Self-dual instanton and nonself-dual instanton-antiinstanton solutions in $d = 4$ Yang-Mills theory

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Abstract

Subjecting the $SU(2)$ Yang–Mills system to azimuthal symmetries in both the $x - y$ and the $z - t$ planes results in a residual subsystem described by a $U(1)$ Higgs like model with two complex scalar fields on the quarter plane. The resulting instantons are labeled by integers (m, n_1, n_2) with topological charges $q = \frac{1}{2}[1 - (-1)^m]n_1 n_2$. Solutions are constructed numerically for $m = 1, 2, 3$ and a range of $n_1 = n_2 = n$. It is found that only the $m = 1$ instantons are self-dual, the $m > 1$ configurations describing composite instanton-antiinstanton lumps.

1 Introduction

After the discovery of the BPST instantons by Belavin *et al* [1], which are unit topological charge spherically symmetric solutions to the $SU(2)$ Yang-Mills (YM) equations, Witten [2] constructed higher charge axially symmetric instantons. Multi-instantons subject to no symmetries were subsequently constructed by 't Hooft and by Jackiw *et al* [3] for the same system, while the most general instantons for arbitrary gauge group $SU(N)$ were classified by Atiyah *et al* [4]. All these instantons are solutions to the first order self-duality equations.

Solutions to the second order Euler-Lagrange equations which are not self-dual, while interesting for their own sake, are of great physical importance. The most important such solutions are the zero topological charge instanton–antiinstantons, whose putative role has been studied from the eariest stages of instanton physics, in particular in the construction of dilute (or otherwise) instanton gases [5, 6]. To date however no such exact solutions were constructed, although the forces between an instanton and an antiinstanton were considered [7] long ago for approximate field configurations.

The existence of non self-dual instantons has been proved in [8], and for the $SU(2)$ YM system in [9, 10]. In the latter, the residual actions studied are one dimensional as a result

of the imposition of quadrupole symmetry in [9], and by applying "equivariant geometry" in [10]. However, the proof in [9] does not cover the existence of non self-dual instantons with topological charges ± 1.

In this Letter, we present both self-dual and non self-dual instantons of the $SU(2)$ YM model, subjected to azimuthal symmetries both in the $x-y$ and the $z-t$ planes. The residual system in our case is an axially symmetric $U(1)$ Higgs like model in the $\rho - \sigma$ quarter plane[1], as distinct from Witten's case [2] where it is a $U(1)$ Higgs model on the half plane (r, t). We find finite action solutions labeled by a triple of integers (m, n_1, n_2) with topological charges $q = \frac{1}{2}[1 - (-1)^m]n_1 n_2$. We find that only for $m = 1$ are the instantons self-dual, all those labeled by $m \geq 2$ being non self-dual. All our solutions are constructed numerically.

2   The model

2.1   Imposition of symmetry and residual action

The usual $SU(2)$ Yang-Mills (YM) action density in 4 Euclidean dimensions

$$\mathcal{L} = \frac{1}{16\pi^2} \operatorname{Tr} \mathcal{F}_{\mu\nu}^2 \quad , \quad \text{with} \ \ \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu], \tag{1}$$

will be subjected to two successive axial symmetries. The normalisation used in (1) is chosen such that the spherically symmetric BPST [1] instanton has unit topological charge, in the spherically symmetric limit of our Ansatz to be stated below. We denote the Euclidean four dimensional coordinates as $x_\mu = (x, y; z, t) \equiv (x_\alpha; x_i)$, with $\alpha = 1, 2$ and $i = 3, 4$, and use the following parametrisation

$$x_\alpha = r \sin\theta \, \hat{x}_\alpha \equiv \rho \, \hat{x}_\alpha, \quad x_i = r \cos\theta \, \hat{x}_i \equiv \sigma \hat{x}_i, \tag{2}$$

where $r^2 = |x_\mu|^2 = |x_\alpha|^2 + |x_i|^2$, with the unit vectors appearing in (2) parametrised as $\hat{x}_\alpha = (\cos\varphi_1, \sin\varphi_1)$, $\hat{x}_i = (\cos\varphi_2, \sin\varphi_2)$, with $0 \leq \theta \leq \frac{\pi}{2}$ spanning the quarter plane, and the two azimuthal angles $0 \leq \varphi_1 \leq 2\pi$ and $0 \leq \varphi_2 \leq 2\pi$.

