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Applications of the Superconformal Index for Protected Operators and q-Hypergeometric Identities to $\mathcal{N}=1$ Dual Theories

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The results of Römelsberger for a $\mathcal{N}=1$ superconformal index counting protected operators, satisfying a BPS condition and which cannot be combined to form long multiplets, are analysed further. The index is expressible in terms of single particle superconformal characters for $\mathcal{N}=1$ scalar and vector multiplets. For SQCD, involving $SU(N_c)$ gauge groups and appropriate numbers of flavours N_f, the formula used to construct the index may be proved to give identical results for theories linked by Seiberg duality using recently proved theorems for q-series elliptic hypergeometric integrals. The discussion is also extended to Kutasov-Schwimmer dual theories in the large N_c, N_f limit and to dual theories with $Sp(N)$ and $SO(N)$ gauge groups. For the former, a transformation identity for elliptic hypergeometric integrals directly verifies that the index is the same for the electric and magnetic theories. For $SO(N)$ theories the corresponding result may also be obtained from the same basic identity. An expansion of the index to several orders is also obtained in a form where the detailed protected operator content may be read off. Relevant mathematical results are reviewed.

Keywords: $\mathcal{N}=1$ Superconformal Symmetry, Seiberg Duality, Characters, Superconformal Index, q-Series

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1. Introduction

A remarkable new insight into the dynamics of supersymmetric quantum field theories was the discovery by Seiberg in the 1990's of dualities analogous to those in soluble two dimensional integrable models [1], for a textbook discussion see [2]. For a $\mathcal{N}=1$ gauge theory with gauge group G and a suitable number N_f of chiral matter 'quark' fields, belonging the fundamental representation of G and transforming under a flavour symmetry group F, there is a duality between the initial 'electric' theory and an associated 'magnetic' theory with a dual gauge group $\tilde{G}$ but the same flavour symmetry F. In the dual magnetic theory, besides the appropriate 'quark' fields, the matter fields also include chiral 'mesons' to match with the corresponding electric theory. Both electric and magnetic theories are asymptotically free but they have a common IR fixed point realising a non-trivial interacting $\mathcal{N}=1$ superconformal theory. As usual in dual theories the strong coupling regime of the electric theory corresponds to the weak coupling regime of the magnetic one, and vice-versa. In the canonical example $G=SU(N_c)$ and $F=SU(N_f)\times SU(N_f)\times U(1)_B\times U(1)_R$ and with $\frac{3}{2}N_f\leq N_c\leq 3N_f$ then $\tilde{G}=SU(N_f-N_c)$. Each conjectured duality is justified by many non-trivial consistency checks. The original Seiberg dualities have also been extended to different gauge groups [3] and theories with further fields [4,5,6] showing the existence of a plethora of superconformal IR fixed points in $\mathcal{N}=1$ supersymmetric field theories linked by RG flows after introducing mass terms or other relevant perturbations.

More recently the detailed operator content of four dimensional superconformal gauge theories has been intensively investigated. A critical issue is to distinguish between protected operators satisfying a BPS condition and whose scale dimensions Δ saturate an associated unitarity bound and those operators which are not so constrained with a scale dimension determined by the detailed dynamics. In $\mathcal{N}=4$ theories the former belong to short or semi-short supermultiplets while the latter form long multiplets with Δ depending on g the coupling so that they may disappear from the spectrum in the strong coupling limit. Since semi-short multiplets may combine to form long multiplets which gain anomalous dimensions in perturbation theory the counting of protected operators, satisfying BPS constraints, is a not an immediately straightforward issue. In [7] Kinney *et al* formulated an index for general $\mathcal{N}$ superconformal theories such that contributions from any combinations of multiplets forming a long multiplet cancel and hence only protected operators are relevant. The index is then a topological invariant under smooth deformations preserving superconformality and was calculated in [7] to give the same results for $\mathcal{N}=4$ theories both at weak coupling and also at strong coupling through the AdS/CFT correspondence, see also [8]. The index in various sectors may also be obtained [9] by considering suitable limits of partition functions for counting gauge singlet operators where the relevant characters involve the supertrace, or equivalently contain a factor $(-1)^F$, and the limit ensures no long multiplet contribution. These results were applied also in [9] to discuss $\mathcal{N}=4$ theories with an $SU(N)$ gauge group in the large N limit.

For $\mathcal{N}=1$ theories the basic contributions to the index are expressible as $SU(2,1)$ characters. For such theories Römelsberger [10,11] also constructed an index which is essentially equivalent to that of [7] in this case. Römelsberger further gave a prescription for determining the index at the non trivial IR fixed points related by Seiberg duality and then showed that there was a very non-trivial matching of the two independent electric and magnetic expressions for the index by considering a series expansion up to a certain order in particular cases. In general to calculate the index it is necessary to identify a supercharge $\mathcal{Q}$, with associated adjoint[1] $\mathcal{Q}^+$, such that

$$\{\mathcal{Q},\mathcal{Q}^+\}=2\mathcal{H}\,,\qquad \mathcal{Q}^2=0\,,\tag{1.1}$$

so that $\mathcal{H}$ has a positive semi-definite spectrum. The index is then formed by the supertrace for states belonging to the kernel of $\mathcal{H}$ and so belonging to the cohomology of $\mathcal{Q},\mathcal{Q}^+$. The generators commuting with $\mathcal{Q},\mathcal{Q}^+$ in $\mathcal{N}=1$ theories are then

$$\mathcal{M}_A{}^B=\begin{pmatrix} M_\alpha{}^\beta+\frac{1}{2}\delta_\alpha{}^\beta\mathcal{R} & \mathcal{P}_\alpha \\ -\bar{\mathcal{P}}^\beta & -\mathcal{R}\end{pmatrix}\,,\qquad \bar{\mathcal{P}}^\beta=(\mathcal{P}_\beta)^+\,,\tag{1.2}$$

which satisfy the Lie algebra for $SU(2,1)$, $[\mathcal{M}_A{}^B,\mathcal{M}_C{}^D]=\delta_C{}^B\mathcal{M}_A{}^D-\delta_A{}^D\mathcal{M}_C{}^B$. In (1.2) $M_\alpha{}^\beta=(M_\beta{}^\alpha)^+$ contains the generators $J_3,J_\pm$ for the $SU(2)$ subgroup acting on chiral spinors while

$$\mathcal{R}=R+2\bar{J}_3+c\mathcal{H}\,,\tag{1.3}$$

with R the generator for $U(1)_R$. The index may then be defined by

$$I(t,x)=\mathrm{tr}_{\ker\mathcal{H}}\big((-1)^F t^{\mathcal{R}}x^{2J_3}\big)\,,\tag{1.4}$$

although this may be extended by further variables related to additional symmetries.

In the prescription of Römelsberger [11] for $\mathcal{N}=1$ superconformal theories the index is first determined on 'single particle states' giving

$$\begin{aligned} i(t,x,h,g)={}&\frac{2t^2-t(x+x^{-1})}{(1-tx)(1-tx^{-1})}\,\chi_{\mathrm{adj.}}(g)\\ &+\sum_i\frac{t^{r_i}\chi_{R_F,i}(h)\chi_{R_G,i}(g)-t^{2-r_i}\chi_{\bar{R}_F,i}(h)\chi_{\bar{R}_G,i}(g)}{(1-tx)(1-tx^{-1})}\,, \end{aligned}\tag{1.5}$$

[1] The adjoint here is defined, for a space of states formed by local field operators ϕ acting on $|0\rangle$, by a scalar product determined by the two point functions for ϕ. It differs from the usual conjugation so that, for any operator $\mathcal{O}$, $\mathcal{O}^+=U^{-1}\mathcal{O}^\dagger U$ for $U^\dagger=U$, [9]. Thus for the dilation operator $H^+=H$ although $H^\dagger=-H$.

which depends also on the symmetry group elements $g \in G$, $h \in F$. In (1.5) the first term represents the contribution for gauge fields belonging to the adjoint representation of G and the sum corresponds to chiral matter fields φ_i transforming under gauge group representations $R_{G,i}$, a flavour symmetry representations $R_{F,i}$, with $\chi_{R_F,i}(h), \chi_{R_G,i}(g)$ the appropriate characters. The terms proportional to t^{r_i} and t^{2-r_i} result from a chiral scalar with R-charge r_i and the fermion descendant, with $\bar{\jmath} = \frac{1}{2}$, of the conjugate anti-chiral partner with R-charge $-r_i$. In order to determine the index for all gauge singlet operators, as relevant for confining theories, this is then inserted into the 'plethystic' exponential [12] giving

$$I(t,x,h) = \int_G \mathrm{d}\mu(g)\, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\, i\big(t^n, x^n, h^n, g^n\big) \right), \tag{1.6}$$

for $\mathrm{d}\mu(g)$ the G invariant measure. A unitary superconformal representation would require in (1.5) $r_i \geq \frac{2}{3}$ with $r_i = \frac{2}{3}$ corresponding to a free field. In confining theories for chiral scalars belonging to non trivial representations of the gauge group this may be relaxed although it is necessary here that $r_i + r_j \geq \frac{2}{3}$ if $R_{G,i} \times R_{G,j}$ contains the identity representation and there is a corresponding composite gauge singlet $\varphi_i \cdot \varphi_j$, unless this operator is coupled to a dynamical field in the superpotential and so is constrained by equations of motion. In general we assume here unitary positive energy representations of $SU(2,1)$ requiring therefore $0 < r_i < 1$.

The interpretation of I as a superconformal index requires that the result for (1.6) should have an expansion of the form

$$I(t,x,h) = \sum_{q,j,R_F} n_{q,j,R_F}\, \frac{t^q\, \chi_{2j+1}(x)}{(1-tx)(1-tx^{-1})}\, \chi_{R_F}(h)\,, \tag{1.7}$$

where χ_{2j+1} are $SU(2)$ characters, and with n_{q,j,R_F} integer coefficients which determine the spectrum of protected operators in the $\mathcal{N}=1$ superconformal theory. Contributions to the sum in (1.7) for different supermultiplets are found in appendix A. Long multiplets are absent but contributions are present for chiral operators when $q = r$, the R-charge, with sign $(-1)^{2j}$ but there may also be contributions for other protected operators when $q = 2 + 2\bar{\jmath} + r$ and for sign $-(-1)^{2j+2\bar{\jmath}}$.

Despite generating formulae for the index which are in impressive agreement for dual superconformal theories the status of the results for the $\mathcal{N}=1$ superconformal index given by (1.5) and (1.6) is nevertheless not immediately clear, even for theories with no superpotential. Unlike the discussion in [7] for the $\mathcal{N}=4$ case there is no continuous link between the free case and the strong coupling limit, which is relevant for an IR fixed point, while preserving superconformal symmetry so that the index is well defined. The index formula in the asymptotically free limit gives different results since then $r_i = \frac{2}{3}$ for all φ_i.

Nevertheless we explore the consequences of the formulae for the index given by (1.5) and (1.6) in a significant number of examples and verify in many cases that the same result is obtained for both the electric and magnetic theories linked by Seiberg duality and its extensions, and hence develop the tests in [11] further. In general this requires non trivial identities for the group integrals in (1.6) for G and its dual $\tilde{G}$ which are then equivalent to identities for q-hypergeometric elliptic integrals. In some cases the magnetic theory is such that the dual gauge group $\tilde{G}$ is trivial. The expression for the magnetic index then requires no group integration so that showing the index identity requires the evaluation of the integral defining the index in the electric theory.

A particular example arises for $N_c = 2$, $N_f = 3$, which is perhaps the simplest non trivial case. The electric theory defines a contour integral in one variable while the magnetic theory provides an explicit evaluation. However, verifying this is very non trivial, a special case is related to a result found by Nassrallah and Rahman for an extension of the usual beta integral [13]. A generalisation of the Nassrallah-Rahman theorem by Spiridonov [14], involving elliptic gamma functions, is shown here to be directly equivalent to the required $N_c = 2$, $N_f = 3$ superconformal index identity. This provides an important clue as to the appropriate mathematical context for showing how the electric and magnetic indices are equal in more general cases. Identities obtained by Rains [15] linking multi-dimensional q-hypergeometric integrals, which reduce to the results of Spiridonov in special cases, are sufficient to prove compatibility of the formulae for indices obtained by applying (1.5) and (1.6) with Seiberg duality in a wide range of cases.

The applicability of these results depends crucially on the detailed form of (1.5) and (1.6). For the chiral matter fields a general term in (1.5) has the form

$$i_S(p,q,y) = \frac{t^r z - t^{2-r} z^{-1}}{(1-tx)(1-tx^{-1})} = \frac{y - pq/y}{(1-p)(1-q)}\,, \quad p = tx,\ q = tx^{-1}\,,\ y = t^r z\,, \tag{1.8}$$

and then in (1.6)

$$\Gamma(y;p,q) = \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n}\, i_S(p^n,q^n,y^n)\Big) = \prod_{j,k\geq 0} \frac{1-y^{-1}p^{j+1}q^{k+1}}{1-y\,p^j\,q^k}\,, \tag{1.9}$$

where $\Gamma(y;p,q)$ is an elliptic Gamma function and we assume p,q real and $0 \leq p,q < 1$. Furthermore for the gauge field part of (1.5) we may define

$$i_V(p,q) = \frac{2t^2 - t(x+x^{-1})}{(1-tx)(1-tx^{-1})} = -\frac{p}{1-p} - \frac{q}{1-q} = 1 - \frac{1-pq}{(1-p)(1-q)}\,, \tag{1.10}$$

and then apply

$$\begin{aligned}
\exp\bigg(\sum_{n=1}^{\infty}\frac{1}{n}\,i_V(p^n,q^n)(z^n+z^{-n})\bigg) &= \frac{\theta(z;p)\,\theta(z;q)}{(1-z)^2} \\
&= \frac{1}{(1-z)(1-z^{-1})\,\Gamma(z;p,q)\Gamma(z^{-1};p,q)}\,, \qquad (1.11) \\
\exp\bigg(\sum_{n=1}^{\infty}\frac{1}{n}\,i_V(p^n,q^n)\bigg) &= (p;p)\,(q;q)\,,
\end{aligned}$$

where the theta function and $(p;p)$ are infinite products defined by

$$\theta(z;p)=\textstyle\prod_{j\geq 0}(1-zp^j)(1-z^{-1}p^{j+1})\,,\quad (x;p)=\prod_{j\geq 0}(1-xp^j)\,. \qquad (1.12)$$

The detailed discussion in this paper is as follows. In section 2 the superconformal transformation properties of $\mathcal{N}=1$ chiral scalar and vector multiplets are described. For free theories it is shown how expressions for the index are constructed which are in accord with the results (1.5) and (1.6) given above but with the R-charge restricted to its free field value. In section 3, the dual Seiberg and Kutasov-Schwimmer theories, with $SU(N_c)$ gauge groups and $SU(N_f)\times SU(N_f)$ flavour symmetry, are reviewed and the single particle indices are obtained by applying (1.5). The multi-particle indices for these theories which are given by (1.6) are then shown to agree in a certain large N_c, N_f limit in section 4. The case of Seiberg duality for $(N_c,N_f)=(2,3)$ is discussed in detail in section 5. Section 6 extends to the general (N_c,N_f) case where a theorem due to Rains is shown to demonstrate that the results for the index in the electric and magnetic theories are identical. Section 7 consider dual theories with $Sp(2N)$ gauge groups. With similar constructions the index is shown to agree for both theories as a consequence of a related theorem. As in section 6 the final result depends on non trivial integral identities. We also discuss in section 8 dual theories with $SO(N)$ gauge groups where the chiral matter fields belong to the vector representation. The resulting elliptic hypergeometric integrals are similar in form to the previous cases and the required identities can be found by expressing them in terms of the corresponding integrals for the $Sp(2N)$ and using the associated identity proven by Rains. We also consider an expansion in one simple case and verify that the result is in accord with (1.7) to the order calculated.

Various appendices with miscellaneous mathematical details are included. Appendix A gives a discussion of $\mathcal{N}=1$ superconformal representation theory and derives expressions for the characters for different representations. The limits which are appropriate for the index and which are relevant for section 2 are also discussed. In appendix B we summarise some general results for group characters which are used in the main text while in appendix C we show how some corrections to the large N limit discussed in section 4 can be calculated. Appendix D describes some properties of the essential elliptic Gamma functions introduced in (1.9) and (1.12). Identities given here are used in appendix E to outline how the single variable elliptic hypergeometric integral, that gives the index in the simple example for the electric theory when $N_c = 2$, $N_f = 3$, may be evaluated in agreement with the result determined by the corresponding magnetic theory. Although a special case, the methods used in this calculation are illustrative of those necessary to obtain more general results.

2. $\mathcal{N} = 1$ Superconformal Transformations and Chiral Fields

The $\mathcal{N} = 1$ superconformal algebra contains besides the usual supercharges, $Q_\alpha, \bar{Q}_{\dot\alpha}$, $\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2P_{\alpha\dot\alpha}$, also their conformal partners, $S^\alpha, \bar{S}^{\dot\alpha}$, $\{\bar{S}^{\dot\alpha}, S^\alpha\} = 2K^{\dot\alpha\alpha}$, the generator of special conformal transformations. For a superconformal primary field $\mathcal{O}$ then $|\mathcal{O}\rangle = \mathcal{O}(0)|0\rangle$ is annihilated by $S^\alpha, \bar{S}^{\dot\alpha}$ and forms a lowest weight state for a supermultiplet. The state has scale dimension Δ and R-charge r if $[H, \mathcal{O}(0)] = \Delta\mathcal{O}(0)$, $[R, \mathcal{O}] = r\mathcal{O}$, and the supermultiplet then has a basis formed by the action of $Q_\alpha, \bar{Q}_{\dot\alpha}, P_{\alpha\dot\alpha}$ on $|\mathcal{O}\rangle$. A chiral field is such that $\bar{Q}_{\dot\alpha}|\mathcal{O}\rangle = 0$. As a consequence of $\{\bar{S}^{\dot\alpha}, \bar{Q}_{\dot\beta}\}|\mathcal{O}\rangle = 0$ the scale dimension is then determined by the R-charge

$$\Delta = \tfrac{3}{2} r \, , \tag{2.1}$$

and $\mathcal{O}$ must belong only to a $(j, 0)$ spin representation.

For a chiral scalar field φ the action of the chiral supercharges Q_α, S^α is then

$$\begin{gathered}
\big[Q_\alpha, \varphi\big] = \psi_\alpha \, , \qquad \big\{Q_\alpha, \psi_\beta\big\} = \varepsilon_{\alpha\beta} F \, , \qquad \big[Q_\alpha, F\big] = 0 \, , \\
\big\{S^\beta, \psi_\alpha\big\} = 6r \, \delta_\alpha{}^\beta \varphi \, , \qquad \big[S^\alpha, F\big] = -2(3r-2) \, \varepsilon^{\alpha\beta} \psi_\beta \, ,
\end{gathered} \tag{2.2}$$

where the S action is determined by consistency with the superconformal algebra. Furthermore for $\bar{Q}_{\dot\alpha}$ the algebra also requires

$$\big\{\bar{Q}_{\dot\alpha}, \psi_\alpha\big\} = 2i \, \partial_{\alpha\dot\alpha} \varphi \, , \qquad \big[\bar{Q}_{\dot\alpha}, F\big] = 2i \, \varepsilon^{\beta\alpha} \partial_{\alpha\dot\alpha} \psi_\beta \, . \tag{2.3}$$

For a chiral $(\frac{1}{2}, 0)$ spinor field λ_α we have similarly

$$\begin{aligned}
\big\{Q_\alpha, \lambda_\beta\big\} &= f_{\alpha\beta} + \varepsilon_{\alpha\beta} \, iD \, , & \big[Q_\alpha, f_{\beta\gamma}\big] &= \varepsilon_{\alpha\beta} \, \mu_\gamma + \varepsilon_{\alpha\gamma} \, \mu_\beta \, , \\
\big[Q_\alpha, D\big] &= i \, \mu_\alpha \, , & \big\{Q_\alpha, \mu_\beta\big\} &= 0 \, , \\
\big[S^\gamma, f_{\alpha\beta}\big] &= 2(3r+1) \, \delta_{(\alpha}{}^\gamma \lambda_{\beta)} \, , & \big[S^\beta, D\big] &= 3(r-1) i \, \varepsilon^{\beta\alpha} \lambda_\alpha \, , \\
\big\{S^\beta, \mu_\alpha\big\} &= -3(r-1) \, \varepsilon^{\beta\gamma} f_{\alpha\gamma} - (3r+1) i \, \delta_\alpha{}^\beta D \, ,
\end{aligned} \tag{2.4}$$

with $f_{\alpha\beta} = f_{\beta\alpha}$, and

$$\begin{aligned} &\big[\bar{Q}_{\dot\alpha}, f_{\alpha\beta}\big] = 2i\,\partial_{(\alpha\dot\alpha}\lambda_{\beta)}\,, \qquad \big[\bar{Q}_{\dot\alpha}, D\big] = \varepsilon^{\beta\alpha}\partial_{\alpha\dot\alpha}\lambda_\beta\,, \\ &\big\{\bar{Q}_{\dot\alpha}, \mu_\alpha\big\} = i\,\varepsilon^{\beta\gamma}\partial_{\gamma\dot\alpha}f_{\alpha\beta} + \partial_{\alpha\dot\alpha}D\,. \end{aligned} \tag{2.5}$$

For each chiral multiplet there is a corresponding anti-chiral partner obtained by conjugation when $(\varphi, \psi_\alpha, F) \to (\overline{\varphi}, \overline{\psi}_{\dot\alpha}, \overline{F})$, $(\lambda_\alpha, f_{\alpha\beta}, D, \mu_\alpha) \to (\overline{\lambda}_{\dot\alpha}, \overline{f}_{\dot\alpha\dot\beta}, \overline{D}, \overline{\mu}_{\dot\alpha})$ and when the R-charges change sign.

For the spinor multiplet, with transformations given by (2.4), (2.5) and their conjugates, we may impose the reality condition

$$D = \overline{D}\,. \tag{2.6}$$

By considering $[Q_\alpha, D]$ we must then have

$$\mu_\alpha = i\,\varepsilon^{\dot\beta\dot\alpha}\partial_{\alpha\dot\alpha}\overline{\lambda}_{\dot\beta}\,, \tag{2.7}$$

and using this to calculate $\{\mu_\alpha, \bar{Q}_{\dot\alpha}\}$ and $\{\mu_\alpha, S^\beta\} = 2\delta_\alpha{}^\beta\varepsilon^{\dot\beta\dot\alpha}\{\bar{Q}_{\dot\alpha}, \overline{\lambda}_{\dot\beta}\}$ it is also necessary for consistency that

$$\varepsilon^{\beta\gamma}\partial_{\gamma\dot\alpha}f_{\alpha\beta} + \varepsilon^{\dot\gamma\dot\beta}\partial_{\alpha\dot\gamma}\overline{f}_{\dot\alpha\dot\beta} = 0\,, \qquad r = 1\,. \tag{2.8}$$

The equation for $f_{\alpha\beta}, \overline{f}_{\dot\alpha\dot\beta}$ is identical with the abelian Bianchi identity for a field strength $F_{\alpha\dot\alpha,\beta\dot\beta} = \varepsilon_{\alpha\beta}\,\overline{f}_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}\,f_{\alpha\beta}$ and the condition $r = 1$, ensuring $f_{\alpha\beta}, \overline{f}_{\dot\alpha\dot\beta}$ and D have vanishing R-charge, shows that no anomalous dimensions are possible with this restriction. The requirement (2.6) of course ensures that the chiral spinor multiplet and its anti-chiral conjugate form the superconformal multiplet for a gauge field.

