

An ab initio Calculation of the Universal Equation of State for the $O(N)$ Model

Denjoe O'Connor¹, J A Santiago² and C R Stephens³

¹ School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

²Centro de Investigación en Matemáticas. Universidad Autónoma del Estado de Hidalgo, Pachuca 42184, México

³ Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, México DF 04510, México

E-mail: denjoe@stp.dias.ie, sgarcia@uaeh.edu.mx,
stephens@nucleares.unam.mx

Abstract. Using an Environmentally Friendly Renormalization Group we derive an *ab initio* universal scaling form for the equation of state for the $O(N)$ model, $y = f(x)$, that exhibits all required analyticity properties in the limits $x \rightarrow 0$, $x \rightarrow \infty$ and $x \rightarrow -1$. Unlike current methodologies based on a phenomenological scaling ansatz the scaling function is derived solely from the underlying Landau-Ginzburg-Wilson Hamiltonian and depends only on the three Wilson functions γ_λ , γ_φ and γ_{φ^2} which exhibit a non-trivial crossover between the Wilson-Fisher fixed point and the strong coupling fixed point associated with the Goldstone modes on the coexistence curve. We give explicit results for $N = 2, 3$ and 4 to one-loop order and compare with known results.

PACS numbers: 64.60.Ak,

1. Introduction

The equation of state for the $O(N)$ model has been an object of intense scrutiny over the last 30-40 years (see, for instance, [1] and [2] for recent reviews). It exhibits crossover behaviour between three distinct asymptotic regimes - the critical region approached along the critical isotherm, the critical region approached along the critical isochore, and, finally, the coexistence curve. It is the difficulty of encapsulating these distinct scaling behaviours within one overall scaling function that has prevented its *ab initio* derivation from an underlying microscopic model.

The first attempts at such an ab initio calculation, using the renormalization group (RG) and an ε -expansion [3], foundered on the fact that they did not exhibit Griffiths analyticity in the large $x = t/\varphi^{1/\beta}$ limit. This was due to the fact that the expansion was around a particular fixed point - the Wilson-Fisher fixed point. However, to obtain a universal equation of state, valid in the entire phase diagram, using the RG and *without*

phenomenological input, the φ -dependent crossover between this fixed point and the strong coupling fixed point associated with the massless Goldstone excitations on the coexistence curve must be accessed and controlled. In the latter regime the non-linear σ model gives a good description [5]. However, this model only accounts for the Goldstone bosons and does not offer a full description of the phase diagram.

As pure first principles calculations using a Landau-Ginzburg-Wilson Hamiltonian and the RG have not been capable of obtaining an equation of state valid in all asymptotic regimes, the current “state of the art” [1, 2] is to base calculations on a parametrised phenomenological scaling form [6] that has the correct analyticity properties. Instead of the magnetisation, φ , reduced temperature, t , and magnetic field, H , new variables, θ and R , are introduced, the relation between them being

$$\varphi = m_0 R^\beta m(\theta) \qquad t = R(1 - \theta^2) \qquad H = h_0 R^{\beta\delta} h(\theta)$$

The particular functional dependence on R ensures that Griffith’s analyticity is preserved. However, the two functions $m(\theta)$ and $h(\theta)$ are arbitrary. Hence, any particular choice constitutes a pure ansatz. Thus, Schofield’s scaling form gives a large class of models, all of which, by construction, are consistent with Griffith’s analyticity. How a particular microscopic model is related to this large class depends on how these undetermined functions are represented, the canonical approach being to represent them as polynomials. Thus, there is no unique map between a microscopic model and a member of the class of Schofield parametrisations.

In this new parametrisation the scaling function of the equation of state is given by

$$f(x) = \left(\frac{m(\theta)}{m(1)}\right)^{-\delta} \frac{h(\theta)}{h(1)} \quad \text{where} \quad x = \frac{1 - \theta^2}{\theta_0^2 - 1} \left(\frac{m(\theta_0)}{m(\theta)}\right)^{1/\beta}. \quad (1)$$

Most current field theoretic formulations for determining the equation of state (see [2] for a comprehensive review) rely on such formulations. The drawback is that the underlying microscopic theory is not used to determine the functional form of $m(\theta)$ and $h(\theta)$. Rather, an ansatz is made as to the general functional form, which depends on certain unknown parameters, and then the underlying microscopic theory is used to fix them. The most common ansatz is that the functions are polynomials in θ . The coefficients of the powers of θ are then determined by calculating the values of certain observables independently from the underlying microscopic theory.

