

Non-locality of non-Abelian anyons

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Topological systems, such as fractional quantum Hall liquids, promise to successfully combat environmental decoherence while performing quantum computation. These highly correlated systems can support non-Abelian anyonic quasiparticles that can encode exotic entangled states. To reveal the non-local character of these encoded states we demonstrate the violation of suitable Bell inequalities. We provide an explicit recipe for the preparation, manipulation and measurement of the desired correlations for a large class of topological models. This proposal gives an operational measure of non-locality for anyonic states and it opens up the possibility to violate the Bell inequalities in quantum Hall liquids or spin lattices.

Quantum mechanics is a non-local theory: it allows for correlations between distant systems that cannot be explained in terms of a local preparation. Many believed that non-locality was due to incompleteness of quantum theory. Einstein, Podolsky and Rosen (EPR) in their seminal work [1] aimed to resolve this by introducing *local hidden variables*. Their values would complement the information supplied by quantum mechanics, thus restoring locality. Bell inequalities aim at validating or rejecting this view from experimental data [2]. To date, unlike local hidden variable (LHV) theories, the predictions of quantum mechanics have been consistent with all Bell tests.

Particle exchange gives a striking example of non-locality in quantum mechanics. For bosons and fermions, one can deal with the exchange interactions by imposing non-local constraints on the form of the wave function. More generally, the wave function can transform in a nontrivial representation of the fundamental group of configuration space when particles are adiabatically exchanged [3]. For planar systems, this group is the braid group and particles transforming in nontrivial braid group representations have been dubbed *anyons* [4]. Anyonic exchange interactions are topological in nature and do not change on variation of the distance between the particles, or the metric of the spacetime manifold. One mechanism for such interactions is the Aharonov Bohm effect [5].

It is natural to ask if the non-local correlations of anyons can in principle be explained using local hidden variables. In this paper we answer this question for a number of anyon models, by constructing a Bell test for anyonic degrees of freedom and showing violation of the associated Bell inequalities.

Anyons can be split into two main categories, they can be Abelian or non-Abelian. Given labels $\{a_j\}$ for the different types of anyons we assign fusion rules that determine the outcome of bringing two anyons together, $a_i \times a_j = \sum_k N_{a_i a_j}^{a_k} a_k$. Here $N_{ab}^c \in \mathbb{N}$ counts the number of ways of combining a and b to obtain c . Non-Abelian anyons have $\sum_c N_{ab}^c \geq 2$ for some pair a, b , while in the Abelian case, the labels of the fused anyons determine a unique outcome which can be determined in a unique way.

In a physical system with anyons, the low energy part

of the Hilbert space can be thought of as a tensor product $\mathcal{H} = \mathcal{H}_{\text{local}} \otimes \mathcal{H}_{\text{non-local}}$, where the first factor describes local degrees of freedom, which we will ignore, and the second describes topological degrees of freedom associated with the anyons. These topological degrees of freedom may arise as a result of nontrivial topology of the space supporting the anyons. For Abelian anyons, this is in fact the only possibility; in the Abelian toric code models [6] for instance, the non-local degrees of freedom are described by elements of the first homology groups of the surface with finite group coefficients. In principle, one probe non-local correlations in these topological degrees of freedom, but the observables involved would need to be non-local themselves [7].

For non-Abelian anyons, even on a contractible surface, there are non-local degrees of freedom associated with the different fusion outcomes. A number of proposals have been made on how the associated quantum numbers, or topological charges, might be measured by interferometry, see [8], [9] for further references and [10] for an overview of the measurement theory. We will not go into the details of interferometric measurements here, but rather just assume that we can do projective measurements onto the various fusion channels. The non-local Hilbert state space of n anyons (a_1, a_2, \dots, a_n) with total charge c has $\dim(\mathcal{H}_{\text{non-local}}) = \sum_{b_1, b_2, \dots, b_{n-2}} N_{a_1 a_2}^{b_1} N_{b_1 a_3}^{b_2} N_{b_2 a_4}^{b_3} \dots N_{b_{n-2} a_n}^c$. This Hilbert space usually does not admit a tensor product structure, e.g. the dimension could be prime, and thus does not obviously fit the usual paradigm for tests of non-locality. Nevertheless, we show that topological interactions can indeed be used to demonstrate non-locality in the EPR sense. In order to do this, we consider two classes of anyonic theories: the $SU(2)_k$ models, including the Fibonacci model [9], and a model based on discrete gauge theory [11, 12]. These models are important both for their potential to process quantum information fault tolerantly and for their viability for experimental realization. For these cases the fusion spaces are at most one dimensional, i.e. $N_{ab}^c < 2$ for all (a, b, c) . Non-commuting measurements project onto different ways of combining particles a, b, c to yield d . Measurement bases are labelled by the intermediate products x and x' obtained by fusing a, b, c

and are related by the recoupling formula: $|(abc) \rightarrow d; x\rangle = \sum_{x'} (F_{abc}^d)^{x'} |(a(bc) \rightarrow d; x')\rangle$.

To date, most Bell tests have been performed on entangled light beams, but there is certainly an interest in showing that material media can be used to demonstrate non-locality. Experiments involving a photon and an atom, two atoms, or even kaons have been proposed or even carried out [13]. The schemes we are proposing contribute to this effort of using ever new media for Bell tests. As we shall see, some of them could be implemented in a fractional quantum Hall liquid [9], while others can be associated with arrays of Josephson junctions [14] or atoms in optical lattices [15].