For the free chiral scalar field we have

$$F = 0 \quad \Rightarrow \quad \varepsilon^{\beta\alpha}\partial_{\alpha\dot\alpha}\psi_\beta = 0\,, \;\; \partial^2\varphi = 0\,, \quad r = \tfrac{2}{3}\,, \tag{2.9}$$

as a consequence of the algebra, (2.2), (2.3). For a free spinor multiplet from (2.4), (2.5)

$$D = \mu_\alpha = 0 \quad \Rightarrow \quad \varepsilon^{\beta\alpha}\partial_{\alpha\dot\alpha}\lambda_\beta = 0\,, \quad \varepsilon^{\beta\alpha}\partial_{\alpha\dot\alpha}f_{\beta\gamma} = 0\,, \quad r = 1\,, \tag{2.10}$$

which clearly are in accord with (2.8).

For the construction of a superconformal index as described in the introduction we identify in (1.1)

$$\mathcal{Q} = \bar{Q}_1\,, \qquad \mathcal{Q}^+ = -\bar{S}^1\,, \qquad \mathcal{H} = H - 2\bar{J}_3 - \tfrac{3}{2}R\,. \tag{2.11}$$

The commuting operators formed from the generators of the superconformal group $SU(2,2|1)$ and which form the generators of the subgroup $SU(2,1)$ as in (1.2) are then

$$\mathcal{P}_\alpha = \tfrac{1}{2} P_{\alpha 2} \, , \qquad \bar{\mathcal{P}}^\beta = -\tfrac{1}{2} K^{2\beta} \, , \tag{2.12}$$

and since $[\mathcal{P}_\alpha, \bar{\mathcal{P}}^\beta] = M_\alpha{}^\beta + \delta_\alpha{}^\beta (H + \bar{J}_3)$ we have

$$\mathcal{R} = R + 2\bar{J}_3 + \tfrac{2}{3}\mathcal{H} \, , \tag{2.13}$$

as in (1.3).

For free fields it is then straightforward to find the results for the index as defined in (1.4). For the chiral scalar and its conjugate then $[\mathcal{Q},\mathcal{Q}^+,\varphi] = 0$, $\{\mathcal{Q},\mathcal{Q}^+,\overline{\psi}_2\} = 0$ so that the subspace annihilated by $\mathcal{Q},\mathcal{Q}^+$, and belonging to the kernel of $\mathcal{H}$, has a basis

$$\mathcal{V}_S = \big\{ P_{12}{}^n P_{22}{}^m |\varphi\rangle , P_{12}{}^n P_{22}{}^m) |\overline{\psi}_2\rangle \big\} \, , \quad n,m = 0,1,2,\dots \, . \tag{2.14}$$

where $\mathcal{R}$ has eigenvalues $(\frac{2}{3}+n+m, \frac{4}{3}+n+m)$ and $2J_3$ $(n-m,n-m)$. Hence evaluating (1.4) on the space spanned by $\mathcal{V}_S$ gives

$$\mathrm{str}_{\mathcal{V}_S}\big(t^{\mathcal{R}} x^{2J_3}\big) = \frac{t^{\frac{2}{3}} - t^{\frac{4}{3}}}{(1-tx)(1-tx^{-1})} \, , \tag{2.15}$$

where the two terms arise from the chiral and anti-chiral fields respectively. For the free vector multiplet $\{\mathcal{Q},\mathcal{Q}^+,\lambda_\alpha\} = 0$, $[\mathcal{Q},\mathcal{Q}^+,\overline{f}_{22}] = 0$ but taking into account the equation of motion

$$\partial_{22}\lambda_1 = \partial_{12}\lambda_2 \, . \tag{2.16}$$

the corresponding basis has the form

$$\mathcal{V}_V = \big\{ P_{12}{}^n P_{22}{}^m |\lambda_1\rangle , P_{22}{}^m |\lambda_2\rangle , P_{12}{}^n P_{22}{}^m) |\overline{f}_{22}\rangle \big\} \, , \quad n,m = 0,1,2,\dots \, . \tag{2.17}$$

Hence

$$\begin{aligned} \mathrm{str}_{\mathcal{V}_V}\big(t^{\mathcal{R}} x^{2J_3}\big) &= -\frac{t\,x}{(1-tx)(1-tx{-}1)} - \frac{t\,x^{-1}}{1-tx^{-1}} + \frac{t^2}{(1-tx)(1-tx^{-1})} \\ &= \frac{2t^2 - t\,\chi_2(x)}{(1-tx)(1-tx^{-1})} \, , \qquad \chi_2(x) = x + x^{-1} \, . \end{aligned} \tag{2.18}$$

These results correspond to appropriate $SU(2,1)$ characters as shown in appendix A. If the chiral field φ forms a representation space for a representation R_S of a internal symmetry group $\mathcal{G}$ while its anti-chiral partner belongs to the conjugate representation

$\bar{R}_S$, and the vector multiplet transforms under the self-conjugate representation R_V,then (2.15) and (2.18) can be extended to

$$\begin{aligned} i_S(p,q,g) &= \frac{1}{(1-p)(1-q)}\big((p\,q)^{\frac{1}{3}}\,\chi_{\mathcal{G},R_S}(g) - (p\,q)^{\frac{2}{3}}\,\chi_{\mathcal{G},\bar{R}_S}(g)\big)\,, \\ i_V(p,q,g) &= -\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\chi_{\mathcal{G},R_V}(g)\,, \end{aligned} \tag{2.19}$$

where we introduce the variables $p = tx, q = tx^{-1}$ as in (1.8) and $\chi_{\mathcal{G},R_S}(g)$, $\chi_{\mathcal{G},\bar{R}_S}(g)$ and $\chi_{\mathcal{G},R_V}(g)$ are corresponding group characters evaluated at $g \in \mathcal{G}$. The general expression in (1.5) is an extension to take into account general R-charges for chiral fields.

3. Indices for Seiberg and Kutasov-Schwimmer Duality

For application to Seiberg duality [1], we first consider the usual $\mathcal{N}=1$ SQCD electric theory with the overall symmetry group $\mathcal{G}_E = U(1)_R \times U(1)_B \times SU(N_f) \times SU(N_f) \times SU(N_c)$, where the generator of $U(1)_B$ is the baryon number charge and $U(1)_R$ is generated by the R-charge and is part of the superconformal group at a fixed point, $SU(N_f) \times SU(N_f)$ is the flavour symmetry group while $SU(N_c)$ is the colour gauge group. For such supersymmetric versions of QCD there are two chiral scalar multiplets Q, $\tilde{Q}$, belonging the fundamental f, anti-fundamental $\bar{f}$ representations of $SU(N_c)$, each carrying baryon number, and a vector multiplet V, in the adjoint. The representation content for all fields is detailed in Table 1, where we have defined

$$\tilde{N}_c = N_f - N_c\,. \tag{3.1}$$

Table 1: Seiberg Electric Theory

Field	$SU(N_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	1	$\tilde{N}_c/N_f$
$\tilde{Q}$	$\bar{f}$	1	$\bar{f}$	-1	$\tilde{N}_c/N_f$
V	adj.	1	1	0	1

The characters $\chi_R(g)$ for $g \in SU(N_c)$ and $\chi_R(h)$ for $h \in SU(N_f) \times SU(N_f)$ are functions of the complex eigenvalues of g, h for which we adopt the abbreviated notation,

$$\mathrm{z} = (z_1,\ldots,z_{N_c})\,,\ \textstyle\prod_i z_i = 1\,, \quad \mathrm{y} = (y_1,\ldots,y_{N_f})\,,\ \tilde{\mathrm{y}} = (\tilde{y}_1,\ldots,\tilde{y}_{N_f})\,, \prod_i y_i = \prod_i \tilde{y}_i = 1\,. \tag{3.2}$$

For $SU(n)$ the required characters, as functions of $\mathrm{x} = (x_1, \dots, x_n)$ with $\prod_i x_i = 1$, are then

$$\begin{aligned} \chi_{SU(n),f}(\mathrm{z}) = p_n(\mathrm{x}) \equiv \textstyle\sum_{j=1}^n x_i \,, \qquad \chi_{SU(n),\bar f}(\mathrm{x}) = p_n(\mathrm{x}^{-1}) \,, \\ \chi_{SU(n),\mathrm{adj.}}(\mathrm{x}) = \sum_{1\le i,j\le n} x_i/x_j - 1 = p_n(\mathrm{x})p_n(\mathrm{x}^{-1}) - 1 \,, \end{aligned} \tag{3.3}$$

using the notation $\mathrm{x}^{-1} = ({x_1}^{-1}, \dots, {x_n}^{-1})$.

Applying (3.3) for $SU(N_c)$ and $SU(N_f)$ the expression given by (1.5) for the single particle index, with v corresponding to $U(1)_B$, becomes

$$\begin{aligned} & i_E(p, q, v, \mathrm{y}, \tilde{\mathrm{y}}, \mathrm{z}) \\ & = -\Big(\frac{p}{1-p} + \frac{q}{1-q}\Big)\Big(p_{N_c}(\mathrm{z})\, p_{N_c}(\mathrm{z}^{-1}) - 1\Big) \\ & \quad + \frac{1}{(1-p)(1-q)}\bigg((p\,q)^{\frac12 r}\, v\, p_{N_f}(\mathrm{y})\, p_{N_c}(\mathrm{z}) - (p\,q)^{1-\frac12 r}\, \frac{1}{v}\, p_{N_f}(\mathrm{y}^{-1})\, p_{N_c}(\mathrm{z}^{-1}) \\ & \qquad\qquad + (p\,q)^{\frac12 r}\, \frac{1}{v}\, p_{N_f}(\tilde{\mathrm{y}}\,)\, p_{N_c}(\mathrm{z}^{-1}) - (p\,q)^{1-\frac12 r}\, v\, p_{N_f}(\tilde{\mathrm{y}}^{-1})\, p_{N_c}(\mathrm{z})\bigg) \,, \end{aligned} \tag{3.4}$$

where

$$r = 1 - \frac{N_c}{N_f} \,. \tag{3.5}$$

For the dual magnetic theory, whereby the overall symmetry group becomes $\mathcal{G}_M = U(1)_R \times U(1)_B \times SU(N_f) \times SU(N_f) \times SU(\tilde N_c)$ with $\tilde N_c$ as in (3.1), we have, again, two fundamental scalar multiplets q, $\tilde q$, a $SU(\tilde N_c)$ adjoint vector multiplet $\tilde V$ along with an extra colour singlet scalar multiplet M with representations and R-charges as in Table 2. The consistency of the choices in Tables 1 and 2 is determined by applying 't Hooft anomaly matching conditions.

Table 2: Seiberg Magnetic Theory

Field	$SU(\tilde N_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
q	f	$\bar f$	1	$N_c/\tilde N_c$	N_c/N_f
$\tilde q$	$\bar f$	1	f	$-N_c/\tilde N_c$	N_c/N_f
$\tilde V$	adj.	1	1	0	1
M	1	f	$\bar f$	0	$2\tilde N_c/N_f$

Applying (1.5) the single particle index for the magnetic theory becomes, in a similar fashion to (3.4), but, for characters for $SU(\tilde N_c)$, replacing z by $\tilde{\mathrm{z}}$

$$i_M(p, q, v, \mathrm{y}, \tilde{\mathrm{y}}, \tilde{\mathrm{z}})$$

$$
\begin{aligned}
&= -\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\left(p_{\tilde N_c}(\tilde{\mathrm z})\,p_{\tilde N_c}(\tilde{\mathrm z}^{-1})-1\right)\\
&\quad+\frac{1}{(1-p)(1-q)}\bigg((p\,q)^{\frac12(1-r)}\,\tilde v\,p_{N_f}(\mathrm y)\,p_{\tilde N_c}(\tilde{\mathrm z})-(p\,q)^{\frac12(1+r)}\,\frac{1}{\tilde v}\,p_{N_f}(\mathrm y^{-1})\,p_{\tilde N_c}(\tilde{\mathrm z}^{-1})\\
&\qquad\qquad+(p\,q)^{\frac12(1-r)}\,\frac{1}{\tilde v}\,p_{N_f}(\tilde{\mathrm y})\,p_{\tilde N_c}(\tilde{\mathrm z}^{-1})-(p\,q)^{\frac12(1+r)}\,\tilde v\,p_{N_f}(\tilde{\mathrm y}^{-1})\,p_{\tilde N_c}(\tilde{\mathrm z})\\
&\qquad\qquad+(p\,q)^{r}\,p_{N_f}(\mathrm y)\,p_{N_f}(\tilde{\mathrm y}^{-1})-(p\,q)^{1-r}\,p_{N_f}(\mathrm y^{-1})\,p_{N_f}(\tilde{\mathrm y})\bigg)\,,
\end{aligned}
\tag{3.6}
$$

with r as in (3.5) and the $U(1)_B$ assignments requiring

$$
\tilde v^{\tilde N_c}=v^{N_c}\,. \tag{3.7}
$$

For Kutasov-Schwimmer dual models [5], the overall symmetry groups are similar to the Seiberg dual theories considered above but there are additional chiral matter fields. In the electric theory there is an extra scalar multiplet X transforming according to the adjoint for $SU(N_c)$. For the dual magnetic theory the $SU(\tilde N_c)$ gauge group now has

$$
\tilde N_c = kN_f - N_c\,, \quad \text{for} \quad k=1,2,\dots\,, \tag{3.8}
$$

and there is also an extra $SU(\tilde N_c)$ adjoint scalar multiplet $\tilde X$ along with now k gauge singlet scalar multiplets, M_j, $j=1,\dots,k$. For $k=1$, these examples reduce to the Seiberg dual theories as X, $\tilde X$ then decouple. The field content in the electric and magnetic theories are outlined in Tables 3 and 4.

Table 3: Kutasov-Schwimmer Electric Theory

Field	$SU(N_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	1	$1-\frac{2}{k+1}\frac{N_c}{N_f}$
$\tilde Q$	$\bar f$	1	$\bar f$	-1	$1-\frac{2}{k+1}\frac{N_c}{N_f}$
V	adj.	1	1	0	1
X	adj.	1	1	0	$\frac{2}{k+1}$

This time the electric theory single particle index is given by,

$$
\begin{aligned}
&i_E(p,q,v,\mathrm y,\tilde{\mathrm y},\mathrm z)\\
&= -\left(\frac{p}{1-p}+\frac{q}{1-q}-\frac{1}{(1-p)(1-q)}\big((p\,q)^{s}-(p\,q)^{1-s}\big)\right)\left(p_{N_c}(\mathrm z)\,p_{N_c}(\mathrm z^{-1})-1\right)\\
&\quad+\frac{1}{(1-p)(1-q)}\bigg((p\,q)^{\frac12 r}\,v\,p_{N_f}(\mathrm y)\,p_{N_c}(\mathrm z)-(p\,q)^{1-\frac12 r}\,\frac{1}{v}\,p_{N_f}(\mathrm y^{-1})\,p_{N_c}(\mathrm z^{-1})\\
&\qquad\qquad+(p\,q)^{\frac12 r}\,\frac{1}{v}\,p_{N_f}(\tilde{\mathrm y})\,p_{N_c}(\mathrm z^{-1})-(p\,q)^{1-\frac12 r}\,v\,p_{N_f}(\tilde{\mathrm y}^{-1})\,p_{N_c}(\mathrm z)\bigg)\,,
\end{aligned}
\tag{3.9}
$$

where now

$$r = 1 - \frac{2}{k+1}\frac{N_c}{N_f}\,, \qquad s = \frac{1}{k+1}\,. \tag{3.10}$$

Table 4: Kutasov-Schwimmer Magnetic Theory

Field	$SU(\tilde{N}_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
q	f	$\bar{f}$	1	$N_c/\tilde{N}_c$	$1-\frac{2}{k+1}\frac{\tilde{N}_c}{N_f}$
$\tilde{q}$	$\bar{f}$	1	f	$-N_c/\tilde{N}_c$	$1-\frac{2}{k+1}\frac{\tilde{N}_c}{N_f}$
$\tilde{V}$	adj.	1	1	0	1
$M_j,\ j=1,\dots k$	1	f	$\bar{f}$	0	$2-\frac{4}{k+1}\frac{N_c}{N_f}+\frac{2}{k+1}(j-1)$
$\tilde{X}$	adj.	1	1	0	$\frac{2}{k+1}$

The magnetic theory single particle index involves a sum over contributions corresponding to M_j of the form $\sum_{j=1}^{k}(p\,q)^{r+s(j-1)}$ which is easily summed giving

$$\begin{aligned}
&i_M(p,q,v,\mathrm{y},\tilde{\mathrm{y}},\tilde{\mathrm{z}})\\
&= -\bigg(\frac{p}{1-p}+\frac{q}{1-q}-\frac{1}{(1-p)(1-q)}\big((p\,q)^s-(p\,q)^{1-s}\big)\bigg)\Big(p_{\tilde{N}_c}(\tilde{\mathrm{z}})\,p_{\tilde{N}_c}(\tilde{\mathrm{z}}^{-1})-1\Big)\\
&\quad+\frac{1}{(1-p)(1-q)}\bigg((p\,q)^{s-\frac{1}{2}r}\,\tilde{v}\,p_{N_f}(\mathrm{y})\,p_{\tilde{N}_c}(\tilde{\mathrm{z}})-(p\,q)^{1-s+\frac{1}{2}r}\,\frac{1}{\tilde{v}}\,p_{N_f}(\mathrm{y}^{-1})\,p_{\tilde{N}_c}(\tilde{\mathrm{z}}^{-1})\\
&\qquad+(p\,q)^{s-\frac{1}{2}r}\,\frac{1}{\tilde{v}}\,p_{N_f}(\tilde{\mathrm{y}}\,)\,p_{\tilde{N}_c}(\tilde{\mathrm{z}}^{-1})-(p\,q)^{1-s+\frac{1}{2}r}\,\tilde{v}\,p_{N_f}(\tilde{\mathrm{y}}^{-1})\,p_{\tilde{N}_c}(\tilde{\mathrm{z}})\\
&\qquad+\frac{1-(p\,q)^{1-s}}{1-(p\,q)^s}\Big((p\,q)^r\,p_{N_f}(\mathrm{y})\,p_{N_f}(\tilde{\mathrm{y}}^{-1})-(p\,q)^{2s-r}\,p_{N_f}(\mathrm{y}^{-1})\,p_{N_f}(\tilde{\mathrm{y}})\Big)\bigg)\,.
\end{aligned} \tag{3.11}$$

with the definitions (3.10) and requiring (3.7) once more. When $k=1$, $s=\frac{1}{2}$ and (3.9) and (3.11) reduce to (3.4) and (3.6).

There are important differences between the Seiberg dual theories and those described by Kutasov and Schwimmer, except in the special case $k=1$ when the operators $X,\tilde{X}$ decouple. In the former case there is no superpotential and so no operator relations to take into account. Requiring the colour singlet operators $Q\tilde{Q}$ and $q\tilde{q}$ to both satisfy the superconformal unitarity bound requires in (3.4) and (3.6) that $r, 1-r \geq \frac{1}{3}$ which corresponds, using (3.5), to the conformal window $\frac{3}{2}N_c \leq N_f \leq 3N_c$. In the Kutasov-Schwimmer electric theory the corresponding condition for the operator $Q\tilde{Q}$ also gives $r \geq \frac{1}{3}$ or with (3.10) $N_f \geq 3N_c/(k+1)$. In the magnetic theory there is no similar restriction for $q\tilde{q}$ since the superpotential implies that it satisfies operator relations in this case.

4. Large N_f, N_c Limits

We now show that the multi-particle index given by (1.6) with i replaced by $i_E(p,q,v,\mathrm{y},\tilde{\mathrm{y}},\mathrm{z})$ with $G=SU(N_c)$ and also by $i_M(p,q,v,\mathrm{y},\tilde{\mathrm{y}},\mathrm{z})$ with $G=SU(\tilde{N}_c)$ agree in the large N_c limit, requiring N_f/N_c fixed so that $\tilde{N}_c$ is also large. This holds in the general Kutasov-Schwimmer dual theories which includes the Seiberg dualities as a special case.

Each single particle index above, (3.4), (3.6), (3.9) and (3.11), may be expressed in the generic form,

$$i(\mathrm{t},\mathrm{z}) = f(\mathrm{t})\big(p_N(\mathrm{z})p_N(\mathrm{z}^{-1})-1\big) + g(\mathrm{t})p_N(\mathrm{z}) + \bar{g}(\mathrm{t})p_N(\mathrm{z}^{-1}) + h(\mathrm{t})\,, \tag{4.1}$$

for $f,g,\bar{g},h$ functions of appropriate variables t and $\mathrm{z}=(z_1,\dots,z_N)$. Inserting $i(\mathrm{t},\mathrm{z})$ into (1.6), with $G=SU(N)$, the leading term in the large N expansion may be obtained by extending the methods used in [16] for $g=\bar{g}=0$. An alternative approach following [7] is also discussed subsequently.

The method in [16] relies on the critical observation that power symmetric polynomials are orthogonal up to contributions which disappear in the large N limit. For $p_N(\mathrm{z}^n)=\sum_{i=1}^N {z_i}^n$, power symmetric polynomials, which are labelled by $\underline{a}=(a_1,a_2,\dots)$ $a_i=0,1,\dots$, are defined by

$$p_{\underline{a}}(\mathrm{z}) = p_{(a_1,a_2,\dots)}(\mathrm{z}) = p_N(\mathrm{z})^{a_1}p_N(\mathrm{z}^2)^{a_2}\cdots\,. \tag{4.2}$$

These obey the orthogonality relation,

$$\int_{SU(N)} \mathrm{d}\mu(\mathrm{z})\, p_{\underline{a}}(\mathrm{z})\, p_{\underline{a}'}(\mathrm{z}^{-1}) = z_{\underline{a}}\,\delta_{\underline{a}\underline{a}'}\,, \qquad |\underline{a}|,|\underline{a}'|<N\,, \tag{4.3}$$

where

$$z_{\underline{a}} = z_{(a_1,a_2,\dots)} = \prod_{n\geq 1} n^{a_n}a_n!\,, \qquad |\underline{a}| = \textstyle\sum_n na_n\,. \tag{4.4}$$

In consequence (4.3) becomes exact for any $\underline{a},\underline{a}'$ in the large N limit.

