Preprint typeset in JHEP style - PAPER VERSION

TIFR/TH/07-33
DIAS-STP-07-20
KEK-TH-1104
arXiv:0712.0646

The instability of intersecting fuzzy spheres

Takehiro Azuma[a,b], **Subrata Bal**[c] **and Jun Nishimura**[a,d]

[a] *Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), 1-1 Oho, Tsukuba 305-0801, Japan*
[b] *Department of Theoretical Physics, Tata Institute of Fundamental Research (TIFR), Homi Bhabha Road, Mumbai, 400-005, India*
[c] *School of Theoretical Physics, Dublin Institute for Advanced Studies (DIAS), 10 Burlington Road, Dublin 8, Ireland*
[d] *Department of Particle and Nuclear Physics, Graduate University for Advanced Studies (SOKENDAI), 1-1 Oho, Tsukuba 305-0801, Japan*
`azuma@theory.tifr.res.in, sbal@stp.dias.ie, jnishi@post.kek.jp`

Abstract: We discuss the classical and quantum stability of general configurations representing many fuzzy spheres in dimensionally reduced Yang-Mills-Chern-Simons models with and without supersymmetry. By performing one-loop perturbative calculations around such configurations, we find that intersecting fuzzy spheres are classically unstable in the class of models studied in this paper. We also discuss the large-N limit of the one-loop effective action as a function of the distance of fuzzy spheres. This shows, in particular, that concentric fuzzy spheres with different radii, which are identified with the 't Hooft-Polyakov monopoles, are perturbatively stable in the bosonic model and in the $D = 10$ supersymmetric model.

Keywords: Matrix Models, Non-Commutative Geometry, Chern-Simons Theories.

Contents

1. Introduction

Fuzzy spheres [1] are simple compact noncommutative manifolds and have been studied extensively from various motivations. First it is expected that the noncommutative geometry provides a crucial link to string theory and quantum gravity. Indeed the Yang-Mills theory on noncommutative geometry is shown to emerge from a certain low-energy limit of string theory [2]. There is also an independent observation that the space-time uncertainty relation, which is naturally realized by noncommutative geometry, can be derived from some general assumptions on the underlying theory of quantum gravity [3]. One may also use fuzzy sphere as a regularization scheme alternative to the lattice regularization [4]. Unlike the lattice, fuzzy sphere preserves the continuous symmetries of the space-time considered, and the well-known problem of chiral symmetry and supersymmetry in lattice theories may become easier to overcome.

As expected from the Myers effect [5] in string theory, fuzzy spheres appear as classical solutions [6] in matrix models with a Chern-Simons term[1]. The perturbative properties of the fuzzy spheres in matrix models have been studied in refs. [8–11]. One can actually use matrix models to define a regularized field theory on a fuzzy sphere [9,10]. Such an approach has been successful in the case of noncommutative torus [12], where nonperturbative studies have produced various important results [13]. These matrix models belong to the class of so-called large-N reduced models, which are believed to provide a constructive definition of superstring and M theories. For instance, the IIB matrix model [14], which can be obtained by dimensional reduction of 10d $\mathcal{N} = 1$ super Yang-Mills theory, is proposed as a constructive definition of type IIB superstring theory. In this model the space-time

[1]Such models appear also in the context of superstring theory in the so-called pp-wave background [7].

is represented by the eigenvalues of bosonic matrices, and hence treated as a dynamical object. The dynamical generation of 4d space-time has been discussed in refs. [15].

In ref. [16] we performed the first non-perturbative study of the dimensionally reduced Yang-Mills-Chern-Simons (YMCS) model, which incorporates the fuzzy sphere as a classical solution. When the coefficient of the Chern-Simons term (α) is large, the fuzzy sphere appears as the true vacuum. However, as we decrease α, the fuzzy sphere becomes unstable at some critical point, and the system undergoes a first-order phase transition. At small α, the large-N behavior of the model becomes qualitatively the same as in the pure Yang-Mills model ($\alpha = 0$). This work has triggered extensive studies of fuzzy spheres based on Monte Carlo simulation [17, 18]. See refs. [19] for studies on other fuzzy manifolds.

In fact the dimensionally reduced YMCS model also has classical solutions which describe many fuzzy spheres. These configurations include concentric fuzzy spheres as well as intersecting ones. Concentric fuzzy spheres appear in the context of the dynamical generation of non-trivial gauge groups in matrix models [20]. They are also used to construct $\mathbf{R} \times \mathrm{S}^3$ geometry [21], which is important in the context of the AdS/CFT correspondence. Intersecting fuzzy spheres, on the other hand, is interesting from the viewpoint of the brane world scenario [22].

The aim of this paper is to study the classical and quantum stability of such configurations. At the one-loop level, it suffices to consider the interaction between two fuzzy spheres which have different radii and centers in general. The one-loop effective action around the two-fuzzy-sphere configuration has been calculated previously in ref. [8], and the asymptotic behaviors for large separation and for small separation have been discussed. We extend this study in the following directions. Firstly we discuss the classical instability of intersecting fuzzy spheres, which appears for intermediate separation and hence was completely overlooked in ref. [8]. Secondly we discuss the quantum (in)stability of separate fuzzy spheres. Here our results include the results in ref. [8], but we further take the large-N limit and confirm that the conclusion remains unaltered. Thirdly we extend our analysis to higher-dimensional supersymmetric models, and find in particular that the quantum instability for concentric fuzzy spheres disappears for $D = 10$. This is interesting since concentric fuzzy spheres with different radii are identified with the 't Hooft-Polyakov monopoles [23] (See also ref. [24].). Such configurations are perturbatively stable in the bosonic model, but not always in the supersymmetric models.

From the string theoretical viewpoint [6], the classical instability corresponds to the appearance of tachyons in the spectrum of an open string connecting the intersecting fuzzy spheres. The process of the tachyon condensation can be studied by Monte Carlo simulation as in ref. [16], but we do not pursue it here. The quantum instability for large separation, on the other hand, is due to the attractive force induced by the closed string propagation between the fuzzy spheres.

