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Triviality from the Exact Renormalization Group

Oliver J. Rosten*
Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

Using the exact renormalization group, it is shown that no physically acceptable non-trivial fixed points, with positive anomalous dimension, exist for (i) O(N) scalar field theory in four or more dimensions, (ii) non-compact, pure Abelian gauge theory in any dimension. It is then shown, for both theories in any dimension, that otherwise physically acceptable non-trivial fixed points with negative anomalous dimension are non-unitary. In addition, a very simple demonstration is given, directly from the exact renormalization group, that should a critical fixed point exist for either theory in any dimension, then the two-point correlation function exhibits the expected behaviour.

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I. INTRODUCTION

The renormalizability of quantum field theories in the nonperturbative, Wilsonian sense, is determined by the existence, or otherwise, of critical fixed points and the renormalized trajectories emanating from them [1, 2]. As a consequence of this, low energy effective theories which naïvely appear nonrenormalizable could in fact be descended from some ultraviolet (UV) fixed point. This is the idea behind the asymptotic safety scenario [3]. In this paper, we will rule out such a scenario for (i) O(N) scalar field theory[1] in or above four dimensions, (ii) non-compact[2], pure Abelian gauge theory in any dimension, by showing that no physically acceptable non-trivial fixed points exist in either case. Consequently, triviality of these theories follows from the very well known fact that the respective Gaussian fixed points do not support non-trivial renormalized trajectories.[3]

There are two criteria we use to determine the physical acceptability of a fixed point. The first is 'quasi-locality' [9]: we demand that the action has an all order derivative expansion. Anticipating that we will be working in Euclidean space, the second is that the theory makes sense as a unitary quantum field theory, upon continuation to Minkowski space.

The analysis of the existence, or otherwise, of non-trivial fixed points is split into two parts. First, we consider the case where the fixed point anomalous dimension, $\eta_\star$, is greater than or equal to zero. We will demonstrate that there are no quasi-local fixed points for the theories mentioned above, given the restriction on dimension for the scalar case. In fact, $\eta_\star = 0$ is already covered by Pohlmeyer's theorem [10] which, for both theories, implies that the only scale invariant (i.e. critical fixed point) theory with $\eta_\star = 0$ corresponds to the Gaussian fixed point.

In the case of negative anomalous dimension, our results in fact apply in all dimensions, even in the case of O(N) scalar field theory. It has been known for a long time that exotic Gaussian fixed points without the standard p^2 kinetic piece—going instead as p^{2n} for n an integer greater than one—have negative anomalous dimensions. However, such fixed points can be excluded from our considerations by the requirement that the theory be physically acceptable, since the absence of the standard kinetic term leads to violation of unitarity. It is worth noting that, from a condensed matter point of view, such a requirement is, of course, an irrelevance. However, even in this context these fixed points are still unimportant in O(N) scalar field theory. This is because, as shown by Wegner [11], they have an infinite number of relevant directions for $D \leq 4$ and so, for typical condensed matter systems of interest, it is scarcely possible to approach the critical point.

Nevertheless, this says nothing as to the possible existence of *non-Gaussian* fixed points with negative anomalous dimension. In this paper, we will not show that such fixed points, which satisfy the requirement of having a quasi-local action, do not exist. Rather, it will be shown that should such fixed points exist, then they necessarily violate unitarity. It is well worth noting that, in the vicinity of a nonperturbative fixed point, we cannot rule out a negative anomalous dimensions by the *usual* unitarity arguments (upon continuation to Minkowski space). Given field strength renormalization, Z, these arguments relate the unitarity constraint $0 \leq Z \leq 1$ to a positive anomalous dimension via a perturbative calculation; but there is no reason to believe a perturbative calculation near to a nonperturbative fixed point (see [12] for an interesting discussion on negative anomalous dimensions).

Whilst the results obtained in this paper are unsurprising, they are either complimentary to or stronger than results obtained elsewhere. The first rigourous proof of triviality in scalar field theories (with field denoted by φ) was provided by Aizenman [13], who considered a lattice $\lambda\varphi^4$ model and showed that no interacting continuum limit exists in $D > 4$. This result was confirmed and

*Electronic address: orosten@stp.dias.ie
[1] Non-linear sigma models are not considered.
[2] As opposed to the compact formulation [4, 5]; see section IV.
[3] There are claims that the Gaussian fixed point of scalar theory supports relevant, interacting directions in $D = 4$ [6] but, as convincingly argued by Morris [2, 7, 8], these directions cannot be used to construct a renormalized trajectory.

extended by Fröhlich [14], who also proved that for one or two-component lattice $\lambda\varphi^4$ models in $D = 4$, the only non-trivial continuum limit would have to be asymptotically free, contradicting perturbative expectations. For a review of these ideas, see [15].

The approach taken in this paper is different, both in implementation and philosophy (see section II for a description of the formalism). First of all, everything is done directly in the continuum. Secondly, we are freed from considering a particular model, such as $\lambda\varphi^4$, and analysing whether a continuum limit exists. Rather, the Wilsonian view point is adopted that the bare action of a nonperturbatively renormalizable theory is something which should be *solved* for, not something which should be put in by hand [2]. Along a renormalized trajectory, the bare action is the 'perfect action' [16] in the vicinity of the ultraviolet (UV) fixed point of the theory, and so it is determined by the fixed point action—which must itself be solved for—and the integration constants associated with the relevant (including marginally relevant) directions.

As to addressing the question of the existence of non-trivial fixed points in $\mathrm{O}(N)$ scalar field theories in $D = 4$, there are a number of studies which take, as a starting point, the same formalism that is employed in this paper [6, 17]. However, these earlier works rely on a truncated derivative expansion of the effective action. Nevertheless, despite the truncation, the space of possible interactions is infinite dimensional, so the power of this approach should not be underestimated. Within this approximation scheme, the space of truncated effective actions was scanned, numerically, with the result that the only fixed point found was the Gaussian one. It should be emphasised that this technique is very powerful for uncovering non-trivial fixed points in $D < 4$ and analysing their properties (see [18] for a comprehensive guide to the literature).

With regards to the triviality of non-compact, pure Abelian gauge theory, the strongest results to date are those of Morris [19]. Working in three dimensions (both with and without a Chern-Simons term) he considered truncated effective actions of the form $f(F_{\mu\nu})$, where f is allowed to be any invariant function of its argument. Again, a numerical search was performed, through the infinite dimensional space of truncated effective actions, with only the Gaussian fixed point found. In this paper, physically acceptable non-trivial fixed points are ruled out without any approximation, and in any dimension, representing a dramatic improvement on the current state of the art.

II. FORMALISM

A. The Polchinski Equation

The formalism employed is the exact renormalization group (ERG), which is basically the continuous version of Wilson's RG. Working in D-dimensional Euclidean space and starting from some high energy scale, degrees of freedom are integrated out down to a lower, effective scale denoted by Λ. During this process, the action evolves into the Wilsonian effective action, S_Λ, such that it encodes the effects of the high momentum modes. The ERG equation determines how the Wilsonian effective action varies with Λ. A central ingredient is the ERG kernel, which provides the flow equation with its UV regularization. To this end, we introduce the 'effective propagator',

$$\Delta(p,\Lambda) = \frac{c(p^2/\Lambda^2)}{p^2}, \tag{1}$$

where $c(p^2/\Lambda^2)$ is a UV cutoff function which dies off sufficiently rapidly for $p^2/\Lambda^2 \to \infty$, and for which

$$c(0) = 1. \tag{2}$$

The position-space kernel, $\Delta(x,y)$, is given by the Fourier transform of (1). Note that we shall use p to denote both a four-vector and its modulus, with the meaning hopefully being clear from the context.

Whilst the choice (1) is the typical one, we will temporarily work with

$$\Delta_m(p,\Lambda) = \frac{c(p^2/\Lambda^2)}{p^2 + m^2(\mu)}, \tag{3}$$

where μ is an arbitrary scale and $m^2(\mu)$ is a mass parameter independent of Λ. This mass term is included to provide infrared (IR) regularization though, as will be seen, it can usually be dispensed with.

In what follows, we will work with a single component scalar field, corresponding to the O(1) model (i.e. there is a $\varphi \to -\varphi$ symmetry); generalization to the multi-component case is trivial. We now define the interaction part of the Wilsonian effective action, $S^{\mathrm{I}}_\Lambda[\varphi]$, according to

$$S_\Lambda[\varphi] = \frac{1}{2}\varphi\cdot\Delta_m^{-1}\cdot\varphi + S^{\mathrm{I}}_\Lambda[\varphi]. \tag{4}$$

As usual, we employ the shorthand $A\cdot B \equiv A_x B_x \equiv \int d^D x\, A(x)B(x)$. Similarly, $\varphi\cdot\Delta_m^{-1}\cdot\varphi \equiv \varphi_x \Delta_m^{-1}(x,y)\varphi_y = \int d^D p\,/(2\pi)^D \varphi(p)\Delta_m^{-1}(p)\varphi(-p)$. Henceforth, we will cease to explicitly indicate the Λ dependence of S^{I}, for brevity.