The $SU(2)$ YM field $\mathcal{F}_{\mu\nu}$ will be subjected to two stages of symmetry, in the x_α and the x_i planes, in succession. The **first stage** of symmetry imposition is of cylindrical symmetry in the $x_\alpha = (x_1, x_2)$ plane. Our *cylindrically symmetric* Ansatz is

$$\mathcal{A}_\alpha = \left(\frac{\phi^3 + n_1}{\rho}\right) \Sigma_{\alpha\beta}\hat{x}_\beta + \left(\frac{\phi^1}{\rho}\right)(\varepsilon\hat{x})_\alpha (\varepsilon n^{(1)}) + \beta \Sigma_{\beta 3} - A_\rho^2 \, \hat{x}_\alpha n_\beta^{(1)} \Sigma_{\beta 3} \tag{3}$$

$$+ \left(\frac{\phi^2}{\rho}\right)(\varepsilon\hat{x})_\alpha (\varepsilon n^{(1)})_\beta \, \Sigma_{\beta 4} + A_\rho^1 \, \hat{x}_\alpha \, n_\beta^{(1)} \Sigma_{\beta 4} - A_\rho^3 \, \hat{x}_\alpha \Sigma_{34} \, ,$$

$$\mathcal{A}_i = -A_i^1 \, n_\beta^{(1)} \Sigma_{\beta 3} + A_i^2 \, n_\beta^{(1)} \Sigma_{\beta 4} - A_i^3 \Sigma_{34} \, , \tag{4}$$

in terms of the unit vector $n_\alpha^{(1)} = (\cos n_1\varphi_1, \sin n_1\varphi_1)$ labeled by the vorticity integer n_1, $\varepsilon_{\alpha\beta}$ being the Levi-Civita symbol. The spin matrices $\Sigma_{\mu\nu} = (\Sigma_{\alpha\beta}, \Sigma_{\alpha i}, \Sigma_{ij})$ in (3) and (4) are one or other of the two chiral representations of $SO(4)$, i.e. they are $SU(2)$ matrices.

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[1]With $\rho^2 = x^2 + y^2$ and $\sigma^2 = z^2 + t^2$, where x, y, z and t are rectangular cordinates on E^4.

Using the notation

$$F_{MN} = \partial_M A_N^a - \partial_N A_M^a + \varepsilon^{abc} A_M^b A_N^c \ , \quad D_M \phi^a = \partial_M \phi^a + \varepsilon^{abc} A_M^b \phi^c \ , \tag{5}$$

with $x_M = (x_i, \rho)$ and the internal index $a = 1, 2, 3$, the components of the curvature $\mathcal{F}_{\mu\nu}$ is expressed exclusively in terms of the gauge covariant quantities (5), describing an effective $SO(3)$ YM-Higgs system on the hyperbolic space with coordinates $x_M = (x_i, \rho)$.

Imposition of the **second stage** of symmetry is in the x_i-plane in this hyperbolic space, by subjecting the fields (A_i^a, A_ρ^a) and ϕ^a appearing in (5), to axial symmetry in the $x_i = (x_3, x_4)$ plane. Relabeling the $SO(3)$ internal index as $a = (i', 3)$, i.e. $i', j', .. = 1, 2$, the axially symmetric Ansatz for the residual $SO(3)$ gauge connection is

$$\begin{aligned}
A_i^{i'} &= -a_\sigma \, \hat{x}_i \, (\varepsilon n_{(2)})^{i'} + \left(\frac{\chi^1}{\sigma} \right) (\varepsilon \hat{x})_i \, n_{(2)}^{i'} \ , \\
A_i^3 &= \left(\frac{\chi^2 + n_2}{\sigma} \right) (\varepsilon \hat{x})_i \ , \\
A_\rho^{i'} &= -a_\rho \, (\varepsilon n_{(2)})^{i'} \ , \\
A_\rho^3 &= 0 \ ,
\end{aligned} \tag{6}$$

with the unit vector $n_{(2)}^{i'} = (\cos n_2 \varphi_2, \, \sin n_2 \varphi_2)$ labeled by a second vorticity integer n_2. Subjecting the $SO(3)$ triplet effective Higgs field $\phi^a = (\phi^{i'}, \phi^3)$ to the same symmetry, we have $\phi^{i'} = \xi^1 \, n_{(2)}^{i'}, \quad \phi^3 = \xi^2$.