The rest of this article is organized as follows. In section 2 we discuss the stability of the multi-fuzzy-sphere configurations in the simplest model. In sections 3 and 4 we extend the analysis to the supersymmetric and higher-dimensional models. Section 5 is devoted to a summary and discussions.

2. The $D = 3$ bosonic model

In this section we investigate the properties of multi-fuzzy-sphere configurations in the simplest dimensionally reduced YMCS model. We calculate the one-loop effective action around those configurations, and discuss their stability.

2.1 The model and its classical solutions

We consider the model defined by the action [6, 9]

$$S = N\,\mathrm{tr}\Big(-\frac{1}{4}\,[A_\mu, A_\nu][A_\mu, A_\nu] + \frac{2}{3}\,i\,\alpha\,\epsilon_{\mu\nu\lambda} A_\mu A_\nu A_\lambda \Big) , \tag{2.1}$$

which can be obtained by taking the zero-volume limit of the 3d YMCS theory. The $N \times N$ matrices A_μ are traceless Hermitian, and the Greek indices run over 1 through 3. The existence of the Chern-Simons term makes it possible for the model to have various types of fuzzy-sphere configurations as classical solutions.

The action (2.1) has the SO(3) symmetry, the "translational symmetry" $A_\mu \to A_\mu + \alpha_\mu \mathbf{1}$ and the SU(N) symmetry $A_\mu \to U A_\mu U^\dagger$. In ref. [25] it is shown that the convergence property of the path integral over the non-compact dynamical variables A_μ [26–28] is not affected by the addition of the Chern-Simons term. Therefore, the path integral of this model converges for $N \geq 4$.

The classical equation of motion is obtained as

$$\Big[A_\nu, [A_\nu, A_\mu]\Big] + i\,\alpha\,\epsilon_{\mu\nu\lambda}[A_\nu, A_\lambda] = 0 . \tag{2.2}$$

The general solution takes the form

$$A_\mu = X_\mu \equiv \alpha \bigoplus_{I=1}^{k} \Big(L_\mu^{(n_I)} + x_\mu^{(I)}\,\mathbf{1}_{n_I} \Big) , \tag{2.3}$$

where $L_\mu^{(n)}$ is the representation matrix for the n-dimensional irreducible representation of the SU(2) algebra $[L_\mu^{(n)}, L_\nu^{(n)}] = i\,\epsilon_{\mu\nu\lambda} L_\lambda^{(n)}$, and $\sum_{I=1}^{k} n_I = N$. Due to the identity

$$\sum_{\lambda=1}^{3} \Big(L_\lambda^{(n)} \Big)^2 = \frac{1}{4}\,(n^2 - 1)\,\mathbf{1}_n , \tag{2.4}$$

we may consider the configuration (2.3) as representing k fuzzy spheres with the radii

$$r_I = \frac{1}{2}\sqrt{(n_I)^2 - 1} \tag{2.5}$$

and the center at $x_\mu^{(I)}$. Here and henceforth, we measure the length in units of α. Plugging this solution into (2.1), we obtain the classical part of the effective action

$$W_{\mathrm{cl}} = -\frac{\alpha^4 N}{24} \sum_{I=1}^{k} \Big\{ (n_I)^2 - 1 \Big\} . \tag{2.6}$$

2.2 One-loop effective action

Next we calculate the effective action around the classical solutions. At the one-loop level, it suffices to consider the interaction between two fuzzy spheres. The result for the multi-fuzzy-sphere configuration (2.3) can be readily obtained by summing over all possible pairs of fuzzy spheres. Therefore, we restrict ourselves in what follows to the two-fuzzy-sphere configuration; i.e., the $k = 2$ case in eq. (2.3) given by

$$X_\mu = \alpha \begin{pmatrix} L_\mu^{(n_1)} + x_\mu^{(1)} \mathbf{1}_{n_1} & \\ & L_\mu^{(n_2)} + x_\mu^{(2)} \mathbf{1}_{n_2} \end{pmatrix} . \tag{2.7}$$

Due to the "translational symmetry" mentioned in section 2.1, the result will depend only on the displacement vector

$$\xi_\mu = x_\mu^{(1)} - x_\mu^{(2)} . \tag{2.8}$$

Furthermore, exploiting the SO(3) symmetry, we may restrict ourselves to the case $\xi_\mu = (0, 0, \xi)$ without loss of generality. We will therefore obtain the one-loop effective action as a function of a single parameter ξ.

We expand the original matrices around the background (2.7) as

$$A_\mu = X_\mu + \tilde{A}_\mu . \tag{2.9}$$

We add the gauge fixing term and the ghost term

$$S_{\text{g.f.}} = -\frac{N}{2} \mathrm{tr}\, [X_\mu, A_\mu]^2, \qquad S_{\text{gh}} = -N \mathrm{tr}\, [X_\mu, \bar{c}][A_\mu, c] . \tag{2.10}$$

Plugging (2.9) into the actions (2.1) and (2.10), we obtain the quadratic terms

$$\begin{aligned} S_2 = {} & \frac{1}{2} N \,\mathrm{tr} \left(\tilde{A}_\mu [X_\lambda, [X_\lambda, \tilde{A}_\mu]] \right) + N \,\mathrm{tr} \Big(\bar{c}\, [X_\lambda, [X_\lambda, c]] \Big) \\ & - N \,\mathrm{tr} \Big\{ \Big([X_\mu, X_\nu] - i\alpha \epsilon_{\mu\nu\rho} X_\rho \Big) [\tilde{A}_\mu, \tilde{A}_\nu] \Big\} . \end{aligned} \tag{2.11}$$

