From Vicious Walkers to TASEP.

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Abstract

We propose a model of semi-vicious walkers, which interpolates between the totally asymmetric simple exclusion process and the vicious walkers model, having the two as limiting cases. For this model we calculate the asymptotics of the survival probability for m particles and obtain a scaling function, which describes the transition from one limiting case to another. Then, we use a fluctuation-dissipation relation allowing us to reinterpret the result as the particle current generating function in the totally asymmetric simple exclusion process. Thus we obtain the particle current distribution asymptotically in the large time limit as the number of particles is fixed. The results apply to the large deviation scale as well as to the diffusive scale. In the latter we obtain a new universal distribution, which has a skew non-Gaussian form. For m particles its asymptotic behavior is shown to be $e^{-\frac{y^2}{2m^2}}$ as $y \to -\infty$ and $e^{-\frac{y^2}{2m}} y^{-\frac{m(m-1)}{2}}$ as $y \to \infty$.

1 Introduction

Exact solutions of 1-dimensional (1D) many particle stochastic models [41] have given much insight into the physics of non-equilibrium systems in one dimension [42]. They serve as a testing ground for the macroscopic theories, being able to verify their predictions [3]. Examples are the description of different kinds of non-equilibrium phase transitions[10], calculation of the large deviation functions for the density profile and total particle current[5], verification of the fluctuation dissipation relations [27] and testing of the range of their validity [20].

The range of models is very broad. In the context of the present article we mention two of them. The first one is the lock step model of vicious walkers (VW) that has been introduced in the physical literature by M. Fisher [14] to describe the wetting and melting phenomena. This is a random process defined as many *non-interacting* particles performing random walks on a 1D lattice, whose space-time trajectories are forbidden to cross each other. The

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term non-interacting means that the probability of a particular realization of the process, which meets the latter constraint, is given by the product of the probabilities of the random walks performed by each individual walker. The other realizations, where crossings occur, are assigned zero statistical weight. Such an elimination of a fraction of possible outcomes at every time step violates the probability conservation. A measure of the probability dissipation is the sum of the probabilities of all possible particle configurations at a given time, referred to as the survival probability. Its leading asymptotics for m particles has been shown by M. Fisher to decay with time t as a power law: $t^{-\frac{m(m-1)}{4}}$.

Another model, the totally asymmetric simple exclusion process (TASEP) [29], has been widely discussed in connection to the Kardar-Parisi-Zhang universality class [24]. In contrast to the VW model, this is a model of *interacting* random walks. The interaction prevents particles from jumping to occupied sites. Therefore, similarly to vicious walkers, the statistical ensemble includes only those events in which the space-time trajectories of particles do not cross. The difference is that there is an interaction that changes the statistical weights of particle trajectories when they pass via neighboring sites so that the total probability is conserved. In this case the quantity corresponding to the survival probability is just a probability normalization constant.

Thus, the probability lost after imposing the global non-crossing constraint on the dynamics of non-interacting particles in VW is regained in the TASEP by adding the lacking probability locally at certain steps. In the present paper we consider the two models as limiting cases of a more general interaction, where the added probability is a varying parameter of the model that controls the probability dissipation, such that the probability conservation is restored when it is tuned to the TASEP value. In this connection a natural question arises: what happens with Fisher's asymptotics for the survival probability under such a generalization and, in particular, how does it cross over to the TASEP normalization constant. This is the first question we address in this paper.

Specifically, we propose a semi-vicious walkers (SVW) model, which interpolates between VW and TASEP. It is a model of interacting particles with partial repulsion or attraction, where trajectory crossings are forbidden. The term partial repulsion (attraction) means that the probability for the particle to jump to an occupied site is not equal to zero like in TASEP but can be less (greater) than that of a free particle. At the same time, the non-crossing constraint leads to lack of probability conservation in the same way as in the VW model. The strength of the interaction, which also characterizes the probability dissipation, is a parameter of the model, which has the TASEP and VW as limiting cases at the endpoints of its range.

In this article we obtain the large-time asymptotics of the survival probability. Its limiting case corresponding to VW is given by the above mentioned result of Fisher, which yields the leading power law asymptotics. Later it was reproduced with more mathematical rigor together with the constant prefactor that was obtained for the particular initial configurations, where the particles are separated by equal spaces [25]. In the case of a general initial configuration of walkers this prefactor depends on the initial positions. This case has been studied in [38].

Our results can be roughly divided into two parts. For generic values of the interaction strength, away from the point corresponding to the TASEP, the probability dissipation is finite. It is intuitively clear that the main asymptotics must be similar to the VW one. Indeed, we obtain the Fisher's power law with a constant prefactor that depends on the initial positions of the particles and on the interaction strength. It is shown to diverge in the TASEP limit. The second and probably the most interesting case is the transition region, which interpolates between the VW and TASEP behavior. To probe into this region, we consider a scaling limit of the survival probability, where the large time limit and the TASEP limit of the interaction strength are combined. In this way we obtain a scaling function of a single parameter that controls the transition from VW to TASEP.

The second problem we address is the distribution of the integral particle current in TASEP. A first example of such an exact distribution has been obtained by Derrida and Lebowitz [8], who found the large deviation function for the particle current in the TASEP confined to a ring. A specific property of the finite system is that there is a finite relaxation time, after which the system settles into a non-equilibrium stationary state, independent of initial conditions [18]. Then, the tool used to study integral current fluctuations is, roughly speaking, an analysis of the relaxation of the system subject to a perturbation into the stationary state. Technically it is an analysis of the largest eigenvalue of the perturbed Markov matrix governing the process.

In genuinely infinite systems the situation is more peculiar. In this case there is no characteristic relaxation time scale. When starting away from the stationary state, the latter is never approached. In this case one needs to consider actual time evolution of quantities of interest, which depend on initial conditions. A major breakthrough in this direction has been achieved by Johansson, [21]. He considered the TASEP evolution of an infinite cluster of particles, which initially occupies all sites of the lattice to the left of a fixed site, and calculated the distribution of the number of steps made by an arbitrary particle in this cluster. Johansson's solution has initiated a burst of activity in the field, which exploited deep connections of the TASEP to the theory of random matrix ensembles and the determinantal point processes. Results have been obtained for different initial conditions and extended to many particle joint distributions [34],[37],[39],[13],[4],. Remarkably, in the scaling limit these results provide parameter free universal distributions [35] of the fluctuations measured in the KPZ characteristic scale, which is of order of $t^{1/3}$ as time t grows to infinity [26]. This is in contrast to the diffusive scale $t^{1/2}$, which, according to the Central Limit Theorem (CLT), characterizes the fluctuations of the distance travelled by a free particle [12]. The large deviation limit of the single particle current distribution has been studied in [19] in connection with the fluctuation dissipation relations.

Despite the great success in finding the distributions of single particle currents and their correlation functions, very few results on the integral particle current, i.e. on the distance travelled by all particles, are available for driven diffusive systems. In fact the only known exact result is the above mentioned large deviation function for the integral particle current for the TASEP in a ring [8] and its generalization for the partially asymmetric case [23], [28]. No results beyond the large deviation scale, neither a generalization for an infinite system has been proposed. On the other hand, extensive quantities like the integral current, are important ingredients of the thermodynamics of the models. A knowledge of the character of their fluctuations could be of help for extension of the thermodynamical formalism to irreversible systems. The present paper makes a step in this direction. The problem we solve here is as follows. We study the large time asymptotics of the distribution of the total number of jumps made by a finite number of TASEP particles in an infinite lattice, given an arbitrary initial configuration. The idea that allows us to consider this problem in line with the previous one is the existence of a kind of fluctuation-dissipation relation that unifies the dissipation of probability in SVW and the statistics of fluctuations of the integrated particle current in TASEP. Specifically, an auxiliary parameter, which violates the probability conservation, can be introduced into the evolution operator in TASEP to account for the total number of steps made by particles, see e.g. [8]. This parameter plays a role similar to the one played by the interaction in SVW, the two problems being equivalent after a certain change of variables. Using this fact, we interpret the result obtained for the survival probability in SVW as a generating function of the particle current in TASEP. The latter, in its turn, can be used to reconstruct the form of the current distribution.

Like those for SVW, the results obtained for the TASEP particle current consist of two parts. The generic values of the interaction strength correspond to the distribution of the particle current at the large deviation scale, i.e. describes the deviations of order of time t. It turns out that it has a skew distribution with asymmetric negative and positive tails. These tails are connected by a middle part corresponding to the transition region. The latter yields the current distribution at the diffusive scale, $t^{1/2}$, which is shown to have a skew non-Gaussian form, depending only on the total number of particles, and we suggest to be universal for particles performing a driven diffusion.

One technical remark has to be made about the connection of our solution to the theory of random matrix ensembles. It is this connection which enabled the above mentioned progress in calculating the single particle current distributions and their many particle generalizations. In our solution this connection has also been exploited. Namely, the survival probability in the SVW model at generic values of the interaction strength and exactly at the TASEP point can be calculated in terms the Mehta integrals $I_{m,k}$ with $k = 1/2$ and $k = 1$, which appear as normalization factors in the orthogonal and unitary Gaussian ensembles of random matrices respectively [33]. Note, however, that the scaling function obtained in the transition region for the system of m particles can be reduced to neither of these integrals except of at three limiting points, where it becomes $I_{m,1/2}$,$I_{m,1}$ and $I_{m-1,1}$ respectively. Thus, we obtain a generalized object, which interpolates between these three Mehta integrals, and, therefore, in a sense unifies three different matrix ensembles. To our knowledge no such generalization has appeared in the theory before.

The article is organized as follows. In Section 2 we formulate the SVW model, state the results obtained and discuss their interpretation in terms of the probability distribution of the particle current in TASEP. Sections 3-5 are a technical part, where we prove the results outlined in Section 2. In Section 3 we solve the master equation for the SVW model. In Section 4 we obtain the asymptotic formulas for the transition probabilities. In Section 5 we prove the limiting properties of the function characterizing the SVW to TASEP transition. Section 6 has a summary and conclusions.

2 Model and results

2.1 Semi vicious walkers model

Consider m particles on a 1D infinite lattice. A configuration X of the system is specified by an m-tuple of strictly increasing integers

$$X = \{x_1 < x_2 < \cdots < x_m\}\,, \tag{1}$$

where x_i is the coordinate of i-th particle. The strictly increasing order reflects the exclusion condition, i.e. two particles cannot occupy the same site. We say that the relation $X \leq Y$ holds for particle configurations if

$$x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_m \leq y_m. \tag{2}$$

The SVW model is a random process, which is defined on a set of sequences of configurations $X^0, X^1, \cdots, X^t$, such that

$$X^0 \leq X^1 \cdots \leq X^t. \tag{3}$$

We refer to such a sequence as a trajectory of the system traveled up to time t. Every such trajectory is realized with probability

$$P(X^0, \ldots, X^t) = T(X^t, X^{t-1}) \cdots T(X^2, X^1) T(X^1, X^0) P_0(X^0). \tag{4}$$

$P_0(X)$ is the initial probability of the configuration X and the transition probability $T(X, Y)$, from the configuration Y to X, is defined as follows

$$T(X, Y) = \vartheta(x_m - y_m) \prod_{k=1}^{m-1} \theta\left(x_i - y_i, x_{i+1} - y_i\right), \tag{5}$$

where

$$\vartheta(k) = (1-p)\,\delta_{k,0} + p\delta_{k,1}, \tag{6}$$
$$\theta\left(k, l\right) = \left(1 - p\left(1 - \kappa\delta_{l,1}\right)\right)\delta_{k,0} + p\delta_{k,1}, \tag{7}$$

and

$$0 < p < 1, \tag{8}$$
$$1 - 1/p \leq \kappa \leq 1. \tag{9}$$

This means that at each discrete time step a particle can jump forward with probability p or stay put with probability $1-p$, provided that the next site is empty. If the next site is occupied, the probability for a particle to stay put is $(1 - p(1-\kappa))$. The probability deficit $p(1-\kappa)$, corresponds to the process when the particle jumps to the adjoining occupied site, which is forbidden. This excluded process results in probability dissipation in this model. The form of the transition probabilities corresponds to the backward sequential update, i.e. the particles are updated starting from the m-th particles one by one in backward direction. In particular limiting cases the model reduces to

1. $\kappa = 0$ - VW, a particle jumps forward with probability p or stays with probability $(1-p)$, irrespective of whether the next site is occupied or not. But then those realizations of the process where two particles come to the same site must be removed from the statistical ensemble.

2. $\kappa = 1$ - TASEP, a particle jumps forward with probability p or stays with probability $(1-p)$ provided the adjoining site is empty. When the next site is occupied the particle stays where it is with probability 1.

The TASEP with the backward sequential update was studied in [2] and [37], where it was referred to as a fragmentation model. In the case $\kappa = (1-1/p)$, the probability for a particle to stay where it is when the next site occupied, is zero. Therefore, the trajectories of particles passing via neighboring sites have zero weight, i.e. they are removed from the ensemble as well as those which meet at the same site. Therefore this situation resembles the vicious walks of dimers. The range of κ given in (9) is due to the requirement for $(1-p(1-\kappa))$ to be a probability. Positive values of κ correspond to repulsive interaction, while negative values correspond to an attractive interaction. The domain $\kappa > 1$ is also of interest in connection with the current fluctuation in TASEP, though it does not have a probabilistic meaning in the context of SVW.