In order to build intuition for the anyonic case we describe the general framework by employing distinguishable spin-1/2 particles with non-trivial fusion properties. The fusion rules are interpreted as the angular momentum decomposition of tensor products of vector spaces. Consider a system divided into two spatially non-overlapping subsystems A and B , conveniently labeled as Alice and Bob, each one possessing three spin-1/2 particles, as seen in Fig. 1. First, we perform a joint measurement on the total spin $\vec{S}_{\text{tot}} = \sum_j \vec{s}_j$ and post-select the $S_{\text{tot}} = 0$ outcome that has state space dimension five. Second, we define a set of measurement operators $\{\Upsilon_{1,2}^A, \Upsilon_{2,3}^A, \Upsilon_{4,5}^B, \Upsilon_{5,6}^B\}$, where $\Upsilon_{i,j} = (\vec{s}_i + \vec{s}_j)^2 - \mathbf{1}$. The eigenvalues of $\Upsilon_{i,j}$ are $+1$ in the triplet space and -1 for the singlet and the operators $\Upsilon^{A(B)}$ act on the subsystems $A(B)$. The operator pair within A or B is non-commuting but $[\Upsilon_{i,j}^A, \Upsilon_{j,k}^B] = 0$. Consider the expectation value of the operator

$$W = \Upsilon_{1,2}^A \Upsilon_{4,5}^B + \Upsilon_{1,2}^A \Upsilon_{5,6}^B - \Upsilon_{2,3}^A \Upsilon_{5,6}^B + \Upsilon_{2,3}^A \Upsilon_{4,5}^B. \quad (1)$$

For a classical theory, even in the presence of local hidden variables (LHV) [16], the Bell inequality for W is $|\langle W \rangle_{\text{LHV}}| \leq 2$. This can be derived straightforwardly as follows [17]. Assume independence of the two subsystems (locality) so that the joint probabilities for pairs of outcomes is just the product of the individual probabilities which could depend on a hidden variable λ , drawn from a fixed distribution $p(\lambda)$. For the above quorum of observables with outcomes $\{m_{1,2}^A, m_{2,3}^A, m_{4,5}^B, m_{5,6}^B\} \in \pm 1$ we have

$$(m_{2,3}^A + m_{1,2}^A) m_{4,5}^B - (m_{2,3}^A - m_{1,2}^A) m_{5,6}^B = \pm 2.$$

Hence, in the LHV model, the outcomes must satisfy

$$\begin{aligned} |W_{\text{LHV}}| &= \left| \int d\lambda p(\lambda) \langle (\Upsilon_{1,2}^A(\lambda) \Upsilon_{4,5}^B(\lambda) \right. \\ &\quad \left. + \Upsilon_{1,2}^A(\lambda) \Upsilon_{5,6}^B(\lambda) - \Upsilon_{2,3}^A(\lambda) \Upsilon_{5,6}^B(\lambda) \right. \\ &\quad \left. + \Upsilon_{2,3}^A(\lambda) \Upsilon_{4,5}^B(\lambda)) \rangle \right| \\ &= \left| \int d\lambda p(\lambda) (m_{2,3}^A(\lambda) + m_{1,2}^A(\lambda)) m_{4,5}^B(\lambda) \right. \\ &\quad \left. - (m_{2,3}^A(\lambda) - m_{1,2}^A(\lambda)) m_{5,6}^B(\lambda) \right| \\ &\leq 2 \end{aligned}$$

Quantum mechanically, the maximum value of $|\langle W \rangle|$ is obtained for eigenstates of W with maximum eigenvalue, i.e. $|\langle W \rangle| \leq \sqrt{7}$. For arbitrary operators in Eq. (1) that have the same commutation structure and square to $\mathbf{1}$, quantum mechanics satisfies Tsirelson's inequality [18], $|\langle W \rangle| \leq \sqrt{8}$. Our

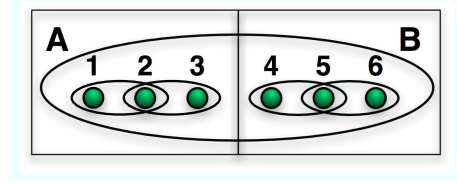


FIG. 1: A Bell type measurement on six particles. First a joint measurement (large oval) on all six particles is made and the result kept if the total charge (or spin) is zero. Alice performs measurements of total charge on pairs 1,2 and 2,3 and Bob performs measurements on pairs 4,5 and 5,6. For some quantum states the correlator $\langle W \rangle$ exceeds the bound set by local hidden variable theories.

aim is to find a violation of the classical upper bound in the subspace of states with $S_{\text{tot}} = 0$. Note that our protocol allows for measuring correlations without the need of a shared reference frame between Alice and Bob [19] thus giving a simple and unambiguous test of Bell inequalities. In the anyonic case treated below the operators $\Upsilon_{i,j}$ also have eigenvalue -1 when the fusion outcome is the vacuum and $+1$ otherwise.

There are two natural orthonormal bases for a three particle system based on the two different orders of fusing the three particles. These are graphically represented by fusion trees in Fig. 2b. The unitary transformation that describes the change from one of these bases to the other is given by the so called F matrices. The basis change for three particles with charges (a, b, c) fusing to d is given by a matrix denoted $F_{a,b,c}^d$. For the case of $SU(2)$ these matrices just describe angular momentum recoupling and their matrix elements are the Wigner 6- j symbols. For six particles with total spin 0, we get four natural product bases from the two pairs of bases for each triple. The contributing particle labels are the spin values $\{0, \frac{1}{2}, 1, \frac{3}{2}\}$ and the only relevant F -matrix for changing between bases is $F_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}}$, which, for the $SU(2)$ case is given by

$$F \equiv F_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

in the basis given by fusion trees with intermediate spins 0 and 1. We find the following set of orthonormal states for the six particle system, in the local fusion basis as defined in Fig. 2,

$$\begin{aligned} |\phi_0\rangle &= |0\rangle_A |0\rangle_B, & |\phi_1\rangle &= |1\rangle_A |0\rangle_B, & |\phi_2\rangle &= |0\rangle_A |1\rangle_B, \\ |\phi_3\rangle &= |1\rangle_A |1\rangle_B, & |\phi_4\rangle &= |1(\frac{1}{2}, \frac{3}{2})\rangle_A |1(\frac{1}{2}, \frac{3}{2})\rangle_B, \end{aligned} \quad (2)$$

where $|x\rangle \equiv |x(\frac{1}{2}, \frac{1}{2})\rangle$ and $|x'\rangle = \sum_x F_x^{x'} |x\rangle$. In spin components we have $|\phi_0\rangle = |\Psi^-\rangle_{1,2} \otimes |\Psi^-\rangle_{3,4} \otimes |\Psi^-\rangle_{5,6}$, where $|\Psi^-\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$, so the state $|\phi_0\rangle$ has three adjacent singlet pairs. Notice that in order to have trivial total charge the local bases occur in pairs that share the same label β as defined in Fig. 2.