Corresponding to the two-fuzzy-sphere configuration (2.7), we decompose the fluctuation matrices as

$$\tilde{A}_\mu = \begin{pmatrix} a_\mu^{(1)} & b_\mu \\ b_\mu^\dagger & a_\mu^{(2)} \end{pmatrix} , \quad \bar{c} = \begin{pmatrix} \bar{c}^{(1)} & \beta \\ \bar{\gamma} & \bar{c}^{(2)} \end{pmatrix} , \quad c = \begin{pmatrix} c^{(1)} & \gamma \\ -\bar{\beta} & c^{(2)} \end{pmatrix} , \tag{2.12}$$

where the first and second diagonal blocks are $n_1 \times n_1$ and $n_2 \times n_2$ matrices, respectively. Plugging (2.7) and (2.12) into (2.11), we obtain

$$\begin{aligned} S_2 &= S_2^{(\text{self})} + S_2^{(\text{int})} , \\ S_2^{(\text{self})} &= N\alpha^2 \sum_{I=1,2} \left[-\frac{1}{2} \mathrm{tr} \left([L_\mu^{(n_I)}, a_\nu^{(I)}]^2 \right) + \mathrm{tr} \Big\{ [L_\mu^{(n_I)}, \bar{c}^{(I)}][L_\nu^{(n_I)}, c^{(I)}] \Big\} \right] , \end{aligned} \tag{2.13}$$

$$S_2^{(\text{int})} = N\alpha^2 \Big\{ b_\mu^\dagger \left(\mathcal{H}^2 \delta_{\mu\nu} - 2i \epsilon_{\mu\nu\lambda} \xi_\lambda \right) b_\nu + \bar{\beta} \mathcal{H}^2 \beta + \bar{\gamma} \mathcal{H}^2 \gamma \Big\} , \tag{2.14}$$

where we have introduced a linear operator

$$\mathcal{H}_\mu = \mathcal{J}_\mu + \xi_\mu \ , \tag{2.15}$$

$$\mathcal{J}_\mu = L_\mu^{(n_1)} \otimes \mathbf{1}_{n_2} + \mathbf{1}_{n_1} \otimes (-L_\mu^{(n_2)*}) \ , \tag{2.16}$$

which act on the space of $n_1 \times n_2$ matrices.

The one-loop effective action W is defined by

$$\mathrm{e}^{-W} = \int da\, db\, dc\, d\bar{c}\, d\beta\, d\bar{\beta}\, d\gamma\, d\bar{\gamma}\, \mathrm{e}^{-S_2} \ . \tag{2.17}$$

In particular, the terms that come from the interaction part of the action are given by

$$W_{\mathrm{int}} = \log\det \frac{(\mathcal{H}^2 + 2\xi)(\mathcal{H}^2 - 2\xi)}{\mathcal{H}^2} \ , \tag{2.18}$$

where we have omitted a ξ-independent term.

Thus the calculation reduces to the eigenvalue problem of the $\mathcal{H}^2$, which is given by

$$\mathcal{H}^2 = \mathcal{J}^2 + 2\,\xi\,\mathcal{J}_3 + \xi^2 \ . \tag{2.19}$$

This can be readily solved [8] by noticing that $\mathcal{J}_\mu$ can be regarded as the total angular momentum operator of the system composed of spins $j_1 = \frac{n_1-1}{2}$ and $j_2 = \frac{n_2-1}{2}$. Hence $\mathcal{J}^2$ and $\mathcal{J}_3$ are simultaneously diagonalizable and their eigenvalues are given by $j(j+1)$ and m, respectively, where j and m take

$$j = j_{\mathrm{min}}, j_{\mathrm{min}}+1, \cdots, j_{\mathrm{max}} \ , \tag{2.20}$$

$$m = -j, -j+1, \cdots, j \tag{2.21}$$

with $j_{\mathrm{min}} = |j_1 - j_2| = \frac{|n_1-n_2|}{2}$ and $j_{\mathrm{max}} = j_1 + j_2 = \frac{n_1+n_2}{2} - 1$. Therefore, the eigenvalues of the operator $\mathcal{H}^2$ are given by

$$h(j,m) = \xi^2 + 2\,\xi\, m + j\,(j+1) \ , \tag{2.22}$$

and the effective action (2.18) is obtained as

$$W_{\mathrm{int}} = \sum_{j=j_{\mathrm{min}}}^{j_{\mathrm{max}}} \log w_j \ , \tag{2.23}$$

$$w_j = \prod_{m=-j}^{j} \frac{h(j,m+1)\,h(j,m-1)}{h(j,m)} = \frac{h(j,j+1)\,h(j,-j-1)}{h(j,j)\,h(j,-j)} \prod_{m=-j}^{j} h(j,m) \ . \tag{2.24}$$

In the case of concentric fuzzy spheres ($\xi = 0$), we get

$$W_{\mathrm{int}} = \sum_{j=j_{\mathrm{min}}}^{j_{\mathrm{max}}} (2j+1) \log\left[j(j+1)\right] \ , \tag{2.25}$$

which agrees with the result of refs. [20, 23].

If we further consider the case with equal radii ($n_1 = n_2 = n$), which corresponds to coinciding fuzzy spheres, we have $j_{\mathrm{min}} = 0$ and $j_{\mathrm{max}} = n-1$. Note that the argument of the log in (2.25) vanishes for $j = 0$, which indicates the appearance of zero modes. The fate of these zero modes is discussed in refs. [16, 23] in detail. In what follows, we therefore exclude this case. Then all the eigenvalues $h(j,m)$ of the operator $\mathcal{H}^2$ are strictly positive.