In the numerical computation, we implemented a **third stage** of symmetry by treating the two azimuthal symmetries imposed in the $x - y$ and the $z - t$ planes on the same footing. Thus we set the two vorticities n_1 and n_2, appearing in (3)-(4) and (6) respectively, equal, $n_1 = n_2 = n$. For the rest of this section however we consider the general case with the two distinct vorticities.

As a result of imposing the two stages of symmetry, the YM field is parametrized by six functions which depend only on ρ and σ. The components of the curvature $(F_{ij}^a, F_{i\rho}^a)$ and the covariant derivatives $(D_\rho \phi^a, D_i \phi^a)$ appearing in (5) are now expressed exclusively in terms of the $SO(2)$ curvature

$$f_{\rho\sigma} = \partial_\rho a_\sigma - \partial_\sigma a_\rho \tag{7}$$

and the covariant derivatives

$$\begin{aligned}
\mathcal{D}_\rho \chi^A &= \partial_\rho \chi^A + a_\rho (\varepsilon \chi)^A \ , \quad \mathcal{D}_\sigma \chi^A = \partial_\sigma \chi^A + a_\sigma (\varepsilon \chi)^A, \\
\mathcal{D}_\rho \xi^A &= \partial_\rho \xi^A + a_\rho (\varepsilon \xi)^A \ , \quad \mathcal{D}_\sigma \xi^A = \partial_\sigma \xi^A + a_\sigma (\varepsilon \xi)^A,
\end{aligned} \tag{8}$$

where we have used the notation $(\chi^1, \chi^2) = \chi^A$, $(\xi^1, \xi^2) = \xi^A$, $A = 1, 2$. The result is a residual $U(1)$ connection (a_ρ, a_σ) interacting covariantly with the scalar fields χ^A and ξ^A, i.e. an Abelian Higgs like model in the quarter plane $\rho - \sigma$, described by a Lagrangean

$$L = \frac{1}{4} \left[\rho\sigma \, f_{\rho\sigma}^2 + \frac{\rho}{\sigma} \left(|\mathcal{D}_\rho \chi^A|^2 + |\mathcal{D}_\sigma \chi^A|^2 \right) + \frac{\sigma}{\rho} \left(|\mathcal{D}_\rho \xi^A|^2 + |\mathcal{D}_\sigma \xi^A|^2 \right) + \frac{1}{\rho\sigma} (\varepsilon^{AB} \chi^A \xi^B)^2 \right] \ . \tag{9}$$

3

This residual action density is a scalar with respect to the *local* $SO(2)$ indices A, B, hence they are manifestly gauge invariant. It describes a $U(1)$ Higgs like model with **two** effective Higgs fields χ^A and ξ^A, coupled minimally to the $U(1)$ gauge connection $a_\mu = (a_\rho, a_\sigma)$. To remove this $U(1)$ gauge freedom we impose the usual gauge condition

$$\partial_\mu a_\mu \equiv \partial_\rho a_\rho + \partial_\sigma a_\sigma = 0 \ . \tag{10}$$

To state compactly the residual self-duality equations which saturate the topological lower bound of (9), we use a new index notation $x_\mu = (\rho, \sigma)$, not to be confused with the index notation used in (1). They are expressed in this notation as

$$f_{\mu\nu} = \frac{1}{\rho\sigma}\varepsilon_{\mu\nu}\varepsilon^{AB}\chi^A\xi^B \ , \tag{11}$$

$$\mathcal{D}_\mu\chi^a = \rho^{-1}\sigma\,\varepsilon_{\mu\nu}\mathcal{D}_\nu\xi^a \ . \tag{12}$$