2.2 The results about the SVW model

Let us consider the quantity

$$\mathcal{P}_t\left(X^0\right) = \sum_{X^0 \le X^1 \cdots \le X^t} P(X^0, \ldots, X^t), \tag{10}$$

where the sum is over all the trajectories of the system starting at the configuration X^0, i.e. $P_0\left(X\right) = \delta_{X,X^0}$. This quantity is the partition function of the statistical ensemble of the trajectories with the statistical weights defined above. On the other hand, if we add the lacking processes allowing the particles to jump to an occupied site, the value of $\mathcal{P}_t\left(X^0\right)$ will have the meaning of probability for all the particles not to meet up to time t. In Fisher's original formulation of such a process, two particles annihilate when getting to the same site. Then, $\mathcal{P}_t\left(X^0\right)$ is the probability for m particles to survive up to time t. Therefore, we refer to this quantity as a survival probability. Below we formulate three theorems, which specify the asymptotic behaviour of the survival probability in the limit of large time for different parts of the range of the parameter κ. The proof of these theorems is the content of Sections 4,5.

Remark 1 Two of the theorems below are stated and proved for complex valued parameter κ and the third one for real $\kappa > 1$. Obviously, the quantity obtained there has a meaning of the survival probability only for real κ varying in the range (9). Consideration of other values of κ is justified by its later interpretation in terms of the generating function of the moments of the total particle current in the TASEP. In the latter case the complex values of κ turn out to be meaningful and useful to reconstruct the total particle current distribution in the TASEP.

2.2.1 The survival probability for SVW

Generic case, $|\kappa| < 1$ In this case the asymptotic behavior of the survival probability $\mathcal{P}_t\left(X^0\right)$ as $t \to \infty$ is given by the following theorem.

Theorem 1 *Let $\kappa \in \mathbb{C}$ be a fixed complex number from the domain $|\kappa| < 1$, and let $\left|x_i^0 - x_j^0\right| < \infty$ for any $i, j = 1, \ldots, m$. Then, as $t \to \infty$ the survival probability $\mathcal{P}_t\left(X^0\right)$ for m particles is*

$$\mathcal{P}_t\left(X^0\right) = A\left(\kappa; X^0\right)\left[tp(1-p)\right]^{-\frac{m(m-1)}{4}}\left[1 + O\left(\left(\log t\right)^3 t^{-1/2}\right)\right], \tag{11}$$

where the prefactor is given by

$$A(\kappa; X^0) = \frac{2^m \prod_{l=1}^m \Gamma\left(l/2+1\right)}{(1-\kappa)^{\frac{m(m-1)}{2}} \pi^{m/2}} \det\left[g_{i,j}(x_m^0 - x_i^0; \kappa)\right]_{1\le i,j\le m} \tag{12}$$

and where the function $g_{i,j}(x;\kappa)$ is defined by

$$g_{i,j}(x;\kappa) = \oint_{C_0} \frac{d\xi}{2\pi \mathrm{i}} \frac{\left(\kappa + \kappa\xi - 1\right)^{i-1}\left(1+\xi\right)^x}{\xi^j}. \tag{13}$$

Thus, in the range $\kappa < 1$, up to the factor $A(X^0;\kappa)$, which captures the dependence on the initial configuration X^0, the survival probability reproduces Fisher's power law. All the dependence on X^0 is in fact hidden in the determinantal part of $A(X^0;\kappa)$. In some particular cases the determinant can be simplified to a more transparent expression. For example, for equidistant initial conditions,

$$x_m^0 - x_i^0 = a\left(m-i\right), \tag{14}$$

where a is a positive integer, it can be calculated explicitly:

$$\det\left[g_{i,j}(x_m^0 - x_i^0;\kappa)\right]_{1\le i,j\le m} = \left(a + \kappa - a\kappa\right)^{\frac{m(m-1)}{2}}. \tag{15}$$

In the limit $\kappa \to 0$ the determinant reduces to

$$\det[g_{i,j}(x_m^0 - x_i^0; 0)] = \prod_{1\le i<j\le m} \frac{x_j^0 - x_i^0}{j-i}. \tag{16}$$

Then, up to rescaling of space and time, one recovers the result [38] for VW:

$$A(0;X^0) = \prod_{1\le i<j\le m}\left(x_j^0 - x_i^0\right)\begin{cases} \pi^{-\frac{m}{4}} 2^{-\frac{m(m-2)}{4}} \prod\limits_{l=1}^{\frac{m}{2}} \frac{1}{(2l-1)!}; & \text{even} m \\ \pi^{\frac{1}{4}-\frac{m}{4}} 2^{-\frac{(m-1)^2}{2}} \prod\limits_{l=1}^{\frac{(m-1)}{2}} \frac{1}{(2l)!}; & \text{odd} m \end{cases}. \tag{17}$$

In the limit $\kappa \to 1$, we have

$$\det\left[g_{i,j}(x_m^0 - x_i^0; 1)\right]_{1\le i,j\le m} = 1. \tag{18}$$

This limit corresponds to the TASEP. Hence the asymptotics must change, as the probability conservation is restored. The signature of this fact is the divergence of the term $A(\kappa;X^0)$ that takes place in this limit. Specifically, (12) and (18) suggests that as κ approaches one, $A\left(\kappa;X^0\right)$ diverges as $(1-\kappa)^{-m(m-1)/2}$. Comparing the exponent of this expression with the one of the time decay $t^{-m(m-1)/4}$, we can guess that the transition takes place on the scale $(1-\kappa) \sim 1/\sqrt{t}$. This hypothesis is justified below.

Generic case, $\kappa > 1$ In this case no values of κ fall into the range (9). Therefore, according to the Remark 1, the result formally obtained for $\mathcal{P}_t\left(X^0\right)$ does not have a probabilistic meaning in terms of SVW. However, it is still meaningful for the description of current fluctuation in the TASEP.

Theorem 2 *Let $\kappa \in (1,\infty)$ be a fixed real number, and let $\left|x_i^0 - x_j^0\right| < \infty$ for any $i,j = 1,\dots,m$. Then, as $t \to \infty$, the survival probability $\mathcal{P}_t\left(X^0\right)$ for m particles is*

$$\begin{aligned} \mathcal{P}_t\left(X^0\right) &= (1-p+\kappa p)^{(m-1)t}\left(1-p+\kappa^{1-m}p\right)^t \\ &\times \frac{[m]_\kappa^{m-1}\kappa^{x_1^0+\dots+x_{m-1}^0-(m-1)x_m^0}}{[m-1]_\kappa!}\left[1+O\left((\log t)^3 t^{-1/2}\right)\right] \end{aligned} \tag{19}$$

Here we use the common notations

$$[m]_q = \frac{1-q^m}{1-q} \tag{20}$$

for the q-number and

$$[m]_q! = [1]_q\cdots[m]_q \tag{21}$$

for the q-factorial. Note that the q-numbers turn to usual numbers $[m]_q \to m$ in the limit $q \to 1$.

Transition regime $\kappa \to 1$ Consider the limit

$$t\to\infty, \kappa\to 1, (1-\kappa)\sqrt{t} = const. \tag{22}$$

We introduce the parameter

$$\alpha = \lim_{t\to\infty}\left[(1-\kappa)\sqrt{tp(1-p)}\right]. \tag{23}$$

Theorem 3 *Let the condition $\left|x_i^0 - x_j^0\right| < \infty$ hold for all $i,j = 1,\dots,m$. Then,in the limit (22), for the parameter $\alpha \in \mathbb{C}$ defined in (23) taking any fixed complex value, $\mathcal{P}_t\left(X^0\right)$ converges to*

$$\mathcal{P}_t = f_m(\alpha)\left[1+O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right], \tag{24}$$

where the function $f_m(\alpha)$ has the form of a multiple integral:

$$\begin{aligned} f_m(\alpha) &= \frac{(-1)^{\frac{m(m-1)}{2}}}{(2\pi)^{\frac{m}{2}}\,2!\cdots(m-2)!} \\ &\times \int_{-\infty}^{\infty} du_1 \int_{u_1}^{\infty} du_2 \cdots \int_{u_{m-1}}^{\infty} du_m \int_0^{\infty} d\nu_2 \cdots \int_0^{\infty} d\nu_m \\ &\times e^{-\frac{1}{2}u_1^2}\prod_{i=2}^m \nu_i^{i-2} e^{-\frac{1}{2}(u_i+\nu_i)^2-\alpha\nu_i}\Delta\left(u_1,\nu_2+u_2,\dots,\nu_m+u_m\right), \end{aligned} \tag{25}$$

where

$$\Delta\left(x_1,\dots,x_m\right) = \prod_{1\le i<j\le m}(x_i - x_j) \tag{26}$$

is the Vandermonde determinant.

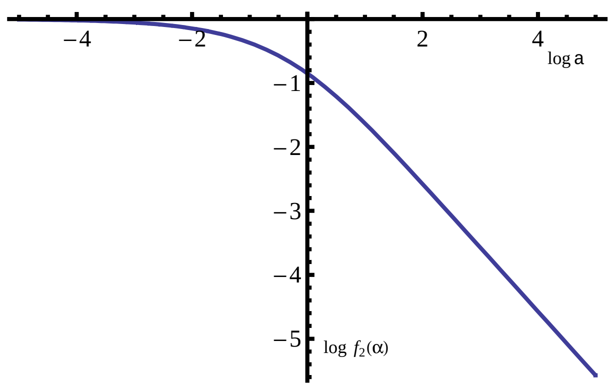

Figure 1: The log-log plot of the function $f_2(\alpha)$ in the range $\alpha > 0$.

The argument of $\mathcal{P}_t$ in (24) can be omitted as the dependence on the initial configuration is lost in the limit under consideration. The limiting behaviors of $f_m(\alpha)$ match the TASEP and VW asymptotics. Indeed, we prove in Section 5 that

$$\lim_{\Re\alpha\to\infty} \alpha^{\frac{m(m-1)}{2}} f_m(\alpha) = \frac{1}{m!}\frac{2^m}{\pi^{m/2}}\prod_{l=1}^{m}\Gamma(l/2+1), \tag{27}$$

$$f_m(0) = 1 \tag{28}$$

$$\lim_{\Re\alpha\to-\infty} e^{-\alpha^2\frac{m(m-1)}{2}} f_m(\alpha) = \frac{m^{m-1}}{(m-1)!} \tag{29}$$

In (27) and (29) the imaginary part of α is implied to take an arbitrary fixed value. The proof of these three limits given in Section 5 is done by reducing $f_m(\alpha)$ to different cases of Mehta integrals, $I_{m,1/2}$, $I_{m.,1}$ and $I_{m-1,1}$ respectively, which are well known in the theory of Gaussian random matrix ensembles. [33].

The above results can be illustrated by the example of the two particle case, $m = 2$, when the integral (25) for $f_2(\alpha)$ simplifies significantly.

$$f_2(\alpha) = e^{\alpha^2}\mathrm{Erfc}(\alpha). \tag{30}$$

Here $\mathrm{Erfc}(\alpha)$ is the complementary Error function

$$\mathrm{Erfc}(\alpha) = \frac{2}{\sqrt{\pi}}\int_{\alpha}^{\infty} dx e^{-x^2}. \tag{31}$$

In Fig. 1 we show how $f_2(\alpha)$ interpolates between the SVW and TASEP limiting cases, which are (27) and (28) respectively.

2.3 Current fluctuations in TASEP

Consider the process with the transition weights $\widetilde{T}(X,Y)$ defined similarly to (5) but where the functions $\vartheta(k)$ and $\theta(k,l)$ are replaced by

$$\widetilde{\vartheta}(k) = (1-\widetilde{p})\,\delta_{k,0} + e^{\gamma}\widetilde{p}\delta_{k,1}, \tag{32}$$

$$\widetilde{\theta}(k,l) = (1-\widetilde{p}(1-\delta_{l,1}))\,\delta_{k,0} + e^{\gamma}\widetilde{p}\delta_{k,1}. \tag{33}$$

Here $0 < \widetilde{p} < 1$ and γ is a complex-valued parameter. It is not difficult to see that these transition weights correspond to the TASEP except that for each particle, the probability $\widetilde{p}$ to jump is multiplied by an additional factor e^{γ}, i.e.

$$e^{\gamma Y_t} P_{TASEP}(X^0, \ldots, X^t) = \widetilde{T}(X^t, X^{t-1}) \cdots \widetilde{T}(X^1, X^0) P_0(X_0), \tag{34}$$

where $P_{TASEP}(X^0, \ldots, X^t)$ is the probability for a sequence of particle configurations $X^0, \ldots, X^t$, to occur in the TASEP for t successive steps and Y_t is the total number of jumps made by all particles in this sequence of configurations. Thus, one can calculate the moment generating function for the cumulative particle current as follows,

$$\left\langle e^{\gamma Y_t} \right\rangle_{TASEP} = \sum_{X^0 \le X^1 \cdots \le X^t} e^{\gamma Y_t} P_{TASEP}(X^0, \ldots, X^t). \tag{35}$$

On the other hand, we can see that, if we define

$$\kappa = e^{-\gamma}, \tag{36}$$

$$p = \frac{\widetilde{p}}{(1-\widetilde{p})\, e^{-\gamma} + \widetilde{p}}, \tag{37}$$

then the following relation exists between the transition weights $\widetilde{T}(X, X')$ defined in (32),(33) and those of SVW, (6),(7),

$$(1-\widetilde{p})^{-m}\, \widetilde{T}(X, X') = (1-p)^{-m}\, T(X, X'). \tag{38}$$

As a result we have

$$\left\langle e^{\gamma Y_t} \right\rangle_{TASEP} = (1 + \widetilde{p}\,(e^{\gamma} - 1))^{mt}\, \mathcal{P}_t\left(X^0\right), \tag{39}$$

where $\mathcal{P}_t\left(X^0\right)$ is the survival probability calculated for the SVW model, and the parameters κ and p of SVW are related to the parameters $\widetilde{p}, \gamma$ of TASEP by (36), (37). The function (39) encodes all information about the distribution of the integrated particle current. Thus, we can apply the Theorems 1-3 to obtain the asymptotic form of this distribution.