2.3 Stability of the classical solutions

First we discuss the classical stability of the two-fuzzy-sphere configuration. Instability can appear only from the operator $\mathcal{H}^2 - 2\xi$. Its eigenvalue $h(j,m) - 2\xi$ is negative if and only if $m = -j$ and $\xi_{j,-} < \xi < \xi_{j,+}$, where

$$\xi_{j,\pm} = (j+1) \pm \sqrt{j+1} \ . \tag{2.26}$$

Since $\xi_{j,\pm}$ increases monotonically with j and $\xi_{j+1,-} < \xi_{j,+}$, we conclude that the two-fuzzy-sphere configuration is unstable for

$$\xi_{j_{\rm min},-} < \xi < \xi_{j_{\rm max},+} \ . \tag{2.27}$$

Note that the both ends of the region is given approximately by

$$\xi_{j_{\rm min},-} \approx |r_1 - r_2| \ , \qquad \xi_{j_{\rm max},+} \approx r_1 + r_2 \tag{2.28}$$

for large n_1 and n_2, which also implies large $N (= n_1 + n_2)$. Therefore, eq. (2.27) implies that intersecting fuzzy spheres are unstable. At finite N, however, the second term in eq. (2.26) is non-negligible, and we have instability even when the two fuzzy spheres are close to intersecting.

Next let us discuss the quantum stability by looking at the ξ-dependence of the one-loop effective action in the region outside (2.27). We take the $N \to \infty$ limit in such a way that the radii r_1, r_2 given by (2.5) and the distance ξ are of the same order. For that purpose, it is convenient to introduce the parameters

$$\nu \equiv \frac{|n_1 - n_2|}{N} \ , \qquad \tilde{\xi} \equiv \frac{\xi}{N} \ , \tag{2.29}$$

which corresponds to

$$\nu \approx \frac{|r_1 - r_2|}{r_1 + r_2} \ , \qquad \tilde{\xi} \approx \frac{\xi}{2\,(r_1 + r_2)} \ . \tag{2.30}$$

We therefore take the large-N limit fixing ν and $\tilde{\xi}$. In that limit the sum over m, which appears after taking the log of (2.24), and the sum over j in (2.23) can be replaced by integrals, and we obtain

$$W_{\rm int} \simeq N^2 \left\{ F\left(\tilde{\xi}, \frac{1}{2}\right) - F\left(\tilde{\xi}, \frac{\nu}{2}\right) \right\} \ , \tag{2.31}$$

$$F(\tilde{\xi}, x) = (2 \log N - 1)\, x^2 + \frac{1}{\tilde{\xi}} \left\{ f\left(x + \tilde{\xi}\right) - f\left(x - \tilde{\xi}\right) \right\} \ , \tag{2.32}$$

$$f(x) = \frac{1}{6} x^3 \log x^2 - \frac{1}{9} x^3 \ . \tag{2.33}$$

Note that this expression is not valid for $\frac{\nu}{2} < \tilde{\xi} < \frac{1}{2}$, which corresponds to the region of classical instability (2.27). Outside that region, $W_{\rm int}$ is a monotonically increasing function of $\tilde{\xi}$ for any $0 < \nu < 1$. The asymptotic behavior of $W_{\rm int}$ is obtained as

$$W_{\rm int} \simeq \begin{cases} \frac{2}{3} N^2 \log(\nu^{-1})\, \tilde{\xi}^2 & \text{for } \tilde{\xi} \ll \frac{\nu}{2} \ , \\ \frac{1}{2} N^2 (1 - \nu^2) \log \tilde{\xi} & \text{for } \tilde{\xi} \gg \frac{1}{2} \ , \end{cases} \tag{2.34}$$

where we have omitted irrelevant constant terms.

When the two fuzzy spheres are located away from each other, they attract each other until they touch and run into the classical instability. When one fuzzy sphere is inside the other, they tend to become concentric. Therefore, concentric fuzzy spheres with different radii, which are identified with the 't Hooft-Polyakov monopoles [23], are perturbatively stable in the bosonic model.

3. Supersymmetric model

In this section we extend our analysis in the previous section to a supersymmetric version of the model (2.1), which is defined by [6, 9]

$$S = N \operatorname{tr} \left(-\frac{1}{4} [A_\mu, A_\nu][A_\mu, A_\nu] + \frac{2}{3} i \, \alpha \, \epsilon_{\mu\nu\lambda} A_\mu A_\nu A_\lambda + \frac{1}{2} \psi_\alpha (\Gamma_\mu)_{\alpha\beta} [A_\mu, \psi_\beta] \right) , \qquad (3.1)$$

where ψ_α ($\alpha = 1, 2$) is a two-component Majorana spinor, each component being a $N \times N$ traceless Hermitian matrix. The 2×2 matrix $\Gamma_\mu = \mathcal{C} \gamma_\mu$ ($\mu = 1, 2, 3$) is a product of the charge conjugation matrix $\mathcal{C}$ and the Euclidean gamma matrix γ_μ. This action has a $\mathcal{N} = 2$ supersymmetry [9]. For $\alpha = 0$, the path integral is known to be divergent [29, 30], which also applies to the $\alpha \neq 0$ case [25]. This problem does not occur in higher dimensional models, which we discuss in the next section. However, even for $D = 3$, we can still make a well-defined perturbative expansion around the general solution (2.3). As in the bosonic case, it suffices to consider the two-fuzzy-sphere configuration (2.7) at the one-loop level. The region of classical instability (2.27) corresponding to intersecting fuzzy spheres remains the same, but we will see below that the one-loop effective action, and hence the issue of quantum stability changes drastically due to the existence of supersymmetry.