One solution of these equations is well known, corresponding to the spherically symmetric unit charge BPST [1] instanton. Expressing the self-duality equations (11)-(12) in terms of the coordinates (r, θ) rather than (ρ, σ), constraining the function $a_\theta = a_\theta(r)$ to be a radial function and assigning the following values of the remaining fuctions (a_r, χ^A, ξ^A):

$$a_r = 0, \ \chi^1 = -\xi^1 = \frac{1}{2}a_\theta \sin 2\theta, \ \chi^2 = -a_\theta \cos^2\theta - 1, \ \xi^2 = -a_\theta \sin^2\theta - 1 \ , \tag{13}$$

these reduce to the single self-duality equation $da_\theta/dr = a_\theta(a_\theta + 2)/r$ yielding

$$a_\theta = -\frac{2r^2}{r^2 + \lambda^2} \tag{14}$$

(λ being the arbitrary scale of the unit charge instanton).

Since our numerical constructions will be carried out using the coordinates (r, θ) we display (9) also as

$$L = \frac{1}{4}\left[r\sin\theta\cos\theta\, f_{r\theta}^2 \ + \ \frac{r\sin\theta}{\cos\theta}\left(|\mathcal{D}_r\chi^A|^2 + \frac{1}{r^2}|\mathcal{D}_\theta\chi^A|^2\right)\right. \tag{15}$$

$$\left. + \ \frac{r\cos\theta}{\sin\theta}\left(|\mathcal{D}_r\xi^A|^2 + \frac{1}{r^2}|\mathcal{D}_\theta\xi^A|^2\right) + \frac{1}{r\sin\theta\cos\theta}(\varepsilon^{AB}\chi^A\xi^B)^2\right] \ ,$$

the total action of these solutions being $S = \int d^4x\sqrt{g}\mathcal{L} = \int_0^\infty \int_0^{\pi/2} dr d\theta L$, with $\mathcal{L}$ the action density (1).

It is worth noting that precisely the same results for the residual gauge connection and reduced action are found by the alternative approach of [11], where the action of the isometries $\partial/\partial\varphi_1$, $\partial/\partial\varphi_2$ on the gauge connection is compensated by suitable gauge transformations.

4

2.2 Boundary conditions

To obtain regular solutions with finite action density we impose at the origin ($r = 0$) the boundary conditions

$$a_r = 0 \quad , \quad a_\theta = 0 \quad , \quad \chi^A = \begin{pmatrix} 0 \\ -n_2 \end{pmatrix} \quad , \quad \xi^A = \begin{pmatrix} 0 \\ -n_1 \end{pmatrix} , \tag{16}$$

which are requested by the analyticity of the ansatz. In order to find finite action solutions, we impose at infinity

$$a_r = 0 \; , \quad a_\theta = -2m \; , \quad \chi^A = (-1)^{m+1} n_2 \begin{pmatrix} \sin 2m\theta \\ \cos 2m\theta \end{pmatrix} \; , \quad \xi^A = -n_1 \begin{pmatrix} \sin 2m\theta \\ \cos 2m\theta \end{pmatrix} , \tag{17}$$

m being a positive integer. Similar considerations lead to the following boundary conditions on the ρ and σ axes:

$$a_r = \frac{1}{n_1} \partial_r \xi^1 \quad , \quad a_\theta = \frac{1}{n_1} \partial_\theta \xi^1 \quad , \quad \chi^1 = 0 \quad , \quad \xi^1 = 0, \; \partial_\theta \chi^2 = 0 \quad , \quad \xi^2 = -n_1 \; , \tag{18}$$

for $\theta = 0$ and

$$a_r = \frac{1}{n_2} \partial_r \chi^1 \quad , \quad a_\theta = \frac{1}{n_2} \partial_\theta \chi^1 \quad , \quad \chi^1 = 0 \quad , \quad \xi^1 = 0, \; \chi^2 = -n_2 \quad , \quad \partial_\theta \xi^2 = 0 \; , \tag{19}$$

for $\theta = \pi/2$, respectively.