$$\begin{aligned} W &= (F^\dagger \sigma^z F \otimes F^\dagger \sigma^z F + F^\dagger \sigma^z F \otimes \sigma^z + \sigma^z \otimes F^\dagger \sigma^z F - \sigma^z \otimes \sigma^z) \\ &\quad \oplus 2|\phi_4\rangle\langle\phi_4| \end{aligned} \quad (3)$$

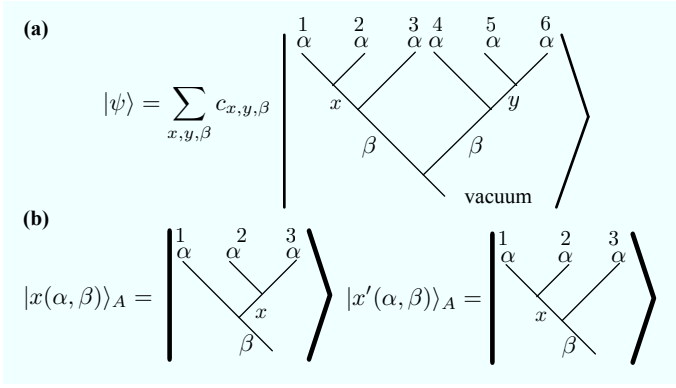


FIG. 2: The state space of anyons in our protocol represented as fusion trees. (a) An arbitrary state of six α type anyons with trivial total charge expanded in terms of fusion outcomes local to A and B . In the models considered here, β is its own antiparticle, but it is straightforward to generalize. (b) A “local” fusion basis satisfying $Y_{2,3}^A |x(\alpha, \beta)_A = \pm |x(\alpha, \beta)_A$ and $Y_{1,2}^A |x'(\alpha, \beta)_A = \pm |x'(\alpha, \beta)_A$ for x a vacuum state or a particle and similarly for B . For each subsystem the bases are related by an F move: $|x'(\alpha, \beta) = \sum_x (F_{\alpha\alpha x}^\beta)_x |x(\alpha, \beta)$.

The Hilbert space splits into different sectors labeled by β , which are conserved by the action of W . There is a four dimensional sector with $\beta = 1/2$ and a one dimensional sector with $\beta = 3/2$, containing $|\phi_4\rangle$. No Bell violation can occur in the $\beta = 3/2$ sector, since the measurement operators commute in that sector. Maximally Bell violating states are thus orthogonal to $|\phi_4\rangle$. For the one parameter family of states

$$|r(a)\rangle = \frac{a}{\sqrt{2}}(|\phi_0\rangle + |\phi_3\rangle) + \frac{\sqrt{1-a^2}}{\sqrt{2}}(|\phi_1\rangle - |\phi_2\rangle), \quad (4)$$

with $(-1 \leq a \leq 1)$ we plot the expectation value $\langle W \rangle$ seen in Fig. 3. The maximal violation ($\langle W \rangle_{\max, \min} = \pm \sqrt{7} \approx \pm 2.6458$) is obtained for $a_{\pm} = \mp \sqrt{(7 \pm 2\sqrt{7})}/14$.

Consider now a two dimensional system with quasiparticle excitations described by $SU(2)_k$ Chern-Simons-Witten theories. The corresponding fusion rules satisfy the addition of angular momentum with the constraints $j_1 \times j_2 \rightarrow j$ only if $j_1, j_2, j \leq k/2$ and $j_1 + j_2 + j \leq k$. It is quickly verified that for $k \geq 3$, the total charge *zero* sector of six particles labeled by spin $1/2$ again has five states, labeled by the same fusion trees as in the $SU(2)$ case. The F matrices will differ, but for our purposes, the only relevant recoupling is still $\tilde{F}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}}$. Computing the quantum $6-j$ symbols, we find (see for instance [20])

$$F \equiv \tilde{F}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{[2]_q} \begin{pmatrix} 1 & \sqrt{[3]_q} \\ \sqrt{[3]_q} & -1 \end{pmatrix}$$

where the quantum integers are defined as $[m]_q = \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}}$ for m integer. For the $SU(2)_k$ theories, $q = e^{\frac{2\pi i}{k+2}}$. In the limit $k \rightarrow \infty$, then $[m]_q \rightarrow m$. As before, we can label the states in the local fusion basis, as in Eq. 2, but with the appropriate F

matrix. The state $|\phi_0\rangle$ is obtained by creating spin-1/2 particle anti-particle pairs at positions (1,2), (3,4), (5,6) out of the vacuum. For the one parameter family of states $|r(a)\rangle$ we find maximal violation at

$$a_{\pm} = -\frac{1}{\sqrt{8\cos(\frac{2\pi}{k+2}) + \cos(\frac{4\pi}{k+2}) + 5}} \left[\cos^2(\frac{2\pi}{k+2}) + 4\cos(\frac{2\pi}{k+2}) + \sqrt{2\cos^4(\frac{\pi}{k+2})(8\cos(\frac{2\pi}{k+2}) + \cos(\frac{4\pi}{k+2}) + 5) + 2} \right]^{\frac{1}{2}} \quad (5)$$

and at $a_{-} = \sqrt{1 - a_{+}^2}$, where

$$\langle W \rangle = \pm \sec^2\left(\frac{\pi}{k+2}\right) \sqrt{4\cos\left(\frac{2\pi}{k+2}\right) + \frac{1}{2}\cos\left(\frac{4\pi}{k+2}\right) + \frac{5}{2}}.$$

It is possible to verify that $k \rightarrow \infty$ corresponds to $SU(2)$. A qualitative difference between anyonic systems and the spin systems discussed before is that, while the z -components of the spins of all particles are in principle measurable, there are not necessarily any observables associated with the z -components of the ‘q-spins’ of the anyons. Only $SU(2)_q$ invariant quantities, such as the total q-spins of groups of anyons, can be observables, or at any rate topologically protected observables. This can be traced back to the superselection rule that says that the total q-spin of all anyons together must be trivial. If it were possible to measure the z -components of every anyons’ q-spin, then the state obtained would no longer be invariant under $SU(2)_q$. In fact, a similar rule would hold for confined particles in gauge theory and so for a better analogy, one may think of the $SU(2)_q$ invariance as begin closer to an $SU(2)$ gauge symmetry rather than spin.