We decompose the fermionic matrix ψ_α as

$$\psi_\alpha = \begin{pmatrix} s_\alpha^{(1)} & t_\alpha \\ t_\alpha^\dagger & s_\alpha^{(2)} \end{pmatrix} . \qquad (3.2)$$

The contribution of the fermions to the quadratic action is

$$S_{2,\mathrm{F}}^{(\mathrm{self})} = \frac{1}{2} \alpha N (\Gamma_\mu)_{\alpha\beta} \sum_{I=1,2} \operatorname{tr} \left\{ s_\alpha^{(I)} [L_\mu^{(I)}, s_\beta^{(I)}] \right\} , \qquad (3.3)$$

$$S_{2,\mathrm{F}}^{(\mathrm{int})} = \alpha N (\Gamma_\mu)_{\alpha\beta} \operatorname{tr} (t_\alpha^\dagger \, \mathcal{H}_\mu t_\beta) . \qquad (3.4)$$

From the integration over the matrix t_α, we get $\det(\Gamma_\mu \mathcal{H}_\mu)$. This gives an extra term

$$W_{\mathrm{int}}^{(\mathrm{F})} = -\log |\det(\Gamma_\mu \mathcal{H}_\mu)| = -\log |\det(\sigma_\mu \mathcal{H}_\mu)| \qquad (3.5)$$

to the effective action (2.18), where σ_μ are the Pauli matrices.

The evaluation of the determinant is slightly more involved than in the bosonic case. For that we add a spin $\frac{1}{2}$ system to the previously considered spin j system. The total angular momentum operator is given by

$$\mathcal{K}_\mu = \mathcal{J}_\mu + \frac{\sigma_\mu}{2} . \qquad (3.6)$$

As the basis of the combined system, we use the eigenstates $|k,n\rangle$ of the operators $\mathcal{K}^2$ and $\mathcal{K}_3$ with the eigenvalues $k(k+1)$ and n, respectively, where $k = j \pm \frac{1}{2}$ and $n = -k, \cdots, k$. We note that

$$\sigma_\mu \mathcal{H}_\mu = \mathcal{K}^2 - j(j+1) - \frac{3}{4} + \xi\, \sigma_3 \ . \tag{3.7}$$

Due to the last term, the operator is not diagonalized with the chosen basis. However, since σ_3 commutes with $\mathcal{K}_3$, the last term only mixes the states with the same n. This means that for $|n| \le j - \frac{1}{2}$, only the two states $|j+\frac{1}{2}, n\rangle$ and $|j-\frac{1}{2}, n\rangle$ are mixed. Using the Clebsch-Gordan coefficients, the corresponding matrix elements are obtained as

$$\left\langle j+\frac{1}{2}, n \middle| \sigma_\mu \mathcal{H}_\mu \middle| j+\frac{1}{2}, n \right\rangle = j + \frac{2n\xi}{2j+1} \ , \tag{3.8}$$

$$\left\langle j\pm\frac{1}{2}, n \middle| \sigma_\mu \mathcal{H}_\mu \middle| j\mp\frac{1}{2}, n \right\rangle = -\frac{2\xi}{2j+1}\sqrt{\left(j-n+\frac{1}{2}\right)\left(j+n+\frac{1}{2}\right)} \ , \tag{3.9}$$

$$\left\langle j-\frac{1}{2}, n \middle| \sigma_\mu \mathcal{H}_\mu \middle| j-\frac{1}{2}, n \right\rangle = -j - 1 - \frac{2n\xi}{2j+1} \ . \tag{3.10}$$

The determinant of the 2×2 matrix is given by $h(j,n)$ using the notation (2.22). For $|n| = j + \frac{1}{2}$, there is no mixing and we get

$$\left\langle j+\frac{1}{2}, \pm\left(j+\frac{1}{2}\right) \middle| \sigma_\mu \mathcal{H}_\mu \middle| j+\frac{1}{2}, \pm\left(j+\frac{1}{2}\right) \right\rangle = j \pm \xi \ . \tag{3.11}$$

Therefore, we obtain

$$W_{\rm int}^{(\rm F)} = -\sum_{j=j_{\rm min}}^{j_{\rm max}} \log w_j^{(\rm F)} \ , \qquad w_j^{(\rm F)} = |\xi^2 - j^2| \prod_{n=-j+\frac{1}{2}}^{j-\frac{1}{2}} h(j,n) \ . \tag{3.12}$$

Adding this to the previous result, we obtain the total effective action (2.23), where w_j is now replaced by

$$w_j = \frac{h(j,j+1)\,h(j,-j-1)}{h(j,j)\,h(j,-j)} \times \left\{ \frac{h(j,j+\frac{1}{2})}{|\xi^2 - j^2|} \prod_{m=-j}^{j} \frac{h(j,m)}{h(j,m+\frac{1}{2})} \right\} \ . \tag{3.13}$$

In the large-N limit, we find that the $\mathrm{O}(N^2)$ term and the $\mathrm{O}(N)$ term vanish exactly due to supersymmetry, and we are left with an $\mathrm{O}(1)$ quantity. Its asymptotic behavior can be obtained as

$$W_{\rm int}^{(\rm SUSY)} \simeq \begin{cases} \log(\nu^{-1}) - 12(\nu^{-2}-1)\tilde{\xi}^2 + \mathrm{O}(\tilde{\xi}^4) & \text{for } \tilde{\xi} \ll \frac{\nu}{2} \ , \\ -(1-\nu^2)\tilde{\xi}^{-2} + \mathrm{O}(\tilde{\xi}^{-4}) & \text{for } \tilde{\xi} \gg \frac{1}{2} \ , \end{cases} \tag{3.14}$$

which should be compared with the results (2.34) for the bosonic case. The factor of N^2 is absent in (3.14), and therefore the interaction is much weaker. The interaction at small $\tilde{\xi}$ is repulsive, which implies that the concentric fuzzy sphere has quantum instability in contrast to the bosonic case.

4. Higher-dimensional models

Let us further extend our analysis to higher dimensional models defined by the action

$$S = N\,\mathrm{tr}\Big(-\frac{1}{4}\,[A_\mu, A_\nu][A_\mu, A_\nu] + \frac{2}{3}\,i\,\alpha\,\epsilon_{abc} A_a A_b A_c + \frac{1}{2}\,\bar{\psi}\,\Gamma_\mu [A_\mu, \psi]\Big)\ , \tag{4.1}$$

where the indices run over $\mu, \nu = 1, 2, \cdots, D$ and $a, b, c = 1, 2, 3$. The dimensionality is limited to $D = 3, 4, 6, 10$, where the matrix model (4.1) has supersymmetry with an appropriate choice of the spinor representations for the fermions. The $D = 4$ case has been studied by Monte Carlo simulation in ref. [18].