The starting point for our analysis is the form of the ERG equation introduced by Polchinski [20]:

$$-\Lambda\partial_\Lambda S^{\mathrm{I}} = \frac{1}{2}\frac{\delta S^{\mathrm{I}}}{\delta\varphi}\cdot\dot{\Delta}_m\cdot\frac{\delta S^{\mathrm{I}}}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\dot{\Delta}_m\cdot\frac{\delta S^{\mathrm{I}}}{\delta\varphi}, \tag{5}$$

where the Λ-derivative is performed at constant φ and the ERG kernel, $\dot{\Delta}_m$, is given by the flow of the effective propagator:

$$\dot{\Delta}_m \equiv -\Lambda\frac{d\Delta_m}{d\Lambda}. \tag{6}$$

In addition to the proofs pertaining to triviality, the techniques developed below allow a very simple derivation of the expected form of the two-point correlation function at a critical fixed point, directly from the ERG.

B. Correlation Functions

We now define the 'dual action', $\mathcal{D}_m[\varphi]$, according to

$$-\mathcal{D}_m[\varphi] = \ln\left\{\exp\left(\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\Delta_m\cdot\frac{\delta}{\delta\varphi}\right)e^{-S^{\mathrm{I}}[\varphi]}\right\}, \quad (7)$$

where the subscript m is to remind us that we have utilized Δ_m in the construction.

Before describing what this object represents, we will compute its flow. It is easy to confirm, using (5) and (6), that

$$-\Lambda\partial_\Lambda\mathcal{D}_m[\varphi] = 0. \quad (8)$$

Thus, the dual action is an invariant of the ERG. Just like the Wilsonian effective action, we can expand the dual action in powers of the fields, thereby defining its vertices, each of which are separately invariants of the ERG:

$$\mathcal{D}_m[\varphi] = \sum_n \frac{1}{n!}\int\frac{d^Dp_1}{(2\pi)^D}\cdots\int\frac{d^Dp_n}{(2\pi)^D}\mathcal{D}_m^{(n)}(p_1,\ldots,p_n) \\ \varphi(p_1)\cdots\varphi(p_n)\hat{\delta}^{(D)}(p_1+\cdots+p_n), \quad (9)$$

where $\hat{\delta}^{(D)}(p) \equiv (2\pi)^D\delta^{(D)}(p)$. At the two-point level, we define $\mathcal{D}_m^{(2)}(p) \equiv \mathcal{D}_m^{(2)}(p,-p)$.

As we will shortly demonstrate, the vertices of the dual action are essentially n-point connected correlation functions. Indeed, from this perspective it is clear why these vertices are ERG invariants: such objects, which incorporate all quantum fluctuations, must be independent of the effective scale, Λ, which is just an introduced as an intermediate step to facilitate the evaluation of the partition function.

To arrive at this interpretation of the dual action, it is useful to introduce a diagrammatic representation, about which three very important points should be made. First, the diagrammatics utilize exact vertices of the Wilsonian effective action, no perturbative expansion of the vertices having been performed. Secondly, the diagrammatic expansion will never be truncated. Thirdly, the dual action exists entirely independently of its diagrammatic representation. Throughout this paper, we will perform various manipulations of the dual action using the diagrammatics. However, it should be emphasised that exactly the same results could be obtained directly from a power expansion of (7), together with a field expansion of the Wilsonian effective action.[4] The point is that, as usual, the diagrammatics provide an intuitive and transparent means of performing these manipulations; but the use of this tool is by no means a necessity.

[4] It is worth pointing out that truncated field expansions are known to suffer deficiencies when looking for fixed points [17], though see [21]. In this paper, no truncation is ever performed, and it is assumed that the conclusions drawn are reliable.

From (7), the dual action comprises all connected diagrams built out of vertices of the interaction part of the Wilsonian effective action and effective propagators (it is the logarithm which, as usual, ensures connectedness). Indeed, this diagrammatic form has been used for some time [22, 23, 24, 25, 26], without realising that it was a representation of (7). A selection of terms contributing to the two-point vertex of the dual action is shown in figure 1.

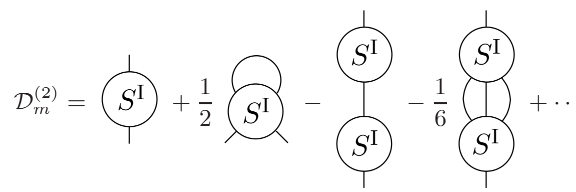

FIG. 1: The first few terms that contribute to $\mathcal{D}^{(2)}$. Momentum arguments have been suppressed. Each of the lobes represents a vertex of the interaction part of the Wilsonian effective action.

It is at this point we see that, by choosing a massive effective propagator, it has been ensured that potentially IR divergent diagrams contributing to $\mathcal{D}_m$ are individually regularized. These divergences have two sources. First, since $\mathcal{D}_m$ contains one-particle reducible (1PR) terms, a massless effective propagator would lead to strongly IR divergent diagrams in any dimension, as the external momenta go to zero. Furthermore, depending on the dimensionality, one-particle irreducible (1PI) diagrams, such as the final diagram of figure 1 (which can also appear as a sub-diagram in 1PR terms), can also possess IR divergences. These latter divergences do not occur exclusively for the external momenta going to zero, but can also occur for other 'special' momenta, whereby the external momenta entering a vertex sum to zero. However, since these divergences are also regularized by using a massive effective propagator, we will take our definition of IR divergences to include them.

Having made such a big deal of the IR regularization of the dual action we now argue that, for most purposes, it is quite legitimate to deal with the vertices constructed from the massless effective propagator. The first point to make is that the dual action plays a very different role from the Wilsonian effective action. In particular, it does not appear as the weight in a partition function where we are integrating over all field configurations. Indeed, if we think of it simply as the object which naturally collects together the vertices $\mathcal{D}_m^{(n)}$, then it does not necessarily matter if the

$$\mathcal{D}^{(n)}(p_1,\ldots,p_n) \equiv \lim_{m(\mu)\to 0}\mathcal{D}_m^{(n)}(p_1,\ldots,p_n) \quad (10)$$

have poles for certain values of their arguments.

With this in mind, consider computing connected n-point correlation functions from the *bare* action.[5] For $n > 2$, the first contribution comes from the n-point bare action vertex, connected to n bare propagators, Δ_b. Since this vertex is pulled down from $e^{-S_{\text{bare}}}$, this contribution comes with a minus sign. Then we must include all other connected contributions with n legs, built out of the bare action. Thus we find that

$$G(p_1, \dots, p_n) = -\mathcal{D}^{(n)}_{\text{bare}}(p_1, \dots, p_n) \prod_{i=1}^{n} \Delta_b(p_i), \qquad n > 2.$$

But, we know that the dual action vertices do not depend on Λ, and so

$$G(p_1, \dots, p_n) = -\mathcal{D}^{(n)}_{m}(p_1, \dots, p_n) \prod_{i=1}^{n} \Delta_b(p_i), \qquad n > 2.$$

Consequently, $\mathcal{D}^{(n)}_m$ is directly related to the n-point connected correlation function and so the limit $m(\mu) \to 0$ makes perfect sense: any IR divergences that now appear are just those we expect from the correlation functions.

Next, let us consider the two-point connected correlation function, $G(p)$, again computed from the bare action. The first contribution to this is just the bare propagator. The full contribution is

$$G(p) = \Delta_b(p) \left[1 - \mathcal{D}^{(2)}_m(p)\Delta_b(p)\right] \qquad (11)$$

where, again, we have recognized that since the two-point dual action vertex is independent of scale, we can evaluate it at the effective, rather than bare, scale.

We now introduce two objects which will play a central role in what follows. First, we define the 1PI components of the $\mathcal{D}^{(n)}_m$, denoted $\overline{\mathcal{D}}^{(n)}_m$. The two-point object, $\overline{\mathcal{D}}^{(2)}_m$, will play a special role. As can be readily seen by considering the diagrammatic expression for $\mathcal{D}^{(2)}_m$ in more detail [22, 25], $\mathcal{D}^{(2)}_m$ is built out of $\overline{\mathcal{D}}^{(2)}_m$ according to a geometric series:

$$\mathcal{D}^{(2)}_m(p) = \frac{\overline{\mathcal{D}}^{(2)}_m(p)}{1 + \Delta_m(p)\overline{\mathcal{D}}^{(2)}_m(p)}. \qquad (12)$$

By inspection, this equation can be inverted:

$$\overline{\mathcal{D}}^{(2)}_m(p) = \frac{\mathcal{D}^{(2)}_m(p)}{1 - \Delta_m(p)\mathcal{D}^{(2)}_m(p)}. \qquad (13)$$

The second key object is the dressed effective propagator, $\tilde{\Delta}_m$:

$$\tilde{\Delta}_m(p) \equiv \frac{1}{\Delta^{-1}_m(p) + \overline{\mathcal{D}}^{(2)}_m(p)}. \qquad (14)$$

[5] The bare action is to be interpreted as in the introduction. For a more detailed discussion see [25], but these subtleties are of no real consequence here.