This result may now be used to evaluate

$$\mathcal{I}(t) = \int_{SU(N)} \mathrm{d}\mu(\mathrm{z})\, \exp\bigg(\sum_{n=1}^\infty \frac{1}{n} i(\mathrm{t}^n,\mathrm{z}^n)\bigg)\,, \tag{4.5}$$

by expanding the exponential

$$\begin{aligned} &\exp\bigg(\sum_{n=1}^\infty \frac{1}{n} i(\mathrm{t}^n,\mathrm{z}^n)\bigg) \\ &= \exp\bigg(\sum_{n=1}^\infty \frac{1}{n}\big(-f(\mathrm{t}^n)+h(\mathrm{t}^n)\big)\bigg) \prod_{n=1}^\infty \sum_{a_n,b_n,\bar{b}_n\geq 0} \frac{1}{n^{a_n+b_n+\bar{b}_n}\,a_n!\,b_n!\,\bar{b}_n!} \\ &\qquad\qquad \times f(\mathrm{t}^n)^{a_n}\,g(\mathrm{t}^n)^{b_n}\,\bar{g}(\mathrm{t}^n)^{\bar{b}_n}\,p_N(\mathrm{z}^n)^{a_n+b_n}\,p_N(\mathrm{z}^{-n})^{a_n+\bar{b}_n}\,, \end{aligned} \tag{4.6}$$

so that, applying (4.3), (4.4) in (4.5), the $SU(N)$ integral gives

$$\begin{aligned} \mathcal{I}(\mathrm{t}) \simeq \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \big(-f(\mathrm{t}^n) + h(\mathrm{t}^n) \big) \Big) \\ \times \prod_{n=1}^{\infty} \sum_{a_n, b_n \geq 0} \frac{(a_n+b_n)!}{n^{b_n}\, a_n!\, b_n!^2}\, f(\mathrm{t}^n)^{a_n} \big(g(\mathrm{t}^n)\, \bar{g}(\mathrm{t}^n) \big)^{b_n} , \end{aligned} \tag{4.7}$$

where the right hand side is exact so long as (4.3) holds and so (4.7) is valid, up to contributions which are negligible for large N. Using $\sum_{r=0}^{\infty} \binom{r+s}{r} x^r = 1/(1-x)^{s+1}$ and $\sum_{s=0}^{\infty} \frac{1}{n^s s!} \frac{y^s}{(1-x)^{s+1}} = \frac{1}{1-x} \exp(\frac{1}{n} \frac{y}{1-x})$ we then easily obtain

$$\mathcal{I}(\mathrm{t}) \simeq \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \Big(\frac{g(\mathrm{t}^n)\, \bar{g}(\mathrm{t}^n)}{1-f(\mathrm{t}^n)} - f(\mathrm{t}^n) + h(\mathrm{t}^n) \Big) \Big) \prod_{n=1}^{\infty} \frac{1}{1-f(\mathrm{t}^n)} \,. \tag{4.8}$$

This result gives the leading large N form for $\mathcal{I}(t)$ if we assume $g, \bar{g}$ are both $\mathrm{O}(N)$ and h is $\mathrm{O}(N^2)$.

Alternatively we can also show how conventional large N techniques give the same result (4.8). For $z_i = e^{i\theta_i}$ the invariant integration over $SU(N)$ has the form

$$\int_{SU(N)} \mathrm{d}\mu(\mathrm{z}) \;=\frac{1}{N}\, \frac{1}{(2\pi)^{N-1}} \int_{-\pi \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_{N-1} \leq \pi} \prod_{i=1}^{N-1} \mathrm{d}\theta_i \quad \prod_{i<j} 4 \sin^2 \tfrac{1}{2} (\theta_i - \theta_j) \,, \tag{4.9}$$

where we impose $\sum_i \theta_i = 0$. The basic integral (4.5) then becomes

$$\mathcal{I}(\mathrm{t}) = \frac{1}{N}\, \frac{1}{(2\pi)^{N-1}} \int_{-\pi \leq \theta_1 \leq \ldots \leq \theta_{N-1} \leq \pi} \prod_{i=1}^{N-1} \mathrm{d}\theta_i \quad e^{-S(\mathrm{t}, \theta)} \,, \tag{4.10}$$

for

$$S(\mathrm{t}, \theta) = \sum_{n=1}^{\infty} \frac{1}{n} \Big\{ \big(1 - f(\mathrm{t}^n) \big) \sum_{i \neq j} e^{in(\theta_i - \theta_j)} - g(\mathrm{t}^n) \sum_i e^{-in\theta_i} - \bar{g}(\mathrm{t}^n) \sum_i e^{in\theta_i} - \hat{h}(\mathrm{t}^n) \Big\} , \tag{4.11}$$

defining for convenience $\hat{h} = h - f$. In the large N limit we assume $\theta_i \to \theta(i/N)$, a continuous monotonic function such that $\sum_i f(\theta_i) \to N \int_0^1 \mathrm{d}x f(\theta(x))$. In (4.10) the product of $\mathrm{d}\theta_i$ integrals then becomes a functional integral $\mathrm{d}[\theta]$. The asymptotic evaluation is obtained by introducing instead of $\theta(x)$ a density function $\rho(\theta)$ defined in terms of $\theta(x)$ by

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \rho(\theta) \,, \tag{4.12}$$

and then defining

$$\rho_n = N \int_{-\pi}^{\pi} \mathrm{d}\theta\, \rho(\theta)\, e^{in\theta} \,, \qquad \rho_0 = N \,, \tag{4.13}$$

we assume

$$\int_{\theta'>0} \mathrm{d}[\theta] \to \int \mathrm{d}[\rho] = \int \prod_{n\geq 1} \frac{n}{\pi} \mathrm{d}^2\rho_n \,, \tag{4.14}$$

normalising to unit group volume. Letting

$$S(\mathrm{t},\theta) \to \tilde{S}(\mathrm{t},\rho) = \sum_{n=1}^{\infty} \frac{1}{n} \Big\{ \big(1 - f(\mathrm{t}^n)\big)\, \rho_n \rho_{-n} - g(\mathrm{t}^n)\, \rho_{-n} - \bar{g}(\mathrm{t}^n)\, \rho_n - h(\mathrm{t}^n) \Big\} \,, \tag{4.15}$$

we obtain

$$\mathcal{I}(\mathrm{t}) \simeq \int \mathrm{d}[\rho]\, e^{-\tilde{S}(\mathrm{t},\rho)} \,, \tag{4.16}$$

which is a straightforward Gaussian functional integral, assuming $1 - f(t) > 0$. The saddle points are

$$\hat{\rho}_n = \frac{g(\mathrm{t}^n)}{1 - f(\mathrm{t}^n)} \,, \quad \hat{\rho}_{-n} = \frac{\bar{g}(\mathrm{t}^n)}{1 - f(\mathrm{t}^n)} \,, \quad n = 1, 2, \ldots \,, \tag{4.17}$$

and it is easy to see that (4.16) reproduces the leading expression shown in (4.8), although it is not so evident that this result is exact for the first few terms in an expansion.

We now apply (4.8) to verify that it gives the same expression for both dual electric and magnetic theories considered in the previous section. Since the Seiberg dual theories are a special case of those considered by Kutasov and Schwimmer we focus on the latter. Comparing (4.1) with (3.9) and (3.11) it is easy to see that f in (4.1) is the same in both cases and that (3.9), (3.11) give

$$1 - f(p,q) = \frac{(1 + (p\,q)^{1-s})(1 - (p\,q)^s)}{(1-p)(1-q)} \,. \tag{4.18}$$

We may then read off from (3.9), comparing with (4.1),

$$\begin{aligned}
g_E(p,q,v,\mathrm{y},\tilde{\mathrm{y}}) &= \frac{v}{(1-p)(1-q)} \Big((p\,q)^{\frac{1}{2}r}\, p_{N_f}(\mathrm{y}) - (p\,q)^{1-\frac{1}{2}r}\, p_{N_f}(\tilde{\mathrm{y}}) \Big) \,, \\
\bar{g}_E(p,q,v,\mathrm{y},\tilde{\mathrm{y}}) &= \frac{v^{-1}}{(1-p)(1-q)} \Big((p\,q)^{\frac{1}{2}r}\, p_{N_f}(\tilde{\mathrm{y}}^{-1}) \ - (p\,q)^{1-\frac{1}{2}r}\, p_{N_f}(\mathrm{y}^{-1}) \Big) \,, \\
h_E(p,q,\mathrm{y},\tilde{\mathrm{y}}) &= 0 \,,
\end{aligned} \tag{4.19}$$

and, from (3.11),

$$\begin{aligned}
g_M(p,q,v,\mathrm{y},\tilde{\mathrm{y}}) &= \frac{\tilde{v}}{(1-p)(1-q)} \Big((p\,q)^{s-\frac{1}{2}r}\, p_{N_f}(\mathrm{y}^{-1}) - (p\,q)^{1-s+\frac{1}{2}r}\, p_{N_f}(\tilde{\mathrm{y}}^{-1}) \Big) \,, \\
\bar{g}_M(p,q,v,\mathrm{y},\tilde{\mathrm{y}}) &= \frac{\tilde{v}^{-1}}{(1-p)(1-q)} \Big((p\,q)^{s-\frac{1}{2}r}\, p_{N_f}(\tilde{\mathrm{y}}) \ - (p\,q)^{1-s+\frac{1}{2}r}\, p_{N_f}(\mathrm{y}) \Big) \,, \\
h_M(p,q,\mathrm{y},\tilde{\mathrm{y}}) &= \frac{1}{(1-p)(1-q)} \frac{1 - (p\,q)^{1-s}}{1 - (p\,q)^s} \\
&\quad \times \Big((p\,q)^r\, p_{N_f}(\mathrm{y})\, p_{N_f}(\tilde{\mathrm{y}}^{-1}) - (p\,q)^{2s-r}\, p_{N_f}(\mathrm{y}^{-1})\, p_{N_f}(\tilde{\mathrm{y}}) \Big) \,,
\end{aligned} \tag{4.20}$$

with the same notation as in (3.10) and also requiring (3.7). From (4.19) and (4.20) we have

$$\begin{aligned} & g_E(p,q,v,\mathrm{y},\tilde{\mathrm{y}})\,\bar{g}_E(p,q,v,\mathrm{y},\tilde{\mathrm{y}}) - g_M(p,q,v,\mathrm{y},\tilde{\mathrm{y}})\,\bar{g}_M(p,q,v,\mathrm{y},\tilde{\mathrm{y}}) \\ & = \frac{1-(p\,q)^{2(1-s)}}{(1-p)^2(1-q)^2}\Big((p\,q)^r\,p_{N_f}(\mathrm{y})\,p_{N_f}(\tilde{\mathrm{y}}^{-1}) - (p\,q)^{2s-r}\,p_{N_f}(\mathrm{y}^{-1})\,p_{N_f}(\tilde{\mathrm{y}})\Big)\,, \end{aligned} \tag{4.21}$$

and hence

$$\frac{g_E(p,q,v,\mathrm{y},\tilde{\mathrm{y}})\,\bar{g}_E(p,q,v,\mathrm{y},\tilde{\mathrm{y}})}{1-f(p,q)} = \frac{g_M(p,q,v,\mathrm{y},\tilde{\mathrm{y}})\,\bar{g}_M(p,q,v,\mathrm{y},\tilde{\mathrm{y}})}{1-f(p,q)} + h_M(p,q,\mathrm{y},\tilde{\mathrm{y}})\,. \tag{4.22}$$

Thus (4.8) demonstrates that the large N limit for the index is the same in both dual and electric theories. In this limit there is no dependence on the $U(1)_B$ variable v since there is no contribution from baryon operators and this limit is also insensitive to the precise dual gauge groups.

Applying (4.8) in this case then gives for the index

$$\begin{aligned} I(p,q,v,\mathrm{y},\tilde{y}) \simeq \exp\bigg(\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{g_E(p^n,q^n,v^n,\mathrm{y}^n,\tilde{\mathrm{y}}^n)\,\bar{g}_E(p^n,q^n,v^n,\mathrm{y}^n,\tilde{\mathrm{y}}^n)}{1-f(p^n,q^n)} - f(p^n,q^n)\Big)\bigg) \\ \times\prod_{n=1}^{\infty}\frac{1}{1-f(p^n,q^n)}\,. \end{aligned} \tag{4.23}$$

The first few terms in the expansion involving operators of low scale dimension are then

$$\begin{aligned} I(tx,tx^{-1},v,\mathrm{y},\tilde{\mathrm{y}}) = {} & 1 + t^{2r}p_{N_f}(\mathrm{y})p_{N_f}(\tilde{\mathrm{y}}^{-1}) + t^{4-2r}p_{N_f}(\mathrm{y}^{-1})p_{N_f}(\tilde{\mathrm{y}}) \\ & - t^2\big(p_{N_f}(\mathrm{y})p_{N_f}(\mathrm{y}^{-1}) + p_{N_f}(\tilde{\mathrm{y}})p_{N_f}(\tilde{\mathrm{y}}^{-1})\big) \\ & + t^{4s} - \big(t^{1+2s} - t^{3-2s}\big)\chi_2(x) + \dots\,, \end{aligned} \tag{4.24}$$

where $\chi_2(x) = x + x^{-1}$ is a $SU(2)$ character corresponding to operators with $j=\frac{1}{2}$. In the Seiberg case, when $s=\frac{1}{2}$ and r is given by (3.5), the results shown in (4.24) are in exact accord with the tables in [11]. The expansion of (4.23) neglects contributions from operators with non-zero baryon charge which first arise at $\mathrm{O}(t^{N_c r})$. In (4.24) the expansion clearly generates integer coefficients, as required in (1.7), to this limited order. Except for the Seiberg case the expression for the index may be expected be modified once constraints on the operator spectrum arising from the superpotential are incorporated.

5. Index Matching for $\mathcal{N}=1$ Superconformal $SU(2)$ Gauge Theories with Three Flavours and its Seiberg Dual

For the Seiberg dual theories analytic proofs of the equality of the index between the electric and magnetic theories are possible for general finite N_c, N_f. These depend crucially on the detailed choice of the dual gauge groups and the assignments of $U(1)_B$ charges and provide very non-trivial tests of duality in this case and also of the framework described here for calculating the index in these theories.

As a simple example in this section we discuss in some detail the example of dual theories when $(N_c, N_f) = (2,3)$. There are various simplifications in this case. Since for $N_c = 2$ there is no distinction between the fundamental representation and its conjugate the flavour symmetry group extends $U(1)_B \times SU(3) \times SU(3) \to SU(6)$. In the electric theory $Q^a = (Q^i, \tilde{Q}_i)$ forms the six dimensional fundamental representation while in the magnetic dual theory $q^{ab} = (\epsilon^{ijk} q_k, \epsilon_{ijk}\tilde{q}^k, M^i{}_j, -M^j{}_i)$ forms the 15 dimensional antisymmetric tensor representation T_A. The index formulae are then more simply given in terms of $SU(6)$ characters which depend on

$$\mathrm{u} = (p\,q)^{\frac{1}{6}}\big(v\mathrm{y}, v^{-1}\tilde{\mathrm{y}}\big)\,, \qquad \textstyle\prod_{a=1}^{6} u_a = pq\,, \tag{5.1}$$

where the rescaling is introduced to ensure i_E, i_M have the form exhibited in (1.8), (1.10). Also in this example $\tilde{N}_c = 1$ so the magnetic theory at the superconformal fixed point is a free theory. From (3.4), since for $N_c = 2$ we may take $\mathrm{z} = (z, z^{-1})$, we then have

$$\begin{aligned} i_E(p,q,\mathrm{u},z) = {} & -\bigg(\frac{p}{1-p} + \frac{q}{1-q}\bigg)\chi_3(z) \\ & + \frac{1}{(1-p)(1-q)}\big(p_6(\mathrm{u}) - p\,q\,p_6(\mathrm{u}^{-1})\big)\chi_2(z)\,, \end{aligned} \tag{5.2}$$

with the $SU(2)$ characters

$$\chi_3(z) = z^2 + 1 + z^{-2}\,, \qquad \chi_2(z) = z + z^{-1}\,. \tag{5.3}$$

Also from (3.6)

$$i_M(p,q,\mathrm{u}) = \frac{1}{(1-p)(1-q)}\big(\chi_{SU(6),T_A}(\mathrm{u}) - p\,q\,\chi_{SU(6),T_A}(\mathrm{u}^{-1})\big)\,, \tag{5.4}$$

where the character for the antisymmetric tensor representation for $SU(n)$ has the form

$$\chi_{SU(n),T_A}(\mathrm{x}) = \sum_{1\le i<j\le n} x_i x_j\,, \qquad \chi_{SU(n),\bar{T}_A}(\mathrm{x}) = \chi_{SU(n),T_A}(\mathrm{x}^{-1})\,. \tag{5.5}$$

For $SU(2)$ the invariant measure becomes

$$\int_{SU(2)} \mathrm{d}\mu(z)\, f(z) = -\frac{1}{4\pi i}\oint \frac{\mathrm{d}z}{z^3}(1-z^2)^2 f(z) = \frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z}(1-z^2)\, f(z)\,, \tag{5.6}$$

for any analytic $f(z) = f(z^{-1})$. Hence we may express the index for the electric theory by using (1.10) with (1.11)

$$
\begin{aligned}
I_E(p,q,\mathrm{u}) &= \int_{SU(2)} \mathrm{d}\mu(z)\, \exp\bigg(\sum_{n=1}^{\infty} \frac{1}{n} i_E\big(p^n, q^n, \mathrm{u}^n, z^n\big)\bigg) \\
&= -\,(p;p)\,(q;q)\, \frac{1}{4\pi i} \oint \frac{\mathrm{d}z}{z^3}\, \theta(z^2;p)\,\theta(z^2;q)\,\mathcal{I}(p,q,\mathrm{u},z)\,,
\end{aligned}
\tag{5.7}
$$

where for $|u_a| < 1$ the contour may be restricted to the unit circle. With the aid of (1.8) and (1.9)

$$
\mathcal{I}(p,q,\mathrm{u},z) = \prod_{a=1}^{6} \Gamma(u_a z;p,q)\,\Gamma(u_a/z;p,q)\,,
\tag{5.8}
$$

or, since from the definition (1.9),

$$
\Gamma(y;p,q)\,\Gamma(pq/y;p,q) = 1\,,
\tag{5.9}
$$

then, with the constraint (5.1), we may also write (5.8) in a form involving just $\hat{\mathrm{u}} = (u_1,\dots,u_5)$

$$
\mathcal{I}(p,q,\mathrm{u},z) = \hat{\mathcal{I}}(p,q,\hat{\mathrm{u}},z) = \frac{\prod_{a=1}^{5} \Gamma(u_a z;p,q)\,\Gamma(u_a/z;p,q)}{\Gamma(\lambda z;p,q)\,\Gamma(\lambda/z;p,q)}\,, \qquad \lambda = \prod_{a=1}^{5} u_a\,,
\tag{5.10}
$$

so that, with $|u_a| < 1,\ a = 1,\dots 5$ and $|\lambda| > pq$,

$$
I_E(p,q,\mathrm{u}) = \mathcal{A}(p,q,\hat{\mathrm{u}}) = -(p;p)\,(q;q)\,\frac{1}{4\pi i}\oint \frac{\mathrm{d}z}{z^3}\,\theta(z^2;p)\,\theta(z^2;q)\,\hat{\mathcal{I}}(p,q,\hat{\mathrm{u}},z)\,.
\tag{5.11}
$$

For the magnetic index there is no integration so that (1.9) gives directly

$$
\begin{aligned}
I_M(p,q,\mathrm{u}) &= \exp\bigg(\sum_{n=1}^{\infty} \frac{1}{n} i_M\big(p^n, q^n, \mathrm{u}^n, z^n\big)\bigg) \\
&= \prod_{1\le a<b\le 6} \Gamma(u_a u_b;p,q) \\
&= \frac{\prod_{1\le a<b\le 5} \Gamma(u_a u_b;p,q)}{\prod_{a=1}^{5} \Gamma(\lambda/u_a;p,q)} = \mathcal{B}(p,q,\hat{\mathrm{u}})\,,
\end{aligned}
\tag{5.12}
$$

where in the last line we have used (5.9) again to write the index in terms of $\hat{\mathrm{u}}$.

An identity obtained by Spiridonov [14] shows that (5.7) and (5.12) are identical. This result is discussed in appendix E, but here we consider on the special case for $p = 0$, which is known in the relevant literature as the Nassrallah-Rahman theorem, and summarise a particular simple proof which may be generalised to show $\mathcal{A}(p,q,\hat{\mathrm{u}}) = \mathcal{B}(p,q,\hat{\mathrm{u}})$.[2]

When $p=0$ (5.7) may be written in the form

$$I_E(0,q,\mathrm{u}) = (q;q)\,\frac{1}{4\pi i}\oint \frac{\mathrm{d}z}{z}\,(z^2;q)(z^{-2};q)\,\mathcal{J}(q,\hat{\mathrm{u}},z) = \mathcal{L}(q,\hat{\mathrm{u}})\,, \tag{5.13}$$

with definitions in (1.12), where

$$\mathcal{J}(q,\hat{\mathrm{u}},z) = \frac{(\lambda z;q)\,(\lambda/z;q)}{\prod_{a=1}^5 (u_a z;q)\,(u_a/z;q)}\,. \tag{5.14}$$

The Nassrallah-Rahman theorem [13] implies essentially that (5.13) is equal to

$$I_M(0,q,\mathrm{u}) = \frac{\prod_{a=1}^5(\lambda/u_a;q)}{\prod_{1\le a<b\le 5}(u_a u_b;q)} = \mathcal{R}(q,\hat{\mathrm{u}})\,. \tag{5.15}$$

If $u_5=0$ the corresponding integral is a well known result first considered by Askey and Wilson, see [17]. A simple proof due to Askey for this result was also extended to the full integral given by (5.13) and (5.15) [18] and involves first finding a q-difference relation satisfied by $\mathcal{J}(q,\hat{\mathrm{u}},z)$, when $u_a \to q u_a$ for a particular a and any z so that it must hold for $\mathcal{L}(q,\hat{\mathrm{u}})$ as well. The essential requirement is that this is also satisfied by $\mathcal{R}(q,\hat{\mathrm{u}})$. The q-difference relation is then shown to allow a proof of the identity $\mathcal{L}(q,\hat{\mathrm{u}}) = \mathcal{R}(q,\hat{\mathrm{u}})$ to be derived from that for some suitable special cases for $\hat{\mathrm{u}}$.