As a classical solution, we take the configuration (2.7) for $\mu = 1, 2, 3$ and $X_\mu = 0$ otherwise.[2] This describes a system of two fuzzy spheres extended in the $1, 2, 3$ directions of the D-dimensional target space. The region of classical instability is the same as (2.27). Outside that region, the interaction part of the one-loop effective action is given as

$$W_{\mathrm{int}}^{(\mathrm{SUSY}),D} = \log\det\left(\mathcal{H}^2 - 2\xi\right)\left(\mathcal{H}^2 + 2\xi\right)(\mathcal{H}^2)^{D-4} - \log\left|\det(\sigma_\alpha \mathcal{H}_a)^{D-2}\right|\ . \tag{4.2}$$

This expression can be evaluated as in the previous section, and the generalization simply amounts to modifying (3.13) by raising the second factor in the parenthesis $\{\ \}$ to the power of $(D-2)$. Thus we obtain

$$W_{\mathrm{int}}^{(\mathrm{SUSY}),D} \simeq \begin{cases} (D-2)\log(\nu^{-1}) + 4\,(D-6)(\nu^{-2}-1)\tilde{\xi}^2 + \mathrm{O}(\tilde{\xi}^4) & \text{for } \tilde{\xi} \ll \frac{\nu}{2}\ , \\ -(1-\nu^2)\tilde{\xi}^{-2} + \mathrm{O}(\tilde{\xi}^{-4}) & \text{for } \tilde{\xi} \gg \frac{1}{2}\ . \end{cases} \tag{4.3}$$

It is interesting that the coefficient of the $\tilde{\xi}^2$ changes its sign at $D = 6$. Therefore, concentric fuzzy spheres are perturbatively stable in the $D = 10$ supersymmetric model.

5. Summary and discussions

In this paper we have discussed the classical and quantum stability of the multi-fuzzy-sphere configurations in the YMCS models using the one-loop effective action. We have shown in general that the configuration becomes unstable when fuzzy spheres intersect. Separate fuzzy spheres attract each other in general, and eventually run into instability upon intersecting. On the other hand, the fate of configurations with one fuzzy sphere located inside another depends on the model. We find that the concentric fuzzy spheres, which correspond to the 't Hooft-Polyakov monopoles, are perturbatively stable in the bosonic model and in the $D = 10$ supersymmetric model.

Our ambitious goal is to investigate the dynamical generation of gauge group, as well as that of the space-time, in nonperturbative formulations of superstring theory such as the IIB matrix model. Refs. [31] present a closely related approach, in which one attempts to obtain the standard model gauge group using fuzzy spheres in the extra dimensions. There, however, the 4d space-time is introduced from the outset as in ordinary field theories. We hope that the dynamical properties of fuzzy spheres studied in this paper will be useful in understanding how our universe has emerged.

[2] In the higher dimensional models, one can separate the two fuzzy spheres also in the $4, 5, \cdots, D$ directions. We do not discuss the results here, since they are not very illuminating.

Acknowledgments

We would like to thank K. Nagao for helpful discussions.

References

[1] J. Madore, *The fuzzy sphere, Class. and Quant. Grav.* **9** (1992) 69.

[2] N. Seiberg and E. Witten, *String theory and noncommutative geometry, J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[3] S. Doplicher, K. Fredenhagen and J.E. Roberts, *The quantum structure of spacetime at the Planck scale and quantum fields, Commun. Math. Phys.* **172** (1995) 187 [hep-th/0303037].

[4] H. Grosse, C. Klimcik and P. Presnajder, *Towards finite quantum field theory in noncommutative geometry, Int. J. Theor. Phys.* **35** (1996) 231 [hep-th/9505175].

[5] R. C. Myers, *Dielectric-branes, J. High Energy Phys.* **12** (1999) 022 [hep-th/9910053].

[6] A. Y. Alekseev, A. Recknagel and V. Schomerus, *Brane dynamics in background fluxes and non-commutative geometry, J. High Energy Phys.* **05** (2000) 010 [hep-th/0003187].

[7] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang Mills, J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021];
G. Bonelli, *Matrix strings in pp-wave backgrounds from deformed super Yang-Mills theory, J. High Energy Phys.* **08** (2002) 022 [hep-th/0205213].

[8] S. Bal and H. Takata, *Interaction between two fuzzy spheres, Int. J. Mod. Phys.* **A 17** (2002) 2445 [hep-th/0108002].

[9] S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, *Noncommutative gauge theory on fuzzy sphere from matrix model, Nucl. Phys.* **B 604** (2001) 121 [hep-th/0101102].

[10] Y. Kitazawa, *Matrix models in homogeneous spaces, Nucl. Phys.* **B 642** (2002) 210 [hep-th/0207115].