2.3 Topological charge

Having determined the values of all the functions on the boundaries of (r, θ)-domain, we proceed to calculate the resulting topological charges. In our normalisation, the topological charge is defined as

$$q = \frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \int \mathrm{Tr} \mathcal{F}_{\mu\nu} \mathcal{F}_{\rho\sigma} \, d^4 x \,, \tag{20}$$

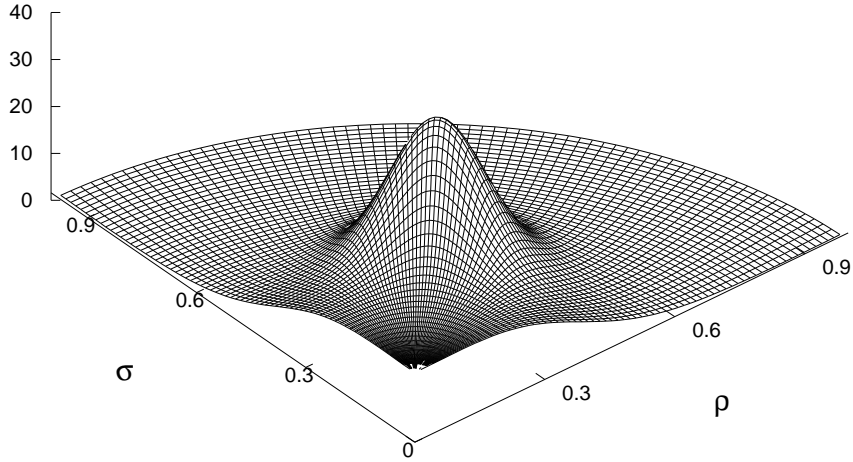which after integration of the azimuthal angles (φ_1, φ_2) reduces to

$$q = \frac{1}{2} \int \left(\varepsilon_{AB} \chi^A \xi^B f_{\rho\sigma} + \mathcal{D}_{[\rho} \chi^A \mathcal{D}_{\sigma]} \xi^A \right) d\rho \, d\sigma \,, \tag{21}$$

which in the index notation used in (11)-(12) can be expressed compactly as

$$q = \frac{1}{2} \varepsilon_{\mu\nu} \int \left(\frac{1}{2} \varepsilon_{AB} \chi^A \xi^B f_{\mu\nu} + \mathcal{D}_\mu \chi^A \mathcal{D}_\nu \xi^A \right) d^2 x \tag{22}$$

$$= \frac{1}{4} \int \varepsilon_{\mu\nu} \partial_\mu (\chi^A \mathcal{D}_\nu \xi^A - \xi^A \mathcal{D}_\nu \chi^A) \, d^2 x \,. \tag{23}$$

The integration in (22) is carried out over the 2 dimensional space $x_\mu = (x_\rho, x_\sigma)$. As expected this is a total divergence expressed by (23).

Figure 1. The action density $\mathcal{L}$ is shown for the $m = 1$, $n = 4$ self-dual instanton (with $\rho = r\sin\theta$ and $\sigma = r\cos\theta$).

Using Stokes' theorem, the two dimensional integral of (23) reduces to the one dimensional line integral

$$q = \frac{1}{4} \int \chi^A \overset{\leftrightarrow}{\mathcal{D}}_\mu \xi^A \, ds_\mu \,, \tag{24}$$

the line integral being taken around the loop

$$q = \frac{1}{4} \left(\int_0^\infty \left(\chi^A \overset{\leftrightarrow}{\mathcal{D}}_\sigma \xi^A \right) \Big|_{\rho=0} d\sigma + \int_0^{\frac{\pi}{2}} \left(\chi^A \overset{\leftrightarrow}{\mathcal{D}}_\theta \xi^A \right) \Big|_{r=\infty} d\theta + \int_\infty^0 \left(\chi^A \overset{\leftrightarrow}{\mathcal{D}}_\rho \xi^A \right) \Big|_{\sigma=0} d\rho \right) . \tag{25}$$

The θ integral over the large quarter circle vanishes since $\mathcal{D}_\theta \chi^A$ and $\mathcal{D}_\theta \xi^A$ both vanish as $r \to \infty$, so the the only contributions come from the σ and ρ integrations. These are immediately evaluated by reading off the appropriate values of χ^A and ξ^A from (18)-(19). The result is

$$q = \frac{1}{2} [1 - (-1)^m] n_1 n_2 \,, \tag{26}$$

such that only for odd m is the Pontryagin charge nonzero and is then always equal to $n_1 n_2$. This is consistent with the description of instanton-antiinstanton chains. Obviously, if zero topological charge solutions exist, they must be non self-dual, as also must the higher odd $m \geq 3$ solutions solutions with nonzero charge since their actions are likely to grow with m while the (nontrivial) lower bound stays the same at $n_1 n_2$. It is likely that the odd $m = 1$ solution does saturate this bound and hence is self-dual. We have verified all these features in our numerical constructions below.