In contrast to the above, in this paper‡, using only the Landau-Ginzburg-Wilson Hamiltonian for the $O(N)$ model and by implementing an Environmentally Friendly RG [8, 9] that tracks the crossover between the fixed points that control the different asymptotic regimes, we derive an universal equation of state that obeys all required analyticity properties and where *no* phenomenological input is required, only the three Wilson functions γ_φ , γ_{φ^2} and γ_λ .

‡ Which is a follow up to [10] which treated the case $N = 1$

2. The Equation of State

In the critical region, the equation of state [11] relates the external magnetic field H , the reduced temperature t , and the magnetization φ ,

$$y = f(x) \quad (2)$$

where the universal scaling function $f(x)$, is normalised such that $f(0) = 1$ on the critical isotherm, and $f(-1) = 0$ on the coexistence curve. The scaling variables y and x are given by $y = B_c^\delta H / \varphi^\delta$ and $x = B^{1/\beta} t / \varphi^{1/\beta}$.

The function $f(x)$ has expansions around the limits $x = 0$ and $x \rightarrow \infty$ given by

$$f(x) = 1 + \sum_{n=1}^{\infty} f_n^0 x^n \quad f(x) = x^\gamma \sum_{n=0}^{\infty} f_n^\infty x^{-2n\beta} \quad (3)$$

In the limit $x \rightarrow \infty$ a natural variable is $z = b_1 \varphi / t^\beta$, where b_1 an amplitude ratio in terms of which the equation of state takes the form $H \propto t^{\beta\delta} F(z)$, where the universal scaling function $F(z)$ for small z has an expansion of the form

$$F(z) = z + \frac{1}{6} z^3 + \sum_{n=3}^{\infty} \frac{r_{2n}}{(2n-1)!} z^{2n-1} \quad (4)$$

where $r_2 = r_4 = 1$ by choice of normalisation. As (4) is an expansion in φ , the constants r_{2n} are related to the $2n$ -point correlation functions at $\varphi = 0$ and hence are very natural observables to calculate in lattice simulations. In the limit $z \rightarrow \infty$, $F(z)$ has an expansion of the form

$$F(z) = z^\delta \sum_{k=0}^{\infty} F_k^\infty z^{-k/\beta} \quad (5)$$

By relating $f(x)$ and $F(z)$ expansion coefficients of the two functions can be related to find

$$f_n^\infty = z_0^{2n+1-\delta} \frac{r_{2n+2}}{F_0^\infty (2n+1)!} \quad (6)$$

$$f_n^0 = \frac{F_n^\infty}{F_0^\infty} z_0^{-n/\beta} \quad (7)$$

where z_0 is the universal zero of the equation of state in terms of the variable z . Thus, we see it is sufficient to know the expansion coefficients of $f(x)$ in the limits $x \rightarrow 0$ and ∞ in order to calculate the asymptotic properties of $F(z)$ and the interesting functions r_{2n} .

Unlike the limits $x \rightarrow 0$ and ∞ , near the coexistence curve, $x \rightarrow -1$, there are no rigorous mathematical arguments as to the analyticity properties of $f(x)$, although there do exist conjectures. In [12], based on an ε -expansion analysis, it was conjectured that $(1+x)$ has a double expansion in powers of y and $y^{(d-2)/2}$ of the form

$$1+x = c_1 y + c_2 y^{1-\epsilon/2} + d_1 y^2 + d_2 y^{2-\epsilon/2} + \dots \quad (8)$$

In three dimensions it predicts an expansion of $(1+x)$ in powers of $y^{1/2}$. Studies of the non-linear σ model lead one to expect a leading behavior of the form

$$f(x) \sim c_f (1+x)^{2/(d-2)} \quad (9)$$

though, as mentioned, the nature of the corrections to this behavior is not well understood, although (8) is one conjecture. In the $1/N$ expansion there is some evidence [2] for logarithmic corrections of the form $\ln(1+x)$ in three dimensions.

3. A Renormalization Group Representation of the Equation of State

In this section we briefly outline the derivation of the equation of state for a theory described by the standard Landau-Ginzburg-Wilson Hamiltonian with $O(N)$ symmetry

$$\mathcal{H}[\varphi] = \int d^d x \left(\frac{1}{2} \nabla \varphi^a \nabla \varphi^a + \frac{1}{2} r(x) \varphi^a \varphi^a + \frac{\lambda_B}{4!} (\varphi^a \varphi^a)^2 \right) \quad (10)$$

with $r = r_c + t_B$, where r_c is the value of r at the critical temperature T_c and $t_B = \Lambda^2 \frac{(T-T_c)}{T_c}$, Λ being the microscopic scale. Due to the Ward identities of the model, it is sufficient to know only the transversal correlation functions $\Gamma_t^{(N,M)}$, as all the other vertex functions can be reconstructed from these. For instance, the equation of state itself is given by $H = \Gamma_t^{(2)} \varphi$.