Substituting (13) into (14) yields

$$\tilde{\Delta}_m(p) = \Delta_m(p) \left[1 - \mathcal{D}^{(2)}_m(p)\Delta_m(p)\right], \qquad (15)$$

and so we see that the dressed effective propagator is a UV regularized version of $G(p)$. Consequently, all the objects $\mathcal{D}^{(n)}_m$ and $\tilde{\Delta}_m$ makes sense in the limit $m(\mu) \to 0$, with any IR divergences having a physical interpretation.

We now recognize that, as usual (see e.g. [27]), the physical mass of our theory is defined by $\Delta^{-1}_m(0) + \overline{\mathcal{D}}^{(2)}_m(0)$. Therefore, IR regularization—should we want it—really means that $\Delta^{-1}_m(0) + \overline{\mathcal{D}}^{(2)}_m(0) \neq 0$. In cases where we have performed the resummation (14), we now interpret any subscript ms to mean that we are constrained to lie on a massive RG trajectory.

For the rest of this paper, we will send $m(\mu) \to 0$. Nevertheless, there are certain circumstances where we should maintain the explicit IR regularization, at least at intermediate stages. This should be done, for example, when inverting (7), to recover the Wilsonian effective action:

$$-S^{\mathrm{I}}[\varphi] = \ln\left[\exp\left(-\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\Delta_m\cdot\frac{\delta}{\delta\varphi}\right)e^{-\mathcal{D}_m[\varphi]}\right]. \qquad (16)$$

Incidentally, this relationship was proven diagrammatically in [25] whereas here it follows trivially, indicating the power of the dual action formalism compared to previous approaches.

C. Generalized ERGs

Our aim now is to attempt to utilize the dual action to investigate the existence of fixed points. Fixed point behaviour is most easily seen by rescaling to dimensionless variables, by dividing all quantities by Λ to the appropriate scaling dimension (by this it is meant, of course, the full scaling dimension, not just the canonical dimension). As it turns out, there is a well known subtlety related to scaling out the anomalous dimension from φ, so we will consider this rescaling first, in isolation. Thus, we make the following transformation:

$$\varphi(x) \to \varphi(x)\sqrt{Z}, \qquad (17)$$

where Z is the field strength renormalization, from which we define the anomalous dimension:

$$\eta \equiv \Lambda \frac{d \ln Z}{d\Lambda}. \qquad (18)$$

The problem with this transformation is that it produces an annoying factor of $1/Z$ on the right-hand side of the flow equation. However, we can remove this factor by utilizing the immense freedom inherent in the ERG. General ERGs are defined according to [28, 29]:

$$-\Lambda\partial_\Lambda e^{-S[\varphi]} = \int_x \frac{\delta}{\delta\varphi(x)}\left(\Psi_x[\varphi]e^{-S[\varphi]}\right). \qquad (19)$$

The total derivative on the right-hand side ensures that the partition function $Z = \int \mathcal{D}\varphi\, e^{-S}$ is invariant under the flow—a fundamental ingredient of any Wilson-inspired ERG equation. The functional, Ψ, parametrizes a general Kadanoff blocking [30] in the continuum and so there is considerable choice in its precise from. We will focus on those blockings for which

$$\Psi_x = \frac{1}{2}\int d^D y\, \dot{\Delta}^{\mathrm{new}}(x,y)\frac{\delta \Sigma}{\delta\varphi(y)}, \tag{20}$$

with $\dot{\Delta}^{\mathrm{new}}$ not yet fixed to be given by either (1) or (3) and

$$\Sigma \equiv S - 2\hat{S},$$

where $\hat{S}$ is the seed action [31, 32, 33, 34, 35, 36]. Whereas we solve the flow equation for the Wilsonian effective action, the seed action serves as an input and, given our choice (20) and a choice of cutoff function, parametrizes the remaining freedom in how modes are integrated out along the flow. The only restrictions on the seed action are that it leads to finite momentum integrals and that it admits an all orders derivative expansion. This latter property, a.k.a. 'quasi-locality' [9], is a fundamental requirement of all ingredients of the ERG equation and so covers $\dot{\Delta}$ and S, as well. Quasi-locality ensures that each ERG step is free of IR divergences or, equivalently, that blocking is performed only over a local patch.

We now choose the new ERG kernel such that, *after* performing the rescaling (17), the flow equation reads:

$$\left(-\Lambda\partial_\Lambda + \frac{\eta}{2}\varphi\cdot\frac{\delta}{\delta\varphi}\right) S = \frac{1}{2}\frac{\delta S}{\delta\varphi}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma}{\delta\varphi}. \tag{21}$$

The next step in the analysis is to define the interaction part of the seed action, analogously to (4):

$$\hat{S}[\varphi] = \frac{1}{2}\varphi\cdot\Delta^{-1}\cdot\varphi + \hat{S}^{\mathrm{I}}[\varphi]. \tag{22}$$

Substituting this into (21) yields, up to a discarded vacuum energy term,

$$\begin{aligned} -\Lambda\partial_\Lambda S^{\mathrm{I}} + \frac{\eta}{2}\varphi\cdot\frac{\delta S}{\delta\varphi} &= \frac{1}{2}\frac{\delta S^{\mathrm{I}}}{\delta\varphi}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma^{\mathrm{I}}}{\delta\varphi} \\ &- \frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma^{\mathrm{I}}}{\delta\varphi} - \varphi\cdot\Delta^{-1}\cdot\dot{\Delta}\cdot\frac{\delta\hat{S}^{\mathrm{I}}}{\delta\varphi} \end{aligned} \tag{23}$$

where $\Sigma^{\mathrm{I}} \equiv S^{\mathrm{I}} - 2\hat{S}^{\mathrm{I}}$. Notice that if we take $\hat{S}^{\mathrm{I}} = 0$—as we are perfectly at liberty to do—then (23) is the rescaled version of Polchinski's equation, modulo the fact that we have gotten rid of the annoying factor of $1/Z$ on the right-hand side. This flow equation was first considered by Ball et al. [37]. The more general version, with a non-zero $\hat{S}^{\mathrm{I}}$, has been considered in [25, 34, 38].

Given the new flow equation, we retain our definition for the dual action (7), despite the fact that the fields have been rescaled according to (17). Using the new flow equation we therefore have:

$$\begin{aligned} -\left(\Lambda\partial_\Lambda + \frac{\eta}{2}\varphi\cdot\frac{\delta}{\delta\varphi}\right)\mathcal{D}[\varphi] &= -\frac{\eta}{2}\varphi\cdot\Delta^{-1}\cdot\varphi \\ + e^{\mathcal{D}}\varphi\cdot\Delta^{-1}\cdot\dot{\Delta}\cdot\exp&\left(\frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\Delta\cdot\frac{\delta}{\delta\varphi}\right)\frac{\delta\hat{S}^{\mathrm{I}}}{\delta\varphi}e^{-S^{\mathrm{I}}}, \end{aligned} \tag{24}$$

where we recall (10). It is well worth noting that the seed action appears only in a single term, all other occurrences having cancelled out. These cancellations were previously demonstrated using elaborate (though increasing sophisticated) diagrammatics [22, 25, 32, 34, 35, 36, 39]; now, however, they follow from a few lines of algebra! This equation obviously simplifies to

$$-\left(\Lambda\partial_\Lambda + \frac{\eta}{2}\varphi\cdot\frac{\delta}{\delta\varphi}\right)\mathcal{D}[\varphi] = -\frac{\eta}{2}\varphi\cdot\Delta^{-1}\cdot\varphi, \tag{25}$$

in the case that we choose $\hat{S}^{\mathrm{I}} = 0$, as we now do. We will comment on this choice further at the end of the paper.

Notice that the relative sign between the two terms on the left-hand side of (25) has flipped, compared to (23). The intuitive reason follows from comparing the solutions of (25) to those of (8). For the dual action vertices with more than two legs, the right-hand side of (25) does not contribute and so we have:

$$\mathcal{D}^{(n>2)}(p_1,\ldots,p_n;\Lambda) = Z^{-n/2}A^{(n)}(p_1,\ldots,p_n), \tag{26}$$

where the $A^{(n)}$ are independent of Λ. We can now see why the η term on the left-hand side of (31) comes with the sign that it does. Consider the equation for $-\Lambda\partial_\Lambda\mathcal{D}$ before any rescalings have been done, i.e. equation (8). After the rescaling (17), we must have that $\mathcal{D}^{(n)} \to \mathcal{D}^{(n)}Z^{-n/2}$, in order that the factors of Z picked up from rescaling φ are properly compensated. This is consistent with the solution (26), which in turn follows from the sign of the η term on the left-hand side of (31).

To conveniently uncover fixed point solutions, we need to complete the rescalings started with (17). To this end, we define the 'RG-time',

$$t \equiv \ln\mu/\Lambda, \tag{27}$$

and also scale out the various canonical dimensions:

$$\varphi(x) \to \varphi(x)\Lambda^{(D-2)/2}, \qquad p_i \to p_i\Lambda. \tag{28}$$

In these units, fixed point solutions follow from the condition

$$\partial_t S_\star[\varphi] = 0 \tag{29}$$

since, if all variables are measured in terms of Λ, independence of Λ implies scale independence. (Subscript $\star$s will be used to denote fixed-point quantities.)