Large deviation function. It follows from the Theorems 1 and 2 and the formula (39) that for fixed $\gamma \in \mathbb{R}$ the asymptotic form of the generating function of the particle current Y_t is as follows

$$\left\langle e^{\gamma Y_t} \right\rangle_{TASEP} \simeq \begin{cases} \dfrac{(1+\widetilde{p}(e^{\gamma}-1))^{mt+\frac{m(m-1)}{2}}}{[te^{\gamma}\widetilde{p}(1-\widetilde{p})]^{\frac{m(m-1)}{4}}} A\left(e^{-\gamma}, X^0\right) & \gamma > 0 \\ (1-\widetilde{p}\,(1-e^{m\gamma}))^{t}\, \dfrac{\left(1-e^{-\gamma m}\right)^{m-1} e^{\gamma \sum_{i=1}^{m-1}(x_m^0 - x_i^0)}}{(1-e^{-\gamma})\cdots\left(1-e^{-\gamma(m-1)}\right)} & \gamma \le 0 \end{cases}. \tag{40}$$

From here we conclude that a scaled cumulant generating function of the random variable Y_t/t exists

$$\lambda(\gamma) \equiv \lim_{t\to\infty} \frac{1}{t} \log \left\langle e^{Y_t \gamma} \right\rangle_{TASEP} = \begin{cases} m \log(1 + \widetilde{p}\,(e^{\gamma} - 1)) & \gamma \ge 0 \\ \log(1 - \widetilde{p}\,(1 - e^{m\gamma})) & \gamma \le 0 \end{cases}.$$

It is convex and differentiable everywhere. Therefore, we refer to the Gärtner-Ellis theorem [15],[9] to show that the random variable v_t satisfies the large deviation principle with a rate function

$$I(v) \equiv \lim_{t\to\infty} \frac{1}{t} \log P(Y_t/t = v) = \sup_{\gamma} (\gamma v - \lambda(\gamma)).$$

The solution of the maximization problem yields

$$I(v) = \begin{cases} mB(v/m) & v \geq m\widetilde{p} \\ B(v/m) & v \leq m\widetilde{p} \end{cases},$$

where $B(v)$ is the usual rate function of the Bernoulli process

$$B(v) = (1-v)\log\frac{1-p}{1-v} + v\log\frac{p}{v}.$$

Central limit theorem scaling. The transition regime corresponds to the following scaling limit

$$t \to \infty, \qquad \gamma \to 0, \qquad \gamma\sqrt{t} = const. \tag{41}$$

To translate the results obtained for this case we consider the random variable

$$y = \lim_{t\to\infty} \frac{Y_t - m\widetilde{p}t}{\sqrt{t\widetilde{p}(1-\widetilde{p})}}, \tag{42}$$

where the convergence is in distribution. Taking the limit (41) in (39), using the Theorem 3 and noting that $\widetilde{p} \to p$ as $\gamma \to 0$ we obtain

$$\langle e^{\alpha y}\rangle_{TASEP} = e^{m\alpha^2/2} f_m(\alpha), \tag{43}$$

where α is an arbitrary complex valued parameter related to γ via -

$$\alpha = \lim_{t\to\infty} \gamma\sqrt{tp(1-p)}. \tag{44}$$

The random variable y is the rescaled deviation of the integrated current Y_t from $m\widetilde{p}t$, i.e. from the average value of Y_t for m non-interacting particles jumping with probability $\widetilde{p}$. Note that y is the variable that, in the case of free non-interacting particles, satisfies the conditions for the applicability of the Central Limit Theorem (CLT). According to the CLT the probability density function (PDF) of y for m independent particles is the Gauss distribution,

$$P_m^{\text{free}}(y) = \exp\left(-y^2/(2m)\right)/\sqrt{2\pi m} \tag{45}$$

Correspondingly, the generating function of its moments is

$$\langle e^{\alpha y}\rangle_{\text{free}} = \exp\left(m\alpha^2/2\right), \tag{46}$$

which is the first factor in the moment generating function (43). Therefore, the form of the second factor, $f_m(\alpha)$, shows how the distribution of y differs from the one for free particles.

The moment generating function contains all the information about the original distribution. In particular, the cumulants of y are given by the derivatives of its logarithm at $\alpha = 0$,

$$\langle y^n \rangle_c = \frac{\partial^n}{\partial \alpha^n} \log \langle e^{\alpha y} \rangle_{TASEP} |_{\alpha=0} . \tag{47}$$

The value of the first derivative, i.e.,

$$\langle y \rangle_c = \langle y \rangle = f'_m(0) , \tag{48}$$

shows how the difference between the mean velocity v_m of the center of mass of the particles and that of free non-interacting particles, which is $\widetilde{p}$, decays with time t,

$$v_m = \frac{\langle Y_t \rangle}{mt} \simeq \widetilde{p} + \frac{\langle y \rangle_c}{\sqrt{t}} \frac{\sqrt{\widetilde{p}(1-\widetilde{p})}}{m} \langle y \rangle_c . \tag{49}$$

A nonzero value of $f'_m(0)$ implies that this difference is of order of $t^{-1/2}$. As the TASEP interaction slows down the particle motion, one expects it to be negative, i.e.

$$f'_m(0) < 0. \tag{50}$$

The second cumulant

$$\langle y^2 \rangle_c = \langle y^2 \rangle - \langle y \rangle^2 = m + f''_m(0) - (f'_m(0))^2 \tag{51}$$

is related to the diffusion constant Δ_m of the center of mass.

$$\Delta_m \equiv \frac{1}{m^2} \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = \frac{\widetilde{p}(1-\widetilde{p})}{m^2} \langle y^2 \rangle_c \tag{52}$$

The next cumulants, e.g.

$$\langle y^3 \rangle_c \equiv \langle y^3 \rangle - 3 \langle y^2 \rangle \langle y \rangle + 2 \langle y \rangle^3 = (\log f_m(\alpha))''' \big|_{\alpha=0} , \tag{53}$$

$$\begin{aligned} \langle y^4 \rangle_c \equiv & \langle y^4 \rangle - 4 \langle y^3 \rangle \langle y \rangle - 3 \langle y^2 \rangle^2 \\ & +12 \langle y \rangle^2 \langle y^2 \rangle - 6 \langle y \rangle^4 = (\log f_m(\alpha))^{(4)} \Big|_{\alpha=0} , \end{aligned} \tag{54}$$

quantify the discrepancy of the distribution from a Gaussian form, being identically zero for the latter.

The asymptotical behavior of the generating function at large absolute values of $\Re\alpha$ can be readily obtained from the ones of $f_m(\alpha)$, (27),(29).

$$\langle e^{\alpha y} \rangle_{TASEP} \simeq \begin{cases} \alpha^{-\frac{1}{2}m(m-1)} e^{\frac{1}{2}m\alpha^2} \frac{2^m \prod_{l=1}^m \Gamma(l/2+1)}{\pi^{m/2} m!}, & \Re\alpha \to \infty \\ e^{\frac{1}{2}\alpha^2 m^2} \frac{m^{m-1}}{(m-1)!}, & \Re\alpha \to -\infty \end{cases} \tag{55}$$

The PDF of the random variable y can be obtained as an inverse Laplace transform of its moment generating function (43)

$$P_m(y) = \int_{\beta - i\infty}^{\beta + i\infty} e^{m\alpha^2/2 - \alpha y} f_m(\alpha) \frac{d\alpha}{2\pi i} \tag{56}$$

As the function $f_m(\alpha)$ is bounded and analytic in any vertical strip of finite width, the parameter β can be chosen arbitrarily. The asymptotic results (55) for the generating function can be used in the integral (56) to evaluate the asymptotics for PDF $P_m(y)$. Choosing $\beta = y/m$ for $y \to \infty$ and $\beta = y/m^2$ for $y \to -\infty$ we obtain

$$P_m(y) \simeq \begin{cases} \left(\frac{m}{y}\right)^{\frac{m(m-1)}{2}} e^{-\frac{y^2}{2m}} \frac{\prod_{l=1}^{m} \Gamma(l/2+1)}{m!\sqrt{2\pi m}} \frac{2^m}{\pi^{m/2}}, & y \to \infty \\ e^{-\frac{y^2}{2m^2}} \frac{m^{m-2}}{\sqrt{2\pi}(m-1)!}, & y \to -\infty \end{cases} . \quad (57)$$

Thus, the form of the distribution $P_m(y)$ is far from being symmetric, having tails of two Gaussian-like functions with different dispersions, m^2 and m, on the left and right respectively, the latter also multiplied by "Fisher's factor" $y^{-m(m-1)/2}$.

Let us compare these result with the data obtained from Monte Carlo simulations. We modelled the TASEP for $m = 2, 3, 4, 5$ particles, which have evolved for $t = 10^6$ time-steps, the statistics having been collected from 10^6 samples. We would like to compare the data obtained for the generating function $\langle e^{\alpha y} \rangle$ and the PDF $P_m(y)$ with our predictions. An explicit evaluation of these functions requires detailed analysis of the function $f_m(\alpha)$, which is given by the multiple integral (25). For arbitrary m this needs a significant calculational effort, which is beyond the goals of the present article. Fortunately, for $m = 2$ the function $f_2(\alpha)$ is simple enough, being given by (30), and we can use it for plotting the generating function and the distribution. In Fig. 2 we show a plot of the logarithm of the $m = 2$ moment generating function, whose analytic expression is

$$\langle e^{\alpha y} \rangle_{TASEP} = e^{2\alpha^2} \left(1 - \mathrm{Erf}(\alpha)\right). \quad (58)$$

It has a skew convex form growing more rapidly to the left than to the right, with a minimum at $\alpha = 0.432752$. One can see good agreement with the numerical data in the central part of the graph. There is some discrepancy at the tails, which can be attributed to the finite-time corrections, i.e. the lack of statistics of large events at the finite period of measurement, which becomes significant when the absolute value of α is large.

The function (58) allows a calculation of any derivatives, and, hence, of any cumulants of the random variable y. In Table 2.3 we show the first four cumulants for $m = 2$, the case of two particles. Their values are in good agreement with the results from the Monte Carlo simulations. In Fig. 3 we show the result

	Analytic	Numerical Analytic	Monte Carlo
$\langle y \rangle_c$	$-2\pi^{-1/2}$	-1.12838	-1.12545
$\langle y^2 \rangle_c$	$4 - 4\pi^{-1}$	2.72676	2.72518
$\langle y^3 \rangle_c$	$4(\pi - 4)\pi^{-3/2}$	-0.616636	-0.617642
$\langle y^4 \rangle_c$	$32(\pi - 3)\pi^{-2}$	0.459083	0.498263

Table 1: Cumulants of the random variable y.

of numerical evaluation of the integral (56) for $m = 2$. There is a very good agreement with the simulation results. At first glance the form of the distribution shown on Fig. 3a appears Gaussian-like. A more accurate impression of the

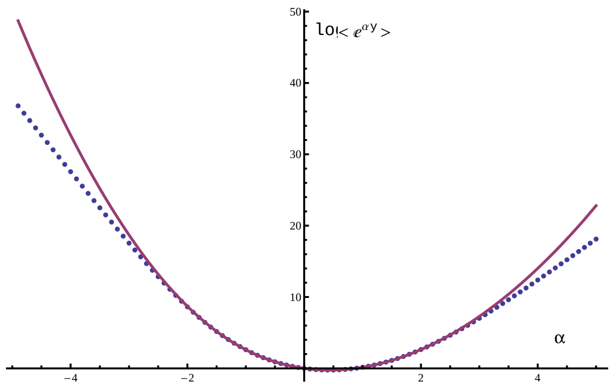

Figure 2: Plot of the logarithm of the moment generating function for $m = 2$. Solid line is the plot of the formula (58). Dotted line is the result of Monte Carlo simulations.

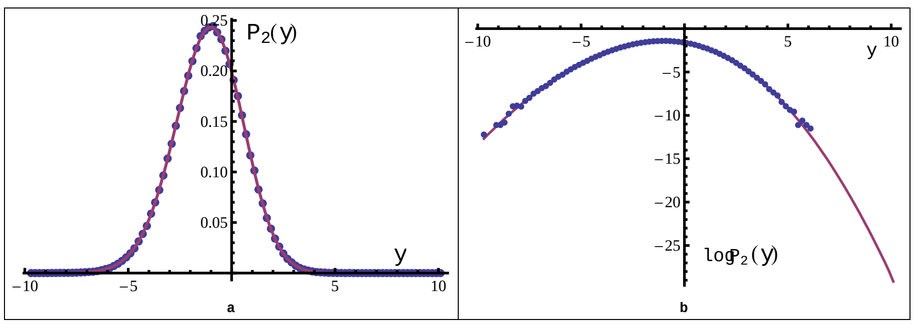

Figure 3: Probability density function $P_m(y)$ for $m = 2$ particles (a) and its logarithm (b). The solid line shows the theoretical predictions for the distributions.

form of the distribution is given by the logarithmic plot of Fig. 3b which shows that the distribution is actually skew, decreasing more rapidly as y grows than as it decreases.