[11] P. Valtancoli, *Stability of the fuzzy sphere solution from matrix model, Int. J. Mod. Phys.* **A 18** (2003) 967 [hep-th/0206075];
T. Imai, Y. Kitazawa, Y. Takayama and D. Tomino, *Quantum corrections on fuzzy sphere, Nucl. Phys.* **B 665** (2003) 520 [hep-th/0303120];
T. Imai and Y. Takayama, *Stability of fuzzy $S^2 \times S^2$ geometry in IIB matrix model, Nucl. Phys.* **B 686** (2004) 248 [hep-th/0312241];
P. Castro-Villarreal, R. Delgadillo-Blando and B. Ydri, *A gauge-invariant UV-IR mixing and the corresponding phase transition for U(1) fields on the fuzzy sphere, Nucl. Phys.* **B 704** (2005) 111 [hep-th/0405201]; *Quantum effective potential for U(1) fields on $S^2_L \times S^2_L$, J. High Energy Phys.* **09** (2005) 066 [hep-th/0506044];
T. Azuma, K. Nagao and J. Nishimura, *Perturbative dynamics of fuzzy spheres at large N, J. High Energy Phys.* **06** (2005) 081 [hep-th/0410263];
H. Kaneko, Y. Kitazawa and D. Tomino, *Stability of fuzzy $S^2 \times S^2 \times S^2$ in IIB type matrix models, Nucl. Phys.* **B 725** (2005) 93 [hep-th/0506033]; *Fuzzy spacetime with SU(3) isometry in IIB matrix model, Phys. Rev.* **D 73** (2006) 066001 [hep-th/0510263];
H. Steinacker, *Quantized gauge theory on the fuzzy sphere as random matrix model, Nucl. Phys.* **B 679** (2004) 66 [hep-th/0307075];

H. Grosse and H. Steinacker, *Finite gauge theory on fuzzy CP^2*, *Nucl. Phys.* **B 707** (2005) 145 [hep-th/0407089];
W. Behr, F. Meyer and H. Steinacker, *Gauge theory on fuzzy $S^2 \times S^2$ and regularization on noncommutative R^4, J. High Energy Phys.* **0507** (2005) 040 [hep-th/0503041];
D. Dou and B. Ydri, *Topology change from quantum instability of gauge theory on fuzzy CP^2*, *Nucl. Phys.* **B 771** (2007) 167 [hep-th/0701160].

[12] J. Ambjørn, Y. M. Makeenko, J. Nishimura and R. J. Szabo, *Finite N matrix models of noncommutative gauge theory, J. High Energy Phys.* **11** (1999) 029 [hep-th/9911041]; *Nonperturbative dynamics of noncommutative gauge theory, Phys. Lett.* **B 480** (2000) 399 [hep-th/0002158]; *Lattice gauge fields and discrete noncommutative Yang-Mills theory, J. High Energy Phys.* **05** (2000) 023 [hep-th/0004147].

[13] W. Bietenholz, F. Hofheinz and J. Nishimura, *The renormalizability of 2D Yang-Mills theory on a non-commutative geometry, J. High Energy Phys.* **09** (2002) 009 [hep-th/0203151]; *Phase diagram and dispersion relation of the non-commutative $\lambda\phi^4$ model in $d = 3$, J. High Energy Phys.* **06** (2004) 042 [hep-th/0404020]; *On the relation between non-commutative field theories at $\theta = \infty$ and large N matrix field theories, J. High Energy Phys.* **05** (2004) 047 [hep-th/0404179];
W. Bietenholz, J. Nishimura, Y. Susaki and J. Volkholz, *A non-perturbative study of 4d U(1) non-commutative gauge theory: The fate of one-loop instability*, *J. High Energy Phys.* **10** (2006) 042 [hep-th/0608072];
J. Ambjørn and S. Catterall, *Stripes from (noncommutative) stars, Phys. Lett.* **B 549** (2002) 253 [hep-lat/0209106];
H. Aoki, J. Nishimura and Y. Susaki, *The index of the overlap Dirac operator on a discretized 2d non-commutative torus*, *J. High Energy Phys.* **02** (2007) 033 [hep-th/0602078]; *Probability distribution of the index in gauge theory on 2d non-commutative geometry*, *J. High Energy Phys.* **10** (2007) 024 [hep-th/0604093].

[14] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large-N reduced model as superstring, Nucl. Phys.* **B 498** (1997) 467 [hep-th/9612115].

[15] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, *Space-time structures from IIB matrix model, Prog. Theor. Phys.* **99** (1998) 713 [hep-th/9802085];
J. Nishimura and G. Vernizzi, *Spontaneous breakdown of Lorentz invariance in IIB matrix model, J. High Energy Phys.* **04** (2000) 015 [hep-th/0003223]; *Brane world generated dynamically from string type IIB matrices, Phys. Rev. Lett.* **85** (2000) 4664 [hep-th/0007022];
J. Nishimura, *Exactly solvable matrix models for the dynamical generation of space-time in superstring theory, Phys. Rev.* **D 65** (2002) 105012 [hep-th/0108070];
K. N. Anagnostopoulos and J. Nishimura, *New approach to the complex-action problem and its application to a nonperturbative study of superstring theory, Phys. Rev.* **D 66** (2002) 106008 [hep-th/0108041];
J. Nishimura and F. Sugino, *Dynamical generation of four-dimensional space-time in the IIB matrix model, J. High Energy Phys.* **05** (2002) 001 [hep-th/0111102];
H. Kawai, S. Kawamoto, T. Kuroki, T. Matsuo and S. Shinohara, *Mean field approximation of IIB matrix model and emergence of four dimensional space-time, Nucl. Phys.* **B 647** (2002) 153 [hep-th/0204240];
T. Aoyama, H. Kawai and Y. Shibusa, *Stability of 4-dimensional space-time from IIB matrix model via improved mean field approximation, Prog. Theor. Phys.* **115** (2006) 1179 [hep-th/0602244];

T. Imai, Y. Kitazawa, Y. Takayama and D. Tomino, *Effective actions of matrix models on homogeneous spaces, Nucl. Phys.* **B 679** (2004) 143 [hep-th/0307007].

[16] T. Azuma, S. Bal, K. Nagao and J. Nishimura, *Nonperturbative studies of fuzzy spheres in a matrix model with the Chern-Simons term, J. High Energy Phys.* **05** (2004) 005 [hep-th/0401038].