Unlike the rescaling (17), the rescalings (28) do not introduce further subtleties concerning the form of the flow equation which now reads:

$$\left(\partial_t + d_\varphi \varphi \cdot \frac{\delta}{\delta\varphi} + \Delta_\partial - D\right) S^{\mathrm{I}} = \frac{\delta S^{\mathrm{I}}}{\delta\varphi} \cdot c' \cdot \frac{\delta S^{\mathrm{I}}}{\delta\varphi} - \frac{\delta}{\delta\varphi} \cdot c' \cdot \frac{\delta S^{\mathrm{I}}}{\delta\varphi} - \frac{\eta}{2}\varphi \cdot \Delta^{-1} \cdot \varphi, \quad (30)$$

where $d_\varphi \equiv (D - 2 + \eta)/2$ is the scaling dimension of the field, a prime denotes a derivative with respect to the argument and Δ_∂ is the 'derivative counting operator' [28, 40] (utterly unrelated to the effective propagator, Δ):

$$\Delta_\partial \equiv D + \int \frac{d^D p}{(2\pi)^D} \varphi(p) p_\mu \frac{\partial}{\partial p_\mu} \frac{\delta}{\delta\varphi(p)}.$$

We can remove the leading D from this expression if we specify that the $\partial/\partial p_\mu$ does not strike the momentum conserving δ-function associated with each vertex.

Equation (25) becomes:

$$\left(\partial_t + \frac{D-2-\eta}{2}\varphi \cdot \frac{\delta}{\delta\varphi} + \Delta_\partial - D\right) \mathcal{D}[\varphi] = -\frac{\eta}{2}\varphi \cdot \Delta^{-1} \cdot \varphi. \quad (31)$$

Given all the rescalings, which in particular mean that $\Delta(p) = c(p^2)/p^2$, it follows from the definition of the dual action (7)—with $m(\mu) = 0$—that the fixed point condition (29) implies

$$\partial_t \mathcal{D}^{(n)}_\star = 0. \quad (32)$$

III. CRITICAL FIXED POINTS IN SCALAR FIELD THEORY

To analyse fixed points using the dual action formalism, let us start by solving (31) for $\mathcal{D}^{(2)}$ at a fixed point:

$$-\frac{2+\eta_\star}{2}\mathcal{D}^{(2)}_\star(p) + p^2 \frac{d\mathcal{D}^{(2)}_\star(p)}{dp^2} = -\frac{\eta_\star}{2}\Delta^{-1}(p). \quad (33)$$

The solution to this equation is

$$\mathcal{D}^{(2)}_\star(p) = -p^{2(1+\eta_\star/2)}\left[\frac{1}{b(\eta_\star)} + \frac{\eta_\star}{2}\int dp^2 \frac{c^{-1}(p^2)}{p^{2(1+\eta_\star/2)}}\right], \quad (34)$$

where $-1/b(\eta_\star)$ is the integration constant (assumed to be finite) and is a functional of the cutoff function. In the case where $\eta_\star \neq 0$, b is defined by the form of $\mathcal{D}^{(2)}_\star(p)$ taken if we perform the indefinite integral by Taylor expanding the cutoff function. For $\eta_\star = 0$, we make a choice such that the leading behaviour in the first case coincides with the behaviour in the second case, as $\eta_\star \to 0$. Thus, for small momentum, we have

$$\mathcal{D}^{(2)}_\star(p) = \begin{cases} -\dfrac{1}{b}p^{2(1+\eta_\star/2)} + \left(p^2 + \text{subleading}\right), & \eta_\star \neq 0, \\ \left(1 - \dfrac{1}{b}\right)p^2, & \eta_\star = 0. \end{cases} \quad (35)$$

Note that the subleading terms are cutoff dependent, not just with regards to their prefactors, but also to their structure. For example, if $\eta_\star = 2$ and $c'(0) \neq 0$, then the subleading piece has a nonpolynomial component $p^4 \ln p^2$, but this is absent altogether if $c'(0) = 0$. However, the real point to make here is that, so long as $\eta_\star < 2$, the subleading term in the brackets is always subleading compared to $bp^{2(1+\eta_\star/2)}$. So, for the case $\eta_\star < 2$, we can perform a sanity check by substituting (35) into (11) to yield

$$G(p) \sim \frac{1}{p^{2(1-\eta_\star/2)}} \sim \tilde{\Delta}_\star(p), \quad (36)$$

for small p, which is precisely the behaviour we expect at a critical fixed point. Whilst there are simple, general arguments as to why such a behaviour is expected (see e.g. [41]), I am unaware of a derivation as simple as this, directly from the ERG (see e.g. [11] for a different ERG derivation of (36) at the critical point of some model). For $\eta_\star \geq 2$, the leading behaviour of $G(p)$ in the small p limit no longer describes critical behaviour and is, indeed, cutoff dependent. We do not consider such cases further.

Note that in the large p limit we have

$$\lim_{p\to\infty} \tilde{\Delta}(p) = \frac{f(p^2)}{p^2}, \quad (37)$$

where $f(p^2)$ is a monotonically decreasing function, related to the cutoff function, with $f(p^2) \geq 0$ for real p [this is most easily seen by using a power law cutoff $c^{-1}(p^2) = 1 + p^{2r}$ in (34) and substituting the result into (15)]. It is worth pointing out that (37) is true irrespective of the sign of $\eta_\star$ and so negative anomalous dimensions cannot obviously be ruled out at non-trivial fixed points, based simply on the form of $\tilde{\Delta}$.

Moving on to dual action vertices with more than two legs, the fixed point equation is

$$\left(n\frac{D-2-\eta_\star}{2} + \sum_{i=1}^{n} p_i \cdot \partial_{p_i} - D\right) \mathcal{D}^{(n>2)}_\star(p_1, \ldots, p_n) = 0. \quad (38)$$

This has solution

$$\mathcal{D}^{(n>2)}_\star(p_1, \ldots, p_n) = P^{(n)}_r(p_1, \ldots, p_n), \quad (39)$$

with $P^{(n)}_r(\xi p_1, \ldots, \xi p_n) = \xi^r P^{(n)}_r(p_1, \ldots, p_n)$ and

$$r = D - n\frac{D-2-\eta_\star}{2}. \quad (40)$$

A. Fixed Points with $\eta_\star \geq 0$

Let us now focus on the case where $2 > \eta_\star \geq 0$. To this end, we now analyse $\overline{\mathcal{D}}^{(2)}_\star(p)$. First, we note from (13) and (35) that the leading behaviour in the small p limit is

$$\overline{\mathcal{D}}^{(2)}_{\star}(p) = \begin{cases} bp^{2(1-\eta_\star/2)} - p^2 + \ldots, & 2 > \eta_\star > 0 \\ (b-1)p^2 + \ldots, & \eta_\star = 0 \end{cases} \quad (41)$$

Secondly, we recognize that we can resum sets of loop diagrams contributing to $\overline{\mathcal{D}}^{(2)}(p)$ such that all internal lines become dressed, as indicated in figure 2.

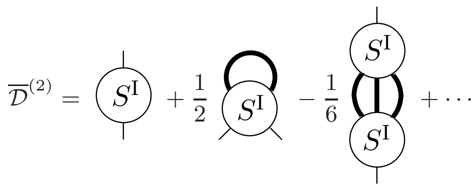

FIG. 2: Resummation of diagrams contributing to $\overline{\mathcal{D}}^{(2)}$: the thick lines represent dressed effective propagators, (14).

Now, consider the following scenarios:

1. $D > 4$, $\eta_\star \geq 0$,
2. $D = 4$ and $\eta_\star > 0$.

Assuming that the Wilsonian effective action vertices are Taylor expandable for small momenta—this being one of our requirements for physical acceptability—it is apparent by power counting that

$$\lim_{p\to 0} \overline{\mathcal{D}}^{(2)}_{\star}(p) \;=\; \text{const}, \quad (42)$$

$$\lim_{p\to 0} \frac{d}{dp^2}\overline{\mathcal{D}}^{(2)}_{\star}(p) \;=\; \text{const}, \quad (43)$$

where both constants are finite[6] (possibly zero). The second relationship follows from considering diagrams like the third one in figure 2. This diagram is the prototype for diagrams whose first derivative with respect to p^2 will diverge for certain dimensions and/or values of $\eta_\star$. In the IR, the leading term from this diagram looks like

$$\int d^D k \int d^D l \, \frac{1}{[k^2(l+p)^2(l+k)^2]^{1-\eta_\star/2}}.$$

[6] We assume that a necessary condition for the (differentiated) sum of diagrams to diverge is that there are (differentiated) individual diagrams which diverge. If there are no such divergences, we expect that the sum of diagrams is either convergent or can be resummed, as is reasonable bearing in mind the relationship of $\overline{\mathcal{D}}^{(2)}$ to the two-point correlation function. Again, it is emphasised that the diagrams' vertices are exact and have not been subject to a perturbative expansion.

Differentiating with respect to p^2 will increase the degree of IR divergence by two, but this is still not enough, given the above conditions on D and $\eta_\star$, to render the term IR divergent as $p \to 0$.