Simulation results obtained for more than two particles can be tested against the asymptotical formulas (57) for the tails of the distribution $P_m(y)$. In Fig. 4 we plot the distributions measured for $m = 2, 3, 4, 5$ particles, (Fig. 4a), and its logarithm, (Fig. 4b), the latter being compared with the graphs of (57).

One can see that for all the four graphs the left tails are perfectly fitted already for rather small values of y. A good fit of the right tail takes place only for $m = 2$. For $m = 3$ only a few data points approach the asymptotical line, i.e. in this case the right asymptotical regime is actually at the borderline of the statistics. In the cases $m = 3, 4$ the statistics available is clearly not good enough to reach the asymptotical regime. Significantly larger evolution time and statistics would have to be considered.

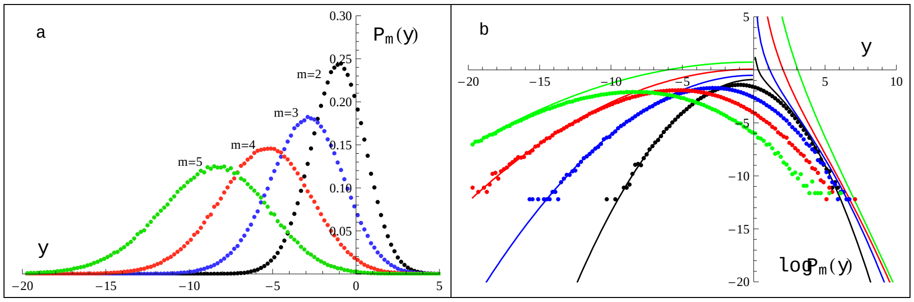

Figure 4: Probability density function $P_m(y)$ for $m = 2, 3, 4, 5$ particles (a) and its logarithm (b). The solid lines show the theoretical asymptotics of the tails of the distributions.

3 The master equation

From now on we consider only the SVW model dependent on the parameters p and κ. Our first step is to calculate the probability $P_t\left(X, X^0\right)$ of transition from the configuration X^0 to X for arbitrary time t:

$$P_t\left(X, X^0\right) = \sum_{X^0 \leq X^1 \cdots \leq X^t \equiv X} P(X^0, \ldots, X^t). \tag{59}$$

The method of finding the transition probability was first developed by Schütz for the continuous time TASEP [40], who used the Bethe Ansatz first applied to the ASEP by Gwa and Spohn [18]. Here we follow a similar procedure. The transition probability obeys the master equation

$$P_t\left(X, X^0\right) = \sum_{X'} T\left(X, X'\right) P_{t-1}(X', X^0); \tag{60}$$

the transition weights $T\left(X, X'\right)$ being defined as above, (5)-(7). The problem of finding the eigenvectors and eigenvalues of the matrix $T\left(X, X'\right)$ can be solved by the Bethe Ansatz technique. As this technique is rather standard and has been reviewed in many monographs, we simply state the results here. For details of similar derivations, the reader can consult for example with the review [5]. As a result we obtain the solution of the left and right eigenvalue problems for the Markov matrix $T\left(X, X'\right)$:

$$\Lambda\left(Z\right)\Psi_Z\left(X\right) = \sum_{X'} T\left(X, X'\right)\Psi_Z\left(X'\right), \tag{61}$$

$$\Lambda\left(Z\right)\overline{\Psi}_Z\left(X\right) = \sum_{X'} T\left(X', X\right)\overline{\Psi}_Z\left(X'\right) \tag{62}$$

parametrized by an m-tuple of complex parameters $Z = \{z_1, \cdots, z_m\}$. The corresponding eigenvalue is expressed in terms of these parameters,

$$\Lambda\left(Z\right) = \prod_{i=1}^{m}\left(1 - p + p/z_i\right), \tag{63}$$

and the eigenvectors are given by the following determinants

$$\Psi_Z(X) = \det\left(z_i^{x_j}(1-\kappa z_i)^{i-j}\right)_{1\le i,j\le m}, \tag{64}$$

$$\overline{\Psi}_Z(X) = \det\left(z_i^{-x_j}(1-\kappa z_i)^{j-i}\right)_{1\le i,j\le m}. \tag{65}$$

It is not difficult to check that these two eigenfunctions can be used to construct the resolution of the identity operator

$$\frac{1}{m!}\oint \Psi_Z(X)\,\overline{\Psi}_Z(X')\prod_{i=1}^{m}\frac{dz_i}{2\pi \mathrm{i} z_i} = \delta_{X,X'}, \tag{66}$$

where the integration over each $z_i, i = 1,\dots,m$, is along a contour of integration which has to satisfy the requirement that the pole of the wave function at $z = 1/\kappa$ has to lie in the exterior. Then the solution of the initial value problem for the master equation is given by

$$P_t\left(X,X^0\right) = \frac{1}{m!}\oint \Lambda^t(Z)\,\Psi_Z(X)\,\overline{\Psi}_Z\left(X^0\right)\prod_{i=1}^{m}\frac{dz_i}{2\pi i z_i}. \tag{67}$$

Finally we end up with the following integral expression for the transition probability

$$P_t\left(X,X^0\right) = \oint \Lambda^t(Z)\prod_{i=1}^{m}\left[\frac{z_i^{x_i-x_m^0}}{(1-\kappa z_i)^{i-1}}\right] \times \det\left(z_i^{x_m^0-x_j^0}(1-\kappa z_i)^{j-1}\right)_{1\le i,j\le m}\prod_{i=1}^{p}\frac{dz_i}{2\pi \mathrm{i} z_i}. \tag{68}$$

The integration can be easily performed by counting the residues. The result is a determinant of an $m\times m$ matrix of the form similar to the one obtained for the discrete time TASEP with backward update ([2] , [37]). Note that in the case of vicious walkers, $\kappa = 0$, the eigenfunctions are of free fermion type

$$\Psi_Z(X) = \det\left(z_i^{x_j}\right)_{1\le i,j\le m}, \tag{69}$$

$$\overline{\Psi}_Z(X) = \det\left(z_i^{-x_j}\right)_{1\le i,j\le m}. \tag{70}$$

and the integration yields the famous Lindström-Gessel-Viennot theorem [30], [16].

$$P_t\left(X,X^0\right) = \det\left[F_0\left(x_i - x_j^0, t\right)\right]_{1\le i,j\le m}, \tag{71}$$

where

$$F_0(x,t) = p^x(1-p)^{t-x}\begin{pmatrix} t \\ x \end{pmatrix}, \tag{72}$$

These formulas serve as a starting point for the asymptotical analysis of the survival probability.

4 Asymptotic form of the survival probability

To obtain the survival probability $\mathcal{P}_t\left(X^0\right)$ for SVW we have to sum the transition probability $P_t\left(X, X^0\right)$ over the set of all final configurations X:

$$\mathcal{P}_t\left(X^0\right) = \sum_{\{X\}} P_t\left(X, X^0\right). \tag{73}$$

We solve this problem in the limit $t \to \infty$. For pedagogical reasons we first outline the derivation for the VW model, which simply reproduce Rubey's results, [38]. The procedure we use amounts to an asymptotical analysis of the expression for $P_t\left(X, X^0\right)$ by means of the saddle point approximation for the integral (67), which reduces the sum over final configurations to known integrals. The main ingredients of the derivation for the VW model are then applied similarly to the SVW model but with some modifications.

4.1 Vicious walkers

In the case of VW ($\kappa = 0$), the integral (68) takes the form

$$\oint \Lambda^t\left(Z\right) \prod_{k=1}^{m} z_k^{x_k - x_m^0} \det(z_i^{x_m^0 - x_j^0})_{1\le i,j\le m} \prod_{l=1}^{m} \frac{dz_l}{2\pi \mathrm{i} z_l}. \tag{74}$$

Here, the determinant under the integral can be expressed in terms of the Vandermonde determinant

$$\Delta\left(Z\right) \equiv \det(z_i^{m-j})_{1\le i,j\le m} = \prod_{1\le i<j\le m} \left(z_i - z_j\right), \tag{75}$$

and the Schur function [31]

$$s_\chi(z_1, \ldots, z_m) \equiv \det\left(z_i^{\chi_j + m - j}\right) / \Delta\left(Z\right) \tag{76}$$

parametrized by the partition $\chi = (\chi_1 \geq \chi_2 \geq \cdots \geq \chi_m \geq 0)$ defined by

$$\chi = (x_m^0 - x_1^0 - m + 1, x_m^0 - x_2^0 - m + 2, \cdots), \tag{77}$$

as follows:

$$\det(z_i^{x_m^0 - x_i^0})_{1\le i,j\le m} = \Delta\left(Z\right) s_\chi\left(Z\right). \tag{78}$$

Thus (74) can be rewritten in the following form

$$P_T\left(X, X^0\right) = \oint \Delta\left(Z\right) s_\chi\left(Z\right) \prod_{i=1}^{m} e^{t h_i(z_i)} \frac{dz_i}{2\pi \mathrm{i} z_i}, \tag{79}$$

where

$$h_i(z) = \log\left(1 - p + p/z\right) + v_i \log z \tag{80}$$

and

$$v_i = \frac{x_i - x_m^0}{t}. \tag{81}$$

Now we are ready to estimate the integral asymptotically as $t \to \infty$. We assume that the differences $\left(x_i^0 - x_j^0\right)$ are kept bounded for any i and j. The saddle point of the function under the integral is defined by the equation

$$h_i'(z_i^*) = 0, \tag{82}$$

which yields

$$z_i^* = \frac{(1-v_i)\,p}{(1-p)\,v_i}. \tag{83}$$

In the vicinity of the saddle point $h_i(z)$ has an expansion

$$\begin{aligned} h_i(z_i^* + \xi) &= \log\left[\left(\frac{1-p}{1-v_i}\right)^{1-v_i}\left(\frac{p}{v_i}\right)^{v_i}\right] \\ &\quad + \frac{1}{2}\left(\frac{1-p}{p}\right)^2 \frac{v_i^3}{1-v_i}\xi^2 + O(\xi^3). \end{aligned} \tag{84}$$

The integration contours can be deformed to a circle centered at 0, crossing the real axis at z_i^*. Writing points on the circle as $z_i = z_i^* e^{\mathrm{i}\phi_i}$, we have

$$\Re(h_i(z_i)) = h(z_i^*) + \frac{1}{2}\log\left[(1 - v_i(1-\cos(\phi_i))^2 + v_i^2\sin^2(\phi_i)\right]. \tag{85}$$

It follows that there is a single maximum at $\phi_i = 0$. Moreover, since all derivatives are bounded provided $v_i < 0$, the saddle-point approximation holds uniformly in x_i. (Note that (74) is zero if $v_i > 1$, and if $v_m = 1$ then the probability p^t can be extracted as a factor, the remaining integral over $z_1, \ldots, z_{m-1}$ being of the same form.) It is easy to see that the contribution to the sum over X from points with $v_i > p + \epsilon$ for any fixed $\epsilon > 0$ is negligible in the limit $t \to \infty$. (Below, we shall see that the effective range of the summation is in fact even smaller.) The saddle-point approximation [11] now yields

$$\begin{aligned} P_t\left(X, X^0\right) = \left(\frac{p}{1-p}\right)^{\frac{m(m-1)}{2}} \prod_{i=1}^{m} \frac{v_i^{1-m} e^{t h_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)}}{\sqrt{2\pi t v_i(1-v_i)}} \times \\ \prod_{1\le i<j\le m} (v_j - v_i)\, s_\chi\left(\frac{(1-v_1)\,p}{(1-p)\,v_1}, \ldots, \frac{(1-v_m)\,p}{(1-p)\,v_m}\right)\left(1 + O\left(\frac{1}{t}\right)\right). \end{aligned} \tag{86}$$

The next step is to perform the summation (73) over the range of the final configurations $X \in \left\{x_1^0 \le x_1 < \cdots < x_m < \infty\right\}$. For this we need to demonstrate that (86) holds uniformly in X. To this end we first show that the main contribution to the sum comes from the domain

$$pt - \sqrt{t}\log t \le x_1 < \cdots < x_m \le \; pt + \sqrt{t}\log t. \tag{87}$$

Indeed, $h_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)$ is a concave function of v_i in the domain $v_i \in (0,1)$ with a single maximum $v_i = p$. It follows then for $|x_i - pt| > \sqrt{t}\log t$

$$e^{t h_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)} < e^{t h_i\left(\frac{1-\sqrt{t}\log t/(1-p)}{1+\sqrt{t}\log t/p}\right)} = e^{-\frac{(\log t)^2}{2p(1-p)}}\left[1 + O\left(\frac{\log t}{\sqrt{t}}\right)\right] \tag{88}$$

All the other factors in (86) are at most of polynomial order in t, while the total number of nonzero terms in the sum of interest (73) is $O\left(t^m\right)$. Therefore, the contribution from the complement of (87) being of order of $O\left(t^s e^{-\frac{(\log t)^2}{2p(1-p)}}\right)$ for some constant s is asymptotically negligible compared to the contribution from (87).