3   Numerical results

Subject to the above boundary conditions (18)-(19) we solve numerically the set of six coupled non-linear elliptic partial differential equations. We employed a compactified radial coordinate

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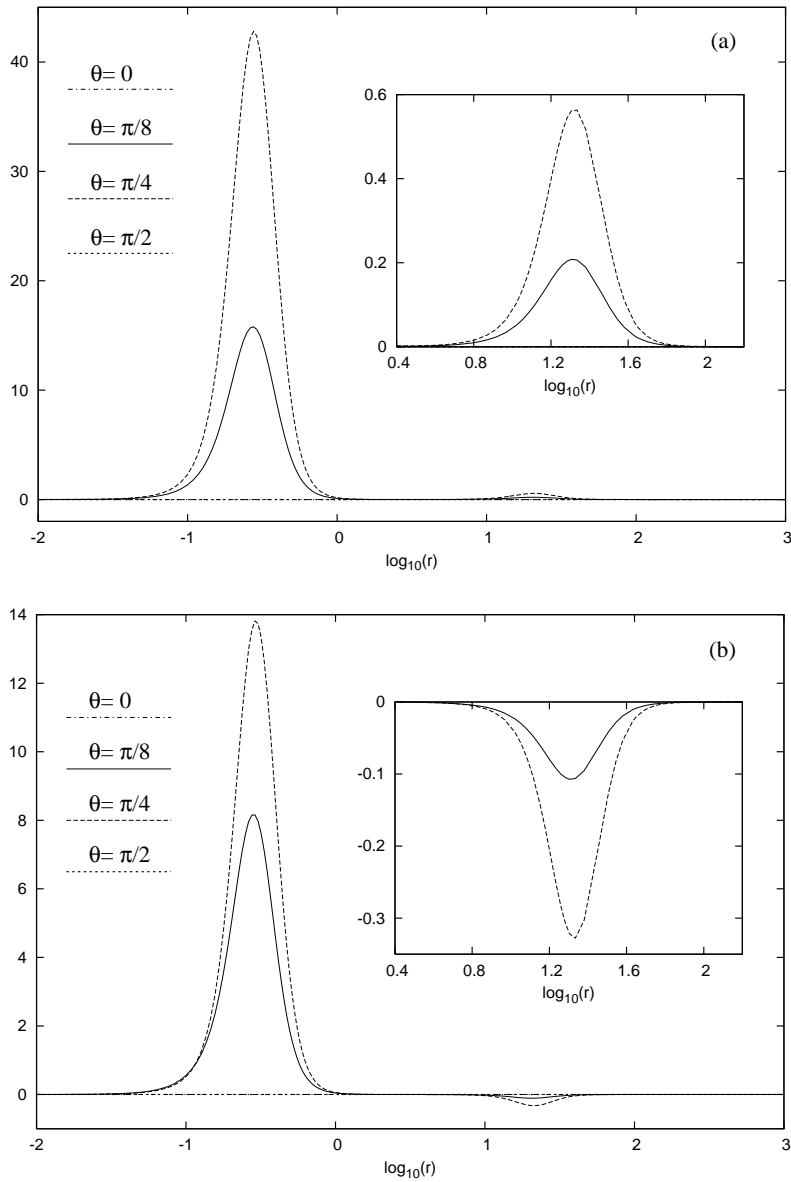
Figure 2. The effective Lagrangean (a) and the topological charge density (b) as given by (15) and (22) respectively, are plotted for the $m = 2$, $n = 2$ instanton-antiinstanton solution.

$x = r/(1 + r)$ in our computations, the equations being discretised on a nonequidistant grid in r and θ with typical grid size 130×60. The numerical calculations were performed with the software package FIDISOL, based on the Newton-Raphson method [12].