More generally, we have the following power counting in the IR. Given I internal lines and V vertices, there are $L = I - V + 1$ loops. If we differentiate with respect to p^2 a total of P times, then the degree of IR divergence is

$$\mathbb{D} \geq D(I - V + 1) - 2(1 - \eta_\star/2)I - 2P,$$

where we understand $\mathbb{D} > 0$ to be IR safe. Now, since all two-point vertices have been absorbed into the dressed effective propagators, and since we have only even-legged vertices, each vertex must have at least four legs. Given that there are two external legs, this implies that

$$I \geq 2V - 1.$$

Consequently [for $D \geq 2(1 - \eta_\star/2)$], we have

$$\mathbb{D} \geq (D-4)V + (2V-1)\eta_\star + 2(1-P).$$

Given the restrictions that either $D > 4$ and $\eta_\star \geq 0$ or $D = 4$ and $\eta_\star > 0$ we see that, both for $P = 0$ and $P = 1$, there are no IR divergent diagrams. Acting on $\overline{\mathcal{D}}^{(2)}(p)$ with further derivatives with respect to p^2, it may be that the limit $p \to 0$ now diverges, but this does not concern us here. Rather, we simply note that we can write

$$\overline{\mathcal{D}}^{(2)}_{\star}(p) \sim \text{const} + \mathcal{O}(p^2) + \text{subleading}, \quad (44)$$

where the subleading terms are not necessarily polynomial in p. For a critical fixed point, we must set the constant piece equal to zero. Comparing the resulting equation with the top line of (41) we deduce the following: (i) in $D > 4$, any critical fixed point with non-negative $\eta_\star$ must have precisely $\eta_\star = 0$ (ii) in $D = 4$ there are no critical fixed points with $\eta_\star > 0$ and so, again, if there are to be any non-trivial fixed points with $\eta_\star \geq 0$, they must saturate the inequality.

At this point, we could conclude our analysis for $\eta_\star \geq 0$ since there is a theorem due to Pohlmeyer which implies that a scale invariant scalar field theory with vanishing anomalous dimension must be trivial [10]. However, since it is simple and instructive to prove, within our approach, that there cannot be any non-trivial critical fixed points with $\eta_\star = 0$ for $D \geq 4$, we will do so.

Let us consider the four-point vertex of the dual action. We start by expressing $\mathcal{D}^{(4)}$ in terms of 1PI pieces:

$$\mathcal{D}^{(4)}(p_1,p_2,p_3,p_4) = \frac{\overline{\mathcal{D}}^{(4)}(p_1,p_2,p_3,p_4)}{\prod_{i=1}^4 \left[1 + \Delta(p_i)\overline{\mathcal{D}}^{(2)}(p_i)\right]}. \quad (45)$$

(Since this formula is generally valid, we have dropped the $\star$.) The resummation of the decorations of the legs into the denominator makes it clear that, since we have just shown that $\overline{\mathcal{D}}^{(2)}_{\star}(p) \sim p^2$, any IR divergences of $\mathcal{D}^{(4)}_{\star}$

must occur within $\overline{\mathcal{D}}_\star^{(4)}$. As with $\overline{\mathcal{D}}^{(2)}$, we can resum classes of loop diagrams contributing to $\overline{\mathcal{D}}^{(4)}$ such that all internal lines become dressed. A selection of the resummed diagrams contributing to $\mathcal{D}^{(4)}$ is shown in figure 3.

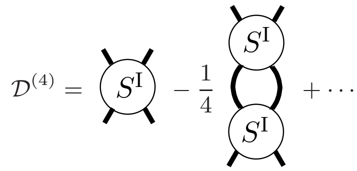

FIG. 3: Resummation of diagrams contributing to $\mathcal{D}^{(4)}$. In the final diagram we implicitly sum over the independent permutations of the external legs. The thick external lines denote decorated legs, as in (45).

From (38) and (40), the fixed point solution for $\mathcal{D}^{(4)}$ is

$$\mathcal{D}_\star^{(4)}(p_1, p_2, p_3, p_4) = P_r^{(4)}(p_1, p_2, p_3, p_4), \tag{46}$$

with

$$r = 4 - D + 2\eta_\star. \tag{47}$$

First let us consider $D > 4$.

The crucial point is that, by the same logic that lead to (44), we have:

$$\overline{\mathcal{D}}_\star^{(4)}(p_1, p_2, p_3, p_4) = c_4 + \text{subleading}, \tag{48}$$

where c_4 is a constant. We can see this by looking at the second diagram on the right-hand side of figure 3 (modulo the dressings on the external legs), which is the prototype for diagrams which possess IR divergences for certain values of D and/or $\eta_\star$. As a direct consequence of (48), it must be that $r \geq 0$. But, given this condition and given $\eta_\star = 0$, (47) has no solutions and so we conclude that $\mathcal{D}_\star^{(4)} = 0$. But, directly from this, it follows that $\mathcal{D}_\star^{(6)} = 0$, also. This is because in $D > 4$ the only contributions to $\mathcal{D}_\star^{(6)}$ which could potentially violate the analogue of (48) comes from the diagram of figure 4. But this term is built from a pair of $\overline{\mathcal{D}}_\star^{(4)}$s, which we have just said vanish! Thus, repeating the same logic that lead to (47), we find that $\mathcal{D}_\star^{(6)} = 0$. By induction, then, we have that $\mathcal{D}_\star^{(n>2)} = 0$ and the only fixed point is the Gaussian one.

In $D = 4$, the argument for the vanishing of $\mathcal{D}_\star^{(4)}$ is only slightly more involved. Consider the fixed point equation for $\mathcal{D}_\star^{(n)}$: returning to (38) there is now a solution for

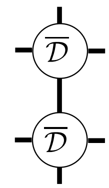

FIG. 4: A potentially strongly IR divergent contribution to $\mathcal{D}^{(6)}$.

the four-point vertex in $D = 4$ with $\eta_\star = 0$ which is potentially compatible with the structure of $\mathcal{D}^{(4)}$:

$$\mathcal{D}_\star^{(4)}(p_1, p_2, p_3, p_4) = c_4. \tag{49}$$

[Whilst contributions of the form e.g. p_1^2/p_2^2 are also solutions of (38), there is no way to generate such a strong IR divergence in $D = 4$, given that $\eta_\star = 0$.] However, it turns out that c_4 must be zero. To see this, consider further building up contributions to the second diagram in figure 3, as shown in figure 5 (we drop the overall combinatoric factor).

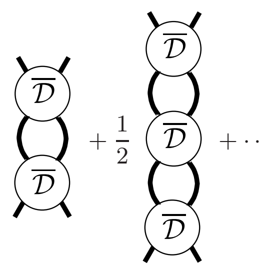

FIG. 5: Further resummation of contributions to $\mathcal{D}^{(4)}$.

The presence of the first term is hopefully obvious enough. However, by unpackaging each of the $\overline{\mathcal{D}}^{(4)}$s, it is apparent that the first diagram contains a copy of the next diagram with a factor of -1. To avoid this double counting, we add the second diagram precisely to remove half of this contribution. (For a discussion of the combinatorics, see [25].) In turn, we should now add a third term etc., as represented by the ellipsis.

Taking the momenta carried by the top pair of legs to be p and q, let us now define

$$I(p,q) \equiv \int d^D k \, \frac{\Delta^2(k)\Delta^2(k+p+q)}{\tilde{\Delta}(k)\tilde{\Delta}(k+p+q)}.$$

Given (45), and since $\mathcal{D}_\star^{(4)} = c_4$, we can resum the diagrams of figure 5 to give[7]

$$\frac{c_4^2 I(p,q)}{1 - \frac{1}{2}c_4 I(p,q)}.$$

This represents a momentum dependent contribution to $\mathcal{D}_\star^{(4)}$ which cannot cancel against anything else. Therefore, since $\mathcal{D}_\star^{(4)}$ is a constant, we deduce that $c_4 = 0$ i.e. $\mathcal{D}_\star^{(4)} = 0$.

Now consider the six-point dual action vertex. The solution to (38), in $D = 4$ with $\eta_\star = 0$, requires that the six-point vertex $\sim 1/\mathrm{mom}^2$. However, such a divergence can only come from the diagram of figure 4, which again vanishes since $\overline{\mathcal{D}}_\star^{(4)} = 0$. Consequently, we conclude that $\overline{\mathcal{D}}_\star^{(6)} = 0$. Proceeding by induction, as before, we find that the only acceptable critical fixed point solution in $D = 4$, given the restriction that $\eta_\star$ is non-negative, is the Gaussian one!

Before moving on, it is instructive to examine the form of the Gaussian solution. In this case, $\overline{\mathcal{D}}_\star^{(2)}(p) = S_\star^{\mathrm{I}(2)}(p)$ and, using (4), we find that

$$S_\star[\varphi] = \frac{1}{2}\varphi \cdot \frac{\Delta^{-1}(p)}{1 - (1 - 1/b)c(p^2)} \cdot \varphi,$$

exactly in agreement with [2]. The reason that there is a line of equivalent Gaussian fixed points, parametrized by b, is due to the reparametrization invariance inherent in the ERG [2, 28, 42]. Note that $b = 1$ corresponds to canonical normalization of the kinetic term.