In the latter one can approximate the function $h_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)$ by the second term of its Taylor expansion at $v_i = p$, which yields

$$
\begin{aligned}
P_t\left(X, X^0\right) &= \frac{1}{(2\pi)^{\frac{m}{2}}\left(tp(1-p)\right)^{\frac{m^2}{2}}} s_\chi(1,\ldots,1) \qquad (89)\\
\prod_{i=1}^m &\exp\left(-\frac{\left(x_i - x_m^0 - pt\right)^2}{2tp(1-p)}\right) \prod_{1\le i<j\le m}\left(x_j - x_i\right)\left[1 + O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right].
\end{aligned}
$$

We now have to evaluate following sum in the limit $t \to \infty$,

$$
\sum_{-\sqrt{t}\log t \le x_1 < \cdots < x_m \le \sqrt{t}\log t} \prod_{i=1}^m e^{-\frac{x_i^2}{2tp(1-p)}} \prod_{1\le i<j\le m}\left(x_j - x_i\right). \qquad (90)
$$

This can be done by means of the following lemma:

Lemma 1 *Let $h : \mathbb{R}^m \to \mathbb{R}$ be a twice differentiable function of at most polynomial growth. Then as $\delta \to 0$,*

$$
\begin{aligned}
&\delta^m \sum_{y_1<\ldots<y_m;\, y_i\in\delta\mathbb{Z}} h(y_1,\ldots,y_n) \prod_{i=1}^m e^{-\frac{1}{2}y_i^2} \qquad (91)\\
&= \int_{-\infty}^{\infty} dy_1 \int_{y_1}^{\infty} dy_2 \ldots \int_{y_{m-1}}^{\infty} dy_m h(y_1,\ldots,y_m) e^{-\frac{1}{2}\sum_{i=1}^m y_i^2} + O(\delta).
\end{aligned}
$$

Proof We subdivide the domain $-\infty < x_1 < \ldots < x_m < \infty$ into hypercubes of the form

$$
B_\delta(y_1,\ldots,y_m) = \{(x_1,\ldots,x_m) : \max|x_i - y_i| \le \frac{\delta}{2}\}, \qquad (92)
$$

where $y_i \in \delta\mathbb{Z}$ and $y_1 < y_2 < \ldots < y_m$. The remaining region is small and its contribution will be estimated shortly. We then write, for $(x_1,\ldots,x_m) \in B_\delta(y_1,\ldots,y_m)$,

$$
\begin{aligned}
&\left|h(x_1,\ldots,x_m)e^{-\frac{1}{2}(x_1^2+\ldots+x_m^2)} - h(y_1,\ldots,y_m)e^{-\frac{2}{2}(y_1^2+\ldots+y_m^2)}\right| \qquad (93)\\
&\le \sup_{(u_1,\ldots,u_m)\in B_\delta(y_1,\ldots,y_m)} \max_{i=1}^m \left|\frac{\partial}{\partial u_i} h(u_1,\ldots,u_m) e^{-\frac{1}{2}(u_1^2+\ldots+u_m^2)}\right| |x_i - y_i|.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\frac{\partial}{\partial u_i} h(u_1,\ldots,u_m) e^{-\frac{1}{2}(u_1^2+\ldots+u_m^2)} \qquad (94)\\
&= \left(\frac{\partial}{\partial u_i} h(u_1,\ldots,u_m) - u_i h(u_1,\ldots,u_m)\right) e^{-\frac{1}{2}(u_1^2+\ldots+u_m^2)}
\end{aligned}
$$

which is easily seen to be bounded by $Ce^{-(y_1^2+\ldots+y_m^2)/2}$ for some constant $C>0$. It follows from the convergence of the sum $\sum_{y\in\delta\mathbb{Z}}\delta e^{-y^2/2}$ uniformly in δ that the difference between the integral over the region

$$\bigcup_{y_1<\ldots<y_m;\, y_i\in\delta\mathbb{Z}} B_\delta(y_1,\ldots,y_m) \tag{95}$$

and the sum is of order δ. There remains the integral over the complementary region, but this is obviously of order δ as the integral converges and the region has width δ.

Then, after going to rescaled variables $y_i = x_i/\sqrt{tp\,(1-p)}$ and writing $\delta = 1/\sqrt{tp(1-p)}$ the sum (90) reduces to the integral

$$(tp\,(1-p))^{\frac{m(m+1)}{4}} \int_{-\infty}^{\infty} dx_1 \cdots \int_{x_{m-2}}^{\infty} dx_{m-1} \int_{x_{m-1}}^{\infty} dx_m \prod_{i=1}^{m} e^{-\frac{1}{2}x_i^2} \prod_{1\le i<j\le m} |x_j - x_i| \tag{96}$$

(the range of summation is extended to $(-\infty \le x_1 < \cdots < x_m \le \infty)$ by the same argument as above. Note that the absolute value signs $|x_j - x_i|$, though redundant in this range, are nevertheless useful as they make the expression symmetric with respect to permutations of the variables $x_1,\ldots,x_m$. One, then, can use this fact to extend the integration to the whole $\mathbb{R}^m$, which yields an additional factor of $m!$, which has to be compensated in the end. As a result we obtain

$$\mathcal{P}_t\left(X^0\right) = \frac{1}{[p(1-p)t]^{m(m-1)/4}} \frac{I_{m,1/2}}{(2\pi)^{m/2}\, m!} s_\chi\,(1,\ldots,1)\left[1+O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right] \tag{97}$$

where

$$\begin{aligned} I_{m,k} \;&\equiv\; \int_{-\infty}^{\infty} dy_p \cdots \int_{-\infty}^{\infty} dy_2 \int_{-\infty}^{\infty} dy_1 \exp\left(-\frac{1}{2}\sum\nolimits_{i=1}^{m} y_i^2\right) \\ &\qquad \times \prod_{1\le i<j\le m} |y_j - y_i|^{2k} \\ &= (2\pi)^{m/2} \prod_{l=1}^{m} \frac{\Gamma\,(lk+1)}{\Gamma(k+1)} \end{aligned} \tag{98}$$

is the Mehta integral [33], which first appeared in the context of Gaussian random matrix ensembles. Finally, one can use the following formula for the Schur function [31]

$$s_\chi\,(1,\ldots,1) = \prod_{1\le i<j\le m} \frac{\chi_i - i - \chi_j + j}{j-i}, \tag{99}$$

resulting in the following expression for the survival probability:

$$\begin{aligned} \mathcal{P}_t\left(X^0\right) \;&=\; \frac{1}{[p(1-p)t]^{m(m-1)/4}} \frac{2^m}{\pi^{m/2}} \prod_{l=1}^{m} \frac{\Gamma\,(l/2+1)}{l!} \\ &\qquad \times \prod_{1\le i<j\le m} \left(x_j^0 - x_i^0\right)\left[1+O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right] \end{aligned} \tag{100}$$

After reexpressing the gamma functions in terms of factorials we obtain the form given in (17).

4.2 Semi-vicious walkers

4.2.1 The case of generic $\kappa \neq 1$

To study the asymptotic behaviour of the survival probability for the case of general κ, one can start with the following integral representation for the transition probability

$$\begin{aligned} P_t\left(X, X^0\right) &= \prod_{i=1}^{m} \oint_{C_0} \frac{dz_i}{2\pi \mathrm{i} z_i} \frac{d\xi_i}{2\pi \mathrm{i}\xi_i} \left(\frac{1-\kappa\xi_i}{1-\kappa z_i}\right)^{i-1} \xi_i^{x_m^0 - x_i^0 + 1} z_i^{x_i - x_m^0} \\ &\times \Lambda^t(Z) \prod_{1\le i,j\le m} \frac{1}{\xi_i - z_j} \prod_{1\le i<j\le m} (z_i - z_j)(\xi_j - \xi_i), \quad (101) \end{aligned}$$

where the integration in each variable is along a small circle around zero, $|z_i| < |\xi_j|$ for any $i, j = 1, \ldots, m$. This representation can be reduced to the form (68) by direct integration over each ξ_j $(j = 1, \ldots, m)$. This is done by summing the contributions to the integral coming from all the poles $\xi_j = z_i$, $i = 1, \ldots, m$.

Though the most of analysis of the large t asymptotics of this expression is similar to the one for VW, one important difference exists. The expressions under the integrals over z_i, $i = 2, \ldots, m$, have singularities at $z_i = 1/\kappa$, the poles of the form $(1-\kappa z_i)^{1-i}$, which can be located between the origin and the saddle point. In this case the contour being deformed to the steepest descent one, crosses this singularity and its contribution must then be extracted from the saddle point contribution. While for $|\kappa| < 1$ this does not affect the asymptotics of the sum over $x_1, \ldots, x_m$, evaluated subsequently, for $|\kappa| > 1$ its contribution turns out to be dominant.

It is, however, technically difficult to calculate the residue at the multiple pole of the complicated expression. To avoid this calculation and to evaluate both cases in one go, we expand the term $(1-\kappa z_i)^{1-i}$ into a series in powers of (κz_i) and then integrate it term by term in the saddle point approximation. As a result we obtain $P_t\left(X, X^0\right)$ in the form of an $(m-1)$ - fold series

$$\begin{aligned} P_t\left(X, X^0\right) = \prod_{k=1}^{m} \oint_{C_0} \frac{d\xi_i}{2\pi \mathrm{i}\xi_k} \left(1-\kappa\xi_k\right)^{i-1} \xi_k^{x_m^0 - x_i^0 + 1} \quad (102) \\ \prod_{1\le i<j\le m} (\xi_j - \xi_i) \sum_{\{n_2,\ldots,n_m\}\in \mathbb{Z}_{\ge 0}^{m-1}} \mathcal{A}\left(\{\xi_i, v_i\}_{i=1}^m, \{n_k\}_{i=2}^m,\right) \end{aligned}$$

where

$$\begin{aligned} &\mathcal{A}\left(\{\xi_i, v_i\}_{i=1}^m, \{n_k\}_{i=2}^m\right) = \\ &\left(\frac{p}{1-p}\right)^{\frac{m(m-1)}{2}} \prod_{i=2}^{m} \kappa^{n_i} \binom{i+n_i-2}{n_i} \quad (103) \\ &\times \prod_{i=1}^{m} \frac{v_i^{1-m} e^{t h_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)}}{\sqrt{2\pi t v_i(1-v_i)}} \prod_{1\le i<j\le m} (v_j - v_i) \end{aligned}$$

$$\times \prod_{1\le i,j\le m} \left(\xi_i - \frac{(1-v_j)\,p}{(1-p)\,v_j}\right)^{-1} \left(1 + O\left(\frac{1}{t}\right)\right),$$

$$\begin{aligned} v_1 &= \frac{x_1 - x_m^0}{t} \\ v_i &= \frac{x_i - x_m^0 + n_i}{t}, i = 2, \dots, m. \end{aligned} \tag{104}$$

The next step is to use this approximation to perform the summation of (102) over the domain $\{x_1^0 < x_1 < x_2 < \cdots < x_m < \infty\}$. The effective range of this summation depends crucially on the behaviour of the other sum in $n_2, \dots, n_m$. Namely, the effective summation range is different depending on whether the value of κ is greater or less than one, when the term κ^{n_i} is decreasing or increasing respectively. We consider these two cases separately.

The case $|\kappa| < 1$ Here κ takes arbitrary complex values in the domain $|\kappa| < 1$. As in the case of vicious walkers, we argue that the exponential part $\exp\left[th_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)\right]$ makes the whole expression negligible beyond the range

$$p - t^{-1/2}\log t \le v_i \le p + t^{-1/2}\log t. \tag{105}$$

In addition, for $n_i > (\log t)^2 / |\log|\kappa||$ we have

$$\kappa^{n_i} \left(\begin{array}{c} i + n_i - 2 \\ n_i \end{array}\right) = O(t^{-\log t}(\log t)^{2(i-2)}), \tag{106}$$

so that we can limit the summation over n_i to $n_i \le (\log t)^2 / |\log|\kappa||$. Therefore, n_i is negligible compared to x_i in the domain (105) and can be neglected in the definition (104) of v_i. In the range (105) we can approximate $\mathcal{A}\left(\{\xi_i, x_i/t\}_{i=1}^m, \{n_k\}_{i=2}^m,\right)$ by the leading term of its Taylor expansion at $v_i = p$, $i = 1, \dots, m$.

$$\begin{aligned} &\mathcal{A}\left(\{\xi_i, x_i/t\}_{i=1}^m, \{n_k\}_{i=2}^m\right) = \\ &\left(\frac{1}{tp\,(1-p)}\right)^{\frac{m(m-1)}{2}} \prod_{i=2}^m \kappa^{n_i} \left(\begin{array}{c} i + n_i - 2 \\ n_i \end{array}\right) \\ &\times \prod_{i=1}^m \frac{e^{-\frac{(x_i - pt)^2}{2p(1-p)t}}}{\sqrt{2\pi tp(1-p)}} \prod_{1\le i<j\le m} (x_j - x_i) \\ &\times \prod_{i=1}^m (\xi_i - 1)^{-m} \left(1 + O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right) \end{aligned} \tag{107}$$

One can see that the terms dependent on $\{x_i\}$ and $\{n_i\}$ decouple and the terms dependent on $n_2, \dots, n_m$ can be summed up.

$$\sum_{\{n_2,\dots,n_m\}\in\mathbb{Z}_{\ge 0}^{m-1}} \prod_{i=2}^m \kappa^{n_i} \left(\begin{array}{c} i + n_i - 2 \\ n_i \end{array}\right) = (1-\kappa)^{-\frac{m(m-1)}{2}} \tag{108}$$