It was shown by Freedman *et al.* [21] that the anyonic theories with $k \geq 3$, $k \neq 4$ are universal for quantum computation. Hence, for those theories, the Bell violating states can be obtained by topological braiding operations alone acting, for example, on the fiducial state $|\phi_0\rangle$. We now check to see if it is possible to generate a state which violates the inequality for the $k = 2$ case. The $SU(2)_2$ anyons are believed to exist in the $\nu = 5/2$ plateau of the fractional quantum Hall effect [?] up to charge factors that affect the Abelian part of braiding. They come in three varieties, the vacuum, 1, the fermion, ψ , and the non-Abelian anyon, σ , that satisfy the non-trivial fusion rules, $\sigma \times \sigma = 1 + \psi$, $\sigma \times \psi = \sigma$ and $\psi \times \psi = 1$. The counter-clockwise exchange of two σ particles, which fuse to either 1 or ψ , results in the matrix evolution $R = 1 \oplus i$ expressed in the basis labeled by the fusion channels $\{1, \psi\}$. The state evolution produced by the exchange of particles with no immediate fusion channel is found by employing the recoupling matrix F . Expressed in the basis $\{|x(\sigma, \sigma)\rangle_A |y(\sigma, \sigma)\rangle_B; x, y \in \{1, \psi\}\}$, we have the following representation of the generators of the braid group \mathcal{B}_6

$$B_1 = e^{-i\frac{\pi}{4}\sigma^x} \otimes \mathbf{1}_2, B_2 = e^{-i\frac{\pi}{4}\sigma^z} \otimes \mathbf{1}_2, B_3 = e^{-i\frac{\pi}{4}\sigma^x \otimes \sigma^z}, \quad (6)$$

$$B_4 = \mathbf{1}_2 \otimes e^{-i\frac{\pi}{4}\sigma^x}, B_5 = \mathbf{1}_2 \otimes e^{-i\frac{\pi}{4}\sigma^z}$$

where B_j results from the exchange of j and $j+1$ particles in a counterclockwise manner. As a simple initial state we can consider $|\phi_0\rangle$ that is produced from (1,2), (3,4) and (5,6) pairs created from the vacuum.

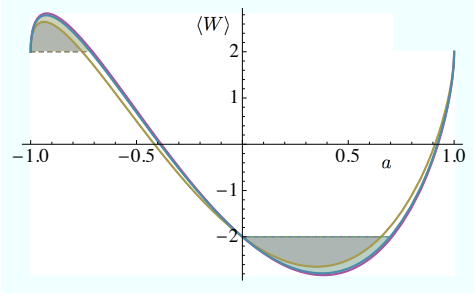


FIG. 3: The expectation value of the Bell witness W as a function of the amplitude of mixing for the total charge zero states $|r(a)\rangle$ in Eqs. (4). The yellow, blue, and red lines correspond to an $SU(2)$, $SO(3)_3$, $SU(2)_2$ theory with six spin-1/2, σ , τ particles, respectively. The shaded region corresponds to states which violate the inequality derived for local hidden variable models.

The braid group generators, B_j , are in the Clifford group, so we cannot generate a dense set in $SU(4)$ by braiding alone[31]. But can we still obtain Bell violating states? In Refs. [24, 25] a LHV model was introduced for a pair of qubits that exactly reproduces the set of allowed operations in the present model. There are two distinct configurations of shared vacuum pairs (up to relabeling of particles by Alice or Bob) both of which can be obtained from $|\phi_0\rangle$ by braiding. Hence it is not possible to build Bell violating states starting out from three shared vacuum pairs using topologically protected operations alone.

Despite the impossibility of producing a Bell violating state from $|\phi_0\rangle$ by topological gates, one can in fact straightforwardly obtain a maximally Bell violating state using non-topological gates[32]. Let us employ the non-Clifford gate $D = e^{-i\frac{\pi}{8}\sigma^z} \otimes \mathbf{1}$. This can be implemented by bringing the 2 and 3 σ anyons nearby, thus shifting the energy of the fermionic fusion channel [27] such that a relative phase $e^{i\pi/4}$ is accumulated on that channel. From these operations one can build the controlled phase gate in the following way $CP = e^{i\pi/4} B_2 B_1 B_2 B_3^{-1} B_2^{-1} B_1^{-1} B_5$. A simple searching algorithm provides us with the sequence that produces $|r(a_-)\rangle = -CPB_3 B_4 DB_2 B_3 |\phi_0\rangle$, with $\langle W \rangle = -2\sqrt{2}$, thus saturating the Tsirelson bound.

In fact, it is indeed possible to realize a maximally violating state without braiding at all. Consider a state $|\phi'_0\rangle$ given by a distribution of singlet pairs on (1,6), (2,5) and (3,4). This state $|\phi'_0\rangle$ is related to the fiducial distribution of pairs by the following braid word $|\phi'_0\rangle = \frac{1}{\sqrt{2}}(|0'\rangle_A |0\rangle_B + |1'\rangle_A |1\rangle_B) = B_2^{-1} B_3^{-1} B_5 B_4 B_3 B_2 |\phi_0\rangle$. This state would be maximally violating if we could measure in arbitrary local bases. For our fixed measurement quorum $|\phi'_0\rangle$ is related to the maximally violating state by $e^{-i\frac{\pi}{8}\sigma^y} \otimes \mathbf{1} |\phi'_0\rangle = |r(a_+)\rangle$ and hence it suffices

to implement the local unitary $e^{-i\frac{\pi}{8}\sigma^y} = e^{-i\frac{\pi}{4}\sigma^z} e^{-i\frac{\pi}{8}\sigma^x} e^{i\frac{\pi}{4}\sigma^z}$ on Alice's side to obtain a Bell violation. The z rotation is simply achieved by bringing anyons 2 and 3 near each other as above, and similarly the x rotation is performed by pushing 1 and 2 together. Note that the maximal Bell violation in these two constructions actually saturates the Tsirelson inequality, making the $SU(2)_2$ case at the same time 'maximally quantum mechanical' and 'topologically classical'.