The required q-difference relation is obtained from

$$\mathcal{J}(q,qu_1,u_2,\dots,u_5,z) = \frac{(1-u_1 z)(1-u_1/z)}{(1-\lambda z)(1-\lambda/z)}\,\mathcal{J}(q,\hat{\mathrm{u}},z)\,, \tag{5.16}$$

---

[2] Even for $p=q=0$, and taking also $u_6=0$, the identities are not entirely trivial. In this limit $i_E(0,0,\mathrm{u},z) = (p_5(\hat{\mathrm{u}})-\lambda)\chi_2(z)$ and $i_M(0,0,\mathrm{u}) = \sum_{1\le a<b\le 5} u_a u_b - \sum_{1\le a\le 5}\lambda/u_a$. Hence

$$\begin{aligned} I_E(0,0,\mathrm{u}) &= \frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z}(1-z^2)\,\frac{(1-\lambda z)(1-\lambda/z)}{\prod_{1\le a\le 5}(1-u_a z)(1-u_a/z)} = \sum_b \frac{(1-\lambda u_b)(1-\lambda/u_b)}{\prod_{a\ne b}(1-u_a u_b)(1-u_a/u_b)} \\ &= \frac{\prod_a (1-\lambda/u_a)}{\prod_{a<b}(1-u_a u_b)} = I_M(0,0,\mathrm{u})\,. \end{aligned}$$

This result may be expanded in terms of Schur polynomials as

$$I_E(0,0,\mathrm{u}) = \sum_{n\ge 0}\big(s_{(n,n,0,0,0,0)}(\mathrm{u}) + s_{(n-3,n-3,2,2,2,0)}(\mathrm{u}) - s_{(n-1,n-2,1,1,1,0)}(\mathrm{u})\big)\,,$$

where we set $u_6=0$ and the three terms contribute for $n\ge 0,5,3$ respectively. This matches the leading terms in the expansion given in [11].

and then using the identity

$$
\begin{aligned}
&u_2(1-u_1z)(1-u_1/z)(1-\lambda u_2)(1-\lambda/u_2) - u_1(1-u_2z)(1-u_2/z)(1-\lambda u_1)(1-\lambda/u_1) \\
&\quad = -(u_1-u_2)(1-u_1u_2)(1-\lambda z)(1-\lambda/z)\,,
\end{aligned}
\tag{5.17}
$$

to show that $\mathcal{J}(q,\hat{\mathrm{u}},z)$ satisfies

$$
\begin{aligned}
&u_2(1-\lambda u_2)(1-\lambda/u_2)\,\mathcal{J}(q,qu_1,u_2,\dots,u_5,z) - u_1(1-\lambda u_1)(1-\lambda/u_1)\,\mathcal{J}(q,u_1,qu_2,\dots,u_5,z) \\
&= -(u_1-u_2)(1-u_1u_2)\,\mathcal{J}(q,\hat{\mathrm{u}},z)\,.
\end{aligned}
\tag{5.18}
$$

Clearly from (5.13) $\mathcal{L}(q,\hat{\mathrm{u}})$ satisfies the same q-difference relation. Also we have from (5.15)

$$
\mathcal{R}(q,qu_1,u_2,\dots,u_5) = \prod_{a=2}^{5}\frac{1-u_1u_a}{1-\lambda/u_a}\,\mathcal{R}(q,\hat{\mathrm{u}})\,,
\tag{5.19}
$$

and in this case using the identity, for λ as in (5.10),

$$
\begin{aligned}
&u_2(1-\lambda u_2)\textstyle\prod_{a\neq 1}(1-u_1u_a) - u_1(1-\lambda u_1)\prod_{a\neq 2}(1-u_2u_a) \\
&= -(u_1-u_2)(1-u_1u_2)\textstyle\prod_{a=3}^{5}(1-u_a/\lambda)\,,
\end{aligned}
\tag{5.20}
$$

it is then easy to show that, as well as $\mathcal{L}(q,\hat{\mathrm{u}})$, $\mathcal{R}(q,\hat{\mathrm{u}})$ also satisfies (5.18).

For the special case chosen in [18], $\hat{\mathrm{u}}_0 = (u,1,-1,q^{\frac{1}{2}},-q^{\frac{1}{2}})$, we then have

$$
(z^2;q)(z^{-2};q)\,\mathcal{J}(q,\hat{\mathrm{u}}_0,z) = \frac{1}{(1-uz)(1-u/z)}\,,
\tag{5.21}
$$

using the identity $(z;q)(-z;q)(q^{\frac{1}{2}}z,q)(-q^{\frac{1}{2}}z,q) = (z^2,q)$, and it is easy to calculate the contour integral in (5.13) giving

$$
\mathcal{L}(q,\hat{\mathrm{u}}_0) = \frac{(q,q)}{2(1-u^2)}\,.
\tag{5.22}
$$

The same result holds from (5.15) for $\mathcal{R}(q,\hat{\mathrm{u}}_0)$ using $(-q;q)(q,q^2)=1$. The q-difference relation implies $\mathcal{L}(q,\hat{\mathrm{u}}_n) = \mathcal{R}(q,\hat{\mathrm{u}}_n)$ for $\hat{\mathrm{u}}_n = (u,q^n,-1,q^{\frac{1}{2}},-q^{\frac{1}{2}})$. Analyticity ensures that equality must hold for any u_1,u_2 and further similar discussion extends this to any $\hat{\mathrm{u}}$.

6. Index Matching for $\mathcal{N}=1$ Superconformal $SU(N_c)$ Gauge Theories with N_f Flavours and its Seiberg Dual

In this section we show how the matching between the multi-particle indices for the general (N_c,N_f) case of Seiberg duality boils down to a theorem for the transformations of certain elliptic hypergeometric integrals, due to Rains [15]. The exact results here apply just to the Seiberg dual theories described in section 3.

For the invariant integral over $SU(n)$ of any symmetric function $f(\mathrm{x})$, $\mathrm{x}=(x_1,\dots,x_n)$, we have, equivalent to (4.9),

$$\int_{SU(n)} \mathrm{d}\mu(\mathrm{x})\, f(\mathrm{x}) = \frac{1}{n!}\int_{\mathbb{T}_{n-1}} \prod_{j=1}^{n-1}\frac{\mathrm{d}x_j}{2\pi i x_j}\, \Delta(\mathrm{x})\Delta(\mathrm{x}^{-1})\, f(\mathrm{x})\Big|_{\prod_{j=1}^n x_j=1}\,, \tag{6.1}$$

for $\mathbb{T}_{n-1}=S^1\times\ldots\times S^1$ the unit torus and where the Vandermonde determinant is, as usual,

$$\Delta(\mathrm{x}) = \prod_{1\le i<j\le n}(x_i-x_j)\,. \tag{6.2}$$

For application here it is convenient to rescale the $SU(N_f)\times SU(N_f)$ variables

$$(p\,q)^{\frac12 r} v\,\mathrm{y}\to\mathrm{y}\,, \qquad (p\,q)^{-\frac12 r} v\,\tilde{\mathrm{y}}\to\tilde{\mathrm{y}}\,, \tag{6.3}$$

where now

$$\prod_{j=1}^{N_f} y_j = (p\,q)^{\frac12\tilde N_c} v^{N_f} = \lambda^{\tilde N_c}\,, \qquad \prod_{j=1}^{N_f}\tilde y_j = (p\,q)^{-\frac12\tilde N_c} v^{N_f} = \tilde\lambda^{\tilde N_c}\,, \tag{6.4}$$

and then (3.4) becomes

$$\begin{aligned} i_E(p,q,\mathrm{y},\tilde{\mathrm{y}},\mathrm{z}) = &-\Big(\frac{p}{1-p}+\frac{q}{1-q}\Big)\Big(\sum_{1\le i,j\le N_c} z_i/z_j - 1\Big)\\ &+\frac{1}{(1-p)(1-q)}\sum_{i=1}^{N_f}\sum_{j=1}^{N_c}\Big(\big(y_i - p\,q\,\tilde y_i\big)z_j + \big(\tilde y_i{}^{-1} - p\,q\,y_i{}^{-1}\big)z_j{}^{-1}\Big)\,. \end{aligned} \tag{6.5}$$

Hence, using (1.9) and (1.11),

$$\begin{aligned} &\exp\Big(\sum_{n=1}^\infty \frac1n i_E\big(p^n,q^n,\mathrm{y}^n,\tilde{\mathrm{y}}^n,\mathrm{z}^n\big)\Big)\\ &= \frac{1}{\Delta(\mathrm{z})\Delta(\mathrm{z}^{-1})}\,\frac{(p;p)^{N_c-1}\,(q;q)^{N_c-1}}{\prod_{1\le i<j\le N_c}\Gamma(z_i/z_j,z_j/z_i;p,q)}\prod_{1\le i\le N_f}\prod_{1\le j\le N_c}\Gamma\big(y_iz_j,1/(\tilde y_i z_j);p,q\big)\,, \end{aligned} \tag{6.6}$$

where we use

$$\prod_{\substack{1\le i,j\le N_c\\ i\ne j}}(1-z_i/z_j) = \Delta(\mathrm{z})\Delta(\mathrm{z}^{-1})\,, \tag{6.7}$$

and adopt the notation

$$\Gamma(x_1,\ldots,x_n;p,q)=\Gamma(x_1;p,q)\cdots\Gamma(x_n;p,q)\,. \tag{6.8}$$

Applying (6.1) for $SU(N_c)$ and (6.6), the expression (1.6) for the electric index becomes

$$\begin{aligned}&I_E(p,q,\mathrm{y},\tilde{\mathrm{y}})_{SU(N_c)}\\&=(p;p)^{N_c-1}(q;q)^{N_c-1}\frac{1}{N_c!}\int\prod_{j=1}^{N_c-1}\frac{\mathrm{d}z_j}{2\pi i z_j}\frac{\prod_{1\le i\le N_f}\prod_{1\le j\le N_c}\Gamma\big(y_iz_j,1/(\tilde y_i z_j);p,q\big)}{\prod_{1\le i<j\le N_c}\Gamma\big(z_i/z_j,z_j/z_i;p,q\big)}\bigg|_{\prod_{j=1}^{N_c}z_j=1}\,,\end{aligned}\tag{6.9}$$

which is solely in terms of elliptic gamma functions. The denominator in (6.9) is naturally associated with the root system A_{N_c-1}, which is expressible in terms of orthonormal unit vectors as the $N_c(N_c-1)$ roots $\pm(e_i-e_j)$, $1\le i<j\le N_c$, where we map the root e_i-e_j to the Γ function depending on z_i/z_j.

For the magnetic dual theory then rewriting (3.6) with the rescaling (6.3) and the definitions (6.4),

$$\begin{aligned}&i_M(p,q,\mathrm{y},\tilde{\mathrm{y}},\tilde{\mathrm{z}})\\&=-\Big(\frac{p}{1-p}+\frac{q}{1-q}\Big)\Big(\sum_{1\le i,j\le\tilde N_c}\tilde z_i/\tilde z_j-1\Big)\\&\quad+\frac{1}{(1-p)(1-q)}\bigg(\sum_{i=1}^{N_f}\sum_{j=1}^{\tilde N_c}\Big(\big(\lambda\,y_i^{-1}-p\,q\,\tilde\lambda\,\tilde y_i^{-1}\big)\,\tilde z_j+\big(\tilde\lambda^{-1}\tilde y_i-p\,q\,\lambda^{-1}y_i\big)\,\tilde z_j^{-1}\Big)\\&\qquad\qquad+\sum_{i,j=1}^{N_f}\Big(y_i\,\tilde y_j^{-1}-p\,q\,y_i^{-1}\,\tilde y_j\Big)\bigg)\,.\end{aligned}\tag{6.10}$$

Hence following the same route as that leading to (6.9)

$$\begin{aligned}&I_M(p,q,\mathrm{y},\tilde{\mathrm{y}})_{SU(\tilde N_c)}\\&=\frac{1}{\tilde N_c!}\int\prod_{j=1}^{\tilde N_c-1}\frac{\mathrm{d}\tilde z_j}{2\pi i\tilde z_j}\,\Delta(\mathrm{z})\Delta(\mathrm{z}^{-1})\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}i_M(p^n,q^n,\mathrm{y}^n,\tilde{\mathrm{y}}^n,\tilde{\mathrm{z}}^n)\Big)\bigg|_{\prod_{j=1}^{\tilde N_c}\tilde z_j=1}\\&=\prod_{1\le i,j\le N_f}\Gamma(y_i/\tilde y_j;p,q)\;(p;p)^{\tilde N_c-1}\,(q;q)^{\tilde N_c-1}\\&\quad\times\frac{1}{\tilde N_c!}\int\prod_{j=1}^{\tilde N_c-1}\frac{\mathrm{d}\tilde z_j}{2\pi i\tilde z_j}\,\frac{\prod_{1\le i\le N_f}\prod_{1\le j\le\tilde N_c}\Gamma\big(\lambda\tilde z_j/y_i,\tilde\lambda^{-1}\tilde y_i/\tilde z_j;p,q\big)}{\prod_{1\le i<j\le\tilde N_c}\Gamma\big(\tilde z_i/\tilde z_j,\tilde z_j/\tilde z_i;p,q\big)}\bigg|_{\prod_{j=1}^{\tilde N_c}\tilde z_j=1}\\&=\prod_{1\le i,j\le N_f}\Gamma(y_i/\tilde y_j;p,q)\;I_E(p,q,\lambda\mathrm{y}^{-1},\tilde\lambda\tilde{\mathrm{y}}^{-1})_{SU(\tilde N_c)}\,.\end{aligned}\tag{6.11}$$

The essential requirement is the electric and magnetic theories are identical at the IR superconformal fixed point so that

$$I_E(p,q,\mathrm{y},\tilde{\mathrm{y}})_{SU(N_c)} = I_M(p,q,\mathrm{y},\tilde{\mathrm{y}})_{SU(\tilde{N}_c)}\,. \tag{6.12}$$

The integrals appearing in (6.9) and (6.11) are just those considered by Rains [15]. The right hand side of (6.9) for $n = N_c - 1,\ m = \tilde{N}_c - 1$ defines the elliptic hypergeometric integral $I^{(m)}_{A_n}\big(\mathrm{y};\tilde{\mathrm{y}}^{-1};p,q\big)$, depending on $(m+n+2)$-dimensional vectors $\mathrm{y},\tilde{\mathrm{y}}$. Theorem 4.1 of [15] requires

$$\begin{gathered} I^{(m)}_{A_n}\big(\mathrm{y};\tilde{\mathrm{y}}^{-1};p,q\big) = \prod_{1\le i,j\le m+n+4} \Gamma\big(y_i/\tilde{y}_j\,;p,q\big)\ I^{(n)}_{A_m}\big(Y^{\frac{1}{m+1}}\mathrm{y}^{-1};\tilde{\mathrm{y}}/\tilde{Y}^{\frac{1}{m+1}}\,;p,q\big)\,, \\ \text{for}\ \ Y=\textstyle\prod_i y_i\,, \qquad \tilde{Y}=\prod_i \tilde{y}_i\,, \qquad Y/\tilde{Y} = (p\,q)^{m+1}\,, \end{gathered} \tag{6.13}$$

implying then exactly (6.12). Furthermore from [15]

$$I^{(0)}_{A_n}\big(\mathrm{y};\tilde{\mathrm{y}}^{-1}\,;p,q\big) = \prod_{1\le i,j\le n+4} \Gamma\big(y_i/\tilde{y}_j\,;p,q\big) \prod_{1\le i\le n+2} \Gamma\big(Y\mathrm{y}^{-1},\tilde{\mathrm{y}}/\tilde{Y}\,;p,q\big)\,, \tag{6.14}$$

with $Y,\tilde{Y}$ as in (6.13). This evaluation of the integral applies when the magnetic gauge group is trivial.

The detailed expressions in both (6.9) and (6.11) depend on the precise details of the dual gauge groups and assignments of $U(1)_B$ charges for each theory so this result is a significant test of the details of Seiberg duality for these theories. This is in contrast to the large N_f, N_c expansions of section 5 where many such details were irrelevant. The proof of the theorem in [15], see also [19], relating these integrals is non trivial and does not involve any straightforward transformations between each side, it requires demonstrating the result for particular special cases which are then argued to form a dense set.

7. Indices for Dual Theories with $Sp(2N)$ Gauge Group

Duality extends to $\mathcal{N}=1$ supersymmetric gauge theories with other gauge groups. In this section, we consider a gauge group $G = Sp(2N)$, with a matter sector consisting of $2N_f$ chiral scalar fields Q, belonging to the $2N$ dimensional fundamental representation of the gauge group. The corresponding flavour symmetry group $F = SU(2N_f)\times U(1)_R$. The vector multiplet V of course belongs to the $N(2N+1)$ dimensional adjoint representation. The overall representation content is summarised in Table 5.

Table 5: Electric $Sp(2N)$ Gauge Theory

Field	$Sp(2N)$	$SU(2N_f)$	$U(1)_R$
Q	f	f	$1-(N+1)/N_f$
V	adj.	1	1

The dual theory is a $Sp(2\tilde{N})$ gauge theory again, where

$$\tilde{N} = N_f - N - 2\,, \tag{7.1}$$

and with the same flavour symmetry group F. The field content consists of $2N_f$ scalar multiplets q, in the $2\tilde{N}$ dimensional fundamental representation, a vector multiplet $\tilde{V}$, in the $\tilde{N}(2\tilde{N}+1)$ dimensional adjoint representation, and a gauge singlet scalar multiplet M belonging to the antisymmetric tensor representation T_A of dimension $N_f(2N_f-1)$ [20]. The representation content is as in Table 6.

Table 6: Magnetic $Sp(2\tilde{N})$ Gauge Theory

Field	$Sp(2\tilde{N})$	$SU(2N_f)$	$U(1)_R$
q	f	$\bar{f}$	$(N+1)/N_f$
$\tilde{V}$	adj.	1	1
M	1	T_A	$2(\tilde{N}+1)/N_f$

Imposing $r \geq \frac{1}{3}$ for both Q, q leads to the conformal window $\frac{3}{2}(N+1) \leq N_f \leq 3(N+1)$.

The single particle index in each case, $i_E(p,q,\mathrm{y},\mathrm{z})$ and $i_M(p,q,\mathrm{y},\tilde{\mathrm{z}})$, may be straightforwardly formed by applying (1.5), using (3.3) and (5.5) for $SU(2N_f)$ characters $\chi_{SU(2N_f)}(\mathrm{y})$ with $\mathrm{y} = (y_1, \dots, y_{2N_f})$. The required $Sp(2N)$ and $Sp(2\tilde{N})$ characters are obtained from the following results for $Sp(2n)$ in general

$$\begin{aligned} \chi_{Sp(2n),f}(\mathrm{x}) &= \sum_{i=1}^{n} \big(x_i + x_i{}^{-1}\big)\,, \\ \chi_{Sp(2n),\mathrm{adj.}}(\mathrm{x}) &= \sum_{1\leq i<j\leq n} \big(x_i\, x_j + x_i\, x_j{}^{-1} + x_i{}^{-1}\, x_j + x_i{}^{-1}\, x_j{}^{-1}\big) + \sum_{i=1}^{n} \big(x_i^2 + x_i{}^{-2}\big) + n\,. \end{aligned} \tag{7.2}$$

For invariant integration over $Sp(2n)$ of any symmetric $f(\mathrm{x})$ we also have

$$\int_{Sp(2n)} \mathrm{d}\mu(\mathrm{x})\, f(\mathrm{x}) = \frac{(-1)^n}{2^n n!} \int_{\mathbb{T}_n} \prod_{j=1}^{n} \frac{\mathrm{d}x_j}{2\pi i x_j} \prod_{j=1}^{n} \big(x_j - x_j{}^{-1}\big)^2 \Delta(\mathrm{x}+\mathrm{x}^{-1})^2\, f(\mathrm{x})\,. \tag{7.3}$$

Assuming the rescaling

$$(p\,q)^{(\tilde{N}+1)/2N_f}\,\mathrm{y} \to \mathrm{y} \quad \Rightarrow \quad \prod_{i=1}^{2N_f} y_i = (p\,q)^{\tilde{N}+1}\,, \tag{7.4}$$

the single particle index then becomes

$$
\begin{aligned}
i_E(p,q,\mathrm{y},\mathrm{z}) = & -\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\chi_{Sp(2N),\mathrm{adj.}}(\mathrm{z}) \\
& + \frac{1}{(1-p)(1-q)}\sum_{i=1}^{2N_f}\big(y_i - p\,q/y_i\big)\,\chi_{Sp(2N),f}(\mathrm{z})\,.
\end{aligned}
\tag{7.5}
$$

As a consequence of (7.3) the result (1.6) for the electric index is expressible as a multi-contour integral

$$
\begin{aligned}
& I_E(p,q,\mathrm{y})_{Sp(2N)} \\
& = \frac{(-1)^N}{2^N N!}\int\prod_{j=1}^{N}\frac{\mathrm{d}z_j}{2\pi i z_j}\prod_{j=1}^{N}\big(z_j - {z_j}^{-1}\big)^2\Delta(\mathrm{z}+\mathrm{z}^{-1})^2\,\exp\bigg(\sum_{n=1}^{\infty}\frac{1}{n}i_E(p^n,q^n,\mathrm{y}^n,\mathrm{z}^n)\bigg)\,.
\end{aligned}
\tag{7.6}
$$

Using (1.10), (1.11) and (7.2), we may write, with the notation (6.8),

$$
\begin{aligned}
& \exp\bigg(-\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{p^n}{1-p^n}+\frac{q^n}{1-q^n}\Big)\chi_{Sp(2N),\mathrm{adj.}}(\mathrm{z}^n)\bigg) \\
& = (-1)^N (p;p)^N\,(q;q)^N\frac{1}{\Delta\big(\mathrm{z}+\mathrm{z}^{-1}\big)^2\prod_{1\le j\le N}(z_j-{z_j}^{-1})^2} \\
& \quad \times \frac{1}{\prod_{1\le i<j\le N}\Gamma\big(z_iz_j, z_i/z_j, z_j/z_i, 1/(z_iz_j);p,q\big)\prod_{1\le j\le N}\Gamma\big(z_j^2,1/z_j^2;p,q\big)}\,,
\end{aligned}
\tag{7.7}
$$

where the inverse $Sp(2N)$ measure is generated by

$$
\begin{aligned}
& \prod_{1\le i<j\le N}(1-z_iz_j)(1-z_i/z_j)(1-z_j/z_i)(1-1/z_iz_j) = \Delta(\mathrm{z}+\mathrm{z}^{-1})^2\,, \\
& \prod_{i=1}^{N}(1-z_i^2)(1-{z_i}^{-2}) = (-1)^N\prod_{i=1}^{N}(z_i-{z_i}^{-1})^2\,.
\end{aligned}
\tag{7.8}
$$

Hence (7.5) becomes

$$
\begin{aligned}
& I_E(p,q,\mathrm{y})_{Sp(2N)} = (p;p)^N\,(q;q)^N\,\frac{1}{2^N N!} \\
& \times\int\prod_{1\le j\le N}\frac{\mathrm{d}z_j}{2\pi i z_j}\,\frac{\prod_{1\le i\le 2N_f}\prod_{1\le j\le N}\Gamma\big(y_iz_j, y_i/z_j;p,q\big)}{\prod_{1\le i<j\le N}\Gamma\big(z_iz_j, z_i/z_j, z_j/z_i, 1/(z_iz_j);p,q\big)\prod_{1\le j\le N}\Gamma\big(z_j^2,1/z_j^2;p,q\big)}\,,
\end{aligned}
\tag{7.9}
$$

where the integrand now involves only elliptic gamma functions in a similar manner to (6.9). In this case the factors in the integrand denominator are associated with the roots for C_N, $\pm e_i \pm e_j$, $i \neq j$, $\pm 2e_i$ for $i,j = 1.\dots N$.