The remaining sum over x_i for $i = 1, \dots, m$ is transformed to an integral using Lemma 1:

$$\sum_{x_1^0 < x_1 < x_2 < \cdots < x_m} \sum_{\{n_2,\dots,n_m\} \in \mathbb{Z}_{\geq 0}^{m-1}} \mathcal{A}\left(\{\xi_i, x_i/t\}_{i=1}^m, \{n_k\}_{i=2}^m\right) \tag{109}$$

$$= (tp(1-p))^{-\frac{m(m-1)}{4}} (2\pi)^{-\frac{m}{2}} (1-\kappa)^{-\frac{m(m-1)}{2}} \prod_{i=1}^m (\xi_i - 1)^{-m}$$
$$\times \int_{-\infty}^{\infty} dy_1 e^{-y_1^2/2} \int_{y_1}^{\infty} dy_2 e^{-y_1^2/2} \cdots \int_{y_{m-1}}^{\infty} dy_m e^{-y_1^2/2} \prod_{1 \leq i < j \leq m} (y_j - y_i)$$
$$\times \left[1 + O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right].$$

Combining (73), (102) and (109) we obtain

$$\begin{aligned} \mathcal{P}_t\left(X^0\right) &= \frac{1}{[p(1-p)t]^{m(m-1)/4}} \frac{I_{m,1/2}}{(2\pi)^{\frac{m}{2}} m!} \\ &\times \prod_{i=1}^m \oint_{C_{|\xi_i|=r>1}} \frac{d\xi_i}{2\pi \mathrm{i} \xi_i} \left(\frac{1-\kappa\xi_i}{1-\kappa}\right)^{i-1} \prod_{1 \leq i < j \leq m} (\xi_j - \xi_i) \\ &\times \prod_{i=1}^m (\xi_i - 1)^{-m} \xi_i^{x_m^0 - x_i^0 + 1} \left[1 + O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right], \end{aligned} \tag{110}$$

where $I_{m,1/2}$ is the Mehta integral defined in (98). Writing the above product of integrals in determinant form and using the definition of $I_{m,1/2}$ we arrive at the final result

$$\begin{aligned} \mathcal{P}_t\left(X^0\right) &\simeq \frac{2^m}{\pi^{m/2}} [p(1-p)t]^{-\frac{m(m-1)}{4}} (1-\kappa)^{-\frac{m(m-1)}{2}} \\ &\times \prod_{l=1}^m \Gamma(l/2+1) \det\left[\left(g_{i,j}(x_m^0 - x_i^0)\right)_{i,j=1}^m\right]. \end{aligned} \tag{111}$$

Here the function $g_{i,j}(x)$ is defined as follows

$$g_{i,j}(x) = \oint_{C_0} \frac{d\xi}{2\pi \mathrm{i}} \frac{(\kappa + \kappa\xi - 1)^{i-1} (1+\xi)^x}{\xi^j}. \tag{112}$$

The case $\kappa > 1$ Let κ be a real number, $\kappa > 1$. We return to the formulas (102,103). The crucial distinction from the case $|\kappa| < 1$ is that the presence of exponentially growing terms κ^{n_i} affects the range of values of $v_1, \dots, v_m$, which make the major contribution to the final sum of (102). Indeed, we can write

$$\kappa^{n_i} = e^{tv_i \log \kappa} \kappa^{-x_i + x_m^0}. \tag{113}$$

Therefore, if we keep v_i fixed, the sum over x_i is rapidly converging. At the same time the maximum of the v_i-dependent exponential part of the r.h.s. of (102) is shifted due to the appearance of the additional term $tv_i \log \kappa$. In a sense, the roles of the variables $x_2, \ldots, x_m$ and $n_2, \ldots, n_m$ are interchanged compared to the case $\kappa < 1$.

Consequently, instead of summing over $n_2, \ldots, n_m$ and then over $x_1, \ldots, x_m$ we go to the variables $v_1, \ldots, v_m$, (104), and $x_2, \ldots, x_m$ and evaluate the sum over the latter first.

$$\sum_{x_1^0 < x_1 < x_2 < \cdots < x_m} \sum_{\{n_2,\ldots,n_m\} \in \mathbb{Z}_{\geq 0}^{m-1}} \mathcal{A}\left(\{\xi_i, v_i\}_{i=1}^m, \{n_k\}_{i=2}^m\right) \tag{114}$$

$$= \sum_{\{v_1,\ldots,v_m\} \in t^{-1}\mathbb{Z}_{\geq x_1^0}^m} \sum_{x_2=x_1+1}^{v_2 t + x_m^0} \cdots \sum_{x_m = x_{m-1}+1}^{v_m t + x_m^0} \mathcal{A}\left(\{\xi_i, v_i\}_{i=1}^m, \left\{v_i - \left(x_i - x_m^0\right)/t\right\}_{i=2}^m\right)$$

Collecting the factors of $\mathcal{A}\left(\{\xi_i, v_i\}_{i=1}^m, \left\{v_i - \left(x_i - x_m^0\right)/t\right\}_{i=2}^m\right)$ dependent on $x_2, \ldots, x_m$ we can evaluate the sum over these variables

$$\sum_{x_2=x_1+1}^{v_2 t + x_m^0} \cdots \sum_{x_m = x_{m-1}+1}^{v_m t + x_m^0} \prod_{i=2}^m \kappa^{-x_i + x_m^0} \binom{tv_i - x_i + x_m^0 + i - 2}{tv_i - x_i + x_m^0}$$
$$= \frac{\kappa^{(m-1)\left(x_m^0 - x_1\right)}}{(\kappa - 1)\cdots(\kappa^{m-1} - 1)} \prod_{i=2}^m \frac{(tv_i)^{i-2}}{(i-2)!} \left(1 + O\left(\frac{1}{t}\right)\right). \tag{115}$$

Here we extended the upper limit of all the summations to infinity, which yields a correction of order of $\kappa^{-v_i t}$, and we used Stirling's formula to approximate the binomial coefficient

$$\binom{i+n}{n} = \frac{n^i}{i!}\left(1 + O\left(\frac{1}{n}\right)\right). \tag{116}$$

We also imply that the value of v_i in the effective summation range is finite and positive. Indeed, the range of summation over v_i is defined as above by the requirement that the exponential parts of $\mathcal{A}\left(\{\xi_i, v_i\}_{i=1}^m, \left\{v_i - \left(x_i - x_m^0\right)/t\right\}_{i=2}^m\right)$ are not too small. Specifically, the exponentiated expressions are

$$\exp\left\{t\left[h_i\left(\frac{(1-v_i)\,p}{(1-p)\,v_i}\right) + v_i \log\kappa\right]\right\} \tag{117}$$

for $i = 2, \ldots, m$, and

$$\exp\left\{t\left[h_1\left(\frac{(1-v_i)\,p}{(1-p)\,v_i}\right) - (m-1)\,v_1 \log\kappa\right]\right\}, \tag{118}$$

the term $t\,(1-m)\,v_1 \log\kappa = \log\left(\kappa^{(m-1)\left(x_m^0 - x_1\right)}\right)$ in the latter coming from the result of the summation over $x_2, \ldots, x_m$, (115). For a real positive κ the major contribution to the sums over $v_1, \ldots, v_m$ comes from the neighborhood of the maxima of the exponentiated expressions

$$|v_i - u_i| < t^{-1/2} \log t \tag{119}$$

where the maxima u_i are located at

$$u_1 = \mathrm{u}(\kappa^{1-m}) \tag{120}$$

and

$$u_i = \mathrm{u}(\kappa) \tag{121}$$

for $i = 2, \ldots, m$ where

$$\mathrm{u}(x) = \frac{px}{1 + (x-1)\,p}. \tag{122}$$

Then we can follow the above procedure to evaluate the sums over $v_1, \ldots, v_m$. Going from the sum to an integral over the variables

$$y_i = \sqrt{t}\frac{v_i - u_i}{\sqrt{u_i\,(1-u_i)}}. \tag{123}$$

we arrive at the integral expression

$$\begin{aligned}
& \sum_{\{v_1,\ldots,v_m\}\in t^{-1}\mathbb{Z}^m_{\geq x_1^0}} \sum_{x_2=x_1+1}^{v_2 t + x_m^0} \cdots \sum_{x_m = x_{m-1}+1}^{v_m t + x_m^0} \\
& \mathcal{A}\left(\{\xi_i, v_i\}_{i=1}^m, \left\{v_i - \left(x_i - x_m^0\right)/t\right\}_{i=2}^m\right) \\
= \; & (1-p+\kappa p)^{(m-1)t}\left(1-p+\kappa^{1-m}p\right)^t \\
& \times \frac{(u-u_1)^{m-1}(1-u)^{\frac{(m-1)(m-2)}{2}}}{u^{m(m-1)/2}u_1^{m-1}}\left(\frac{p}{1-p}\right)^{\frac{m(m-1)}{2}} \\
& \times \frac{\prod_{k=1}^m \left(\xi_k - \frac{(1-u)p}{(1-p)u}\right)^{1-m}\left(\xi_k - \frac{(1-u_1)p}{(1-p)u_1}\right)^{-1}}{(2\pi)^{m/2}\prod_{i=2}^m\left[(i-2)!\left(\kappa^{i-1}-1\right)\right]} \\
& \times \int_{-\infty}^{+\infty} dy_1 e^{-y_1^2/2} \cdots \int_{-\infty}^{+\infty} dy_m e^{-y_m^2/2} \prod_{2\leq l<j\leq m}(y_j - y_l) \\
& \times \prod_{s=2}^m \left(y_s + \frac{u\sqrt{t}}{\sqrt{u(1-u)}}\right)^{s-2}\left(1 + O\left(\frac{1}{t}\right)\right).
\end{aligned} \tag{124}$$

Note that we keep the leading terms of the Taylor expansion in $(v_i - u_i)$, $i = 1, \ldots, m$, everywhere under the integral except the product in the last line, where we keep the terms of two subsequent orders. The reason for the latter is that the leading terms of the multipliers cancel due to the antisymmetry of the rest of the expression in $v_2, \ldots, v_m$. Therefore, what contributes is the antisymmetric part of this line, that is $\prod_{2\leq l<j\leq m}^m (y_l - y_j)\,/\,(m-1)!$, which contains only the terms of the same order. Inserting it, we again arrive at the Mehta integral $I_{m-1,1}$ over $(m-1)$, variables $y_2, \ldots, y_m$, while the integral over y_1 decouple being just the Laplace integral. After substitution of the explicit form of u and u_1 the r.h.s. of (124) becomes

$$\frac{(1-p+\kappa p)^{(m-1)t}\left(1-p+\kappa^{1-m}p\right)^t(\kappa^m-1)^{m-1}\kappa^{-\frac{m(m-1)}{2}}}{\prod_{i=2}^m\left[(i-1)!\left(\kappa^{i-1}-1\right)\right]}$$

$$\times I_{m-1,1}\left(2\pi\right)^{-\frac{m-1}{2}}\prod_{k=1}^{m}\left(\xi_k-\frac{1}{\kappa}\right)^{1-m}\left(\xi_k-\kappa^{m-1}\right)^{-1}. \tag{125}$$

This formula together with (73), (102) and (98) yields

$$\begin{aligned}\mathcal{P}_t\left(X^0\right) &= \left(1-p+\kappa p\right)^{(m-1)t}\left(1-p+\kappa^{1-m}p\right)^{t} \\ &\times \frac{\left(\kappa^m-1\right)^{m-1}\left(-1\right)^{\frac{m(m-1)}{2}}}{\prod_{i=1}^{m-1}\left(\kappa^i-1\right)}\prod_{k=1}^{m}\oint_{C_{r>\kappa^{m-1}}}\frac{d\xi_k}{2\pi\mathrm{i}} \\ &\times \frac{\xi_k^{x_m^0-x_k^0}}{\left(\xi_k-\kappa^{-1}\right)^{m-k}\left(\xi_k-\kappa^{m-1}\right)}\prod_{1\le i<j\le m}(\xi_j-\xi_i).\end{aligned} \tag{126}$$

The integration over ξ_k, $k=1,\ldots,m$ is performed over a circle encircling the singularities of the expression under the integral, i.e. $|\xi_k|>\kappa^{m-1}$. First we integrate over ξ_m, then over ξ_{m-1}, $\xi_{m-2},\cdots,\xi_1$. It turns out that, if we integrate in this order, the expression under the integral being evaluated will contain only one simple pole at each step. As a result we arrive at the simple one term expression

$$\begin{aligned}&\prod_{k=1}^{m}\oint_{C_{r>\kappa^{m-1}}}\frac{d\xi_k}{2\pi\mathrm{i}}\frac{\xi_k^{x_m^0-x_k^0}}{\left(\xi_k-\kappa^{-1}\right)^{m-k}\left(\xi_k-\kappa^{m-1}\right)}\prod_{1\le i<j\le m}(\xi_j-\xi_i) \\ =\; &\left(-1\right)^{\frac{m(m-1)}{2}}\kappa^{\sum_{k=1}^{m-1}\left(x_k^0-x_m^0\right)}\end{aligned} \tag{127}$$

This formula together with (126) results in (19).

Remark 2 An extension of the approximation technique used to complex values of κ is problematic for $|\kappa|>1$. The reason is that in this case the critical points $u_1,\ldots,u_m$ are away from the real axis. It follows then that their contribution can be exponentially smaller the other corrections appearing.