Let us turn now to Fibonacci anyons from the $SO(3)_3$ theory. This is the theory obtained from $SU(2)_3$ but using only integer spin particles: the vacuum 1 and the non-Abelian anyon τ , with non-trivial fusion rule $\tau \times \tau = 1 + \tau$. All particles are their own anti-particles and the quantum dimensions are $d_1 = 1$ and $d_\tau = \phi \equiv (1 + \sqrt{5})/2$. The relevant recoupling matrix is

$$F = F_{\tau\tau\tau}^\tau = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}$$

expressed in the basis of 1 and τ . The dimension of the topological Hilbert space of $m+1$ type τ anyons with total charge zero is f_m , the m th Fibonacci number, hence there are five states in the fusion space. These states can be decomposed into superpositions of products of local basis states as in Eq. 2 where $|0\rangle = |1(\tau, \tau)\rangle$ and $|1\rangle = |\tau(\tau, \tau)\rangle$ and $|\phi_4\rangle = |\tau(\tau, 1)\rangle_A |\tau(\tau, 1)\rangle_B$. The state $|\phi_0\rangle$ is the state obtained by creating type τ particle anti-particle pairs on (1,2), (3,4), (5,6) out of the vacuum. For the one parameter family of states $|r(a)\rangle$ we find the same maximal violation as in Eq. 5 for $SU(2)_3$: $\langle W \rangle = \pm 2\sqrt{-7 + 4\sqrt{5}} \approx \pm 2.7887$. This is not very surprising, considering that the $SU(2)_3$ theory is equivalent to the product of the Fibonacci theory and an Abelian theory with \mathbb{Z}_2 fusion rules (see for instance [28]). The action under braiding is represented by the matrix $R_{\tau\tau} = e^{i4\pi/5} \oplus e^{i7\pi/5}$ expressed in the basis $\{1, \tau\}$. We obtain the following representation of the generators of the braid group \mathcal{B}_6 expressed in the basis $\{|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle\} \sqcup |\phi_4\rangle$:

$$\begin{aligned} B_1 &= [FR_{\tau\tau}F^{-1} \otimes \mathbf{1}_2] \oplus (e^{i7\pi/5}) \\ B_2 &= [R_{\tau\tau} \otimes \mathbf{1}_2] \oplus (e^{i7\pi/5}) \\ B_3 &= O^\dagger [e^{i4\pi/5} \oplus e^{i7\pi/5} \oplus e^{i7\pi/5} \oplus \begin{pmatrix} M_1^1 & M_0^1 \\ M_1^0 & M_0^0 \end{pmatrix}] O \quad (7) \\ B_4 &= [\mathbf{1}_2 \otimes FR_{\tau\tau}F^{-1}] \oplus (e^{i7\pi/5}) \\ B_5 &= [\mathbf{1}_2 \otimes R_{\tau\tau}] \oplus (e^{i7\pi/5}) \end{aligned}$$

where O maps the product basis to the basis $\{|\phi_j\rangle\}_{j=0}^4$, and $M = (F_{\tau\tau}^\tau)^{-1} R_{\tau\tau} F_{\tau\tau}^\tau$. A length 25 braid word produces a Bell violation: $|\Psi\rangle = [B_3 B_4^{-1} B_1^{-1} B_3^{-1} B_2^{-1}]^5 |\phi_0\rangle$ with $\langle \Psi | W | \Psi \rangle = 2.5310$.

In the models above, measurements by Alice and Bob had two outcomes for the two fusion products of the anyons. To accommodate more outcomes we can use higher dimensional Bell witnesses [29]. We demonstrate how this works in another anyonic model with excitations in one to one correspondence with irreducible representations of a Hopf algebra, $D(G)$, the quantum double of a finite group G [11, 12]. The

particles can carry electric and magnetic charge and are labeled by $\Pi_{R(N_{[\alpha]})}^{[\alpha]}$ where $[\alpha]$ denotes a conjugacy class of G which labels the magnetic charge, and $R(N_{[\alpha]})$ denotes a unitary irreducible representation R of the centralizer of an element in the conjugacy class $[\alpha]$, which labels the electric charge. The dimension of the carrier space for each irreducible representation, which equals the quantum dimension of the particle $\Pi_{R(N_{[\alpha]})}^{[\alpha]}$ is $d_{R(N_{[\alpha]})}^{[\alpha]} = |[\alpha]| |R(N_{[\alpha]})|$. The quantum dimensions satisfy the sum rule $\sum (d_{R(N_{[\alpha]})}^{[\alpha]})^2 = |G|^2$. We focus on the simplest non-Abelian finite group, S_3 , the group of permutations on three objects. Elements of S_3 are organized into three conjugacy classes: $[e] = \{e\}$ the identity element, $[t] = \{t_0, t_1, t_2\}$ the transpositions, and $[c] = \{c_+, c_-\}$ the cyclic permutations. The 8 irreducible representations for $D(S_3)$ are

$$\begin{aligned} \Pi_{R_1^+}^{[e]} \quad d = 1 \quad (\text{vacuum}) \\ \Pi_{\beta_0}^{[c]}, \Pi_{\gamma_0}^{[t]} \quad d = 2, 3 \quad (\text{pure magnetic charges}) \\ \Pi_{R_1^-}^{[e]}, \Pi_{R_2^-}^{[e]} \quad d = 1, 2 \quad (\text{pure electric charges}) \\ \Pi_{\beta_1}^{[c]}, \Pi_{\beta_2}^{[c]}, \Pi_{\gamma_1}^{[t]} \quad d = 2, 2, 3 \quad (\text{dyonic combinations}) \end{aligned} \quad (8)$$