Due to the existence of large fluctuations in the critical regime a renormalization of the microscopic bare parameters of the form

$$t(m_t, \kappa) = Z_{\varphi^2}^{-1}(\kappa) t_B(m_t) \quad (11)$$

$$\lambda(\kappa) = Z_\lambda(\kappa) \lambda_B \quad (12)$$

$$\varphi(\kappa) = Z_\varphi^{-1/2}(\kappa) \varphi_B \quad (13)$$

must be imposed, where κ is an arbitrary renormalization scale and m_t is the inverse transverse correlation length. The renormalized parameters satisfy the differential equations

$$\kappa \frac{dt(\kappa)}{d\kappa} = \gamma_{\varphi^2}(\kappa) t(\kappa) \quad \text{where} \quad \gamma_{\varphi^2}(\kappa) = -\kappa \frac{d}{d\kappa} \ln Z_{\varphi^2}|_c \quad (14)$$

$$\kappa \frac{d\lambda(\kappa)}{d\kappa} = \gamma_\lambda(\kappa) \lambda(\kappa) \quad \text{where} \quad \gamma_\lambda(\kappa) = \kappa \frac{d}{d\kappa} \ln Z_\lambda|_c \quad (15)$$

$$\kappa \frac{d\varphi(\kappa)}{d\kappa} = -\frac{1}{2} \gamma_\varphi(\kappa) \varphi(\kappa) \quad \text{where} \quad \gamma_\varphi(\kappa) = \kappa \frac{d}{d\kappa} \ln Z_\varphi|_c \quad (16)$$

where the right hand side are the Wilson functions associated with this coordinate transformation and the derivative is taken along an appropriately chosen curve in the phase diagram, which we here denote by c .

Integration of the RG equation for any multiplicatively renormalizable $\Gamma_t^{(N,M)}$ yields

$$\Gamma_t^{(N,M)}(t, \lambda, \varphi) = e^{\int_\kappa^{m_t} (\frac{N}{2} \gamma_\varphi - M \gamma_{\varphi^2}) \frac{dx}{x}} \Gamma_t^{(N,M)}(t(\kappa), \lambda(\kappa), \varphi(\kappa)) \quad (17)$$

The renormalization constants Z_φ , Z_{φ^2} and Z_λ are fixed by imposing the explicitly magnetization dependent normalization conditions on the transverse correlation functions

$$\partial_{p^2} \Gamma_t^{(2)}(p, t(\kappa, \kappa), \lambda(\kappa), \varphi(\kappa), \kappa) \Big|_{p^2=0} = 1 \quad (18)$$

$$\Gamma_t^{(2,1)}(0, t(\kappa, \kappa), \lambda(\kappa), \varphi(\kappa), \kappa) = 1 \quad (19)$$

$$\Gamma_t^{(4)}(0, t(\kappa, \kappa), \lambda(\kappa), \varphi(\kappa), \kappa) = \lambda. \quad (20)$$

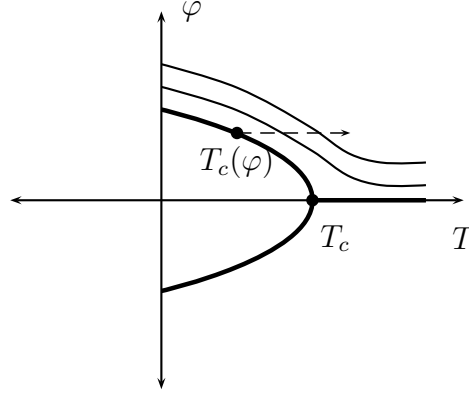


Figure 1. Integrating along curves of constant φ (or t) in the phase diagram starting from the coexistence curve.

while the condition

$$\kappa^2 = \Gamma_t^{(2)}(0, t(\kappa, \kappa), \lambda(\kappa), \varphi(\kappa), \kappa), \quad (21)$$

serves as a gauge fixing condition that relates the sliding renormalization scale κ to the physical temperature t and magnetization φ . Physically, κ is a fiducial value of the non-linear scaling field m_t .