B. Fixed Points with $\eta_\star < 0$

The simplest fixed points with negative anomalous dimension to deal with are the exotic Gaussian fixed points found by Wegner [11]. To recover these, we set $\mathcal{D}_\star^{(n>2)} = 0$ and note that at a Gaussian fixed point we have $\overline{\mathcal{D}}_\star^{(2)}(p) = S_\star^{\mathrm{I}(2)}(p)$. Defining $\gamma_\star \equiv -\eta_\star$ we can read off the leading behaviour in the small momentum limit from (41) by recognizing that this comes from the first line, but with the relative importance of the first two terms interchanged:

$$S_\star^{\mathrm{I}(2)}(p) = -p^2 + bp^{2(1+\gamma_\star/2)} + \dots. \tag{50}$$

Now, the crucial point about the $-p^2$ term is that, as is apparent from (4), it removes the standard kinetic term, $\frac{1}{2}\varphi \cdot p^2 \cdot \varphi$, from the full Wilsonian effective action. Thus we find that, for small p,

$$S_\star^{(2)}(p) \sim p^{2\gamma_\star/2}, \tag{51}$$

where, to ensure locality, we must take $\gamma_\star/2$ to be an integer. This removal of the standard kinetic term is a generic feature of fixed points with negative anomalous dimension, as we will now see.

The vital property of fixed points with negative anomalous dimension, which we will now exploit, is that

$$\lim_{p\to 0} \frac{d}{dp^2}\overline{\mathcal{D}}_\star^{(2)}(p) = -1, \tag{52}$$

completely independently of the shape of the cutoff function. Note that for fixed points with positive anomalous dimension, the right-hand side of (52) instead diverges and so the logic which we now use does not apply. This is a good job, as otherwise we would rule out physically acceptable fixed points which we know to exist for $D < 4$.

In what follows, it will be useful to define

$$z \equiv \lim_{p\to 0} \frac{d}{dp^2} S^{\mathrm{I}(2)}(p). \tag{53}$$

Note that z is just a number. Similarly to the above discussion, if $z > -1$, then the *full* action has a p^2 kinetic term with the right sign. In this case, z is a free parameter corresponding to the normalization of the field, with $z = 0$ being canonical normalization. Let us denote the remaining contributions to the left-hand side of (52) by W.

There are now two cases to consider: $W = 0$ and $W \neq 0$. Note that the first case includes Wegner's Gaussian fixed points but could, in principle, include non-Gaussian fixed points. This could happen if all $n > 2$-point vertices come with sufficiently high powers of momenta on each of their legs. Either way, such fixed points are ruled out by our requirement of unitarity, upon continuation to Minkowski space. This just leaves the case where

$$W = -z - 1 \tag{54}$$

which, as we again emphasise, must be true independently of the shape of the cutoff function.

With this observation in mind we notice that, whilst the characteristic scale of the cutoff function $c(k^2)$ is $k^2 \sim 1$, the freedom in the shape of the cutoff function means that we can readily suppress modes considerably before or after this point. For example, $c(x) = e^{-x}$ and $c(x) = \exp(e - \exp e^x)$ are both perfectly legitimate cutoff functions, but which effectively suppress modes above somewhat different values of $x \equiv k^2$. Now, imagine a cutoff profile which is essentially flat up to $k^2 \sim 1$ and then falls off very rapidly. (Polchinski-like ERG equations need careful treatment for a sharp cutoff [43], but we can get arbitrarily close to this limit without running into difficulties.) Next, consider a cutoff profile of the same general shape, but which cuts off modes at a scale δk earlier. Since W is independent of the shape of the cutoff profile, this tells us that there can be no net contributions from the various loop integrals which involve momenta in the range $1 - \delta k \leq k \leq 1$. Repeating this

[7] Modulo the factor of 1/2, the resummation works just the same as in (13).

argument, it becomes clear that W cannot receive contributions from any range of loop momenta. This leaves the only potential contributions coming from when the loop momenta are precisely equal to zero. It is tempting to say that such contributions must have zero support but this is not true, as it is quite possible that individual terms contributing to $d\overline{\mathcal{D}}^{(2)}_{\star}(p)/dp^2$ diverge as $p \to 0$.

However, inspired by the resummations shown in figure 5, let us resum the diagrams contributing to $\overline{\mathcal{D}}^{(2)}_{\star}(p)$, yet further, as shown in figure 6.

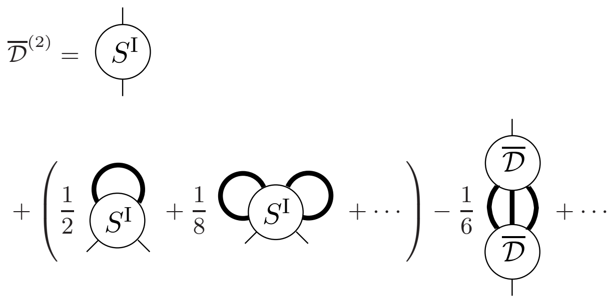

FIG. 6: Further resummation of diagrams contributing to $\overline{\mathcal{D}}^{(2)}$. The brackets contain a sequence of terms with a single vertex decorated by an increasing number of $\tilde{\Delta}$s. The second ellipsis represents diagrams built out of $\overline{\mathcal{D}}^{(n>2)}$ vertices.

As before, there are corrections to the final term in figure 6, to avoid overcounting. However, the crucial point is that all such contributions, as well as the rest of those included in the second ellipsis, are built out of the 1PI dual action vertices (with more than two legs) and dressed effective propagators. Now, we know from the above arguments that we can analyse the various contributions to the left-hand side of (52) for small loop momenta. In this case, it is straightforward to show that the diagrams built out of $\overline{\mathcal{D}}^{(n>2)}$ vertices and dressed effective propagators go, after differentiation with respect to p^2, precisely as

$$p^{2(\gamma_\star/2)}$$

at a fixed point and, therefore, vanish in the $p \to 0$ limit. The argument for this goes as follows.

First, we need to determine the number of powers of momenta, $\overline{r}_n$, carried by the $\overline{\mathcal{D}}^{(n>2)}_{\star}$. Let us start by recalling from (40) that, for the $\mathcal{D}^{(n>2)}_{\star}$, we have

$$r_n = D - n\frac{D-2+\gamma_\star}{2}, \qquad (55)$$

where we now explicitly tag r with n. To go from r_n to $\overline{r}_n$ we will strip off the leg decorations from $\mathcal{D}^{(n>2)}$ and, to this end, define $\mathcal{D}'^{(n>2)}$ via

$$\mathcal{D}^{(n>2)}(p_1,\ldots,p_n) = \frac{\mathcal{D}'^{(n>2)}(p_1,\ldots,p_n)}{\prod_{i=1}^n \left[1+\Delta(p_i)\overline{\mathcal{D}}^{(2)}(p_i)\right]}. \qquad (56)$$

Notice that $\mathcal{D}'^{(4)} = \overline{\mathcal{D}}^{(4)}$ but, beyond the four point level, there are additional contributions (see for example figure 4). However, one of the contributions to $\mathcal{D}'^{(n>2)}$ is always $\overline{\mathcal{D}}^{(n>2)}$ and so, from (55) and (56), it is apparent that

$$\overline{r}_n = r'_n = D - n\frac{D-2-\gamma_\star}{2}. \qquad (57)$$

Thus, we can interpret each $\overline{\mathcal{D}}^{(n>2)}_{\star}$ vertex as carrying $n(\gamma_\star + 2 - D)/2$ powers of momentum per leg, plus an additional D powers. Consider, then, the small momentum behaviour, $\mathcal{R}$, of a diagram contributing to $\lim_{p\to 0} d\overline{\mathcal{D}}^{(2)}_{\star}(p)/dp^2$ built out of V $\overline{\mathcal{D}}^{(n)}_{\star}$ vertices and I dressed effective propagators. Totting up the dependencies from the loop integrals, the dressed effective propagators, the vertices, and remembering that we differentiate with respect to p^2 we have:

$$\begin{aligned}\mathcal{R} &= D(I-V+1) - I(2+\gamma_\star) \\ &\qquad + (I+1)(2+\gamma_\star - D) + VD - 2 \\ &= \gamma_\star.\end{aligned}$$

Therefore, diagrams of this type do indeed behave like $p^{2\gamma_\star/2}$, as claimed, and so do not contribute to W.

Consequently, the only contributions to W come from the diagrams enclosed by the brackets in figure 6, which are most certainly IR safe for $p \to 0$. Given the independence of W on the cutoff function, these diagrams neither receive contributions from any range of loop momenta, nor have support for zero loop momenta. Thus, there are no fixed points with $W \neq 0$. Finally, then, the only fixed points with negative anomalous dimension are those for which $z = -1$ and these correspond to non-unitary theories, upon continuation to Minkowski space.

IV. NON-COMPACT, PURE ABELIAN GAUGE THEORY

Exactly the same methodology can be applied to demonstrate that there are no physically acceptable non-trivial fixed points in non-compact, pure Abelian gauge theory, but this time in any dimension. In a lattice formulation, compactness refers to the gauge connection, A_μ, being valued on a circle, as opposed to the real line. In the continuum limit, the compact case supports the existence of field configurations corresponding to monopoles.