A complete derivation of the fusion rules for this model is given in [30]. In the toric code realization of these anyon models, the quantum dimensions $d_{R(N_{[\alpha]})}^{[\alpha]} = |[\alpha]| |R(N_{[\alpha]})|$ actually count local degrees of freedom associated with the anyon. In the discrete gauge theory context, these degrees of freedom are also present in the description of the system, but some of them are gauge. A single particle's electric charge and magnetic charge can always be measured locally (or at least within a region of size characteristic of the particles), by braiding with other locally prepared charge pairs and measuring the outcome of fusion of the pairs. Truly non-local properties are contained in the fusion space. To explore this we pick a fusion subalgebra of $D(S_3)$: $\{\Pi_{R_1^+}^{[e]}, \Pi_{R_1^-}^{[e]}, \Pi_{\beta_0}^{[c]}\}$ which we label for convenience $\{1, \Lambda, \Phi\}$. The non trivial fusion rules are

$$\Lambda \times \Lambda = 1, \quad \Lambda \times \Phi = \Phi, \quad \Phi \times \Phi = 1 + \Lambda + \Phi.$$

These fusion rules are the same as the fusion rules for the representations of S_3 itself and also the same as the fusion rules of the integer spin sectors of $SU(2)_4$. The particles are their own anti-particles. The magnetic charge Φ with quantum dimension 2 carries non-Abelian statistics and the fusion of n such particles gives: $\Phi \times n = \frac{1}{3}(2^{n-1} + (-1)^n)(1 + \Lambda) + \frac{1}{3}(2^n + (-1)^{n-1})\Phi$. As before, we will work in the superselection sector with total trivial charge. The smallest number of particles in this sector that could hope to violate a Bell inequality should have fusion space dimension ≥ 4 . If we are to pick measurement operators for Alice and Bob that measure total charge on pairs of Φ particles and we want two non commuting operators on each side then we require at least six particles in total. Exactly six particles suffices, giving Hilbert space dimension eleven for the vacuum sector.

Either by using the representation theory of $D(S_3)$, or by solving the pentagon and hexagon equations directly, we find the following recoupling and braid matrices, expressed in the basis $\{1, \Lambda, \Phi\}$,

$$F \equiv F_{\Phi\Phi\Phi}^{\Phi} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}; \quad R \equiv R_{\Phi\Phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We notice immediately that R has eigenvalues ± 1 , so that we will end up with a representation of the permutation group when 'braiding' the anyons. Nevertheless, these anyons are not bosons or fermions, since this representation is non-Abelian. The fact that we have a permutation group representation does signal the fact that braiding in this theory is not universal for quantum computation. This is in fact a general property of braiding in discrete gauge theories.

A basis of the eleven dimensional vacuum sector of the six anyon Hilbert space can be given in terms of superpositions of products of local basis states, as defined in figure 2,

$$\begin{aligned} & \{\sum_y F_1^y |y(\Phi, \Phi)\rangle_A |1(\Phi, \Phi)\rangle_B, \sum_y F_\Lambda^y |y(\Phi, \Phi)\rangle_A |1(\Phi, \Phi)\rangle_B \\ & \sum_y F_\Phi^y |y(\Phi, \Phi)\rangle_A |1(\Phi, \Phi)\rangle_B, \sum_y F_1^y |y(\Phi, \Phi)\rangle_A |\Lambda(\Phi, \Phi)\rangle_B \\ & \sum_y F_\Lambda^y |y(\Phi, \Phi)\rangle_A |\Lambda(\Phi, \Phi)\rangle_B, \sum_y F_\Phi^y |y(\Phi, \Phi)\rangle_A |\Lambda(\Phi, \Phi)\rangle_B \\ & \sum_y F_1^y |y(\Phi, \Phi)\rangle_A |\Phi(\Phi, \Phi)\rangle_B, \sum_y F_\Lambda^y |y(\Phi, \Phi)\rangle_A |\Phi(\Phi, \Phi)\rangle_B \\ & \sum_y F_\Phi^y |y(\Phi, \Phi)\rangle_A |\Phi(\Phi, \Phi)\rangle_B, |\Phi(\Phi, \Lambda)\rangle_A |\Phi(\Phi, \Lambda)\rangle_B \\ & |\Phi(\Phi, 1)\rangle_A |\Phi(\Phi, 1)\rangle_B\} = \{|\phi_j\rangle\}_{j=0}^{10}. \end{aligned}$$

The state $|\phi_0\rangle$ is obtained by creating type Φ particle anti-particle pairs on $(1, 2), (3, 4), (5, 6)$ out of the vacuum. Each such vacuum magnetic charge pair is written: $|\Phi, \Phi; (i, j)\rangle = \frac{1}{\sqrt{2}}(|c_+, c_-; (i, j)\rangle + |c_-, c_+; (i, j)\rangle)$.

Now in analogy to the cases studied for $SU(2)_k$, we could look for a Bell like inequality but using measurement operators with three outcomes. Let Alice have one operator $\Upsilon_{1,2}^A$ which measures the outcome of total charge for particles 1 and 2 with outcomes $\{1, \Lambda, \Phi\}$ and another, non commuting operator, $\Upsilon_{2,3}^A$ that measures total charge for particles 2 and 3 with outcomes $\{1, \Lambda, \Phi\}$. In other words, $\Upsilon_{1,2}^A$ is a measurement in the basis $\{(F_{\alpha\alpha\alpha}^\beta)^\dagger |y(\alpha, \beta)\rangle_A\}$ with outcome $m_{1,2}^A = y \in \{1, \Lambda, \Phi\}$ and $\Upsilon_{2,3}^A$ is a measurement in the basis $\{|y(\alpha, \beta)\rangle_A\}$ with outcome $m_{2,3}^A = y$. Similarly, let Bob have two measurement operators, $\Upsilon_{4,5}^B$ that measures in the basis $\{(F_{\alpha\alpha\alpha}^\beta)^\dagger |y(\alpha, \beta)\rangle_B\}$ with outcome $m_{4,5}^B = y$, and $\Upsilon_{5,6}^B$ which measures onto the basis $\{|y(\alpha, \beta)\rangle_B\}$ with outcome $m_{5,6}^B = y$. Now $(F_{\Phi\Phi\Phi}^1)_x^y = \delta_{x,\Phi} \delta_{y,\Phi} = (F_{\Phi\Phi\Phi}^\Lambda)_x^y$, so in the subspace of $\{|\phi_9\rangle, |\phi_{10}\rangle\}$, the measurement operators all commute. These states cannot yield a Bell violation and we can focus on the states in the 9 dimensional orthogonal subspace which is isomorphic to the Hilbert space of two three dimensional particles (qutrits).