[17] X. Martin, *A matrix phase for the ϕ^4 scalar field on the fuzzy sphere*, *J. High Energy Phys.* **04** (2004) 077 [hep-th/0402230];
N. Kawahara, J. Nishimura and K. Yoshida, *Dynamical aspects of the plane-wave matrix model at finite temperature*, *J. High Energy Phys.* **06** (2006) 052 [hep-th/0601170];
D. O'Connor and B. Ydri, *Monte Carlo simulation of a NC gauge theory on the fuzzy sphere*, *J. High Energy Phys.* **11** (2006) 016 [hep-lat/0606013];
M. Panero, *Numerical simulations of a non-commutative theory: The scalar model on the fuzzy sphere*, *J. High Energy Phys.* **05** (2007) 082 [hep-th/0608202]; *Quantum field theory in a non-commutative space: Theoretical predictions and numerical results on the fuzzy sphere*, *SIGMA* **2** (2006) 081 [hep-th/0609205];
N. Kawahara, J. Nishimura and S. Takeuchi, *Exact fuzzy sphere thermodynamics in matrix quantum mechanics*, *J. High Energy Phys.* **05** (2007) 091 [arXiv:0704.3183].

[18] K. N. Anagnostopoulos, T. Azuma, K. Nagao and J. Nishimura, *Impact of supersymmetry on the nonperturbative dynamics of fuzzy spheres, J. High Energy Phys.* **09** (2005) 046 [hep-th/0506062].

[19] T. Azuma, S. Bal, K. Nagao and J. Nishimura, *Absence of a fuzzy S^4 phase in the dimensionally reduced 5d Yang-Mills-Chern-Simons model, J. High Energy Phys.* **07** (2004) 066 [hep-th/0405096]; *Dynamical aspects of the fuzzy CP^2 in the large N reduced model with a cubic term, J. High Energy Phys.* **05** (2006) 061 [hep-th/0405277]; *Perturbative versus nonperturbative dynamics of the fuzzy $S^2 \times S^2$, J. High Energy Phys.* **09** (2005) 047 [hep-th/0506205].

[20] T. Azuma, S. Bal and J. Nishimura, *Dynamical generation of gauge groups in the massive Yang-Mills-Chern-Simons matrix model, Phys. Rev.* **D 72** (2005) 066005 [hep-th/0504217];
T. Aoyama, T. Kuroki and Y. Shibusa, *Dynamical generation of non-Abelian gauge group via the improved perturbation theory, Phys. Rev.* **D 74** (2006) 106004 [hep-th/0608031].

[21] H. Kaneko, Y. Kitazawa and K. Matsumoto, *Effective Actions of IIB Matrix Model on S^3*, *Phys. Rev.* **D 76** (2007) 084024 [arXiv:0706.1708];
G. Ishiki, S. Shimasaki, Y. Takayama and A. Tsuchiya, *Embedding of theories with SU(2|4) symmetry into the plane wave matrix model*, *J. High Energy Phys.* **11** (2006) 089 [hep-th/0610038];
T. Ishii, G. Ishiki, S. Shimasaki and A. Tsuchiya, *T-duality, fiber bundles and matrices*, *J. High Energy Phys.* **05** (2007) 014 [hep-th/0703021];
G. Grignani, L. Griguolo, N. Mori and D. Seminara, *Thermodynamics of theories with sixteen supercharges in non-trivial vacua*, *J. High Energy Phys.* **10** (2007) 068 [arXiv:0707.0052].

[22] R. Blumenhagen, M. Cvetic, P. Langacker and G. Shiu, *Toward realistic intersecting D-brane models*, *Ann. Rev. Nucl. Part. Sci.* **55** (2005) 71 [hep-th/0502005].

[23] H. Aoki, S. Iso, T. Maeda and K. Nagao, *Dynamical generation of a nontrivial index on the fuzzy 2-sphere, Phys. Rev.* **D 71** (2005) 045017 [Erratum:*ibid.* 069905] [hep-th/0412052].

[24] H. Aoki, S. Iso and T. Maeda, *On the Ginsparg-Wilson Dirac operator in the monopole backgrounds on the fuzzy 2-sphere*, *Phys. Rev.* **D 75** (2007) 085021 [hep-th/0610125].

[25] P. Austing and J. F. Wheater, *Adding a Myers term to the IIB matrix model, J. High Energy Phys.* **11** (2003) 009 [hep-th/0310170].

[26] W. Krauth and M. Staudacher, *Finite Yang-Mills integrals*, *Phys. Lett.* **B 435** (1998) 350 [hep-th/9804199].

[27] T. Hotta, J. Nishimura and A. Tsuchiya, *Dynamical aspects of large N reduced models, Nucl. Phys.* **B 545** (1999) 543, [hep-th/9811220].

[28] P. Austing and J.F. Wheater, *The convergence of Yang-Mills integrals*, *J. High Energy Phys.* **02** (2001) 028 [hep-th/0101071].

[29] W. Krauth, H. Nicolai and M. Staudacher, *Monte Carlo approach to M-theory*, *Phys. Lett.* **B 431** (1998) 31 [hep-th/9803117].

[30] P. Austing and J.F. Wheater, *Convergent Yang-Mills matrix theories*, *J. High Energy Phys.* **04** (2001) 019 [hep-th/0103159].

[31] P. Aschieri, J. Madore, P. Manousselis and G. Zoupanos, *Dimensional reduction over fuzzy coset spaces, J. High Energy Phys.* **04** (2004) 034 [hep-th/0310072]; *Unified theories from fuzzy extra dimensions, Fortschr. Phys.* **52** (2004) 718 [hep-th/0401200];
P. Aschieri, T. Grammatikopoulos, H. Steinacker and G. Zoupanos, *Dynamical generation of fuzzy extra dimensions, dimensional reduction and symmetry breaking*, *J. High Energy Phys.* **09** (2006) 026 [hep-th/0606021];
H. Steinacker and G. Zoupanos, *Fermions on spontaneously generated spherical extra dimensions*, *J. High Energy Phys.* **09** (2007) 017 [arXiv:0706.0398].