Besides m_t , the other non-linear scaling field we use to parametrise our results is

$$m_\varphi^2 = \frac{1}{3} \frac{\Gamma_t^{(4)} \varphi^2}{\partial_{p^2} \Gamma_t^{(2)}|_{p^2=0}} \quad (22)$$

which is a RG invariant. It represents the anisotropy in the masses of the longitudinal and transverse modes and is related to the stiffness constant $\rho_s = \varphi^2 \partial_{p^2} \Gamma_t^{(2)}|_{p^2=0}$ via $m_\varphi^2 = \frac{1}{3} \lambda \rho_s$. With this renormalization prescription one may determine the equation of state in terms of the non-linear scaling fields m_t and m_φ , as the transverse and longitudinal propagators that appear in all perturbative diagrams can be parametrised in terms of them.

We now give a short review of the general methodology used to derive the equation of state as developed in reference [9, 10]. The equation can be found by integrating $d\Gamma_t^{(2)}(t, \varphi)$ along a curve of constant φ (see figure (1)), where in $dt = d\Gamma_t^{(2)}/\Gamma_t^{(2,1)}$, the right hand side is written in the coordinate system (m_t, m_φ) . We note the following points [10]:

1) We take $m_t = \kappa$ as the flow variable of the RG and hold φ constant. The non-linear scaling variable, m_φ , is then κ dependent through its definition (22).

2) We integrate dt along a curve of constant φ and fix the boundary condition on the coexistence curve, where $m_t = 0$, whence we may write $(T - T_c(\varphi)) = t + \Delta$, where $\Delta = (T_c - T_c(\varphi))$ is the temperature shift that measures the distance between the critical point and the point on the coexistence curve, $T_c(\varphi)$, see figure (1). One finds

$$A_1(1+x) = \mathcal{F}(z) \quad (23)$$

where $z = m_t/m_\varphi$ and the scaling variable $x = B^{1/\beta}t/\varphi^{1/\beta}$, B being a amplitude. The universal scaling function $\mathcal{F}(z)$ is given by

$$\mathcal{F}(z) = \int_0^z \frac{2(2-\gamma_\varphi)}{2-\gamma_\lambda+\gamma_\varphi} e^{-\int_\infty^x 2\left(\frac{\Delta\gamma_\varphi 2^{-\frac{\Delta\gamma_\lambda}{2\beta} + \frac{\Delta\gamma_\varphi}{2\beta}}\right) \frac{dy}{y}} x^{\frac{1}{\beta}} \frac{dx}{x} \quad (24)$$

where we have defined $\Delta\gamma_i = (\gamma_i - \gamma_i^{\text{WF}})$, WF denoting the Wilson-Fisher fixed point.

3) The equation of state is obtained using $H = \Gamma_t^{(2)}\varphi$ and the RG equation for $\Gamma_t^{(2)}$. One obtains $H/\varphi^\delta = A_3^{-1}\mathcal{G}(z)$, with the universal function

$$\mathcal{G}(z) = e^{\frac{\gamma}{\beta} \int_\infty^z \frac{\Delta\gamma_\lambda - \Delta\gamma_\varphi}{2-\gamma_\lambda+\gamma_\varphi} \frac{dy}{y}} e^{-\int_\infty^z \frac{2\Delta\gamma_\varphi}{2-\gamma_\lambda+\gamma_\varphi} \frac{dy}{y}} \quad (25)$$

Defining the scaling variable $y = A_3^{-1}(H/\varphi^\delta)$ we then obtain the universal equation of state in the form (2) with

$$f(x) = \frac{1}{A_3} \mathcal{G}(\mathcal{F}^{-1}(A_1(1+x))). \quad (26)$$

4) The amplitudes A_1 and A_3 are both universal and can be determined as $A_1 = \mathcal{F}(z_c)$ and $A_3 = \mathcal{G}(z_c)$, where z_c is the value of z that corresponds to the critical isotherm.

In order to determine the expansion coefficients f_n^0 and f_n^∞ , as introduced in section 1, one requires the Taylor expansion of $f(x)$ around $x = 0$ and $x = \infty$. In terms of our parametric representation, $d^n f(x)/dx^n$ can be expressed using $d/dx = (dz/dx)d/dz$, where $dz/dx = A_1/(d\mathcal{F}(z)/dz)$ hence,

$$\frac{d^n f(x)}{dx^n} = \frac{A_1}{A_3} \left(\left(\frac{d\mathcal{F}(z)}{dz} \right)^{-1} \frac{d}{dz} \right)^n \mathcal{G}(z) \quad (27)$$

which needs to be evaluated at the points of interest $z = z_c$ ($x = 0$), $z = \infty$ ($x = \infty$) and $z = 0$ ($x = -1$).