In Ref. [29] it was shown how to construct Bell inequalities for bipartite systems of equal but arbitrary finite dimension. In particular for two qutrits the authors introduce the witness I_3 which for all LHV theories satisfies $|\langle I_3 \rangle| \leq 2$, whereas for

quantum mechanical systems $|\langle I_3 \rangle| \leq 4$. To simplify notation let us introduce the projectors $\pi_y \equiv |y(\Phi, \Phi)\rangle\langle y(\Phi, \Phi)|$ and $\tilde{\pi}_y \equiv F^\dagger |y(\Phi, \Phi)\rangle\langle y(\Phi, \Phi)| F$. For the quorum of observables above, the Bell witness I_3 is:

$$\begin{aligned}
I_3 = & \tilde{\pi}_1 \otimes \tilde{\pi}_1 + \tilde{\pi}_\Lambda \otimes \tilde{\pi}_\Lambda + \tilde{\pi}_\Phi \otimes \tilde{\pi}_\Phi + \pi_\Phi \otimes \tilde{\pi}_1 + \pi_1 \otimes \tilde{\pi}_\Lambda \\
& + \pi_\Lambda \otimes \tilde{\pi}_\Phi + \pi_1 \otimes \pi_1 + \pi_\Lambda \otimes \pi_\Lambda + \pi_\Phi \otimes \pi_\Phi + \tilde{\pi}_1 \otimes \pi_1 \\
& + \tilde{\pi}_\Lambda \otimes \pi_\Lambda + \tilde{\pi}_\Phi \otimes \pi_\Phi - \tilde{\pi}_1 \otimes \tilde{\pi}_\Lambda + \tilde{\pi}_\Lambda \otimes \tilde{\pi}_\Phi + \tilde{\pi}_\Phi \otimes \tilde{\pi}_1 \\
& - \pi_1 \otimes \tilde{\pi}_1 + \pi_\Lambda \otimes \tilde{\pi}_\Lambda + \pi_\Phi \otimes \tilde{\pi}_\Phi - \pi_1 \otimes \pi_\Lambda + \pi_\Lambda \otimes \pi_\Phi \\
& + \pi_\Phi \otimes \pi_1 - \tilde{\pi}_\Lambda \otimes \pi_1 + \tilde{\pi}_\Phi \otimes \pi_\Lambda + \tilde{\pi}_1 \otimes \pi_\Phi \\
& + 2|\Phi(\Phi, \Lambda)\rangle\langle\Phi(\Phi, \Lambda)| \otimes |\Phi(\Phi, \Lambda)\rangle\langle\Phi(\Phi, \Lambda)| \\
& + 2|\Phi(\Phi, 1)\rangle\langle\Phi(\Phi, 1)| \otimes |\Phi(\Phi, 1)\rangle\langle\Phi(\Phi, 1)|
\end{aligned} \tag{9}$$

The state with the largest violation has $\langle I_3 \rangle = -2.5216$.

We obtain the following representation of the generators for \mathcal{B}_6 expressed in the basis $\{|x(\Phi, \Phi)\rangle_A |y(\Phi, \Phi)\rangle_B; x, y \in \{1, \Lambda, \Phi\}\} \sqcup \{|\Phi(\Phi, \Lambda)\rangle_A |\Phi(\Phi, \Lambda)\rangle_B \sqcup |\Phi(\Phi, 1)\rangle_A |\Phi(\Phi, 1)\rangle_B\}$:

$$\begin{aligned}
B_1 = & [FRF^{-1} \otimes \mathbf{1}_3] \oplus R_{\Phi\Phi}^\Phi \oplus R_{\Phi\Phi}^\Phi, B_2 = [R \otimes \mathbf{1}_3] \oplus R_{\Phi\Phi}^\Phi \oplus R_{\Phi\Phi}^\Phi \\
B_3 = & O^\dagger \left[R_{\Phi\Phi}^1 \oplus R_{\Phi\Phi}^\Lambda \oplus R_{\Phi\Phi}^\Phi \oplus R_{\Phi\Phi}^\Lambda \oplus R_{\Phi\Phi}^1 \oplus R_{\Phi\Phi}^\Phi \oplus \right. \\
& \left. R_{\Phi\Phi}^\Phi \oplus R_{\Phi\Phi}^\Phi \oplus \begin{pmatrix} M_\Phi^\Phi & M_\Lambda^\Phi & M_1^\Phi \\ M_\Phi^\Lambda & M_\Lambda^\Lambda & M_1^\Lambda \\ M_\Phi^1 & M_\Lambda^1 & M_1^1 \end{pmatrix} \right] O \\
B_4 = & [\mathbf{1}_3 \otimes FRF^{-1}] \oplus R_{\Phi\Phi}^\Phi \oplus R_{\Phi\Phi}^\Phi, B_5 = [\mathbf{1}_3 \otimes R] \oplus R_{\Phi\Phi}^\Phi \oplus R_{\Phi\Phi}^\Phi
\end{aligned} \tag{10}$$