We focus on just the non-compact case, for which a manifestly gauge invariant flow equation is provided by [44]:

$$-\Lambda\partial_\Lambda S[A] = \frac{1}{2}\frac{\delta S}{\delta A}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma}{\delta A} - \frac{1}{2}\frac{\delta}{\delta A}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma}{\delta A}, \qquad (58)$$

where the dots now include a contraction of the Lorentz indices, wherever these indices are suppressed. Manifest gauge invariance follows since, for the Abelian Abelian symmetry, $\delta/\delta A$ is gauge invariant. Gauge invariance of the functional derivatives also means that the flow equation can be regularized just as in the scalar case, simply by introducing a cutoff function $c(p^2/\Lambda^2)$.

Seeing as we have not fixed the gauge, it is important that the effective propagator is properly interpreted. The point is that, first and foremost, the object $\dot{\Delta}$ in the flow equation should be thought of as an ERG kernel, which provides UV regularization, ensuring that the flow equation is well defined; and if the flow equation is well defined, then we are happy. In scalar field theory, it just so happens that $\dot{\Delta}$ can be directly interpreted as the flow of a UV regularized propagator. In manifestly gauge invariant ERGs, the interpretation must be different [35, 36], since we cannot define a propagator in the usual way, having never fixed the gauge. However, the integrated ERG kernel plays a role which is analogous to the usual propagator, albeit residing inside ERG diagrams, rather than Feynman diagrams. Indeed, this is the reason for the terminology 'effective propagator'. Actually, in the current case of *pure* Abelian gauge theory, these considerations are essentially irrelevant, as we will see below.

With these points in mind, we define

$$S[A] = \frac{1}{2} A_\mu \cdot \Delta^{-1}_{\mu\nu} \cdot A_\nu + S^{\mathrm{I}}[A] \tag{59}$$

and also

$$\Delta(p) \equiv \frac{c(p^2)}{p^2}. \tag{60}$$

The consequence of manifest gauge invariance is that $\Delta(p)$ is the inverse of $\Delta^{-1}_{\mu\nu}(p)$ only inverse in the transverse space:

$$\Delta^{-1}_{\mu\nu}(p)\Delta(p) = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} = \frac{\Box_{\mu\nu}(p)}{p^2}, \tag{61}$$

where $\Box_{\mu\nu}(p) \equiv p^2\delta_{\mu\nu} - p_\mu p_\nu$. However, in the pure Abelian case, the final term, 'the gauge remainder' [31], has no effect since gauge invariance implies that all vertices satisfy

$$p_{\mu_i} S_{\mu_1\cdots\mu_i\cdots\mu_n}(p_1, \ldots, p_i, \ldots, p_n) = 0, \qquad \forall i. \tag{62}$$

Consequently, whenever a gauge remainder appears inside a diagram, the diagram is annihilated.

As before, we define the dual action according to

$$-\mathcal{D}[A] = \ln\left\{\exp\left(\frac{1}{2}\frac{\delta}{\delta A}\cdot\Delta\cdot\frac{\delta}{\delta A}\right) e^{-S^{\mathrm{I}}[A]}\right\}, \tag{63}$$

and, as before, the dual action is an ERG invariant. Similarly, we can introduce the 1PI contributions to the dual action vertices, so long as we properly take account of the fact the the two-point dual action vertex is transverse. Thus, defining w according to

$$\mathcal{D}^{(2)}_{\mu\nu}(p) \equiv w(p)\Box_{\mu\nu}(p), \tag{64}$$

and defining $\overline{w}$ as the corresponding 1PI piece, the same logic that led to (12) yields:

$$w(p) = \frac{\overline{w}(p)}{1 + c(p^2)\overline{w}(p)}. \tag{65}$$

Noting on account of (62) that, when building up internal lines according to $\Delta(p)\left[\delta_{\mu\nu} - \Delta(p)\overline{w}(p)\Box_{\mu\nu}(p) + \cdots\right]$, all $p_\mu p_\nu$ contributions can be effectively set to zero, we define the dressed effective propagator according to:

$$\tilde{\Delta}(p) \equiv \frac{1}{p^2\left[c^{-1}(p^2) + \overline{w}(p)\right]}. \tag{66}$$

Exactly as in the scalar case, the next step is to rescale to dimensionless variables, having changed flow equation to avoid annoying factors of $1/Z$ on the right-hand side. Taking the seed action to possess only a kinetic term, the rescaled flow equation for the dual action reads:

$$\begin{aligned}\left(\partial_t + \frac{D-2-\eta}{2} A\cdot\frac{\delta}{\delta A} + \Delta_\partial - D\right)\mathcal{D}[A] \\ = -\frac{\eta}{2} A_\mu \cdot \Delta^{-1}_{\mu\nu} \cdot A_\nu.\end{aligned} \tag{67}$$

When computing $\mathcal{D}^{(2)}_\star(p)$, we must remember to account for the $\Box_{\mu\nu}(p)$ buried in the right-hand side. Employing (64), we find that

$$w_\star(p) = p^{2\eta_\star/2}\left[b - \frac{\eta_\star}{2}\int dp^2 \frac{c^{-1}(p^2)}{p^{2(1+\eta_\star/2)}}\right]. \tag{68}$$

We now follow through the logic used in the scalar case but note that, on account of (62) we have that, for $\eta_\star \geq 0$,

$$\lim_{p\to 0} \overline{w}_\star(p) = \text{const} \tag{69}$$

in any dimension. However, since for $2 > \eta_\star \geq 0$ equations (65) and (68) imply that

$$\lim_{p\to 0} \overline{w}_\star(p) \sim p^{-2\eta_\star/2},$$

we conclude that the anomalous dimension for a putative critical fixed point with non-negative $\eta_\star$ is zero. Utilizing the Ward identity, it is straightforward to show that this implies that $\mathcal{D}^{(4)}_\star$ vanishes, but now in any dimension. As before, we can proceed by induction to show that all $\mathcal{D}^{(>2)}_\star$ vanish. The analysis of fixed points with negative anomalous dimension exactly mirrors the scalar case.

Note that, in three dimensions, the analysis is unchanged by the presence of a Chern-Simons term (see [19] for a discussion of the effects of such a term in the context of the ERG).

V. CONCLUSION

The proofs presented in this paper rely crucially on the introduction of the 'dual action', given by (7) and (63) for scalar field theory and non-compact, pure Abelian gauge theory, respectively. This object has a natural interpretation in terms of correlation functions and, as a consequence, is an invariant of the ERG. It follows from this that the behaviour of the dual action vertices under rescalings of their momenta is directly related to the full scaling dimension of the field. Nevertheless, without somehow fixing the anomalous dimension, not much more can be said. The crucial point is that, for the examples studied in this paper, the behaviour of the two-point dual action vertex forces the anomalous dimension at a critical fixed point to be either zero or negative. In the former case, Pohlmeyer's theorem immediately implies that the only critical fixed point theory is trivial. In the latter case, whilst it might be that quasi-local critical fixed points exist, they necessarily violate unitarity.

An obvious set of questions to ask now is whether these methods can be extended to say something about:

1. critical fixed points in scalar field theory for $D < 4$;
2. other theories.

With regards to the first question it is certainly straightforward to identify the Wilson-Fisher fixed point using an ϵ-expansion (this has been explicitly checked). Whether any new approximation scheme and/or some refinement of existing approximation schemes is possible within the framework of this paper is left as an open question.

With regards to the second question, there are several major obstructions to applying the methods here to more complicated theories. First of all, recall that if the seed action has interactions then the flow of the dual action picks up an extra term on the right-hand side [see (24)]. In the cases considered in this paper, this term could be dropped, leaving a very simple equation for the dual action. However, in any theory where the two-point function is related to any higher point vertices by some symmetry—including, of course, QED and Yang-Mills theories—the seed action must possess interaction terms, in order that this symmetry be preserved by the flow equation. In these situations, the dual action is no longer an invariant of the ERG, and it is hard to see how to proceed. Indeed, if we use a direct generalization of the dual action proposed in this paper for these theories, then it does not even satisfy the right symmetries. (Although in retrospect, it is apparent that what would be the $\mathcal{O}(p^2)\times$nonpolynomial part of the two-point dual action vertex was used to extract the β-function in both QED [26, 35], SU(N) Yang-Mills [22, 32, 33], and QCD [36].) Whether the formalism can be adapted to cope with these issues is also left to the future.

However, there are at least a couple of places where one might be able to apply the techniques of this paper. First of all, it is reasonably straightforward to supersymmetrize the construction and analyse the existence of non-trivial fixed points in theories of a scalar chiral superfield [45]. Actually, using general arguments, it has recently been proven that for an asymptotic safety scenario to exist for the Wess-Zumino model, the associated non-trivial fixed point must have a negative anomalous dimension [46]. This can almost certainly be ruled out using the methodology of this paper. More generally, an asymptotic safety scenario for theories of a scalar chiral superfield—including those without a three-point contribution to the superpotential—most likely does not exist.