4. Results

4.1. The one loop Wilson functions

To one loop order the bare vertex functions are given by

$$\Gamma_t^{(2)} = p^2 + r + \frac{\lambda}{6}\varphi^2 + \frac{\lambda}{2} \text{loop} + \frac{\lambda}{6}(N-1) \text{loop} \quad (28)$$

$$\Gamma_t^{(2,1)} = 1 - \frac{\lambda}{2} \left(\text{loop} + \frac{N-1}{3} \text{loop} \right) \quad (29)$$

$$\Gamma_t^{(4)} = \lambda - \frac{3}{2}\lambda^2 \left(\text{loop} + \frac{N-1}{9} \text{loop} \right) \quad (30)$$

where the dashed line in the loops denotes the transversal propagator and the continuous line the longitudinal one, i.e., $\text{loop} = \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2+t+\frac{\lambda}{2}\varphi^2}$ and $\text{loop} = \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2+t+\frac{\lambda}{6}\varphi^2}$.

Using (18), (19) and (20), the renormalization constants can be determined and hence the Wilson functions

$$\gamma_\lambda = -\frac{3}{2}\lambda\kappa\frac{d}{d\kappa}\left(\text{loop} + \frac{N-1}{9}\text{loop}\right) \quad (31)$$

$$\gamma_\varphi = 0 \quad (32)$$

$$\gamma_{\varphi^2} = -\frac{\lambda}{2}\kappa\frac{d}{d\kappa}\left(\text{loop} + \frac{N-1}{3}\text{loop}\right) \quad (33)$$

The running dimensionless coupling λ satisfies

$$z\frac{d\lambda(z)}{dz} = -\varepsilon\lambda + c_d\lambda^2(z)\left(\left(1 + \frac{1}{z^2}\right)^{\frac{d-6}{2}} + \frac{(N-1)}{9}\right) \quad (34)$$

Taking the initial condition $\lambda(z_0) = \lambda$, in the limit $z_0 \rightarrow \infty$, $\lambda \rightarrow \infty$ one arrives at the universal separatrix solution \S

$$\lambda(z) = \left(c_d\left(\left(1 + \frac{1}{z^2}\right)^{\frac{d-6}{2}} + \frac{(N-1)}{9}\right)\right)^{-1} \quad (35)$$

On the separatrix

$$\gamma_\lambda = (4-d)\left(\frac{\left(1 + \frac{1}{z^2}\right)^{\frac{d-6}{2}} + \frac{(N-1)}{9}}{\left(1 + \frac{1}{z^2}\right)^{\frac{d-4}{2}} + \frac{(N-1)}{9}}\right) \quad (36)$$

$$\gamma_{\varphi^2} = (4-d)\left(\frac{\left(1 + \frac{1}{z^2}\right)^{\frac{d-6}{2}} + \frac{(N-1)}{3}}{3\left(1 + \frac{1}{z^2}\right)^{\frac{d-4}{2}} + \frac{(N-1)}{3}}\right) \quad (37)$$

$$\gamma_\varphi = 0 \quad (38)$$

With the Wilson functions in hand we can calculate the scaling functions. In three dimensions the scaling function $\mathcal{G}(z)$ is analytic, while the function $\mathcal{F}(z)$ can be written as an integral. Explicitly,

$$\mathcal{G}(z) = z^4 \prod_{i=1}^3 \left(\frac{\sqrt{1+1/z^2} - r_i}{1-r_i}\right)^{\frac{12}{12+(N-1)r_i}} \quad (39)$$

$$\begin{aligned} \mathcal{F}(z) &= \int_0^z \frac{4}{2-\gamma_\lambda} \left(\frac{1+\sqrt{1+1/x^2}}{2}\right)^{\frac{12(N-1)}{(N-10)(N+8)}} \\ &\times \prod_{i=1}^3 \left(\frac{\sqrt{1+1/x^2} - r_i}{1-r_i}\right)^{n_i} x^{\frac{20+N}{8+N}} dx \end{aligned} \quad (40)$$

where r_i satisfies the cubic equation $(N-1)r_i^3 + 18r_i^2 - 9 = 0$ and $n_i = \frac{9(1-N)+(161+2N-N^2)r_i+9(N-1)r_i^2}{12r_i+(N-1)r_i^2} \frac{4}{(10-N)(N+8)}$.

In the limit $z \rightarrow \infty$, which corresponds to approaching the critical point along the critical isochore, the Wilson-Fisher fixed point is approached and $\gamma_i \rightarrow \gamma_i^{\text{WF}}$ with, at one loop, $\gamma_\lambda = (4-d)$ and $\gamma_{\varphi^2} = (4-d)(N+2)/(N+8)$. On the contrary, in the limit $z \rightarrow 0$, which corresponds to approaching the critical point along the coexistence

\S This solution may also be reached by choosing the initial coupling to be on the separatrix solution at $z = z_0$.

curve $x \rightarrow -1$, the strong-coupling fixed point is approached and $\gamma_i \rightarrow \gamma_i^{\text{SC}}$. There, the Goldstone bosons dominate and $\gamma_\lambda = \gamma_{\varphi^2} = (4 - d)$. Finally, the critical isotherm, $x = 0$, is reached in the limit $z \rightarrow z_c$.