For the corresponding magnetic theory the single particle index becomes, using (5.5),

$$
\begin{aligned}
i_M(p,q,\mathrm{y},\tilde{\mathrm{z}}) = & -\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\chi_{Sp(2\tilde{N}),\mathrm{adj.}}(\tilde{\mathrm{z}}) \\
& + \frac{1}{(1-p)(1-q)}\Bigg((p\,q)^{\frac12}\sum_{i=1}^{2N_f}\left(y_i^{-1}-y_i\right)\chi_{Sp(2\tilde{N}),f}(\tilde{\mathrm{z}}) \\
& \qquad + \sum_{1\le i<j\le 2N_f}\left(y_iy_j - p\,q/(y_iy_j)\right)\Bigg).
\end{aligned} \tag{7.10}
$$

The magnetic index is then

$$
\begin{aligned}
I_M(p,q,\mathrm{y})_{Sp(2\tilde{N})} = & \frac{(-1)^{\tilde{N}}}{2^{\tilde{N}}\tilde{N}!}\int\prod_{j=1}^{\tilde{N}}\frac{\mathrm{d}\tilde{z}_j}{2\pi i\tilde{z}_j}\prod_{j=1}^{\tilde{N}}\left(\tilde{z}_j-\tilde{z}_j{}^{-1}\right)^2\Delta(\tilde{\mathrm{z}}+\tilde{z}^{-1})^2 \\
& \times\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}i_M(p^n,q^n,\mathrm{y}^n,\tilde{\mathrm{z}}^n)\right).
\end{aligned} \tag{7.11}
$$

and, in the same fashion as (7.9) was obtained, we now have

$$
\begin{aligned}
& I_M(p,q,\mathrm{y})_{Sp(2\tilde{N})} = \prod_{1\le i<j\le 2N_f}\Gamma\left(y_i\,y_j;p,q\right)\,(p;p)^{\tilde{N}}\,(q;q)^{\tilde{N}}\,\frac{1}{2^{\tilde{N}}\tilde{N}!} \\
& \times\int\prod_{1\le j\le\tilde{N}}\frac{\mathrm{d}\tilde{z}_j}{2\pi i\tilde{z}_j}\frac{\prod_{1\le i\le 2N_f}\prod_{1\le j\le\tilde{N}}\Gamma\left(t\tilde{z}_j/y_i,t/(y_i\tilde{z}_j);p,q\right)}{\prod_{1\le i<j\le\tilde{N}}\Gamma\left(\tilde{z}_i\tilde{z}_j,\tilde{z}_i/\tilde{z}_j,\tilde{z}_j/\tilde{z}_i,1/(\tilde{z}_i\tilde{z}_j);p,q\right)\prod_{1\le j\le\tilde{N}}\Gamma\left(\tilde{z}_j{}^2,1/\tilde{z}_j{}^2;p,q\right)} \\
& = \prod_{1\le i<j\le 2N_f}\Gamma\left(y_i\,y_j;p,q\right)\,I_E(p,q,t\mathrm{y}^{-1})_{Sp(2\tilde{N})},
\end{aligned} \tag{7.12}
$$

for $t=(p\,q)^{\frac12}$.

Again, happily, the relevant integrals were considered by Rains [15]. The right hand side of (7.9) for $n=N$, $m=\tilde{N}$ defines the elliptic hypergeometric integral

$$
\begin{aligned}
& I^{(m)}_{BC_n}(\mathrm{y};p,q) = (p;p)^n\,(q;q)^n\,\frac{1}{2^n n!} \\
& \times\int\prod_{1\le j\le n}\frac{\mathrm{d}z_j}{2\pi i z_j}\frac{\prod_{1\le i\le 2(m+n+2)}\prod_{1\le j\le n}\Gamma\left(y_iz_j,y_i/z_j;p,q\right)}{\prod_{1\le i<j\le n}\Gamma\left(z_iz_j,z_i/z_j,z_j/z_i,1/(z_iz_j);p,q\right)\prod_{1\le j\le n}\Gamma\left(z_j^2,1/z_j^2;p,q\right)},
\end{aligned} \tag{7.13}
$$

depending on a $2(m+n+2)$-dimensional vector y. Theorem 3.1 of [15] requires

$$
I^{(m)}_{BC_n}(\mathrm{y}\,;p,q) = \prod_{1\le i<j\le 2(m+n+2)}\Gamma\left(y_i\,y_j;p,q\right)\,I^{(n)}_{BC_m}\left(\sqrt{pq}\;\mathrm{y}^{-1};p,q\right),\ \text{ for }\prod_i y_i=(p\,q)^{m+1}. \tag{7.14}
$$

This then implies $I_E(p,q,\mathrm{y})_{Sp(2N)}$ and $I_M(p,q,\mathrm{y})_{Sp(\tilde{N})}$ in (7.9) and (7.12) are equal. In this case $I^{(m)}_{BC_0}(\mathrm{y};p,q)=1$. Applying the transformation twice leads to the identity as a consequence of (5.9).

8. Indices for Dual Theories with $SO(N)$ Gauge Groups

The original paper on duality [1], see also [21], discussed additionally $\mathcal{N}=1$ theories with orthogonal gauge groups with N_f chiral quark fields in the vector representation, so that the flavour symmetry group $F = SU(N_f) \times U(1)_R$. The adjoint representation here has dimension $\frac{1}{2}N(N-1)$. The overall representation content is summarised in Table 7.

Table 7: Electric $SO(N)$ Gauge Theory

Field	$SO(N)$	$SU(N_f)$	$U(1)_R$
Q	vec.	f	$1-(N-2)/N_f$
V	adj.	1	1

The dual theory is also a $SO(\tilde{N})$ gauge theory, where

$$\tilde{N} = N_f - N + 4\,, \tag{8.1}$$

and with the same flavour symmetry group F. The field content consists of N_f scalar multiplets q, in the vector representation, a vector multiplet $\tilde{V}$, in the $\frac{1}{2}\tilde{N}(\tilde{N}-1)$ dimensional adjoint representation, and a gauge singlet scalar multiplet M belonging to the symmetric tensor representation T_S of dimension $\frac{1}{2}N_f(N_f+1)$. The representation content is as in Table 8.

Table 8: Magnetic $SO(\tilde{N})$ Gauge Theory

Field	$SO(\tilde{N})$	$SU(N_f)$	$U(1)_R$
q	vec.	$\bar{f}$	$(N-2)/N_f$
$\tilde{V}$	adj.	1	1
M	1	T_S	$2-2(N-2)/N_f$

Imposing $r \geq \frac{1}{3}$ for both Q, q leads to the conformal window $\frac{3}{2}(N-2) \leq N_f \leq 3(N-2)$.

For characters for $SO(N)$ it is necessary to distinguish according to whether N is even or odd. For $\mathrm{x} = (x_1, \dots, x_n)$ the relevant results are

$$\begin{aligned} \chi_{SO(2n),\text{vec.}}(\mathrm{x}) &= \sum_{i=1}^{n} \left(x_i + x_i{}^{-1}\right), \\ \chi_{SO(2n),\text{adj.}}(\mathrm{x}) &= \sum_{1\leq i<j\leq n} \left(x_i\, x_j + x_i\, x_j{}^{-1} + x_i{}^{-1}\, x_j + x_i{}^{-1}\, x_j{}^{-1}\right) + n\,, \end{aligned} \tag{8.2}$$

and

$$
\begin{aligned}
\chi_{SO(2n+1),\text{vec.}}(\mathrm{x}) &= \sum_{i=1}^{n} \left(x_i + x_i{}^{-1}\right) + 1\,, \\
\chi_{SO(2n+1),\text{adj.}}(\mathrm{x}) &= \sum_{1\le i<j\le n} \left(x_i\, x_j + x_i\, x_j{}^{-1} + x_i{}^{-1}\, x_j + x_i{}^{-1}\, x_j{}^{-1}\right) + \sum_{i=1}^{n} \left(x_i + x_i{}^{-1}\right) + n\,.
\end{aligned} \tag{8.3}
$$

We also require

$$
\chi_{SU(n),T_S}(\mathrm{x}) = \sum_{1\le i<j\le n} x_i x_j + \sum_{i=1}^{n} x_i{}^2\,, \qquad \chi_{SU(n),\bar{T}_S}(\mathrm{x}) = \chi_{SU(n),T_S}(\mathrm{x}^{-1})\,. \tag{8.4}
$$

For invariant integration over $SO(N)$ of any symmetric $f(\mathrm{x})$ we also have

$$
\begin{aligned}
\int_{SO(2n)} \mathrm{d}\mu(\mathrm{x})\, f(\mathrm{x}) &= \frac{1}{2^{n-1} n!} \int_{\mathbb{T}_n} \prod_{j=1}^{n} \frac{\mathrm{d}x_j}{2\pi i x_j}\, \Delta(\mathrm{x}+\mathrm{x}^{-1})^2\, f(\mathrm{x})\,, \\
\int_{SO(2n+1)} \mathrm{d}\mu(\mathrm{x})\, f(\mathrm{x}) &= \frac{(-1)^n}{2^n n!} \int_{\mathbb{T}_n} \prod_{j=1}^{n} \frac{\mathrm{d}x_j}{2\pi i x_j}\, \prod_{j=1}^{n} \left(x_j{}^{\frac12} - x_j{}^{-\frac12}\right)^2 \Delta(\mathrm{x}+\mathrm{x}^{-1})^2\; f(\mathrm{x})\,.
\end{aligned} \tag{8.5}
$$

The single particle indices (1.5) are obtained in a similar fashion as previously

$$
\begin{aligned}
i_E(p,q,\mathrm{y},\mathrm{z}) = {} & -\left(\frac{p}{1-p} + \frac{q}{1-q}\right) \chi_{SO(N),\text{adj.}}(\mathrm{z}) \\
& + \frac{1}{(1-p)(1-q)} \sum_{i=1}^{N_f} \left(y_i - p\,q/y_i\right) \chi_{SO(N),\text{vec.}}(\mathrm{z})\,,
\end{aligned} \tag{8.6}
$$

and

$$
\begin{aligned}
i_M(p,q,\mathrm{y},\tilde{\mathrm{z}}) = {} & -\left(\frac{p}{1-p} + \frac{q}{1-q}\right) \chi_{SO(\tilde N),\text{adj.}}(\tilde{\mathrm{z}}) \\
& + \frac{1}{(1-p)(1-q)} \Bigg((p\,q)^{\frac12} \sum_{i=1}^{N_f} \left(y_i^{-1} - y_i\right) \chi_{SO(\tilde N),\text{vec.}}(\tilde{\mathrm{z}}) \\
& \qquad + \sum_{1\le i<j\le N_f} \left(y_i y_j - p\,q/(y_i y_j)\right) + \sum_{i=1}^{N_f} \left(y_i{}^2 - p\,q/y_i{}^2\right) \Bigg)\,,
\end{aligned} \tag{8.7}
$$

where y has been rescaled so that

$$
\prod_{i=1}^{N_f} y_i = (p\,q)^{\frac12 (N_f - N + 2)}\,. \tag{8.8}
$$

The integral formulae for the index are then generated very much as before. The adjoint characters in (8.2) and (8.3) generate contributions which cancel the integration measures in (8.5) by using (7.8) once more. Hence, taking $N = 2n$ and $N = 2n+1$,

$$\begin{aligned} I_E(p,q,\mathrm{y})_{SO(2n)} = {}& (p;p)^n\,(q;q)^n\,\frac{1}{2^{n-1}n!} \\ & \times \int \prod_{1\le j\le n} \frac{\mathrm{d}z_j}{2\pi i z_j}\,\frac{\prod_{1\le i\le N_f}\prod_{1\le j\le n}\Gamma\big(y_i z_j, y_i/z_j;p,q\big)}{\prod_{1\le i<j\le n}\Gamma\big(z_i z_j, z_i/z_j, z_j/z_i, 1/(z_i z_j);p,q\big)}\,, \end{aligned} \tag{8.9}$$

and

$$\begin{aligned} I_E(p,q,\mathrm{y})_{SO(2n+1)} = {}& (p;p)^n\,(q;q)^n \prod_{1\le i\le N_F}\Gamma(y_i;p,q)\,\frac{1}{2^n n!} \\ \times \int \prod_{1\le j\le n} \frac{\mathrm{d}z_j}{2\pi i z_j} & \frac{\prod_{1\le i\le N_f}\prod_{1\le j\le n}\Gamma\big(y_i z_j, y_i/z_j;p,q\big)}{\prod_{1\le i<j\le n}\Gamma\big(z_i z_j, z_i/z_j, z_j/z_i, 1/(z_i z_j);p,q\big)\prod_{1\le j\le n}\Gamma\big(z_j,1/z_j;p,q\big)}\,. \end{aligned} \tag{8.10}$$

In (8.9) and (8.10) the factors in the integrand denominator may be matched with the roots for D_N, $\pm e_i \pm e_j$, $i<j$, and B_N, $\pm e_i \pm e_j$, $i<j$, $\pm e_j$, respectively.

For the corresponding magnetic theory the results are very similar except for contributions involving the meson field M, which are obtained from the last line of (8.7). The results are expressed concisely as

$$I_M(p,q,\mathrm{y})_{SO(\tilde N)} = \prod_{1\le i<j\le N_f}\Gamma\big(y_i\,y_j;p,q\big)\prod_{1\le i\le N_f}\Gamma\big(y_i^2;p,q\big)\,I_E(p,q,\sqrt{pq}\,\mathrm{y}^{-1})_{SO(\tilde N)}\,. \tag{8.11}$$

The required identity is then

$$I_E(p,q,\mathrm{y})_{SO(N)} = I_M(p,q,\mathrm{y})_{SO(\tilde N)}\,,\ \text{ for } \textstyle\prod_i y_i = (pq)^{\frac12\tilde N-1}\,,\ N_f = N+\tilde N-4\,. \tag{8.12}$$

The relation (8.12) involving B_N and D_N multi-variable elliptic beta integrals can be reduced to a special case of (7.14) by virtue of an argument due to Rains [22]. It is easy to verify

$$\Gamma(z^2;p,q) = \textstyle\prod_a \Gamma(z u_a;p,q)\,, \qquad \mathrm{u} = \big(1,-1,p^{\frac12},-p^{\frac12},q^{\frac12},-q^{\frac12},(pq)^{\frac12},-(pq)^{\frac12}\big)\,, \tag{8.13}$$

With the definition (8.9) we may then express the index in terms of $I^{(m)}_{BC_n}$ as in (7.13) by

$$I_E(p,q,\mathrm{y})_{SO(2n)} = \begin{cases} 2\,I^{(\frac12(N_f+4)-n)}_{BC_n}(\mathrm{y},\mathrm{u};p,q)\,, & N_f \text{ even;} \\ 2\,I^{(\frac12(N_f+3)-n)}_{BC_n}(\mathrm{y},\mathrm{v};p,q)\,, & N_f \text{ odd;} \end{cases} \tag{8.14}$$

where, noting that $\Gamma((pq)^{\frac{1}{2}}z_j,(pq)^{\frac{1}{2}}/z_j;p,q)=1$ as a consequence of (5.9), we also define

$$\mathrm{v}=\big(1,-1,p^{\frac{1}{2}},-p^{\frac{1}{2}},q^{\frac{1}{2}},-q^{\frac{1}{2}},-(pq)^{\frac{1}{2}}\big)\,. \tag{8.15}$$

In a similar vein starting from (8.10) we may also write

$$I_E(p,q,\mathrm{y})_{SO(2n+1)}=\begin{cases}\prod_{i=1}^{N_F}\Gamma(y_i;p,q)\ I_{BC_n}^{(\frac{1}{2}(N_f+4)-n)}(\mathrm{y},\mathrm{u}';p,q)\,, & N_f \text{ even};\\ \prod_{i=1}^{N_F}\Gamma(y_i;p,q)\ I_{BC_n}^{(\frac{1}{2}(N_f+5)-n)}(\mathrm{y},\mathrm{v}';p,q)\,, & N_f \text{ odd};\end{cases} \tag{8.16}$$

for

$$\begin{aligned}\mathrm{u}'&=\big(-1,p^{\frac{1}{2}},-p^{\frac{1}{2}},q^{\frac{1}{2}},-q^{\frac{1}{2}},-(pq)^{\frac{1}{2}}\big)\,,\\ \mathrm{v}'&=\big(-1,p^{\frac{1}{2}},-p^{\frac{1}{2}},q^{\frac{1}{2}},-q^{\frac{1}{2}},-(pq)^{\frac{1}{2}},(pq)^{\frac{1}{2}}\big)\,.\end{aligned} \tag{8.17}$$

The necessary identity to ensure (8.11) and (8.12) then follows from (7.14), taking into account $\mathrm{v}'\sim\sqrt{pq}\,\mathrm{v}^{-1}$ and the results

$$\begin{aligned}\textstyle\prod_a\Gamma(y_iu_a;p,q)&=\Gamma(y_i^2;p,q)\,, \qquad \textstyle\prod_a\Gamma(y_iv_a;p,q)=\Gamma(y_i^2;p,q)\,\Gamma(\sqrt{pq}/y_i;p,q)\,,\\ \textstyle\prod_a\Gamma(y_iu'_a;p,q)&=\Gamma(y_i^2;p,q)\,\frac{\Gamma(\sqrt{pq}/y_i;p,q)}{\Gamma(y_i;p,q)}\,, \qquad \textstyle\prod_a\Gamma(y_iv'_a;p,q)=\frac{\Gamma(y_i^2;p,q)}{\Gamma(y_i;p,q)}\,,\\ \textstyle\prod_{a<b}\Gamma(u_au_b;p,q)&=\textstyle\prod_{a<b}\Gamma(u'_au'_b;p,q)=1\,,\\ 2\textstyle\prod_{a<b}\Gamma(v_av_b;p,q)&=\tfrac{1}{2}\textstyle\prod_{a<b}\Gamma(v'_av'_b;p,q)=1\,.\end{aligned} \tag{8.18}$$

We also test the result in the simple case $N=4$, $N_f=3$, $\tilde{N}=3$ which involves duality between $SO(N)$ gauge theories with even and odd N by considering the first few terms in an expansion. As a result of $SO(4)=SU(2)\times SU(2)/\mathbb{Z}_2$ and $SO(3)=SU(2)/\mathbb{Z}_2$ we have, letting for $SO(4)$ $z_1=uv$, $z_2=u/v$ and for $SO(3)$ $z_1=w^2$, from (8.5)

$$\int_{SO(4)}\mathrm{d}\mu(\mathrm{z})=\int_{SU(2)}\mathrm{d}\mu(v)\int_{SU(2)}\mathrm{d}\mu(u)\,, \qquad \int_{SO(3)}\mathrm{d}\mu(\mathrm{z})=\int_{SU(2)}\mathrm{d}\mu(w)\,, \tag{8.19}$$

since $\Delta(\mathrm{z}+\mathrm{z}^{-1})^2=(1-u^2)^2(1-v^2)^2/u^2v^2$. With $p=tx$, $q=tx^{-1}$ the electric single particle index from (8.6) becomes

$$\begin{aligned}i_E(tx,tx^{-1},\mathrm{y},u,v)=\frac{1}{(1-tx)(1-tx^{-1})}\Big(&\big(2t^2-t\chi_2(x)\big)\big(\chi_3(u)+\chi_3(v)\big)\\ &+\big(p_3(\mathrm{y})-t^2p_3(\mathrm{y}^{-1})\big)\chi_2(u)\chi_2(v)\Big)\,,\end{aligned} \tag{8.20}$$

for

$$\mathrm{y}=(y_1,y_2,y_3)\,, \qquad y_1y_2y_3=t\,, \tag{8.21}$$

and with χ_2, χ_3 defined in (5.3). For the magnetic index from (8.7),

$$
\begin{aligned}
i_M(tx, tx^{-1}, \mathrm{y}, w) = \frac{1}{(1-tx)(1-tx^{-1})} \Big(& \big(2t^2 - t\chi_2(x) + t\, p_3(\mathrm{y}^{-1}) - t\, p_3(y)\big)\chi_3(w) \\
& + s_{(2,0)}(\mathrm{y}) - t^2 s_{(2,0)}(\mathrm{y}^{-1}) \Big) , \\
s_{(2,0)}(\mathrm{y}) = \tfrac{1}{2}\big(p_3(y)^2 + p_3(\mathrm{y}^2)\big) \, . &
\end{aligned}
\tag{8.22}
$$

The required index identity is then from (1.6)

$$
\begin{aligned}
I(tx, tx^{-1}, \mathrm{y}) &= \int_{SU(2)} \mathrm{d}\mu(v) \int_{SU(2)} \mathrm{d}\mu(u) \, \exp\bigg(\sum_{n=1}^{\infty} \frac{1}{n}\, i_E\big(t^n, x^n, \mathrm{y}^n, u^n, v^n\big)\bigg) \\
&= \int_{SU(2)} \mathrm{d}\mu(w) \, \exp\bigg(\sum_{n=1}^{\infty} \frac{1}{n}\, i_M\big(t^n, x^n, \mathrm{y}^n, w^n\big)\bigg) \, .
\end{aligned}
\tag{8.23}
$$