4.2.2 The limiting case $\kappa\to1$

Now we consider the limiting case

$$t\to\infty,\kappa\to1,(1-\kappa)\sqrt{t}=const. \tag{128}$$

We start with the formula (68) and expand the term $(1-\kappa z_i)^{-i+1}$ into its Taylor series,

$$\begin{aligned}P_t\left(X,X^0\right) &= \sum_{\{n_i\}\in\mathbb{Z}^m_{\ge0}}\prod_{i=2}^{m}\kappa^{n_i}\begin{pmatrix} i+n_i-2 \\ n_i\end{pmatrix} \\ &\times\det\left(\oint\frac{dz_i}{2\pi\mathrm{i}z_i}\left(1-p+\frac{p}{z_i}\right)^{t}z_i^{x_i+n_i-x_j^0}(1-\kappa z_i)^{j-1}\right)_{1\le i,j\le m}.\end{aligned} \tag{129}$$

The integral in the determinant can be evaluated in the saddle point approximation, the analysis being similar to the one above, (80)-(86), with the same function $h_i(z)$, (80) except that v_i now depends on n_i:

$$v_i=\frac{x_i+n_i-x_m^0}{t}. \tag{130}$$

What is special about the limit $\kappa \to 1$ is that the saddle point can coincide with a zero of the factor $(1-\kappa z)^j$ within the effective range of the summation over v_i. Therefore, instead of expanding this term into a Taylor series, we leave it in the integral as is, while the rest can be expanded around the saddle point as usual. Then we use the following formula for Hermite polynomials (see [17], formula 3.462.4),

$$\int_{-\infty}^{\infty} e^{-x^2} (x-\beta)^n \, dx = \sqrt{\pi} \left(\frac{\mathrm{i}}{2}\right)^n H_n(\mathrm{i}\beta). \tag{131}$$

As a result we obtain

$$\begin{aligned} \oint \frac{dz}{2\pi \mathrm{i} z} e^{t h_i(z)} z^{x_m^0 - x_j^0} (1-\kappa z)^{j-1} &= e^{t h_i(z^*)} (z^*)^{x_m^0 - x_j^0 - 1} \\ &\times \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{-\frac{1}{2} t |h_i''(z^*)| \xi^2} (1-\kappa(z^* + \mathrm{i}\xi))^{j-1} \\ = \quad & \frac{e^{t h_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)}}{\sqrt{\pi}(2t)^{j/2}} H_{j-1}\left(\sqrt{\frac{t v_i (1-v_i)}{2}} \left(\frac{1}{\kappa} \frac{(1-p) v_i}{(1-v_i) p} - 1\right)\right) \\ & \times \kappa^{j-1} \left(\frac{p}{1-p}\right)^{j + x_m^0 - x_j^0} \frac{(1-v_i)^{x_m^0 - x_j^0 - j/2}}{v_i^{x_m^0 - x_j^0 - 3j/2}} \left(1 + O\left(\frac{1}{t}\right)\right). \end{aligned} \tag{132}$$

Next, we argue that the dominant range of the summation over X and $\{n_i\}_{i=1}^m$ is the domain

$$pt - \sqrt{t} \log t \le x_i + n_i \le pt + \sqrt{t} \log t, \tag{133}$$

where x_i varies within the range

$$x_1^0 \le x_1 \le \cdots \le x_m \le t, \tag{134}$$

and

$$0 \le n_i < \infty. \tag{135}$$

To this end, consider the integral (132) for some particular i and j. After expanding $(1-\kappa z_i)^j$ into a binomial sum, it becomes a finite sum of terms like

$$(1-p)^{t-(n_i+x_i-x_j^0+k)} \, p^{n_i+x_i-x_j^0+k} \begin{pmatrix} t \\ n_i + x_i - x_j^0 + k \end{pmatrix}, \tag{136}$$

where k is a finite integer, $0 \le k \le j$. Beyond the range (133) this can be estimated using Stirling's formula to be $O(t^{-1/2} e^{-\frac{(\log t)^2}{2p(1-p)}})$. The summation over x_i, which includes at most t nonzero terms, multiplies this estimate by a factor of t. Finally the summation over n_i yields an additional factor $(1-\kappa)^{-i}$, with the result that the order of the contribution from outside the domain (133) is

$$O\left((1-\kappa)^{-i} t^{1/2} e^{-\frac{(\log t)^2}{2p(1-p)}}\right). \tag{137}$$

Below, the leading term of the sum of interest will be shown to decay at most as a power of t. Therefore, when κ is such that $(1-\kappa) = O(t^{-s})$ with any fixed $s > 0$, the term (137) is asymptotically negligible.

One can approximate (132) using the Taylor formula, which yields

$$\det\left[H_{j-1}\left(\sqrt{\frac{tp\,(1-p)}{2}}\left(\frac{v_i-p}{p\,(1-p)}-1+\frac{1}{\kappa}\right)\right)\right]_{i,j=1}^{m}$$
$$\times\frac{\prod_{i=1}^{m}e^{-t\frac{(v_i-p)^2}{2p(1-p)}}}{\pi^{\frac{m}{2}}\left[2tp\,(p-1)\right]^{\frac{m(m+1)}{4}}}\left(1+O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right). \qquad (138)$$

The determinant can be simplified by adding to every line a multiple of the lines below it, such that all the terms of the Hermite polynomials except the highest cancel:

$$\det\left[H_{j-1}\left(a_i\right)\right]_{i,j=1}^{m}=(-2)^{\frac{m(m-1)}{2}}\,\Delta(a_1,\ldots,a_m) \qquad (139)$$

Thus, the survival probability takes the following form

$$\begin{aligned}\mathcal{P}_t\left(X^0\right) &= \sum_{\{x_i\}_{i=1}^m}\sum_{\{n_i\}_{i=2}^m}\frac{(-1)^{\frac{m(m-1)}{2}}}{(2\pi)^{\frac{m}{2}}\,t^{\frac{m(m+1)}{2}}\left[p\,(p-1)\right]^{\frac{m^2}{2}}}\\ &\times \prod_{i=2}^{m}\kappa^{n_i}e^{-\frac{\left(x_i+n_i-x_m^0-pt\right)^2}{2p(1-p)t}}\binom{i+n_i-2}{n_i} \qquad (140)\\ &\times \Delta(x_1,x_2+n_2,\ldots,x_m+n_m)\left[1+O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right],\end{aligned}$$

where the summation is over the domains of $\{x_i\}_{i=1}^m$ and $\{n_i\}_{i=2}^m$ defined by the inequalities (133)-(135). Due to the presence of the Gaussian factor $\exp(-t(v_i-p)^2/(2p(1-p)t))$ the sum over n_i converges uniformly in x_i. Therefore we can interchange the order of summations over x_i and over n_i. This allows us to apply Lemma 1 first to the variables x_i and then also to the variables n_i. As the characteristic scale of n_i is of order $\sqrt{t}$ we can use the approximation (116) for the binomial coefficient, where the correction term yields an error of the order $O\left(t^{-1/2}\right)$ in the final result. To write down the final limiting formula as $t\to\infty$ for $\mathcal{P}\left(X^0\right)$ we introduce the rescaled variables

$$u_i = \frac{(x_i-pt)}{\sqrt{tp\,(1-p)}}, \qquad (141)$$

$$\nu_i = \frac{n_i}{\sqrt{tp\,(1-p)}}. \qquad (142)$$

and the transition parameter α, (23), which is constant in the limit under consideration. The formula (140) then takes the form (24) , where $f_m\,(\alpha)$ is given by (25).

5 Asymptotic behaviour of $f_m\,(\alpha)$

In this section we evaluate the limiting behaviour of $f_m(\alpha)$ for $\alpha\to\infty$ and $\alpha\to-\infty$ and its value at $\alpha=0$. In the latter case it is just the probability normalization of the TASEP, so $f_m\,(0)$ must be equal 1. The limit $\alpha\to-\infty$

has no probabilistic meaning, but it can be considered a particular limit of the generating function of the rescaled particle current in the TASEP: see Section 2. Let us introduce the notation

$$\begin{aligned} J_m(\alpha) &= \int_{-\infty}^{\infty} du_1 \int_{u_1}^{\infty} du_2 \cdots \int_{u_{m-1}}^{\infty} du_m \int_0^{\infty} d\nu_2 \cdots \int_0^{\infty} d\nu_m \\ &\times \quad e^{-\frac{1}{2}u_1^2} \prod_{i=2}^{m} \nu_i^{i-2} e^{-\frac{1}{2}(u_i+\nu_i)^2 - \alpha\nu_i} \Delta\left(u_1, \nu_2+u_2, \ldots, \nu_m+u_m\right). \end{aligned} \tag{143}$$

for the multiple integral entering into the expression of $f_m(\alpha)$. Then we have

$$f_m(\alpha) = \frac{(-1)^{\frac{m(m-1)}{2}}}{(2\pi)^{\frac{m}{2}}\, 2! \cdots (m-2)!} J_m(\alpha). \tag{144}$$

The form of $J_m(\alpha)$ is reminiscent of the multiple integrals which appear in the theory of Gaussian random matrix ensembles. The following three lemmas show that in the three limiting cases $J_m(\alpha)$ can be explicitly evaluated in the form of Mehta integrals.

Lemma 2

$$\lim_{\alpha\to\infty} \alpha^{\frac{m(m-1)}{2}} J_m(\alpha) = I_{m,1/2} \frac{(-1)^{\frac{m(m-1)}{2}}\, 2! \cdots (m-2)}{m!} \tag{145}$$

where $I_{m,1/2}$ is the Mehta integral defined in (98).

Proof Let us make a variable change under the integral (25) introducing new integration variables

$$\varphi_1 = u_1, \tag{146}$$

$$\varphi_i = \nu_i + u_i, i = 2, \ldots, m, \tag{147}$$

$$\mu_{i-1} = \alpha\nu_i, i = 2, \ldots, m. \tag{148}$$

In the new variables the integral (25) can be written as

$$J_m(\alpha) = \frac{1}{\alpha^{m(m-1)/2}} \prod_{i=1}^{m-1} \int_0^{\infty} d\mu_i \mu_i^{i-1} e^{-\mu_i} g(\mu_1, \ldots, \mu_{m-1}; \alpha), \tag{149}$$

where we introduce the notation

$$\begin{aligned} & g(\mu_1, \ldots, \mu_{m-1}; \alpha) \\ &= \int_{-\infty}^{\infty} d\varphi_1 \int_{\varphi_1+\frac{\mu_1}{\alpha}}^{\infty} d\varphi_2 \int_{\varphi_2+\frac{\mu_2-\mu_1}{\alpha}}^{\infty} d\varphi_3 \qquad (150) \\ & \cdots \int_{\varphi_{m-1}+\frac{\mu_{m-1}-\mu_{m-2}}{\alpha}}^{\infty} d\varphi_m e^{-\frac{1}{2}\left(\varphi_m^2+\cdots+\varphi_1^2\right)} \Delta\left(\varphi_1, \ldots, \varphi_m\right). \qquad (151) \end{aligned}$$

The function $g(\mu_1, \ldots, \mu_{m-1}; \alpha)$ is bounded uniformly in $\alpha \in \mathbb{R}$.

$$|g(\mu_1, \ldots, \mu_{m-1}; \alpha)| \leq I_{m,1/2}, \tag{152}$$

which can be shown by replacing the Vandermonde determinant under the integral by its absolute value and extending the lower integration limits to minus infinity. Therefore the function under the integral in (149) is uniformly bounded and integrable. By the dominating convergence theorem one can interchange the limit $\alpha \to \infty$ and integration. Then, for the function $g(\mu_2, \ldots, \mu_m; \alpha)$ we have

$$\lim_{\alpha \to \infty} g(\mu_2, \ldots, \mu_m; \alpha) = I_{m,1/2} \frac{(-1)^{\frac{m(m-1)}{2}}}{m!}. \tag{153}$$

Remarkably the limiting value does not depend on the variables $\{\mu_1, \ldots, \mu_{m-1}\}$. Therefore the integration in (149) can be performed independently for each $i = 2, \ldots, m$, each resulting in $(i-1)!$. This yields (145).

Lemma 3

$$J_m(0) = \frac{(-1)^{\frac{m(m-1)}{2}}}{(m-1)!m!} I_{m,1} \tag{154}$$

Proof Let us make the variable change

$$\begin{aligned} \chi_1 &= u_1 \\ \chi_i &= \nu_i + u_i, i = 2, \ldots, m. \end{aligned} \tag{155}$$

Then the integral takes the form

$$J_m(0) = \int_{-\infty}^{\infty} d\chi_1 \int_{\chi_1}^{\infty} du_2 \int_{u_2}^{\infty} du_3 \cdots \int_{u_{m-1}}^{\infty} du_m \int_{u_2}^{\infty} d\chi_2 \tag{156}$$

$$\cdots \int_{u_m}^{\infty} d\chi_m e^{-\frac{1}{2}\chi_1^2} \prod_{i=2}^{m} (\chi_i - u_i)^{i-2} e^{-\frac{1}{2}\chi_i^2} \Delta(\chi_1, \ldots, \chi_m).$$