With (40) and (39) we can plot the scaling function, $f(x)$, for the full universal equation of state, as seen in Figures 2, 3 and 4 for the cases $N = 2, 3$ and 4 respectively.

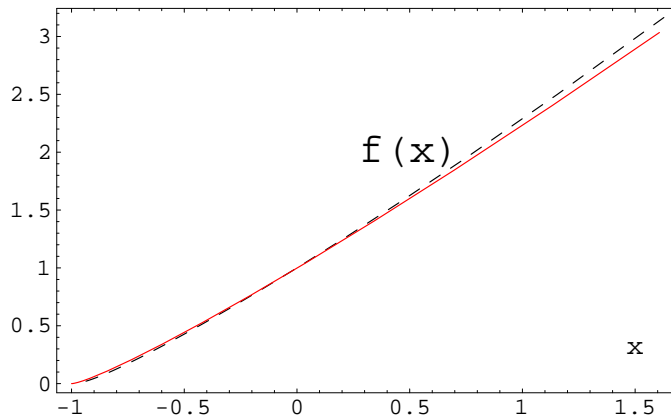


Figure 2. The scaling function $f(x)$ for the three-dimensional XY model. The dashed line was taken from reference [15].

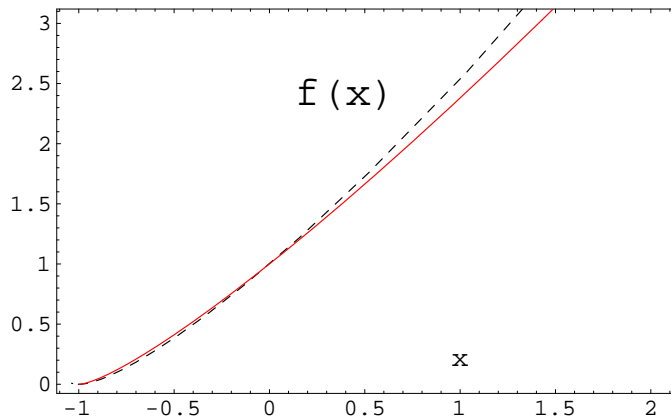


Figure 3. The scaling function $f(x)$ for the three-dimensional Heisenberg model. The dashed line was taken from reference [13].

4.2. The limit $z \rightarrow \infty$

In the limit $z \rightarrow \infty$, the Wilson functions can be expanded as power series in z^{-2}

$$\gamma_i(z) = \gamma_i^{\text{WF}} + \sum_{n=1}^{\infty} a_i(n) z^{-2n} \quad (41)$$

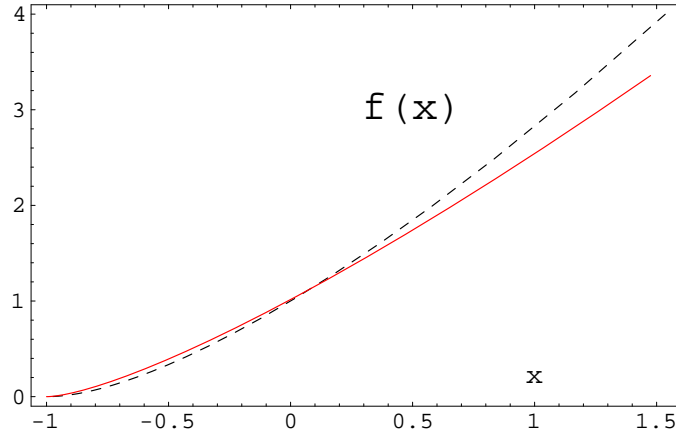


Figure 4. The scaling function $f(x)$ for the three-dimensional $O(4)$ model. The dashed line was taken from reference [14].

Hence, the universal scaling functions \mathcal{F} and \mathcal{G} can also be written as power series in z^{-2} . This is true in a diagrammatic expansion to all orders, not just at one loop. In the limit $z \rightarrow 0$, $\gamma_i \rightarrow \gamma_i^{SC}$ but the nature of the corrections is not obvious. At the one-loop level, from (36), one can see that the leading corrections to the strong coupling fixed point values will be $z^{(4-d)/2}$.