where O maps the product basis to the basis $\{|\phi_j\rangle\}_{j=0}^{10}$, and $M = F^{-1}RF$. Here $B_j^2 = \mathbf{1}_{11} \forall j$, so we have the permutation group S_6 , as mentioned before. We compute the action on states consisting of vacuum magnetic charge pairs. The state $|\phi_0\rangle = |\Phi, \Phi; (1, 2)\rangle |\Phi, \Phi; (3, 4)\rangle |\Phi, \Phi; (5, 6)\rangle$ is the fiducial state, and the other distinct configuration of vacuum magnetic charge pairs is $|\Phi, \Phi; (1, 4)\rangle |\Phi, \Phi; (2, 5)\rangle |\Phi, \Phi; (3, 6)\rangle = B_3 B_4 B_2 B_1 |\phi_0\rangle$, hence it suffices to consider the orbit of $|\phi_0\rangle$. An exhaustive search through $6! = 720$ braid words corresponding to all distinct permutations in S_6 finds that, while $\langle I_3 \rangle$ is not constant under braiding, we do find that in all cases $|\langle I_3 \rangle| \leq 2$. Hence we require some operation beyond braiding to produce a violation of LHV under our protocol. Even if we restrict to non-topologically protected operations that just involve interacting pairs of particles, we can indeed produce a Bell violating state. Consider the family of states $|\phi'\rangle = D_{3,4}(\alpha_1, \alpha_2) D_{1,2}(\alpha_3, \alpha_4) D_{2,3}(\alpha_5, \alpha_6) B_1 B_5 B_3 B_2 B_3 B_4 |\phi_0\rangle$ where $D_{i,j}(\alpha, \beta)$ is the non-topologically protected gate obtained by bringing anyons i and j of type Φ nearby each other and allowing them to interact for a time such that the fusion channel $\Phi \times \Phi \rightarrow \Lambda$ accumulates a phase $e^{i\alpha}$ and the fusion channel $\Phi \times \Phi \rightarrow \Phi$ accumulates a phase $e^{i\beta}$. Optimizing $|\langle I_3 \rangle|$ over the interaction phases, we find a violation $\langle \phi' | I_3 | \phi' \rangle = 2.0512$ for the angles: $\alpha_1 = 0.7943, \alpha_2 = 0.3989, \alpha_3 = 3.5531, \alpha_4 = 0.9257, \alpha_5 = -0.8525, \alpha_6 = 0.1036$. No systematic attempt was made to optimize the violation over other braid words and it likely stronger violations could be found.

We have described a protocol to reveal non-locality in several classes of non-Abelian anyonic theories. The need for

at least six anyons shared between two parties arises because each party needs three anyons in order to have two non-commuting topologically protected observables. It is possible this could be reduced using a shared resource which fixes a common gauge, akin to using a shared reference frames to reveal non-locality in mode entanglement with bosons [19]. The size of the maximum violation depends on the recoupling matrices F and the ability to generate Bell violating states beginning from 3 vacuum charge pairs depends on the power of the braiding operations. It is intriguing to ask whether one could find intermediate anyonic theories which have the power to generate Bell violating states by topologically protected gates, but are not universal for topological quantum computation. Finally, it would be very interesting to have an experimental demonstration of (some of) the schemes presented here. We believe that the required effort would not be significantly higher than that necessary to perform non-abelian interferometry.

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- [1] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47**, 777 (1935).
 - [2] J. S. Bell, *Speakable and Unsayable in Quantum Mechanics*, Cambridge University Press (1987).
 - [3] J. M. Leinaas and J. Myrheim, Nuovo Cim. B **37**, 1 (1977).
 - [4] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982).
 - [5] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
 - [6] A. Kitaev, Annals of Physics **303** 2 (2003).
 - [7] For example, in the toric code, $\mathcal{H}_{\text{non-local}}$ is isomorphic to the space of two qubits and single qubit observables correspond to non contractible string operators, see e.g. M.-G. Hu., D.-L. Deng, and J.-L. Chen, arXiv:0810.1157.
 - [8] S. Das Sarma, M. Freedman, and C. Nayak, Phys. Rev. Lett. **94**, 166802 (2005).
 - [9] S. Das Sarma, M. Freedman, C. Nayak, S.H. Simon, and A. Stern, Rev. Mod. Phys. (to appear); arXiv:0707.1889.
 - [10] P. Bonderson, K. Shtengel, and J.K. Slingerland, Annals of Physics **323**, 2709-2755, (2008).
 - [11] F. A. Bais, P. van Driel, and M. de Wild Propitius, Phys. Lett. B **280**, 63 (1992).
 - [12] M. de Wild Propitius and F. A. Bais, in *Particles and Fields*, edited by G. Semenoff and L. Vinet (Springer Verlag, New York, 1998), CRM Series in Mathematical Physics, pp. 353-439, hep-th/9511201.
 - [13] D. L. Moehring, M. J. Madsen, B. B. Blinov, and C. Monroe, Phys. Rev. Lett. **93**, 090410 (2004). D. N. Matsukevich, P. Maunz, D. L. Moehring, S. Olmschenk, and C. Monroe, Phys. Rev. Lett. **100**, 150404 (2008). R. A. Bertlmann and B. C. Hiesmayr, Phys. Rev. A **63**, 062112 (2001).
 - [14] B. Douçot, L. B. Ioffe, and J. Vidal, Phys. Rev. B **69**, 214501 (2004).
 - [15] M. Aguado, G.K. Brennen, F. Verstraete, and J.I. Cirac, arXiv:0802.3163.
 - [16] See e.g. N. Gisin, arXiv quant-ph/0702021 and references therein.

- [17] J. F. Clauser, M.A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. **23**, 880-884 (1969).
- [18] B. S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980).
- [19] S.D. Bartlett, T. Rudolph, and R. Spekkens, Rev. Mod. Phys. **79**, 555 (2007).
- [20] J.K. Slingerland and F.A. Bais, Nucl.Phys. B **612**, 229-290 (2001).
- [21] M. Freedman, M. Larsen, and Z. Wang, Commun. Math. Phys. **227**, 605 (2002); *ibid* **228**, 177 (2002).
- [22] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
- [23] S. Bravyi, Phys. Rev. A **73**, 042313 (2006).
- [24] R.W. Spekkens, Phys. Rev. A **75**, 032110 (2007).
- [25] K.S. Gibbons, M.J. Hoffman, and W.K. Wootters, Phys. Rev. A **70**, 062101 (2004).
- [26] C. Zhang, S. Tewari, and S. das Sarma, Phys. Rev. Lett. **99**, 220502 (2007).
- [27] V. Lahtinen, *et al.*, Annals of Physics **323**, 2286 (2008).
- [28] P. Bonderson, *Non-Abelian Anyons and Interferometry*, PhD thesis, California Institute of Technology, Pasadena, 2007.
- [29] D. Collins, *et al.*, Phys. Rev. Lett. **88**, 040404 (2002).
- [30] M. de Wild Propitius, Topological interactions in broken gauge theories. PhD thesis, University of Amsterdam, 1995.
- [31] Complementing topological operations by some noisy non-topological operations, one can achieve universality [23]
- [32] For an alternative construction of this kind, but using the more standard tensor product structure of 4-anyon qubits, giving a total of 8 anyons, see [26].