It is straightforward to expand (8.23) where we may use $\chi_2(u^n) = \chi_{n+1}(u) - \chi_{n-1}(u)$, $\chi_3(u^n) = \chi_{2n+1}(u) - \chi_{2n-1}(u) + 1$ and apply standard $SU(2)$ tensor product rules to decompose products of χ_n into single characters. The $SU(2)$ integrals can then be evaluated using orthonormality of characters or equivalently just by evaluating residues. The index has an expansion

$$
I(tx, tx^{-1}, \mathrm{y}) = 1 + \sum_{n>0} f_n(t, x, \mathrm{y}) \, , \qquad f_n(t, x, \mathrm{y}) = \mathrm{O}\big(t^{\frac{1}{3}n}\big) \, ,
\tag{8.24}
$$

where from (8.21) $\mathrm{y} = \mathrm{O}(t^{\frac{1}{3}})$. We have checked that both the electric and magnetic contributions to (8.23) are the same up to $\mathrm{O}(t^4)$ and give the following non zero terms, in terms of $SU(3)$ Schur polynomials $s_{(\lambda,\mu)}(\mathrm{y})$ and $SU(2)$ characters $\chi_{2j+1}(x)$,

$$
\begin{aligned}
f_2(t,x,\mathrm{y}) &= s_{(2,0)}(\mathrm{y})\,, \\
f_4(t,x,\mathrm{y}) &= s_{(4,0)}(\mathrm{y}) + s_{(2,2)}(\mathrm{y})\,, \\
f_5(t,x,\mathrm{y}) &= t\chi_2(x)\big(s_{(2,0)}(\mathrm{y}) - s_{(1,1)}(\mathrm{y})\big)\,, \\
f_6(t,x,\mathrm{y}) &= s_{(6,0)}(\mathrm{y}) + s_{(4,2)}(\mathrm{y}) - t\, s_{(2,1)}(\mathrm{y}) + 2t^2\,, \\
f_7(t,x,\mathrm{y}) &= t\,\chi_2(x)\big(s_{(4,0)}(\mathrm{y}) + s_{(2,2)}(\mathrm{y})\big)\,, \\
f_8(t,x,\mathrm{y}) &= s_{(8,0)}(\mathrm{y}) + s_{(6,2)}(\mathrm{y}) + s_{(4,4)}(\mathrm{y}) - t\, s_{(4,1)}(\mathrm{y}) - t\, s_{(3,2)}(\mathrm{y}) - t^2 s_{(1,1)}(\mathrm{y}) \\
&\quad + t^2\chi_3(x)\big(s_{(2,0)}(\mathrm{y}) - s_{(1,1)}(\mathrm{y})\big)\,, \\
f_9(t,x,\mathrm{y}) &= t\,\chi_2(x)\big(s_{(6,0)}(\mathrm{y}) + 2s_{(4,2)}(\mathrm{y}) - s_{(3,3)}(\mathrm{y}) + t\, s_{(2,1)}(\mathrm{y}) + t^2\big)\,, \\
f_{10}(t,x,\mathrm{y}) &= s_{(10,0)}(\mathrm{y}) + s_{(8,2)}(\mathrm{y}) + s_{(6,4)}(\mathrm{y}) \\
&\quad - ts_{(6,1)}(\mathrm{y}) - ts_{(5,2)}(\mathrm{y}) - ts_{(4,3)}(\mathrm{y}) - 2t^2 s_{(3,1)}(\mathrm{y}) \\
&\quad + t^2\chi_3(x)\big(2s_{(4,0)}(\mathrm{y}) - s_{(3,1)}(\mathrm{y}) + 2s_{(2,2)}(\mathrm{y}) - t\, s_{(1,0)}(\mathrm{y})\big)\,, \\
f_{11}(t,x,\mathrm{y}) &= t\,\chi_2(x)\big(s_{(8,0)}(\mathrm{y}) + 2s_{(6,2)}(\mathrm{y}) + s_{(4,4)}(\mathrm{y}) + t^2 s_{(2,0)}(\mathrm{y}) - t^2 s_{(1,1)}(\mathrm{y})\big) \\
&\quad + t^3\chi_4(x)\big(s_{(2,0)}(\mathrm{y}) - s_{(1,1)}(\mathrm{y})\big)\,, \\
f_{12}(t,x,\mathrm{y}) &= s_{(12,0)}(\mathrm{y}) + s_{(10,2)}(\mathrm{y}) + s_{(8,4)}(\mathrm{y}) + s_{(6,6)}(\mathrm{y}) \\
&\quad - t\, s_{(8,1)}(\mathrm{y}) - t\, s_{(7,2)}(\mathrm{y}) - t\, s_{(6,3)}(\mathrm{y}) - t\, s_{(5,4)}(\mathrm{y}) \\
&\quad - 2t^2 s_{(5,1)}(\mathrm{y}) - t^2 s_{(4,2)}(\mathrm{y}) + 2t^2 s_{(3,3)}(\mathrm{y}) + 2t^3 s_{(3,0)}(\mathrm{y}) + t^3 s_{(2,1)}(\mathrm{y}) - t^4 \\
&\quad + t^2\chi_3(x)\big(2s_{(6,0)}(\mathrm{y}) + 3s_{(4,2)}(\mathrm{y}) - 2s_{(3,3)}(\mathrm{y}) - t\, s_{(3,0)}(\mathrm{y}) + t\, s_{(2,1)}(\mathrm{y}) + 2t^2\big)\,.
\end{aligned}
\tag{8.25}
$$

These results are sensitive to all terms which are in i_E and i_M in (8.20) and (8.22), and therefore provide good support for the required all orders result (8.23). It is significant to note also that all coefficients are integers in accord with the expectation in (1.7).

9. Conclusions

This paper has demonstrated that the naive prescription for the superconformal index given by (1.5) and (1.6) and using the standard results for dual $\mathcal{N}=1$ gauge theories, where the matter content and its R-charges are determined by careful matching of the spectrum of gauge invariant operators and also matching the 't Hooft anomalies, leads to results which are the same in both dual theories. The exact equality of the two expressions for the index has been shown for theories in which there is no superpotential and then depends on very non-trivial q-series type integral identities, only recently proved, which are only valid for the detailed R-charges and gauge groups determined by the consistency conditions for duality. This remarkable correspondence perhaps lends credence to the results for the index described here following on from Römelsberger [11]. The elliptic hypergeometric functions which are generated by the index, and whose non trivial transformation properties are a necessary requirement for duality, are also relevant to other areas such as quantum integrable systems, [23].

The situation when there is a superpotential, as in the Kutasov-Schwimmer case, is less clear. The operator spectrum is then constrained by equations of motion and the result for the index should be modified. Nevertheless we also verified that the naive formula for the index gave results which agreed in the large N limit and also showed that the leading finite N correction was also consistent. Perhaps physical considerations may suggest novel identities which have not yet been proved. Seiberg duality has been extended to a much wider class of $\mathcal{N}=1$ theories than those considered in this paper, including theories with exceptional gauge groups [24].

A remaining issue concerns the precise derivation of the formula for the index provided by applying (1.5) and (1.6). In particular other than in the free case when $r=\frac{2}{3}$ the results for chiral fields given by (1.5) have not been derived in this paper. For interacting theories it is necessary to consider the superconformal algebra in (2.2) and (2.4) with in general F and D non zero. However, letting for instance $F \to \bar{\varphi}^n$ for some n still enforces $r=\frac{2}{3}$ as a consequence of the commutator $[S^\alpha, F]$. Similar considerations apply for other modifications although the derivatives in (2.3) and (2.5) may be replaced by gauge covariant derivatives by allowing for the algebra to be extended by appropriate gauge transformations. Perhaps further inclusion of internal symmetry transformations is necessary at non trivial superconformal fixed points. This is perhaps suggested by the rescaling of internal symmetry character variables, such as in (6.3), which was a necessary feature of the analysis of the integrals defining the index at the interacting $\mathcal{N}=1$ superconformal fixed points, at which the duality between electric and magnetic theories that is considered in this paper is fully realised.

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Appendix A. $\mathcal{N}=1$ Superconformal Representation Theory and Characters

Using the notation of [9], the generators of the $\mathcal{N}=1$ superconformal group $SU(2,2|1)$ consist of those for Lorentz transformations M_{ab}, translations P_a, special conformal transformations K_a, $a,b=1,\dots,4$, dilatations H, which is the Hamiltonian in radial quantisation of conformal theories, the $U(1)_R$ R-charge R along with supercharges Q_α, $\bar{Q}_{\dot\alpha}$ and their superconformal partners S^α, $\bar{S}^{\dot\alpha}$, $\alpha,\dot\alpha=1,2$. In a spinorial basis $P_{\alpha\dot\alpha}=(\sigma^a)_{\alpha\dot\alpha}P_a$, $K^{\dot\alpha\alpha}=(\bar\sigma^a)^{\dot\alpha\alpha}K_a$, $M_\alpha{}^\beta=-\frac{1}{4}i(\sigma^a\bar\sigma^b)_\alpha{}^\beta M_{ab}$, $\bar{M}^{\dot\alpha}{}_{\dot\beta}=-\frac{1}{4}i(\bar\sigma^a\sigma^b)^{\dot\alpha}{}_{\dot\beta}M_{ab}$[3] . With the notation,

$$\mathcal{M}_{\mathcal{A}}{}^{\mathcal{B}}=\begin{pmatrix} M_\alpha{}^\beta+\frac{1}{2}\delta_\alpha{}^\beta H & \frac{1}{2}P_{\alpha\dot\beta} \\ \frac{1}{2}K^{\dot\alpha\beta} & \bar{M}^{\dot\alpha}{}_{\dot\beta}-\frac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}H \end{pmatrix}, \quad \mathcal{Q}_{\mathcal{A}}=\begin{pmatrix} Q_\alpha \\ \bar{S}^{\dot\alpha}\end{pmatrix}, \quad \bar{\mathcal{Q}}^{\mathcal{B}}=\begin{pmatrix} S^\beta & \bar{Q}_{\dot\beta}\end{pmatrix}, \tag{A.1}$$

the $SU(2,2|1)$ algebra is expressible as

$$\begin{aligned}
\left[\mathcal{M}_{\mathcal{A}}{}^{\mathcal{B}},\mathcal{M}_{\mathcal{C}}{}^{\mathcal{D}}\right]&=\delta_{\mathcal{C}}{}^{\mathcal{B}}\mathcal{M}_{\mathcal{A}}{}^{\mathcal{D}}-\delta_{\mathcal{A}}{}^{\mathcal{D}}\mathcal{M}_{\mathcal{C}}{}^{\mathcal{B}}\,,\\
\left[\mathcal{M}_{\mathcal{A}}{}^{\mathcal{B}},\mathcal{Q}_{\mathcal{C}}\right]&=\delta_{\mathcal{C}}{}^{\mathcal{B}}\mathcal{Q}_{\mathcal{A}}-\tfrac{1}{4}\delta_{\mathcal{A}}{}^{\mathcal{B}}\mathcal{Q}_{\mathcal{C}}\,,\qquad \left[\mathcal{M}_{\mathcal{A}}{}^{\mathcal{B}},\bar{\mathcal{Q}}^{\mathcal{C}}\right]=-\delta_{\mathcal{A}}{}^{\mathcal{C}}\bar{\mathcal{Q}}^{\mathcal{B}}+\tfrac{1}{4}\delta_{\mathcal{A}}{}^{\mathcal{B}}\bar{\mathcal{Q}}^{\mathcal{C}}\,,\\
\left[R,\mathcal{Q}_{\mathcal{A}}\right]&=-\mathcal{Q}_{\mathcal{A}}\,,\qquad \left[R,\bar{\mathcal{Q}}^{\mathcal{B}}\right]=\bar{\mathcal{Q}}^{\mathcal{B}}\,,\\
\left\{\mathcal{Q}_{\mathcal{A}},\bar{\mathcal{Q}}^{\mathcal{B}}\right\}&=4\mathcal{M}_{\mathcal{A}}{}^{\mathcal{B}}+3\delta_{\mathcal{A}}{}^{\mathcal{B}}R\,,\quad \left\{\mathcal{Q}_{\mathcal{A}},\mathcal{Q}_{\mathcal{B}}\right\}=0\,,\quad \left\{\bar{\mathcal{Q}}^{\mathcal{A}},\bar{\mathcal{Q}}^{\mathcal{B}}\right\}=0\,,
\end{aligned} \tag{A.2}$$

for $\delta_{\mathcal{A}}{}^{\mathcal{B}}=\begin{pmatrix}\delta_\alpha{}^\beta & 0\\ 0 & \delta^{\dot\alpha}{}_{\dot\beta}\end{pmatrix}$. In terms of the usual angular momentum generators we have,

$$\left[M_\alpha{}^\beta\right]=\begin{pmatrix} J_3 & J_+\\ J_- & -J_3\end{pmatrix},\qquad \left[\bar{M}^{\dot\beta}{}_{\dot\alpha}\right]=\begin{pmatrix}\bar{J}_3 & \bar{J}_+\\ \bar{J}_- & -\bar{J}_3\end{pmatrix}, \tag{A.3}$$

with $[J_+,J_-]=2J_3$, $[\bar{J}_+,\bar{J}_-]=2\bar{J}_3$.

A generic highest weight primary state for this superalgebra $|\Delta,r,j,\bar\jmath\rangle^{\text{h.w.}}$, which has conformal dimension Δ, belongs to the spin $SU(2)_J\times SU(2)_{\bar J}$ representation $(j,\bar\jmath)$ and has R-symmetry eigenvalue r, satisfies

$$\begin{aligned}
(K^{\dot\alpha\alpha},S^\alpha,\bar{S}^{\dot\alpha},J_+,\bar{J}_+)|\Delta,r,j,\bar\jmath\rangle^{\text{h.w.}}&=0\,,\\
(H,R,J_3,\bar{J}_3)|\Delta,r,j,\bar\jmath\rangle^{\text{h.w.}}&=(\Delta,r,j,\bar\jmath)|\Delta,r,j,\bar\jmath\rangle^{\text{h.w.}}\,.
\end{aligned} \tag{A.4}$$

[3] The standard hermiticity requirements are

$$\left(\mathcal{M}_{\mathcal{A}}{}^{\mathcal{B}}\right)^\dagger=(\tau\mathcal{M}\tau)_{\mathcal{B}}{}^{\mathcal{A}}\,,\quad R^\dagger=R\,,\quad (\mathcal{Q}_{\mathcal{A}})^\dagger=(\bar{\mathcal{Q}}\tau)^{\mathcal{A}}\,,\quad \tau=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

Thus $H^\dagger=-H$ and $(M_\alpha{}^\beta)^\dagger=\bar{M}^{\dot\beta}{}_{\dot\alpha}$, interchanging $SU(2)_J$ and $SU(2)_{\bar J}$.

The corresponding Verma module $\mathcal{V}_{(\Delta,r,j,\bar\jmath)}$ is then spanned by the states

$$\prod_{\alpha,\dot\alpha,\beta,\dot\beta=1,2} (P_{\alpha\dot\alpha})^{N_{\alpha\dot\alpha}}(Q_\beta)^{n_\beta}(\bar Q_{\dot\beta})^{\bar n_{\dot\beta}}(J_-)^N(\bar J_-)^{\bar N}|\Delta,r,j,\bar\jmath\rangle^{\text{h.w.}}\,, \tag{A.5}$$

for $N_{\alpha\dot\alpha}, N,\ \bar N, = 0,1,2,\dots$ and $n_\beta,\ \bar n_{\dot\beta} = 0,1$.

When BPS conditions involving different supercharges are imposed there are truncated Verma modules and Δ is determined in terms of $r,j,\bar\jmath$, although there may also be various other potential constraints on $r,j,\bar\jmath$. For unitary representations the following conditions are relevant, labelled by $t,\bar t$ according to the fraction of the $Q,\bar Q$ supercharges to be omitted from (A.5),

$$\begin{aligned}
&\bar t = \tfrac12 : \Delta = 2+2\bar\jmath+\tfrac32 r\,,\\
&\Big(\bar Q_1 + \frac{1}{2\bar\jmath}\,\bar Q_2\bar J_-\Big)|\Delta,r,j,\bar\jmath\rangle^{\text{h.w.}} = 0\,,\ \bar\jmath>0\,, \qquad \bar Q^2|\Delta,r,j,0\rangle^{\text{h.w.}} = 0\,, && \text{(A.6}a)\\
&t = \tfrac12 : \Delta = 2+2j-\tfrac32 r\,,\\
&\Big(Q_2 - \frac{1}{2j}\,Q_1 J_-\Big)|\Delta,r,j,\bar\jmath\rangle^{\text{h.w.}} = 0\,,\ j>0\,, \qquad Q^2|\Delta,r,0,\bar\jmath\rangle^{\text{h.w.}} = 0\,, && \text{(A.6}b)
\end{aligned}$$

which are referred to as semi-short [25]. The conditions (A.6a) and (A.6b) are equivalent to the descendant states $\bar Q_2|\Delta,r,j,\bar\jmath\rangle^{\text{h.w.}}$ and $Q_1|\Delta,r,j,\bar\jmath\rangle^{\text{h.w.}}$ being annihilated by $\bar Q_1, \bar S^1$ and Q_2, S^2 respectively. Chiral/anti-chiral short multiplets correspond to the BPS conditions

$$\begin{aligned}
&\bar t = 1 : \Delta = \tfrac32 r\,, \qquad &\bar Q_{\dot\alpha}|\Delta,r,j,0\rangle^{\text{h.w.}} = 0\,,\\
&t = 1 : \Delta = -\tfrac32 r\,, \qquad &Q_\alpha|\Delta,r,0,\bar\jmath\rangle^{\text{h.w.}} = 0\,.
\end{aligned} \tag{A.7}$$

Only if there are BPS conditions requiring both $t,\bar t$ non zero is r and hence Δ fixed and the associated supermultiplet is therefore protected.

When $t,\bar t = \frac12$, Q_2, $\bar Q_1$ are omitted from (A.5), if $t,\bar t = 1$ then Q_α, $\bar Q_{\dot\alpha}$ are removed respectively. As a consequence of $\{Q_\alpha,\bar Q_{\dot\alpha}\} = 2P_{\alpha\dot\alpha}$ then for $t,\bar t$ both non zero particular $P_{\alpha\dot\alpha}$ should be removed from (A.5), thus for $t=\bar t=\frac12$ P_{21} is dropped. The corresponding Verma module is denoted by $\mathcal{V}^{t,\bar t}_{(\Delta,r,j,\bar\jmath)}$.

The Verma modules do not form a basis of physical states for a unitary representation, since in particular the action of $J_-,\bar J_-$ in (A.5) is truncated to ensure positivity of the norm. A space with positive norm $\mathcal{H}^{t,\bar t}_{(\Delta,r,j,\bar\jmath)}$ is constructed from the quotient of corresponding Verma module by zero norm sub-modules if $2j, 2\bar\jmath = 0,1,2,\dots$. For unitary representations we also require $\Delta \ge 2+2\bar\jmath+\frac32 r, 2+2j-\frac32 r$ unless one of the BPS conditions in (A.7) hold and accordingly then $\Delta = \frac32 r$ or $-\frac32 r$. As described in [9,26] the characters corresponding to unitary representations are constructed from the formal Verma module characters by symmetrising under the Weyl group for the maximal compact subgroup of the superconformal group, in this case the spin group $SU(2)_J \times SU(2)_{\bar{J}}$ with Weyl group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The characters for the Verma modules $\mathcal{V}^{t,\bar{t}}_{(\Delta,r,j,\bar{\jmath})}$ are expressed in terms of variables $s, u, x, \bar{x}$ so that in a series expansion of the character the zeroth term is $s^{2\Delta} u^r x^{2j} \bar{x}^{2\bar{\jmath}}$ which corresponds to the contribution from the highest weight state. The states in the Verma module in (A.5) correspond to terms with further factors according to $P_{\alpha\dot{\alpha}} \to s^2 x^{\pm 1} \bar{x}^{\mp 1}$, $Q_\alpha \to s\, u^{-1} x^{\pm 1}$, $\bar{Q}_{\dot{\alpha}} \to s\, u\, \bar{x}^{\mp 1}$, where $\alpha = 1,2$ correspond to x, x^{-1} and $\dot{\alpha} = 1,2$ to $\bar{x}^{-1}, \bar{x}$, respectively. For $t = \bar{t} = 0$ the Verma module character, which is written as a formal trace, is then

$$
\begin{aligned}
C_{(\Delta,r,j,\bar{\jmath})}(s,u,x,\bar{x}) &= \widetilde{\mathrm{tr}}_{\mathcal{V}_{(\Delta,r,j,\bar{\jmath})}} \big(s^{2H}\, u^R\, x^{2J_3}\, \bar{x}^{2\bar{J}_3} \big) \\
&= s^{2\Delta} u^r C_j(x) C_{\bar{\jmath}}(\bar{x}) \sum_{\substack{n_{\varepsilon\eta}=0,1,2,\ldots, \\ \varepsilon,\eta=\pm 1}} (s^2 x^\varepsilon \bar{x}^\eta)^{n_{\varepsilon\eta}} \sum_{\substack{\varepsilon,\eta=\pm 1 \\ n_\varepsilon, \bar{n}_\eta = 0,1}} (s\, u\, x^\varepsilon)^{n_\varepsilon} (s\, u^1\, \bar{x}^\eta)^{\bar{n}_{j\eta}} \\
&= s^{2\Delta} u^r\, C_j(x) C_{\bar{\jmath}}(\bar{x})\, P(s,x,\bar{x})\, \mathcal{Q}(su^{-1},x)\, \mathcal{Q}(su,\bar{x})\,,
\end{aligned} \tag{A.8}
$$

where the factors

$$
P(s,x,\bar{x}) = \prod_{\varepsilon,\eta=\pm 1} \frac{1}{(1 - s^2\, x^\varepsilon\, \bar{x}^\eta)}\,, \qquad \mathcal{Q}(s,x) = \prod_{\varepsilon=\pm 1} (1 + s\, x^\varepsilon)\,, \tag{A.9}
$$

arise from the translation generators and supercharges, and also

$$
C_j(x) = \widetilde{\mathrm{tr}}_{\mathcal{V}_j} \big(x^{2J_3} \big) = \sum_{N=0}^{\infty} x^{2j-2N} = \frac{x^{2j+2}}{x^2 - 1}\,, \tag{A.10}
$$

corresponds to the $SU(2)$ Verma module $\mathcal{V}_j = \{ (J_-)^N |j\rangle^{\text{h.w.}} \}$, $J_3 |j\rangle^{\text{h.w.}} = j |j\rangle^{\text{h.w.}}$, $J_+ |j\rangle^{\text{h.w.}} = 0$. With shortening conditions the corresponding Verma module character $C^{t,\bar{t}}_{(\Delta,r,j,\bar{\jmath})}(s,u,x,\bar{x})$ has various factors in (A.8) omitted in accordance with the above discussion. Since the Weyl group is generated by $x \to x^{-1}, \bar{x} \to \bar{x}^{-1}$ the actual characters for physical unitary irreducible representations are then given by

$$
\begin{aligned}
\chi^{t,\bar{t}}_{(\Delta,r,j,\bar{\jmath})}(s,u,x,\bar{x}) &= \mathrm{tr}_{\mathcal{H}^{t,\bar{t}}_{(\Delta,r,j,\bar{\jmath})}} \big(s^{2H}\, u^R\, x^{2J_3}\, \bar{x}^{2\bar{J}_3} \big) \\
&= \sum_{\varepsilon,\eta=\pm 1} C^{t,\bar{t}}_{(\Delta,r,j,\bar{\jmath})}(s,u,x^\varepsilon,\bar{x}^\eta)\,,
\end{aligned} \tag{A.11}
$$

where we may note that

$$
\chi_n(x) = \sum_{\varepsilon=\pm 1} C_{\frac{1}{2}(n-1)}(x^\varepsilon) = \frac{x^n - x^{-n}}{x - x^{-1}}\,, \tag{A.12}
$$

is the usual character for the familiar n-dimensional $SU(2)$ representation. For the supertrace in (A.11) it is sufficient to let $x, \bar{x} \to -x, -\bar{x}$.