To determine the constant A_1 we take the $z \rightarrow \infty$ limit of (40), identify the divergent part with $A_1 x$ and the constant remainder with $A_1 \parallel$. For instance, for $N = 1, 2, 3, 4$ we can write the divergent component of the function \mathcal{F} :

$$I_1^\infty = 4 - 4 \left(1 + \frac{1}{3} \sum_{i=1}^2 \frac{1}{r_i - 1} \right) \frac{1}{z^2} \quad (42)$$

$$I_2^\infty = 4 + \left(-\frac{15}{4} + \frac{1}{10} \sum_{i=1}^3 \frac{9 - 161r_i - 9r_i^2}{r_i(r_i - 1)(r_i + 12)} \right) \frac{1}{z^2} \quad (43)$$

$$I_3^\infty = 4 + 2 \left(-\frac{138}{77} + \frac{4}{77} \sum_{i=1}^3 \frac{18 - 158r_i - 18r_i^2}{r_i(r_i - 1)(2r_i + 12)} \right) \frac{1}{z^2} \quad (44)$$

$$I_4^\infty = 4 + \left(-\frac{7}{2} + \frac{4}{77} \sum_{i=1}^3 \frac{27 - 153r_i - 27r_i^2}{r_i(r_i - 1)(3r_i + 12)} \right) \frac{1}{z^2} \quad (45)$$

The universal amplitudes A_3 can then be obtained using the corresponding equation in 4) in section 3. With these two constants, along with the two scaling functions \mathcal{F} and \mathcal{G} the scaling function of the universal equation of state may be obtained, and, correspondingly, any expansion coefficient, f_n^0 , f_n^∞ or r_n .

In Table 1 we see some results for a variety of expansion coefficients associated with the asymptotic regimes $x \rightarrow 0$ and $x \rightarrow \infty$. We also compare with various other known results. The agreement is as good as one would expect from a one-loop

\parallel If $2(4-d)/3(d-2)$ is a positive integer, n , then this remainder is zero and A_1 cannot be determined by looking at the asymptotic limit $z \rightarrow \infty$. This is a pure artefact of the one-loop approximation, where $\eta = 0$, and has no physical meaning.

approximation. However, of note here is not the precision of the estimates but that they have been obtained from an ab initio calculation where no phenomenological input was necessary. Thus, any expansion coefficient is obtained by an expansion in the appropriate asymptotic limit of the universal functions $\mathcal{F}(z)$ and $\mathcal{G}(z)$ which in their turn depend only on the Wilson functions.

Table 1. Values of some one-loop amplitudes for the 3-dimensional $O(N)$ model. We have written some known results as reported in [2] (see table (21)) for $N=2$; [13] for $N=3$ and [14] for $N=4$.

N	2	3	4
z_c	0.694	0.677	0.656
A_1	0.959	0.934	0.897
A_3	1.022	0.840	0.698
R_+^4	6.35 (7.6(2))	6.06 (7.8(3))	8.86 (7.6(4))
F_0^∞	0.035 (0.0304(3))	0.0348 (0.0266(5))	0.015 (0.0240(5))
R_χ	1.42 (1.41(6))	1.280 (1.31(7))	1.18 (1.12(11))
f_1^∞	0.746	0.789	1.246
f_2^∞	0.641	0.695	1.171
f_3^∞	0.103	0.097	0.151
f_4^∞	-0.005	-0.006	-0.014
c_f	15.6 (15(10))	4.92 (5(3))	2.66 (2.8(1.4))
f_1^0	1.16	1.26 (1.34(5))	1.35 (1.5(8))
f_2^0	0.086	0.143 (0.020(2))	0.2031 (0.33(5))
f_3^0	-0.022	-0.040(-0.10(1))	-0.056 (-0.08(2))
r_6	2.704 (1.951(14))	2.90 (2.1(6))	2.11 (1.79(2))
r_8	2.873 (1.36(9))	2.80 (0.6(2))	1.29 (0.2(4))
r_{10}	-1.58 (-7(5))	- 2.05 (-6(3))	-0.97 (-5(6))

4.3. The limit $z \rightarrow 0$

In the limit $z \rightarrow 0$, corresponding to the coexistence curve, the parametric functions $\mathcal{F}(z)$ and $\mathcal{G}(z)$ can be written as power series in the scaling variable z as $\mathcal{F}(z) = A_1(1+x) = \frac{a}{2}z^2 + \frac{b}{3}z^3 + \frac{c}{4}z^4 + \dots$ and $\mathcal{G}(z) = A_3 y = dz^4 + fz^5 + gz^6 + \dots$. We can invert the equation for $\mathcal{F}(z)$ to obtain the non-linear scaling field z in terms of $(1+x)$ and then, once we substitute into the function $\mathcal{G}(z)$, we obtain the equation of state near the coexistence curve in the form

$$y = c_f(1+x)^2 + c_1(1+x)^{5/2} + c_2(1+x) + \dots \quad (46)$$

or alternatively

$$(1+x) = \tilde{c}_1 y + \tilde{c}_2 y^{1/2} + \tilde{c}_3 y^{3/4} + \dots \quad (47)$$

In Table 2 we give the first expansion coefficients and compare them to the results of [12].