It might also be possible to investigate similar issues in the context of non-commutative scalar field theory. Whilst, from an ERG perspective, we might worry about how to deal with the non-locality inherent in this scenario, it turns out that one can transfer to a matrix basis and construct a matrix version of the Polchinski equation. Indeed, having added a harmonic oscillator term to the action, Grosse and Wulkenhaar used precisely this formalism to demonstrate the perturbative renormalizability of the resulting model [47]. Trivially, then, the dual action can be constructed; the challenge is to understand what the criteria for physical acceptability of (the non-commutative analogue of) fixed points are and what the procedure for constraining $\eta_\star$ becomes, in the matrix base [48]. This could be particularly interesting in light of the claim that there does indeed exist a non-trivial fixed point in the Grosse and Wulkenhaar model [49].

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[1] K. G. Wilson and J. Kogut, "The Renormalization group and the epsilon expansion," *Phys. Rept.* **12** (1974) 75.

[2] T. R. Morris, "Elements of the continuous renormalization group," *Prog. Theor. Phys.* **131** (1998) 395, hep-th/9802039.

[3] S. Weinberg, "Ultraviolet Divergences In Quantum Theories Of Gravitation," in *General Relativity*, S. W. Hawking and W. Israel, eds., p. 790. CUP, 1979.

[4] A. M. Polyakov, "Compact gauge fields and the infrared catastrophe," *Phys. Lett.* **B 59** (1975) 82.

[5] A. M. Polyakov, "Quark Confinement And Topology Of Gauge Groups," *Nucl. Phys.* **B 120** (1977) 429.

[6] K. Halpern and K. Huang, "Non-Trivial Directions for Scalar Fields," *Phys. Rev.* **D 53** (1996) 3252, hep-th/9510240.

[7] T. R. Morris, "On the Fixed-Point Structure of Scalar Fields," *Phys. Rev. Lett.* **77** (1996) 1658,

hep-th/9601128.

[8] T. R. Morris, "Three-dimensional massive scalar field theory and the derivative expansion of the renormalization group," *Nucl. Phys.* **B 495** (1997) 477, hep-th/9612117.

[9] T. R. Morris, "A manifestly gauge invariant exact renormalization group," in *The Exact Renormalization Group*, A. Krasnitz *et al.*, eds., p. 1. World Sci., 1999. hep-th/9810104.

[10] K. Pohlmeyer, "The Jost-Schroer Theorem For Zero-Mass Fields," *Commun. Math. Phys.* **12** (1969) 204.

[11] F. J. Wegner, "The Critical State, General Aspects," in *Phase Transitions and Critical Phenomena, Vol.6*, C. Domb and M. S. Green, eds., p. 7. London, 1976.

[12] K. Higashijima and E. Itou, "Unitarity bound of the wave function renormalization constant," *Prog. Theor. Phys.* **110** (2003) 107, hep-th/0304047.

[13] M. Aizenman, "Proof of The Triviality of φ^4_d Field Theory and Some Mean Field Features of Ising Models For $d > 4$," *Phys. Rev. Lett.* **47** (1981) 1.

[14] J. Frohlich, "On The Triviality of $\lambda\varphi^4_d$ Theories and the Approach to the Critical Point in $d \overset{>}{(-)} 4$ Dimensions," *Nucl. Phys.* **B 200** (1982) 281.

[15] D. J. E. Callaway, "Triviality Pursuit: Can Elementary Scalar Particles Exist?," *Phys. Rept.* **167** (1988) 241.

[16] P. Hasenfratz and F. Niedermayer, "Perfect Lattice Action For Asymptotically Free Theories," *Nucl. Phys.* **B 414** (1994) 785, hep-lat/9308004.

[17] T. R. Morris, "On truncations of the exact renormalization group," *Phys. Lett.* **B 334** (1994) 355, hep-th/9405190.

[18] C. Bagnuls and C. Bervillier, "Exact renormalization group equations: An introductory review," *Phys. Rept.* **348** (2001) 91, hep-th/0002034.

[19] T. R. Morris, "Noncompact pure gauge QED in 3-D is free," *Phys. Lett.* **B 357** (1995) 225, hep-th/9503225.

[20] J. Polchinski, "Renormalization And Effective Lagrangians," *Nucl. Phys.* **B 231** (1984) 269.

[21] K. I. Aoki, K. Morikawa, W. Souma, J. I. Sumi, and H. Terao, "Rapidly converging truncation scheme of the exact renormalization group," *Prog. Theor. Phys.* **99** (1998) 451, hep-th/9803056.

[22] O. J. Rosten, "A manifestly gauge invariant and universal calculus for SU(N) Yang-Mills," *Int. J. Mod. Phys.* **A 21** (2006) 4627, hep-th/0602229.

[23] O. J. Rosten, "General computations without fixing the gauge," *Phys. Rev.* **D 74** (2006) 125006, hep-th/0604183.

[24] O. J. Rosten, "Universality from very general nonperturbative flow equations in QCD," *Phys. Lett.* **B 645** (466) 2007, hep-th/0611323.

[25] O. J. Rosten, "Sensitivity of Nonrenormalizable Trajectories to the Bare Scale," *J. Phys.* **A 41** (2008) 075406, arXiv:0710.3658 [hep-th].

[26] O. J. Rosten, "A Resummable beta-Function for Massless QED," *Phys. Lett.* **B 662** (2008) 237, arXiv:0801.2462 [hep-th].

[27] S. Weinberg, *The Quantum Theory of Fields II*. CUP, 1995.

[28] F. J. Wegner, "Some Invariance Properties of the Renormalization Group," *J. Phys* **C 7** (1974) 2098.

[29] T. R. Morris and J. L. Latorre, "Exact scheme independence," *JHEP* **0011** (2000) 004, hep-th/0008123.

[30] L. P. Kadanoff, "Scaling laws for Ising models near T(c)," *Physics* **2** (1966) 263.

[31] S. Arnone, A. Gatti, and T. R. Morris, "A proposal for a manifestly gauge invariant and universal calculus in Yang-Mills theory," *Phys. Rev.* **D 67** (2003) 085004, hep-th/0209162.

[32] S. Arnone, T. R. Morris, and O. J. Rosten, "A Generalised manifestly gauge invariant exact renormalisation group for SU(N) Yang-Mills," *Eur. Phys. J.* **C 50** (2007) 467, hep-th/0507154.

[33] T. R. Morris and O. J. Rosten, "A manifestly gauge invariant, continuum calculation of the SU(N) Yang-Mills two-loop beta function," *Phys. Rev.* **D 73** (2006) 065003, hep-th/0508026.

[34] S. Arnone, A. Gatti, T. R. Morris, and O. J. Rosten, "Exact scheme independence at two loops," *Phys. Rev.* **D 69** (2004) 065009, hep-th/0309242.

[35] S. Arnone, T. R. Morris, and O. J. Rosten, "Manifestly gauge invariant QED," *JHEP* **0510** (2005) 115, hep-th/0505169.

[36] T. R. Morris and O. J. Rosten, "Manifestly gauge invariant QCD," *J. Phys.* **A 39** (2006) 11657, hep-th/0606189.

[37] R. D. Ball, P. E. Haagensen, J. I. Latorre, and E. Moreno, "Scheme Independence And The Exact Renormalization Group," *Phys. Lett.* **B 347** (1995) 80, hep-th/9411122.

[38] S. Arnone, A. Gatti, and T. R. Morris, "Exact scheme independence at one loop," *JHEP* **0205** (2002) 059, hep-th/0201237.

[39] O. J. Rosten, "A primer for manifestly gauge invariant computations in SU(N) Yang-Mills," *J. Phys.* **A 39** (2006) 8699, hep-th/0507166.

[40] T. R. Morris, "Derivative expansion of the exact renormalization group," *Phys. Lett.* **B 329** (1994) 241, hep-th/9403340.

[41] M. E. Fisher, "Renormalization group theory: Its basis and formulation in statistical physics," *Rev. Mod. Phys.* **70** (1998) 653.

[42] K. G. Wilson and T. L. Bell, "Nonlinear Renormalization Groups," *Phys. Lett.* **B 10** (1974) 3935.

[43] T. R. Morris, "The Exact renormalization group and approximate solutions," *Int. J. Mod. Phys.* **A 9** (1994) 2411, hep-ph/9308265.

[44] T. R. Morris, "A gauge invariant exact renormalization group. I," *Nucl. Phys.* **B 573** (2000) 97, hep-th/9910058.

[45] O. J. Rosten, "Work in Progress."

[46] O. J. Rosten, "Constraints on an Asymptotic Safety Scenario for the Wess-Zumino Model," arXiv:0807.4106 [hep-th].

[47] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4 theory on noncommutative $\mathbb{R}^4$ in the matrix base," *Commun. Math. Phys.* **256** (2005) 305, hep-th/0401128.

[48] O. J. Rosten and C. Sämann, "Work in Progress."

[49] M. Disertori, R. Gurau, J. Magnen, and V. Rivasseau, "Vanishing of beta function of non commutative ϕ^4_4 theory to all orders," *Phys. Lett.* **B 649** (2007) 95, hep-th/0612251.