For long multiplets all states in the Verma module (A.5) contribute and (A.8) and (A.11) give,

$$\chi^{0,0}_{(\Delta,r,j,\bar{j})}(s,u,x,\bar{x}) = s^{2\Delta}\, u^r\, \chi_{2j+1}(x)\, \chi_{2\bar{j}+1}(\bar{x})\, P(s,x,\bar{x})\, \mathcal{Q}(su^{-1},x)\, \mathcal{Q}(su,\bar{x})\,, \tag{A.13}$$

For semi-short multiplets we have,

$$\begin{aligned}
&\chi^{0,\frac12}_{(2\bar{j}+2+\frac32 r,r,j,\bar{j})}(s,u,x,\bar{x}) \\
&= s^{4\bar{j}+4+3r}\, u^r\, \chi_{2j+1}(x)\big(\chi_{2\bar{j}+1}(\bar{x}) + s\,u\,\chi_{2\bar{j}+2}(\bar{x})\big)P(s,x,\bar{x})\,\mathcal{Q}(su^{-1},x)\,, \quad r \ge \tfrac23(j-\bar{j})\,, \\
&\chi^{\frac12,0}_{(2j+2+\frac32 r,-r,j,\bar{j})}(s,u,x,\bar{x}) \\
&= s^{4j+4+3r}\, u^{-r}\,\big(\chi_{2j+1}(x) + s\,u^{-1}\,\chi_{2j+2}(x)\big)\chi_{2\bar{j}+1}(\bar{x})\,P(s,x,\bar{x})\,\mathcal{Q}(su,\bar{x})\,, \quad r \ge \tfrac23(\bar{j}-j)\,.
\end{aligned} \tag{A.14}$$

Similarly, for chiral/anti-chiral short multiplets the superconformal characters are,

$$\begin{aligned}
\chi^{0,1}_{(\frac32 r,r,j,0)}(s,u,x,\bar{x}) &= (s^3u)^r\, \chi_{2j+1}(x)\, P(s,x,\bar{x})\, \mathcal{Q}(su^{-1},x)\,, \quad r \ge \tfrac23(j+1)\,, \\
\chi^{1,0}_{(\frac32 r,-r,0,\bar{j})}(s,u,x,\bar{x}) &= (s^3u^{-1})^r\, \chi_{2\bar{j}+1}(\bar{x})\, P(s,x,\bar{x})\, \mathcal{Q}(su,\bar{x})\,, \quad r \ge \tfrac23(\bar{j}+1)\,.
\end{aligned} \tag{A.15}$$

The characters in (A.15) are a special case of those in (A.14) since

$$\begin{aligned}
\chi^{0,\frac12}_{(1+\frac32 r,r,j,-\frac12)}(s,u,x,\bar{x}) &= \chi^{0,1}_{(\frac32(r+1),r+1,j,0)}(s,u,x,\bar{x})\,, \\
\chi^{\frac12,0}_{(1+\frac32 r,-r,-\frac12,\bar{j})}(s,u,x,\bar{x}) &= \chi^{1,0}_{(\frac32(r+1),-r-1,0,\bar{j})}(s,u,x,\bar{x})\,.
\end{aligned} \tag{A.16}$$

The other cases correspond to protected multiplets. The relevant examples are, for a self-conjugate multiplet involving conserved currents,

$$\begin{aligned}
\chi^{\frac12,\frac12}_{(j+\bar{j}+2,\frac23(j-\bar{j}),j,\bar{j})}(s,u,x,\bar{x}) = u^{\frac23(j-\bar{j})}\big(&\mathcal{D}_{j,\bar{j}}(s,x,\bar{x}) + u^{-1}\,\mathcal{D}_{j+\frac12,\bar{j}}(s,x,\bar{x}) \\
&+ u\,\mathcal{D}_{j,\bar{j}+\frac12}(s,x,\bar{x}) + \mathcal{D}_{j+\frac12,\bar{j}+\frac12}(s,x,\bar{x})\big)\,,
\end{aligned} \tag{A.17}$$

where

$$\mathcal{D}_{j,\bar{j}}(s,x,\bar{x}) = s^{2(j+\bar{j}+2)}\big(\chi_{2j+1}(x)\,\chi_{2\bar{j}+1}(\bar{x}) - s^2\chi_{2j}(x)\,\chi_{2\bar{j}}(\bar{x})\big)P(s,x,\bar{x})\,, \tag{A.18}$$

is the conformal group character for a $(j,\bar{j})$ conserved current in four dimensions [26], and the Dirac multiplet, with its conjugate, for which the characters are

$$\begin{aligned}
\chi^{\frac12,1}_{(j+1,\frac23(j+1),j,0)}(s,u,x,\bar{x}) &= u^{\frac23(j+1)}\big(\mathcal{E}_j(s,x,\bar{x}) + u^{-1}\,\mathcal{E}_{j+\frac12}(s,x,\bar{x})\big)\,, \\
\chi^{1,\frac12}_{(\bar{j}+1,-\frac23(\bar{j}+1),0,\bar{j})}(s,u,x,\bar{x}) &= u^{-\frac23(\bar{j}+1)}\big(\overline{\mathcal{E}}_{\bar{j}}(s,x,\bar{x}) + u\,\overline{\mathcal{E}}_{\bar{j}+\frac12}(s,x,\bar{x})\big)\,,
\end{aligned} \tag{A.19}$$

where

$$\begin{aligned}\mathcal{E}_j(s,x,\bar x) &= s^{2j+2}\big(\chi_{2j+1}(x) - s^2\chi_{2j}(x)\,\chi_2(\bar x) + s^4\chi_{2j-1}(x)\big)P(s,x,\bar x)\,,\\ \overline{\mathcal{E}}_{\bar\jmath}(s,x,\bar x) &= s^{2\bar\jmath+2}\big(\chi_{2\bar\jmath+1}(\bar x) - s^2\chi_2(x)\,\chi_{2\bar\jmath}(\bar x) + s^4\chi_{2\bar\jmath-1}(\bar x)\big)P(s,x,\bar x)\,.\end{aligned} \tag{A.20}$$

The characters in (A.19) correspond to spin-j chiral/spin-$\bar\jmath$ anti-chiral free field representations of the conformal group in four dimensions [26].

At the unitarity threshold the multiplets are reducible which is reflected by

$$\begin{aligned}\chi^{0,0}_{(2\bar\jmath+2+\frac32 r,r,j,\bar\jmath)}(s,u,x,\bar x) &= \chi^{0,\frac12}_{(2\bar\jmath+2+\frac32 r,r,j,\bar\jmath)}(s,u,x,\bar x)\ + \chi^{0,\frac12}_{(2\bar\jmath+\frac52+\frac32 r,r+1,j,\bar\jmath-\frac12)}(s,u,x,\bar x)\,,\\ \chi^{0,\frac12}_{(j+\bar\jmath+2,\frac23(j-\bar\jmath),j,\bar\jmath)}(s,u,x,\bar x) &= \chi^{\frac12,\frac12}_{(j+\bar\jmath+2,\frac23(j-\bar\jmath),j,\bar\jmath)}(s,u,x,\bar x)\\ &\quad + \chi^{\frac12,0}_{(j+\bar\jmath+\frac52,\frac23(j-\bar\jmath)-1,j-\frac12,\bar\jmath)}(s,u,x,\bar x)\,,\end{aligned} \tag{A.21}$$

where we may use (A.16) if j or $\bar\jmath$ are zero.

The results for the index in section 2 are equivalent to setting $1+su\bar x=0$ and then letting $s\to 0$ for fixed $t=s^3u$ and x. From (A.19) we obtain

$$\begin{aligned}\chi^{\frac12,1}_{(j+1,\frac23(j+1),j,0)}(s,u,-x,-su)\big|_{t=s^3u} &= (-1)^{2j}\,t^{\frac23(j+1)}\,\frac{\chi_{2j+1}(x)-t\,\chi_{2j}(x)}{(1-tx)(1-tx^{-1})}\,,\\ \chi^{1,\frac12}_{(j+1,-\frac23(j+1),0,j)}(s,u,-x,-su)\big|_{t=s^3u} &\underset{s\to0}{\longrightarrow} -(-1)^{2j}\,\frac{t^{\frac43(j+1)}}{(1-tx)(1-tx^{-1})}\,.\end{aligned} \tag{A.22}$$

The expressions (2.15) and (2.18) correspond just to the sum of the chiral/anti-chiral contributions in (A.22) for $j=0$ and $j=\frac12$ respectively.

For other characters the limit in (A.22) gives just the following non zero results

$$\begin{aligned}\chi^{0,\frac12}_{(2\bar\jmath+2+\frac32 r,r,j,\bar\jmath)}(s,u,-x,-su)\big|_{t=s^3u} &\underset{s\to0}{\longrightarrow} -(-1)^{2j+2\bar\jmath}\,t^{2\bar\jmath+2+r}\,\frac{\chi_{2j+1}(x)}{(1-tx)(1-tx^{-1})}\,,\\ \chi^{0,1}_{(\frac32 r,r,j,0)}(s,u,-x,-su)\big|_{t=s^3u} &\underset{s\to0}{\longrightarrow} (-1)^{2j}\ t^{r}\,\frac{\chi_{2j+1}(x)}{(1-tx)(1-tx^{-1})}\,,\\ \chi^{\frac12,\frac12}_{(j+\bar\jmath+2,\frac23(j-\bar\jmath),j,\bar\jmath)}(s,u,-x,-su)\big|_{t=s^3u} &\underset{s\to0}{\longrightarrow} -(-1)^{2j+2\bar\jmath}\,t^{\frac23(j+2\bar\jmath+3)}\,\frac{\chi_{2j+1}(x)}{(1-tx)(1-tx^{-1})}\,.\end{aligned} \tag{A.23}$$

The expressions in (A.23) are relevant for disentangling contributions of different operators in the expansion of the index in (1.7).

Appendix B. Characters for Unitary, Symplectic and Orthogonal Groups

We here give general results for characters for the groups discussed in the text and verify orthogonality properties in the case of $SU(N)$.

For $SU(n)$ the characters, depending on $\mathrm{x} = (x_1, \ldots, x_n)$ subject to $\prod_{i=1}^n x_i = 1$, are the well known Schur polynomials,

$$s_{\underline{\lambda}}(\mathrm{x}) = s_{(\lambda_1,\ldots,\lambda_n)}(\mathrm{x}) = \frac{\det\left[{x_i}^{\lambda_j+n-j}\right]}{\det\left[{x_i}^{n-j}\right]}\,, \tag{B.1}$$

where we require $\underline{\lambda}$ to be ordered so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and since, as a consequence of the constraint on $\prod_i x_i$, $s_{(\lambda_1,\ldots,\lambda_n)}(\mathrm{x}) = s_{(\lambda_1+c,\ldots,\lambda_n+c)}(\mathrm{x})$ we may also impose $\lambda_n = 0$. In terms of (3.3), $\chi_{SU(n),f}(\mathrm{x}) = s_{(1,0,\ldots,0)}(\mathrm{x})$, $\chi_{SU(n),\bar{f}}(\mathrm{x}) = s_{(1,\ldots,1,0)}(\mathrm{x})$ and $\chi_{SU(n),\mathrm{adj.}}(\mathrm{x}) = s_{(2,1,\ldots,1,0)}(\mathrm{x})$. For the Vandermonde determinant in (6.2),

$$\Delta(\mathrm{x}) = \det\left[{x_i}^{n-j}\right]\,. \tag{B.2}$$

As a consistency check we may verify orthogonality of Schur polynomials $s_{\underline{\lambda}}(\mathrm{x}), s_{\underline{\lambda}'}(\mathrm{x})$, where both $\underline{\lambda}, \underline{\lambda}'$ are ordered, with respect to the measure (6.1)

$$\begin{aligned} &\int_{SU(n)} \mathrm{d}\mu(\mathrm{x})\, s_{\underline{\lambda}}(\mathrm{x})\, s_{\underline{\lambda}'}(\mathrm{x}) = \frac{1}{n!}\int \prod_{i=1}^{n-1} \frac{\mathrm{d}x_i}{2\pi i x_i}\, \Delta(\mathrm{x})\Delta(\mathrm{x}^{-1})\, s_{\underline{\lambda}}(\mathrm{x})\, s_{\underline{\lambda}'}(\mathrm{x})\Big|_{\prod_{i=1}^n x_i=1} \\ &= \int \left(\prod_{i=1}^{n-1} \frac{\mathrm{d}x_i}{2\pi i x_i}\, {x_i}^{-\lambda_i-n+i}\right) \sum_{\sigma\in\mathcal{S}_n} \mathrm{sign}(\sigma) \prod_{j=1}^{n} (\sigma x_j)^{\lambda'_j+n-j}\Big|_{\prod_{i=1}^n x_i=1} = \delta_{\underline{\lambda},\underline{\lambda}'}\,, \end{aligned} \tag{B.3}$$

where the sum is over $n!$ permutations σ, $\sigma x_j = x_{j^\sigma}$, belonging to $\mathcal{S}_n$ the Weyl group for $SU(n)$. The only non zero term surviving the integration in (B.3) is then for $\sigma = e$, the identity, and only when $\underline{\lambda} = \underline{\lambda}'$.

The Weyl characters for $Sp(2n)$ are also given by the determinantal formula,

$$\widetilde{s}_{(\lambda_1,\ldots,\lambda_n)}(\mathrm{x}) = \frac{\det\left[{x_i}^{\lambda_j+n-j+1} - {x_i}^{-\lambda_j-n+j-1}\right]}{\det\left[{x_i}^{n-j+1} - {x_i}^{-n+j-1}\right]}\,, \tag{B.4}$$

with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. The results in (7.2) correspond to $\chi_{Sp(2n),f}(\mathrm{x}) = \widetilde{s}_{(1,0,\ldots,0)}(\mathrm{x})$, $\chi_{Sp(2n),\mathrm{adj.}}(\mathrm{x}) = \widetilde{s}_{(2,0,\ldots,0)}(\mathrm{x})$. For the denominator in (B.4)

$$\det\left[{x_i}^{n-j+1} - {x_i}^{-n+j-1}\right] = \Delta(\mathrm{x}+\mathrm{x}^{-1}) \prod_{i=1}^{n} (x_i - {x_i}^{-1})\,. \tag{B.5}$$

For $N = 2n$ the characters for $SO(N)$ are given by

$$\hat{s}_{\underline{\lambda}}(\mathrm{x}) = \frac{\det\left[{x_i}^{\lambda_j+n-j} + {x_i}^{-\lambda_j-n+j}\right] + \det\left[{x_i}^{\lambda_j+n-j} - {x_i}^{-\lambda_j-n+j}\right]}{2\,\Delta(\mathrm{x}+\mathrm{x}^{-1})}\,, \tag{B.6}$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq |\lambda_n| \geq 0$ and $\chi_{SO(2n),\mathrm{adj.}}(\mathrm{x}) = \hat{s}_{(1,1,0,\ldots,0)}(\mathrm{x})$. For $N = 2n+1$,

$$\bar{s}_{\underline{\lambda}}(\mathrm{x}) = \frac{\det\left[{x_i}^{\lambda_j+\frac{1}{2}+n-j} + {x_i}^{-\lambda_j-\frac{1}{2}-n+j}\right]}{\Delta(\mathrm{x}+\mathrm{x}^{-1})\, \prod_{i=1}^n ({x_i}^{\frac{1}{2}} - {x_i}^{-\frac{1}{2}})}\,. \tag{B.7}$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$ and $\chi_{SO(2n+1),\mathrm{adj.}}(\mathrm{x}) = \bar{s}_{(1,1,0,\ldots,0)}(\mathrm{x})$.

Appendix C. Finite N Corrections

In section 4 we discussed the leading large N expressions for the index, here we discuss the form of the leading corrections which involve contributions from operators with non zero baryon number. The expansion of the integral defining the index generates power symmetric polynomials $p_{\underline{a}}(\mathrm{z})$ in $\mathrm{z} = (z_1, z_2, \ldots)$ as defined in (4.2). We follow a method described in [16] which relates them to the symmetric Schur polynomials, as defined in (B.1),

$$s_{\underline{\lambda}}(\mathrm{z})\,, \quad \text{where} \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\underline{\lambda})} \geq 1\,, \quad \lambda_{\ell(\underline{\lambda})+1} = 0\,. \tag{C.1}$$

The Schur polynomials are characters of $SU(N)$ when z has N components and $\ell(\underline{\lambda}) \leq N$. In this case also

$$\textstyle\prod_{i=1}^{N} z_i = 1 \quad \Rightarrow \quad s_{\underline{\lambda}}(\mathrm{z}) = s_{\underline{\lambda}+\underline{\rho}_N}(\mathrm{z})\,, \quad \underline{\rho}_N = (1,1,\ldots,1)\,, \ \ell(\underline{\rho}_N) = N\,. \tag{C.2}$$

The power and Schur symmetric polynomials are related by

$$p_{\underline{a}}(\mathrm{z}) = \sum_{\substack{\underline{\lambda} \\ \ell(\underline{\lambda}) \leq N}} \omega_{\underline{a}}{}^{\underline{\lambda}}\, s_{\underline{\lambda}}(\mathrm{z})\,, \qquad |\underline{a}| = |\underline{\lambda}| = \textstyle\sum_n \lambda_n\,. \tag{C.3}$$

The coefficients $\omega_{\underline{a}}{}^{\underline{\lambda}}$ are characters for the symmetric group and they satisfy the completeness relations

$$\sum_{\underline{\lambda}} \omega_{\underline{a}}{}^{\underline{\lambda}}\, \omega_{\underline{b}}{}^{\underline{\lambda}} = z_{\underline{a}}\, \delta_{\underline{a},\underline{b}}\,, \tag{C.4}$$

for $z_{\underline{a}}$ as in (4.4), and (C.3) can be inverted giving

$$s_{\underline{\lambda}}(\mathrm{z}) = \sum_{\underline{a}} \frac{1}{z_{\underline{a}}}\, \omega_{\underline{a}}{}^{\underline{\lambda}}\, p_{\underline{a}}(\mathrm{z})\,. \tag{C.5}$$

The orthogonality relation (B.3) can be extended to, as a consequence of (C.2),

$$\int_{SU(N)} \mathrm{d}\mu(\mathrm{z})\, s_{\underline{\lambda}}(\mathrm{z})\, s_{\underline{\lambda}'}(\mathrm{z}) \bigg|_{\ell(\underline{\lambda}),\ell(\underline{\lambda}')\leq N} = \delta_{\underline{\lambda}',\underline{\lambda}} + \textstyle\sum_{n=1}^{\infty} \big(\delta_{\underline{\lambda}',\underline{\lambda}+n\underline{\rho}_N} + \delta_{\underline{\lambda}'+n\underline{\rho}_N,\underline{\lambda}} \big)\,. \tag{C.6}$$

Hence

$$\int_{SU(N)} \mathrm{d}\mu(\mathrm{z})\, p_{\underline{a}}(\mathrm{z})\, p_{\underline{b}}(\mathrm{z}^{-1}) = \sum_{\substack{\underline{\lambda} \\ \ell(\underline{\lambda}) \leq N}} \Big(\omega_{\underline{a}}{}^{\underline{\lambda}}\, \omega_{\underline{b}}{}^{\underline{\lambda}} + \textstyle\sum_{n=1}^{\infty} \big(\omega_{\underline{a}}{}^{\underline{\lambda}}\, \omega_{\underline{b}}{}^{\underline{\lambda}+n\underline{\rho}_N} + \omega_{\underline{a}}{}^{\underline{\lambda}+n\underline{\rho}_N}\, \omega_{\underline{b}}{}^{\underline{\lambda}} \big) \Big)\,. \tag{C.7}$$

We now consider applying these results to the integral (4.5) for the index, where $i(\mathrm{t},\mathrm{z})$ is given by (4.1) but assuming here for simplicity $h(\mathrm{t}) = f(\mathrm{t})$ (otherwise there is an additional overall factor as in (4.7)). Hence the integral becomes

$$
\begin{aligned}
\mathcal{I}(\mathrm{t}) &= \int_{SU(N)} \mathrm{d}\mu(\mathrm{z}) \, \exp\bigg(\sum_{n=1}^{\infty} \frac{1}{n} \Big(f(\mathrm{t}^n) p_N(\mathrm{z}^n) p_N(\mathrm{z}^{-n}) + g(\mathrm{t}^n) p_N(\mathrm{z}^n) + \bar{g}(\mathrm{t}^n) p_N(\mathrm{z}^{-n}) \Big) \bigg) \\
&= \sum_{\underline{a},\underline{b},\underline{\bar{b}}} \frac{1}{z_{\underline{a}}\, z_{\underline{b}}\, z_{\underline{\bar{b}}}} \; f_{\underline{a}}(\mathrm{t})\, g_{\underline{b}}(\mathrm{t})\, \bar{g}_{\underline{\bar{b}}}(\mathrm{t}) \int_{SU(N)} \mathrm{d}\mu(\mathrm{z}) \; p_{\underline{a}+\underline{b}}(\mathrm{z})\, p_{\underline{a}+\underline{\bar{b}}}(\mathrm{z}^{-1}) \,,
\end{aligned}
\tag{C.8}
$$

with the definitions (4.4) and also

$$
f_{\underline{a}}(\mathrm{t}) = \prod_{n\geq 1} f(\mathrm{t}^n)^{a_n} \,, \quad g_{\underline{b}}(\mathrm{t}) = \prod_{n\geq 1} g(\mathrm{t}^n)^{b_n} \,, \quad \bar{g}_{\underline{\bar{b}}}(\mathrm{t}) = \prod_{n\geq 1} \bar{g}(\mathrm{t}^n)^{\bar{b}_n} \,. \tag{C.9}
$$