Table 2. Values of some one-loop amplitudes for the 3-dimensional $O(N)$ model in the limit $z \rightarrow 0$. The values in parenthesis for \tilde{c}_1 y \tilde{c}_2 were taken from [12], where an ε expansion was used.

N	2	3	4
c_1	-111.6	-17.8	-6.4
c_2	399.8	32.0	7.6
\tilde{c}_1	0.82(0.9)	0.66(0.82)	0.54(0.75)
\tilde{c}_2	0.25(0.1)	0.45(0.18)	0.61(0.25)
\tilde{c}_3	0.46	0.55	0.57

5. Conclusions

Although there exist established methods for calculating the universal equation of state, $f(x)$, while preserving Griffiths analyticity, they are based on a *phenomenological* scaling ansatz that has no underlying microscopic basis. On the other hand it has not been possible to preserve all asymptotic properties of $f(x)$ starting from a Landau-Ginzburg-Wilson Hamiltonian using the RG, due to the fact that the latter involved an expansion around the Wilson-Fisher fixed point. In order to calculate $f(x)$ using RG methods it is necessary to be able to capture φ dependent crossover between the two different fixed points.

In this paper we have developed a RG that captures the crossover between the Wilson-Fisher and strong-coupling fixed points and therefore captures all elements of Griffiths analyticity, accessing all three different scaling regimes. Thus, we have achieved an ab initio calculation of the universal equation of state from an underlying microscopic model with no phenomenological input. The calculation depends only on the three Wilson functions γ_φ , γ_{φ^2} and γ_λ . We used the method to calculate $f(x)$ to one loop for arbitrary N and made a comparison to known results from other methods. The novelty of the present approach is not in the precision of any result for an expansion coefficient, as here we have only worked to one loop, but rather in exhibiting a method that order by order in perturbation theory preserves all necessary analyticity properties of $f(x)$. However, the method can be extended to higher loop order. The chief difficulty in doing so is that the higher order Feynman diagrams must be evaluated numerically (remembering that they are crossover functions not constants) as there are no closed form expressions for them. As the amplitude A_1 involves cancelling two divergent expressions this also is trickier when done numerically. We will return to the question of two-loop calculations in a future publication.

Acknowledgments

CRS and JAS would like to thank DIAS Dublin for hospitality while some of this work was carried out. JAS acknowledges CONACyT Grant 53187-F and PROMEP for

financial support.

References

- [1] Zinn-Justin J 2001 *Phys. Rept.* **344** 159
- [2] Pelissetto A and Vicari E 2002 *Phys. Rept.* **368** 549
- [3] Brézin E Wallace D J and Wilson K G 1973 *Phys. Rev. B* **7** 232
- [4] Griffiths R B 1967 *Phys. Rev.* **158** 176
- [5] Lawrie I D 1981 *J. Phys. A: Math. Gen.* **14** 2489
- [6] Schofield P 1969 *Phys. Rev. Lett.* **22** 606
- [7] Engels J Holtmann S Mendes T and Schulze T 2000 *Phys. Lett. B* **492** 219
- [8] O'Connor D and Stephens C R 1994 *Int. J. Mod. Phys. A* **9** 2805 [Erratum-ibid.1994 A **9** 5851]
O'Connor D and Stephens C R 1991 *Nucl. Phys. B* **360** 297
- [9] O'Connor D and Stephens C R 2002, *Phys. Rept.* **363** 425
- [10] O'Connor D Santiago J A and Stephens C R 2007 *J. Phys. A: Math. Theor.* **40** 901
- [11] Widom B 1965 *J. Chem. Phys.* **43** 3898
- [12] Wallace D J and Zia R K P 1975 *Phys. Rev. B* **12** 5340
- [13] Campostrini M Hasenbusch M Pelissetto A Rossi P and Vicari E 2002 *Phys. Rev. B* **65** 144520
- [14] Toldin F Pelissetto A and Vicari E 2003 *JHEP* **0307** 029
- [15] Campostrini M Pelissetto A Rossi P and Vicari E 2000 *Phys. Rev. B* **62** 5843