The integrals over u_i, for $i = 1, \ldots, m$, can be evaluated step by step. First, for $i = m$, we have

$$\int_{u_{m-1}}^{\infty} du_m \int_{u_m}^{\infty} d\chi_m e^{-\frac{1}{2}\chi_m^2} (\chi_m - u_m)^{m-2} \Delta(\chi_1, \ldots, \chi_m) \tag{157}$$

$$= \frac{1}{m-1} \int_{u_{m-1}}^{\infty} d\chi_m e^{-\frac{1}{2}\chi_m^2} (\chi_m - u_{m-1})^{m-1} \Delta(\chi_1, \ldots, \chi_m),$$

which is done by changing the integration order. In the next step, the integral over u_{m-1} can be calculated by parts:

$$\int_{u_{m-2}}^{\infty} du_{m-1} \int_{u_{m-1}}^{\infty} d\chi_{m-1} \int_{u_{m-1}}^{\infty} d\chi_m e^{-\frac{1}{2}\chi_{m-1}^2 - \frac{1}{2}\chi_m^2}$$

$$\times\left(\chi_{m-1}-u_{m-1}\right)^{m-3}\left(\chi_m-u_{m-1}\right)^{m-1}\Delta\left(\chi_1,\ldots,\chi_m\right)$$

$$= \frac{1}{m-2}[\int_{u_{m-2}}^{\infty} d\chi_{m-1}\int_{u_{m-2}}^{\infty} d\chi_m e^{-\frac{1}{2}\chi_{m-1}^2-\frac{1}{2}\chi_m^2} \tag{158}$$

$$\times\left(\chi_{m-1}-u_{m-2}\right)^{m-2}\left(\chi_m-u_{m-2}\right)^{m-1}\Delta\left(\chi_2,\ldots,\chi_m\right)$$

$$-(m-1)\int_{u_{m-2}}^{\infty} du_{m-1}\int_{u_{m-1}}^{\infty} d\chi_{m-1}\int_{u_{m-1}}^{\infty} d\chi_m e^{-\frac{1}{2}\chi_{m-1}^2-\frac{1}{2}\chi_m^2}$$

$$\times\left(\chi_{m-1}-u_{m-1}\right)^{m-2}\left(\chi_m-u_{m-1}\right)^{m-2}\Delta\left(\chi_1,\ldots,\chi_m\right)].$$

The second term cancels because of the antisymmetry of the Vandermonde determinant with respect to interchange of χ_m and χ_{m-1}. Iterating this procedure we remove $(m-1)$ integrals in the variables $u_2,\ldots,u_m$.

$$\frac{1}{(m-1)!}\int_{-\infty}^{\infty} d\chi_1\int_{\chi_1}^{\infty} d\chi_2\cdots\int_{\chi_1}^{\infty} d\chi_m \tag{159}$$

$$\times e^{-\chi_1^2}\prod_{i=2}^{m}\left(\chi_i-\chi_1\right)^{i-1}e^{-\frac{1}{2}\chi_i^2}\Delta\left(\chi_1,\ldots,\chi_m\right)$$

A symmetrization of the expression under the integral in the variables χ_i, $i = 2,\ldots,m$, yields another Vandermonde determinant. Thus

$$J_m(0) = \frac{(-1)^{\frac{m(m-1)}{2}}}{\left((m-1)!\right)^2}\int_{-\infty}^{\infty} d\chi_1\int_{\chi_1}^{\infty} d\chi_2 \tag{160}$$

$$\cdots\int_{\chi_1}^{\infty} d\chi_m e^{-\frac{1}{2}\left(\chi_1^2+\cdots+\chi_m^2\right)}\left|\Delta\left(\chi_1,\ldots,\chi_m\right)\right|^2.$$

Finally we add this integral to the $(m-1)$ similar ones, obtained by interchanging χ_1 with each of $\chi_2,\ldots,\chi_m$, and divide the sum by m.

$$J_m(0) = \frac{(-1)^{\frac{m(m-1)}{2}}}{(m-1)!m!}\int_{-\infty}^{\infty} d\chi_1 \tag{161}$$

$$\cdots\int_{-\infty}^{\infty} d\chi_m e^{-\frac{1}{2}\left(\chi_1^2+\cdots+\chi_m^2\right)}\left|\Delta\left(\chi_1,\ldots,\chi_m\right)\right|^2$$

This gives us the stated result.

Lemma 4

$$\lim_{\alpha\to-\infty} e^{-\alpha^2 m(m-1)/2}J_m(\alpha) = I_{m-1,1}\frac{\sqrt{2\pi}\,(-1)^{\frac{m(m-1)}{2}}}{\left((m-1)!\right)^2}m^{m-1}. \tag{162}$$

Proof We start from the integral in (25) and make a change of variables as follows,

$$x_i = \nu_i + \alpha + u_1, \quad i = 1, \ldots, m-1 \tag{163}$$
$$s_i = |\alpha| (u_i - u_1), \quad i = 1, \ldots, m-1 \tag{164}$$
$$s_1 = u_1 - \alpha (m-1), \tag{165}$$

which yields the following integral expression,

$$J_m(\alpha) = \frac{e^{\alpha^2 \frac{m(m-1)}{2}}}{|\alpha|^{m-1}} \int_{-\infty}^{\infty} ds_1 e^{-\frac{1}{2}s_1^2} \int_0^{\infty} ds_2 e^{-s_2} \int_{s_2}^{\infty} ds_3 e^{-s_3} \cdots \tag{166}$$
$$\times \int_{s_{m-1}}^{\infty} ds_m e^{-s_m} \int_{s_1+\alpha m}^{\infty} dx_2 \cdots \int_{s_1+\alpha m}^{\infty} dx_m \prod_{i=2}^{m} (x_i - s_1 - \alpha m)^{i-2}$$
$$\times \prod_{i=2}^{m} e^{-\frac{1}{2}\left[x_i^2 + \frac{1}{|\alpha|}\left(2 s_i x_i + \frac{s_i^2}{|\alpha|}\right)\right]} \Delta\left(s_1 + \alpha m, x_2 + \frac{s_2}{|\alpha|} \ldots, x_m + \frac{s_m}{|\alpha|}\right).$$

Due to the presence of the Gaussian and exponential terms, the main contribution to the integral comes from finite values of $s_1, \ldots, s_m$ and $x_2, \ldots, x_m$. Therefore, up to corrections of order of $O(1/|\alpha|)$, we can neglect the terms divided by $|\alpha|$, and extend the lower limits of integration over $x_2, \ldots, x_m$ to $-\infty$. The the integrals over $s_2, \ldots, s_m$ decouple, and we can evaluate them to $1/(m-1)!$. The Vandermonde determinant becomes antisymmetric with respect to permutations of the variables $x_2, \ldots, x_m$. As the integration is over the symmetric domain, we can leave only the antisymmetric part of the rest of the expression. The product $\prod_{i=2}^{m}\left(x_i - s_1 - \frac{\alpha m}{2}\right)^{i-2}$ then results in

$$(-1)^{\frac{(m-1)(m-2)}{2}} \frac{\Delta(x_2 \ldots, x_m)}{(m-1)!}. \tag{167}$$

After collecting the leading order terms from the first argument of

$$\Delta(s_1 + \alpha m, x_2 \ldots, x_m) \simeq (\alpha m)^{m-1} \Delta(x_2 \ldots, x_m), \tag{168}$$

the integral over s_1 decouples as well, and yields $\sqrt{\pi}$. We finally obtain

$$J_m(\alpha) \simeq \frac{e^{\frac{\alpha^2 m(m-1)}{2}} \sqrt{2\pi} (-1)^{\frac{(m-1)(m-2)}{2}}}{|\alpha|^{m-1} ((m-1)!)^2} (\alpha m)^{m-1}$$
$$\times \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_m e^{-\frac{1}{2}\sum_{i=2}^{m} x_i^2} |\Delta(x_2, \ldots, x_m)|^2. \tag{169}$$

Using the definition of the Mehta integral, (98) we obtain (162).

The above lemmas, the formula for the Mehta integral (98) and the definition (144) of $f_m(\alpha)$ establish the results (27)-(29).

6 Discussion of the results and conclusion

The first result obtained in this paper is an expression for the survival probability for m walkers in the SVW model. At the fixed parameter $\kappa < 1$ and $t \to \infty$ the leading asymptotics $t^{-m(m-1)/4}$ coincides with that for the usual VW [14]. This result is intuitively clear. In the case of the VW it is obtained by considering the evolution of independent noninteracting particles and reducing the number of its outcomes by dropping those realizations where the crossings of particle time space trajectories occur. Then, the asymptotics of the survival probability for two walkers $t^{-1/2}$ follows directly from the diffusion law and the method of images. For m walkers, the method of images involves $m(m-1)/2$ reflecting wall planes. Each wall brings with it a factor $t^{-1/2}$ providing total contribution $t^{-m(m-1)/4}$. For any fixed $\kappa < 1$ the events of crossing occur with a finite, though changed, probability, so that this argument still holds. The survival probability changes by a constant factor dependent on κ, leaving Fisher's law unchanged. As κ approaches one this factor diverges, which indicates a qualitative change of behaviour of the survival probability. As the crossings become less probable, it finally saturates to the TASEP normalization constant. The transition between the two regions takes place on the scale $(1-\kappa)\sqrt{t} = const$, where the effect of the diffusive spreading of particles becomes comparable to the one caused by SVW interaction. In this case, the survival probability is expressed by the scaling function $f_m(\alpha)$, which has the SVW and TASEP asymptotics as limiting cases.

The formulas for the survival probability in SVW can be reinterpreted in terms of the moment generating function of the time integrated particle current Y_t in TASEP, which, in turn, can be used to construct the distribution of the same quantity. The cases of generic $\kappa < 1$ and $\kappa > 1$ correspond to the tails of the probability distribution of Y_t at the large deviation scale, characterizing positive, $(Y_t - \langle Y_t \rangle)/t > 0$, and negative, $(Y_t - \langle Y_t \rangle)/t < 0$, deviations respectively. The positive tail has the form specific for the current distribution for m free independent Bernoulli particles, while the form of the negative tail looks like that of the distribution for one Bernoulli particle that makes m steps at a time or m particles jumping one step synchronously. Such asymmetry reveals different mechanisms of positive and negative fluctuations. For positive deviations m particles have to be "accelerated" independently. The main contribution to the negative deviations comes from the events when the first particle decelerates all particles following behind. Our results are to be compared to the ones obtained by Derrida and Lebowitz [8] (see also [7]) for the large deviation function of the TASEP current on a ring. In particular, they have shown that for large m the non-universal tails of the current probability have the form $P(Y_t/t = v) \sim e^{mH_+(v/m)t}$ and $P(Y_t/t = v) \sim e^{H_-(v/m)t}$ for positive and negative deviations respectively. Our results have the same functional form even for finite m with $H_-(v) = H_+(v)$ being a simple large deviation function of the Bernoulli process. Remarkably, such a mechanism survives on the infinite lattice, despite the particles drifting apart from each other in the course of time, so that they meet less and less often. The acceleration-deceleration asymmetry was also observed in the large deviations of the distance travelled by individual particles in TASEP studied in [19] for a general case of particle dependent hopping rates. There the negative large deviations do not depend on the order number of a particle whereas the positive ones do.

The result obtained for the SVW in the transition region $(1-\kappa)\sqrt{t} = const$ provides us with the limiting distribution of the particle current measured at the diffusive scale, $|Y_t - \langle Y_t \rangle| \sim \sqrt{t}$. The distribution obtained is parameter free, dependent only on the number of particles. We expect that it is a universal distribution for the systems of particles performing a driven diffusion, independent of the details of microscopic dynamics. The distribution has a skew, non-Gaussian form, with tails matching the large deviation behaviour asymptotically.

Several directions for future work can be mentioned. First direct continuation of the present paper is an asymptotic study of the function $f_m(\alpha)$ as $m \to \infty$. Considering a certain common scaling with the variable α is expected to give a new universal scaling function characterizing the KPZ class. Note that in the limiting cases this function reduces to Mehta integrals, which are the probability normalization constants for Gaussian random matrix ensembles, namely the Unitary and Orthogonal ensembles. The asymptotic evaluation of these integrals performed in the random matrix theory resulted in non-trivial densities of critical points, from which comes the leading contribution to the integrals [32]. It would be interesting to see how these densities transform from one to another as the parameters vary between the limiting cases corresponding to different ensembles. It would also be interesting to study the TASEP confined in a ring. The starting point of this analysis could be the recently obtained expressions for the Green functions [36]. In this way one could obtain a scaling function that characterizes the behaviour of KPZ interfaces in finite systems.

Another possible development of the present article is a generalization of the above mentioned results for the probability distributions of the distance travelled by an individual particle in the TASEP and the corresponding correlation functions to SVW case. Note that similar results exist also for the VW [1]. In both cases the appropriately rescaled distribution of the distance travelled by a single particle starting from a half filled lattice is shown to converge, to the so-called Tracy-Widom distributions [43], which appear in the random matrix theory as a distribution of maximal eigenvalue in the Gaussian ensembles, unitary in case of TASEP and orthogonal for VW. The SVW model establishes a bridge between these two cases. However, its Green function has neither a Töplitz form like in VW nor a special structure like in TASEP, which allowed Sasamoto, [39], to reinterpret it again as a problem of the VW and finally to represent its evolution as a determinantal point process. Therefore, a significant extension of the existing techniques is in order. In this connection we should mention the recent advance for the Partially Asymmetric Simple Exclusion Process [44]-[46], which was made only on the basis of the Bethe Ansatz solution without use any free fermionic representation like VW.

An interesting example of SVW has been proposed recently by Johansson [22] in his analysis of a domino tilling problem on the Aztec diamond known as the arctic circle problem. It was shown that the domino configurations are in one-to-one correspondence with trajectories of an n-particle process which is defined as follows. At each discrete time step a particle jumps forward with probability q or stays put with probability $p = 1-q$. If the next site is occupied, the probability to stay put is $1 - q(1-\kappa)$ as in the SVW model. In addition, after each step, a particle i can be translated back to the distance s_i with probability q^{s_i} provided that $s_i < X_i - X_{i-1}$ for all i. If one chooses $\kappa = -q$, the model belongs to the free fermion class and its transition probabilities admit a determinant representation. It has been shown in [22] that the position of the first particle is described by the Airy process in the thermodynamic limit for the domain wall boundary conditions in the domino tiling problem. By similarity of the models, one may expect that the extremal statistics of the SVW model also exhibits a kind of Tracy-Widom distribution for appropriate initial conditions.

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