha - 1$, and the boundary conditions (l_x, l_y) are encoded as $\theta_\alpha = \frac{l_\alpha+1}{2}\frac{\pi}{N_\alpha}$. As the fermions do not preseve vorticity, valid excitation energies above the ground state are given by adding pairs of energies $E_{\boldsymbol{k}}$ to E_{gs}.

For the fully periodic vortex free sector there is no need for the Bloch-Messiah reduction although, similarly to [4] (see also note [31]), the BCS state (13) is not valid in the B phase. However, with the fermions we defined above, the problematic point in k-space occurs at $\boldsymbol{k} = (\pi, \pi)$ rather than $\boldsymbol{k} = (0, 0)$. At this point the BCS state (13) vanishes because, $u_{\boldsymbol{k}} = 0$, $v_{\boldsymbol{k}} = -i$ and $c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}} = 0$. However, unlike Read and Green, we cannot propose the singles $\gamma^\dagger_{\pi,\pi}|\,\mathrm{gs}\rangle$ as an alternative because the Bogoliubov fermions do not preserve the plaquette gauge symmetries. Furthermore we observe numerically that the double excitations have too great an energy. This puzzle can be understood if one examines the state

$$|\,\psi_{\pi,\pi}\rangle = \prod_{\boldsymbol{k}\neq(\pi,\pi)} (u_{\boldsymbol{k}} + v_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}})|\,\{Q_{\boldsymbol{q}}, l_x, l_y\}, \{\emptyset\}\rangle \quad (15)$$

with $(l_x, l_y) = (-1, -1)$ and $Q_{\boldsymbol{q}} = 1 \;\; \forall \boldsymbol{q}$. The state (15) is clearly from the vortex-free sector, has even fermion parity, and is an eigenstate of the Hamiltonian with energy $E_{\mathrm{gs}} + E_{\pi,\pi}$. By inspection, it is not difficult to see that the states $|\,\psi_{\boldsymbol{k}}\rangle = \gamma_{\pi,\pi}\gamma^\dagger_{\boldsymbol{k}}|\,\psi_{\pi,\pi}\rangle$ are also vortex-free eigenstates of the system but with energy $E_{\mathrm{gs}} + E_{\boldsymbol{k}}$. It is therefore natural to assume that the ground state of this sector is actually the state $|\,\psi_{\boldsymbol{k}}\rangle$ such that the discretized $E_{\boldsymbol{k}}$ is a minimum. As an aside, we should point out that there is no reason why the construction here can not be applied in 'real' p-wave superconductors, where fermionic electrons and vortices are not dependent in the same way. While this result indicates that the odd fermion single introduced by Read and Green must be degenerate with the even fermion BCS product introduced above, it does not conflict with the general assertion that, for a gapped system, the lowest energy odd fermion state is not part of the groundstate manifold.

This work has been supported by Science Foundation Ireland through the President of Ireland Research Award 05/Y12/1680 and the Principal Investigator Award 08/IN.1/I1961.

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