The integral in (C.7) ensures

$$
\begin{aligned}
\mathcal{I}(\mathrm{t}) = & \sum_{\underline{a},\underline{b},\underline{\bar{b}}} \frac{1}{z_{\underline{a}}\, z_{\underline{b}}\, z_{\underline{\bar{b}}}} \; f_{\underline{a}}(\mathrm{t})\, g_{\underline{b}}(\mathrm{t})\, \bar{g}_{\underline{\bar{b}}}(\mathrm{t}) \\
& \times \sum_{\substack{\underline{\lambda} \\ \ell(\underline{\lambda})\leq N}} \Big(\omega_{\underline{a}+\underline{b}}{}^{\underline{\lambda}} \; \omega_{\underline{a}+\underline{\bar{b}}}{}^{\underline{\lambda}} + \textstyle\sum_{n=1}^{\infty} \big(\omega_{\underline{a}+\underline{b}}{}^{\underline{\lambda}} \; \omega_{\underline{a}+\underline{\bar{b}}}{}^{\underline{\lambda}+n\underline{\rho_N}} + \omega_{\underline{a}+\underline{b}}{}^{\underline{\lambda}+n\underline{\rho_N}} \; \omega_{\underline{a}+\underline{\bar{b}}}{}^{\underline{\lambda}} \big) \Big) \,.
\end{aligned}
\tag{C.10}
$$

Using the completeness relation (C.4) the leading term in (C.10) gives, essentially as in (4.8),

$$
\mathcal{I}_0(\mathrm{t}) = \sum_{\underline{a},\underline{b}} \frac{z_{\underline{a}+\underline{b}}}{z_{\underline{a}}\, {z_{\underline{b}}}^2} \, f_{\underline{a}}(\mathrm{t})\, g_{\underline{b}}(\mathrm{t})\, \bar{g}_{\underline{b}}(\mathrm{t}) = \exp\bigg(\sum_{n=1}^{\infty} \frac{1}{n} \frac{g(\mathrm{t}^n)\, \bar{g}(\mathrm{t}^n)}{1 - f(\mathrm{t}^n)} \bigg) \prod_{n=1}^{\infty} \frac{1}{1 - f(\mathrm{t}^n)} \,. \tag{C.11}
$$

The result (C.10) then shows that

$$
\mathcal{I}(\mathrm{t}) = \mathcal{I}_0(\mathrm{t}) + \mathcal{I}_1(\mathrm{t}) \,, \tag{C.12}
$$

where

$$
\begin{aligned}
\mathcal{I}_1(\mathrm{t}) = & \sum_{\underline{a},\underline{b},\underline{\bar{b}}} \frac{1}{z_{\underline{a}}\, z_{\underline{b}}\, z_{\underline{\bar{b}}}} \; f_{\underline{a}}(\mathrm{t})\, g_{\underline{b}}(\mathrm{t})\, \bar{g}_{\underline{\bar{b}}}(\mathrm{t}) \\
& \times \bigg(\sum_{\substack{\underline{\lambda} \\ \ell(\underline{\lambda})\leq N}} \Big(\textstyle\sum_{n=1}^{\infty} \big(\omega_{\underline{a}+\underline{b}}{}^{\underline{\lambda}} \; \omega_{\underline{a}+\underline{\bar{b}}}{}^{\underline{\lambda}+n\underline{\rho_N}} + \omega_{\underline{a}+\underline{b}}{}^{\underline{\lambda}+n\underline{\rho_N}} \; \omega_{\underline{a}+\underline{\bar{b}}}{}^{\underline{\lambda}} \big) \Big) - \sum_{\substack{\underline{\lambda} \\ \ell(\underline{\lambda})> N}} \omega_{\underline{a}+\underline{b}}{}^{\underline{\lambda}} \; \omega_{\underline{a}+\underline{\bar{b}}}{}^{\underline{\lambda}} \bigg) \,.
\end{aligned}
\tag{C.13}
$$

Any sub-leading terms for large N may then be extracted from the expression (C.13) for $\mathcal{I}_1(\mathrm{t})$. The first non zero term arises for $n=1$ and $\underline{\lambda}=\underline{0}$ when (C.13) reduces to

$$\mathcal{I}_1(\mathrm{t}) \sim \sum_{\underline{b}} \frac{1}{z_{\underline{b}}} \big(g_{\underline{b}}(\mathrm{t}) + \bar{g}_{\underline{b}}(\mathrm{t}) \big)\, \omega_{\underline{b}}{}^{\rho_N} \quad , \qquad |\underline{b}| = N \, . \tag{C.14}$$

We consider here the application of (C.14) to the Seiberg and Kutasov-Schwimmer dual theories, extending the discussion in section 4. Thus we take $N=N_c$ and $N=\tilde{N}_c$ and use the leading results for $g_E(tx,tx^{-1},v,\mathrm{y},\tilde{\mathrm{y}})$, $\bar{g}_E(tx,tx^{-1},v,\mathrm{y},\tilde{\mathrm{y}})$, which are proportional to t^r, from (4.19) and also $g_M(tx,tx^{-1},v,\mathrm{y},\tilde{\mathrm{y}})$ and $\bar{g}_M(tx,tx^{-1},v,\mathrm{y},\tilde{\mathrm{y}})$, which are proportional to t^{2s-r}, from (4.20). This gives

$$\begin{aligned} I_{E,1}(tx,tx^{-1},v,\mathrm{y},\tilde{\mathrm{y}}) &\sim t^{N_c r} \sum_{\underline{b}} \frac{1}{z_{\underline{b}}} \big(v^{\,N_c} p_{\underline{b}}(\mathrm{y}) + v^{-N_c}\, p_{\underline{b}}(\tilde{\mathrm{y}}^{-1}) \big)\, \omega_{\underline{b}}{}^{\rho_{N_c}} \quad , \\ I_{M,1}(tx,tx^{-1},v,\mathrm{y},\tilde{\mathrm{y}}) &\sim t^{\tilde{N}_c(2s-r)} \sum_{\underline{b}} \frac{1}{z_{\underline{b}}} \big(\tilde{v}^{\,\tilde{N}_c} p_{\underline{b}}(\mathrm{y}^{-1}) + \tilde{v}^{-\tilde{N}_c}\, p_{\underline{b}}(\tilde{\mathrm{y}}) \big)\, \omega_{\underline{b}}{}^{\rho_{\tilde{N}_c}} \quad , \end{aligned} \tag{C.15}$$

with $p_{\underline{b}}(\mathrm{y})$ defined as in (C.9).

For Seiberg dual theories then $k=1$, $s=\frac{1}{2}$ and from (3.1), (3.5) and (3.7) the results in (C.15) are proportional to $t^{N_c\tilde{N}_c/N_f} v^{N_c}$ for both electric and magnetic cases. The dependence on $\mathrm{y},\tilde{\mathrm{y}}$ is also compatible using (C.5)

$$\sum_{\underline{b}} \frac{1}{z_{\underline{b}}}\, p_{\underline{b}}(\mathrm{y})\ \omega_{\underline{b}}{}^{\rho_{N_c}} = s_{(1^{N_c})}(\mathrm{y}) = \sum_{\underline{b}} \frac{1}{z_{\underline{b}}}\, p_{\underline{b}}(\mathrm{y}^{-1})\ \omega_{\underline{b}}{}^{\rho_{\tilde{N}_c}} = s_{(1^{\tilde{N}_c})}(\mathrm{y}^{-1}) \, , \tag{C.16}$$

assuming $\prod_i y_i = 1$ and $N_c, \tilde{N}_c \le N_f$.

For Kutasov-Schwimmer dual theories $k=2,3,\dots$ and $\tilde{N}_c$ is as in (3.8) and r,s are given by (3.10). For this case

$$(k+1)N_f\big(N_c r - \tilde{N}_c(2s-r)\big) = (k-1)N_f(kN_f - 2N_c) \, , \quad kN_f - 2N_c = 2\tilde{N}_c - kN_f \, . \tag{C.17}$$

In consequence the powers of t in (C.15) do not match. If $kN_f - 2N_c < 0$ then we must have $N_c > N_f$ and then

$$\sum_{\underline{b}} \frac{1}{z_{\underline{b}}}\, p_{\underline{b}}(\mathrm{y})\ \omega_{\underline{b}}{}^{\rho_{N_c}} = 0 \, , \tag{C.18}$$

so that the leading contribution to $I_{E,1}(tx,tx^{-1},v,\mathrm{y},\tilde{\mathrm{y}})$ in (C.15) vanishes. Conversely if $kN_f - 2N_c > 0$ we must have $\tilde{N}_c > N_f$ and the leading contribution to $I_{M,1}(tx,tx^{-1},v,\mathrm{y},\tilde{\mathrm{y}})$ is absent. In consequence there is no manifest inconsistency between I_E and I_M beyond the large N limit and so perhaps further evidence for matching of the index for the electric and magnetic dual Kutasov-Schwimmer theories.

Appendix D. Useful Identities

We here note some useful properties of the elliptic Gamma functions and other infinite products defined by (1.9) and (1.12). As well as (6.8) we may also define

$$\begin{aligned} (x_1,\ldots,x_n;q) &= (x_1;q)\cdots(x_n;q)\,, \\ \theta(x_1,\ldots,x_n;q) &= \theta(x_1;q)\cdots\theta(x_n;q)\,, \end{aligned} \tag{D.1}$$

for $(x;q), \theta(x;q)$ in (1.12). Useful identities are

$$(x;q) = (x;q^2)\,(xq;q^2)\,, \qquad (x;q^2) = (\sqrt{x};q)\,(-\sqrt{x};q)\,, \tag{D.2}$$

which extend also to $\theta(x;q)$. For the latter we may also note

$$\theta(qx;q) = \theta(x^{-1};q) = -\frac{1}{x}\,\theta(x;q)\,. \tag{D.3}$$

In terms of standard Jacobi theta functions $(q^2,q^2)\,\theta(qe^{2iu},q^2) = \vartheta_4(u,q)$. The Jacobi product identity is equivalent to

$$(q;q)\,\theta(x;q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} x^n\,, \tag{D.4}$$

while the addition formula in the form

$$a\,\theta(ba,ba^{-1},cz,cz^{-1};p) + b\,\theta(cb,cb^{-1},ca,ca^{-1};p) + c\,\theta(ac,ac^{-1},bz,bz^{-1};p) = 0\,, \tag{D.5}$$

with notation as in (D.1), is significant later.

For the elliptic gamma function, properties which prove useful are, besides the reflection formula (5.9),

$$\Gamma(xq;p,q) = \theta(x;p)\,\Gamma(x;p,q)\,, \qquad \Gamma(xp;p,q) = \theta(x;q)\,\Gamma(x;p,q)\,, \tag{D.6}$$

and

$$\Gamma(p;p,q) = (q;q)/(p;p)\,, \qquad \Gamma(q;p,q) = (p;p)/(q;q)\,, \tag{D.7}$$

and

$$\Gamma(-1;p,q) = \frac{1}{2(-q;q)\,(-p;p)}\,, \tag{D.8}$$

so that, using also (D.2),

$$\Gamma(-1;p,q)\,\Gamma(-p;p,q) = \tfrac{1}{2}(p,p^2)^2\,, \qquad \Gamma(-1;p,q)\,\Gamma(-q;p,q) = \tfrac{1}{2}(q,q^2)^2\,. \tag{D.9}$$

With the notation in (6.8) we have

$$\Gamma(z,z^{-1};p,q) = \frac{1}{\theta(z;q)\,\theta(z^{-1};p)}\,, \tag{D.10}$$

which may be rewritten in various forms with the aid of (D.3).

Appendix E. Verification of the Spiridonov Elliptic Beta Integral

We here describe an approach to showing $\mathcal{A}(p,q,\hat{\mathrm{u}}) = \mathcal{B}(p,q,\hat{\mathrm{u}})$, as defined in (5.11) and (5.12), analogous to that outlined for the Nassrallah-Rahman theorem in section 5. From its definition in (5.10) and appendix D, with the notation in (D.1), we have that

$$\hat{\mathcal{I}}(p,q,qu_1,u_2,\dots,z) = \frac{\theta(u_1 z, u_1/z;p)}{\theta(\lambda z,\lambda/z;p)}\,\hat{\mathcal{I}}(p,q,\hat{\mathrm{u}},z)\,, \tag{E.1}$$

so that using the identity, which follows from (D.5) and (D.3),

$$u_2\theta(u_1z,u_1/z,\lambda u_2,\lambda/u_2;p) - u_1\theta(u_2z,u_2/z,\lambda u_1,\lambda/u_1;p) = -u_1\theta(u_1u_2,u_2/u_1,\lambda z,\lambda/z;p)\,, \tag{E.2}$$

(for $p=0$ this reduces to (5.17)) we find that $\hat{\mathcal{I}}(p,q,\hat{\mathrm{u}},z)$ satisfies the q-difference relation

$$\begin{aligned} & u_2\theta(\lambda u_2,\lambda/u_2;p)\,\hat{\mathcal{I}}(p,q,qu_1,u_2,\dots,z) - u_1\theta(\lambda u_1,\lambda/u_1;p)\,\hat{\mathcal{I}}(p,q,u_1,qu_2,\dots,z) \\ & = -u_1\theta(u_1u_2,u_2/u_1;p)\,\hat{\mathcal{I}}(p,q,\hat{\mathrm{u}},z)\,. \end{aligned} \tag{E.3}$$

Since this holds for any z the q-difference relation extends to $\mathcal{A}(p,q,\hat{\mathrm{u}})$.

Similarly,

$$\mathcal{B}(p,q,qu_1,u_2,\dots,u_5) = \prod_{a=1}^{5}\frac{\theta(u_1u_a;p)}{\theta(\lambda/u_a;p)}\,\mathcal{B}(p,q,\hat{\mathrm{u}})\,, \tag{E.4}$$

so that using the identity, which is also equivalent to (D.5),

$$\begin{aligned} & u_2\theta(u_1u_3,u_1u_4,u_1u_5,\lambda u_2;p) - u_1\theta(u_2u_3,u_2u_4,u_2u_5,\lambda u_1;p) \\ & = -u_1\theta(u_2/u_1,\lambda/u_3,\lambda/u_4,\lambda/u_5;p)\,, \end{aligned} \tag{E.5}$$

assuming λ as in (5.10), it is easy to show that $\mathcal{B}(p,q,\hat{\mathrm{u}},z)$ also satisfies (E.3).

The proof is now essentially the same as that described in section 5. $\mathcal{A}(p,q,\hat{\mathrm{u}})$, $\mathcal{B}(p,q,\hat{\mathrm{u}})$ are both are analytic functions in each u_a so it is sufficient to show that they are equal for a particular non zero choice of $\hat{\mathrm{u}}$ and use the q-difference relation to extend this to an infinite discrete set of $\hat{\mathrm{u}}$ which then, by analyticity, implies $\mathcal{A}(p,q,\hat{\mathrm{u}}) = \mathcal{B}(p,q,\hat{\mathrm{u}})$ for arbitrary $\hat{\mathrm{u}}$ so long as both are non singular.

We then consider the same special case chosen for proving the Nassrallah-Rahman theorem, $\hat{\mathrm{u}}_0 = (u,1,-1,q^{\frac12},-q^{\frac12})$. For $\hat{\mathcal{I}}$ as in (5.10) we have

$$\hat{\mathcal{I}}(p,q,\hat{\mathrm{u}}_0,z) = -z^2\,\frac{1}{\theta(z^2;q)\,\theta(z^2;p^2)\,\theta(uz,u/z;p)}\,, \tag{E.6}$$

and in (5.11),

$$\mathcal{A}(p,q,\hat{\mathrm{u}}_0) = (p;p)\,(q;q)\,\frac{1}{4\pi i}\oint \frac{\mathrm{d}z}{z}\,\frac{(z^2p;p^2)\,(z^{-2}p;p^2)}{\theta(uz,u/z;p)}\,. \tag{E.7}$$

Using (D.7), (D.8) and (D.9), we may show, for $\mathcal{B}$ as in (5.12), that

$$\mathcal{B}(p,q,\hat{\mathrm{u}}_0) = \frac{(q;q)}{(p;p)}\,\frac{1}{2\theta(u^2;p^2)}\,. \tag{E.8}$$

Thus to show equality of (E.7) and (E.8) it is necessary to verify that

$$\mathcal{F}(u,p) = \frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z}\,\mathcal{I}_0(u,z,p) = \frac{1}{(p;p)^2}\,\frac{1}{\theta(u^2;p^2)}\,, \quad \mathcal{I}_0(u,z,p) = \frac{\theta(z^2p;p^2)}{\theta(uz,u/z;p)}\,, \tag{E.9}$$

where, requiring $p < |u| < 1$, the z-integration is around the unit circle.

Spiridonov [14] evaluated the integral in (E.9) by using rather non trivial identities. We here present a simpler argument. The integrand $\mathcal{I}_0(u,z,p)$ has poles inside the contour $|z| = 1$ at $z = up^n, p^{n+1}/u$ and outside at $z = p^{-n}/u, up^{-n-1}$, for $n = 0,1,2,\ldots$, and satisfies, from (D.3),

$$\mathcal{I}_0(pu,z,p) = u^2\mathcal{I}_0(u,z,p)\,. \tag{E.10}$$

If we let $u \to pu$ (E.10) would naively imply that a similar relation holds for $\mathcal{F}(u,p)$ but under this change the pole at $z = p/u$ moves outside the contour while the one at $z = u/p$ moves inside. Taking into account the contributions of these poles we get

$$\mathcal{F}(pu,p) = u^2\mathcal{F}(u,p) + \frac{2}{(p;p)^2}\,\frac{1}{\theta(u^2p^2;p^2)} = u^2\bigg(\mathcal{F}(u,p) - \frac{2}{(p;p)^2}\,\frac{1}{\theta(u^2;p^2)}\bigg)\,. \tag{E.11}$$

The form of the integral in (E.9) shows that $\mathcal{F}(u,p)$ has poles solely at $u^2 = p^r$ for some positive or negative integer r, then (E.11) and analyticity implies that $\mathcal{F}(u,p)$ can only have the form given by the result in (E.9).

References

[1] N. Seiberg, *Electric-Magnetic Duality in Supersymmetric Non-Abelian Gauge Theories*, Nucl. Phys. B435 (1995) 129, hep-th/9411149.

[2] J. Terning, *Modern Supersymmetry*, Clarendon Press, Oxford 2006.

[3] K.A. Intriligator, *New RG Fixed Points And Duality In Supersymmetric* $Sp(N_c)$ *And* $SO(N_c)$ *Gauge Theories*, Nucl. Phys. B448 (1995) 187, hep-th/9505051.

[4] D. Kutasov, *A Comment on Duality in* $N = 1$ *Supersymmetric Non-Abelian Gauge Theories*, Phys. Lett. B 351 (1995) 230, hep-th/9503086.

[5] D. Kutasov and A. Schwimmer, *On Duality in Supersymmetric Yang-Mills Theory*, Phys. Lett. B 354 (1995) 315, hep-th/9505004.

[6] D. Kutasov, A. Schwimmer and N. Seiberg, *Chiral Rings, Singularity Theory and Electric-Magnetic Duality*, Nucl. Phys. B459 (1996) 455, hep-th/9510222.

[7] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An Index for* 4 *Dimensional Super Conformal Theories*, Comm. Math. Phys. 275 (2007) 209, hep-th/0510251.

[8] R.A. Janik and M. Trzetrzelew, *Supergravitons from one loop perturbative* $N = 4$ *SYM*, arXiv:0712.2714 [hep-th].

[9] M. Bianchi, F.A. Dolan, P.J. Heslop and H. Osborn, $\mathcal{N} = 4$ *Superconformal Characters and Partition Functions*, Nucl. Phys. B767 (2007) 163, hep-th/0609179.

[10] C. Römelsberger, *Counting chiral primaries in* $\mathcal{N} = 1$, $d = 4$ *superconformal field theories*, Nucl. Phys. B747 (2006) 329, hep-th/0510060.

[11] C. Römelsberger, *Calculating the Superconformal Index and Seiberg Duality*, arXiv:0707.3702 [hep-th].

[12] B. Feng, A. Hanany, Y.H. He, *Counting Gauge Invariants: The Plethystic Program*, JHEP (2007) 0703:090, hep-th/0701063.

[13] B. Nassrallah and M. Rahman, *Projection formulas, a reproducing kernel and a generating function for q-Wilson polynomials*, SIAM J. Math. Anal. 16 (1985) 186; M. Rahman, *An integral representation of a* ${}_{10}\varphi_9$ *and continuous biorthogonal* ${}_{10}\varphi_9$ *rational functions*, Can. J. Math. 38 (1986) 605.

[14] V.P. Spiridonov, *On the Elliptic Beta Function*, Russian Math. Surveys (2001) 56, 185;
An Elliptic Beta Integral, in Proceedings of the Fifth International Conference on Difference Equations, ed. S. Elaydi, Taylor & Francis, London, 2002.

[15] E.M. Rains, *Transformations of Elliptic Hypergeometric Integrals*, Ann. Math., to appear, math/0309252.

[16] F.A. Dolan, *Counting BPS operators in $\mathcal{N}=4$ SYM*, Nucl. Phys. B790 (2008) 432, arXiv:0704.1038 [hep-th].

[17] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 2004.

[18] R. Askey, *Beta Integrals in Ramanujan's Papers*, Ramanujan Revisited, Academic Press, London 1988, pp. 561-590.

[19] V.P. Spiridonov, *Short proofs of elliptic beta integrals*, Ramanujan Journal 13 (2007) 265, math/0408369 [math.CA];
E.M. Rains and V.P. Spiridonov, *Determinants of elliptic hypergeometric integrals*, arXiv:0712.4253 [math.CA].

[20] K.A. Intriligator and P. Pouliot, *Exact Superpotentials, Quantum Vacua and Duality in Supersymmetric $Sp(N_c)$ Gauge Theories*, Phys. Lett. B353 (1995) 471, hep-th/9505006.

[21] K. Intriligator and N. Seiberg, *Duality, Monopoles, Dyons, Confinement and Oblique Confinement in Supersymmetric $SO(N_c)$ Gauge Theories*, Nucl. Phys. B444 (1995) 125, hep-th/9503179.

[22] E.M. Rains, private communication.

[23] V.P. Spiridonov. *Classical Elliptic Hypergeometric Functions and Their Applications*, Rokko Lectures in Mathematics, vol.18, Elliptic Integrable Systems, ed. M. Noumi and K. Takasaki (2005),arXiv:math/0511579 [math.CA].

[24] J. Distler and A. Karch, *N=1 Dualities for Exceptional Gauge Groups and Quantum Global Symmetries*, Fortsch. Phys. 45 (1997) 517, hep-th/9611088;
A. Karch, *More on N=1 Dualities and Exceptional Gauge Groups*, Phys. Lett. B405 (1997) 280, hep-th/9702179.

[25] F.A. Dolan and H. Osborn, *On short and semi-short representations for four-dimensional superconformal symmetry*, Ann. Phys. 307 (2003) 41, hep-th/0209056.

[26] F.A. Dolan, *Character Formulae and Partition Functions in Higher Dimensional Conformal Field Theory*, J. Math. Phys. 47 (2006) 062303, hep-th/0508031.