To simplify the general picture we set $n_1 = n_2 = n$. We have studied solutions for $m = 1$ with $1 \leq n \leq 5$ and for $m = 2$ with $1 \leq n \leq 3$. Our preliminary numerical results indicate that there exist also solutions with $m = 3$, and hence likely for all (m, n).

Since the $m = 1$ solutions are special, we start by discussing their properties. It turns out that all solutions with $m = 1$ are self-dual. This was verified by checking numerically that the solutions of the second order equations satisfy the first order self-duality equations (11)-(12).
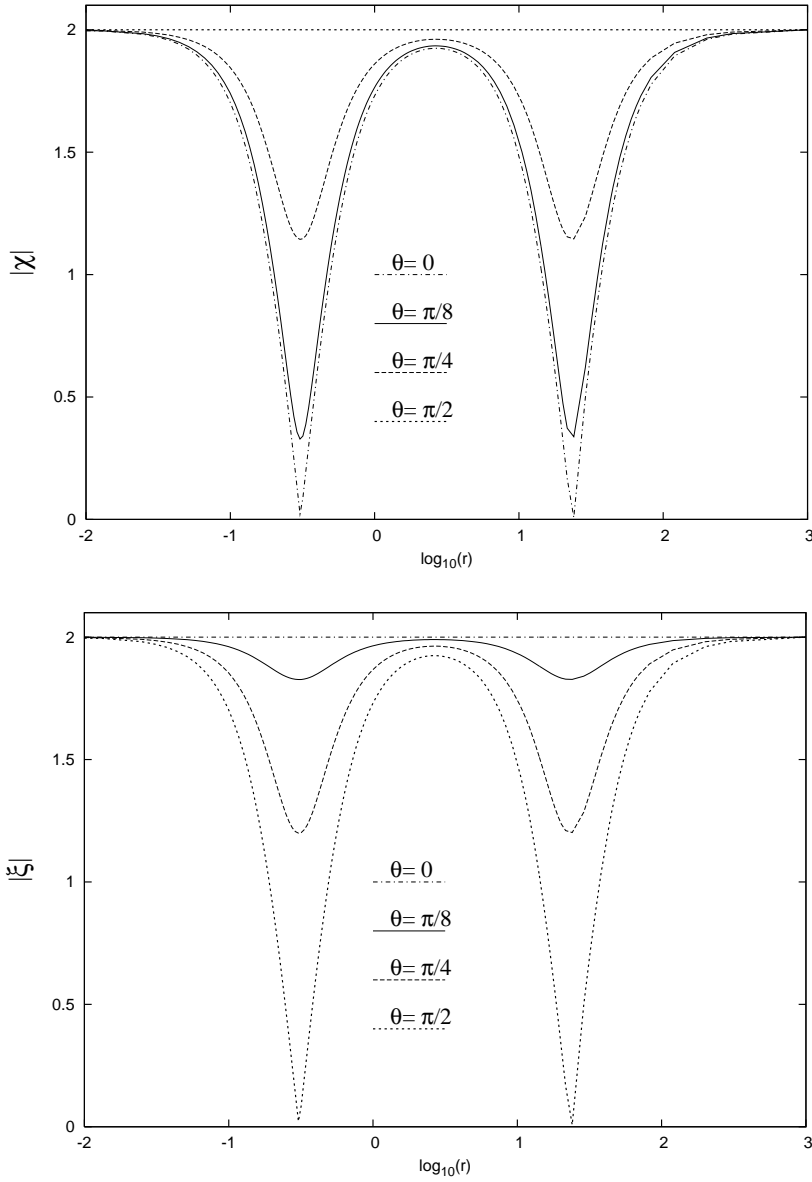
Figure 3. The moduli $|\chi|$ and $|\xi|$ of the effective Higgs fields of the $m = 2$, $n = 2$ solution are shown as a function of r for several angles.

Moreover the $(m = 1, n = 1)$ solution is spherically symmetric, corresponding to the unit charge BPST instanton (13), (14). The $m = 1$ solutions with $n \geq 2$ are axially symmetric and their gauge potentials a_r, a_θ, χ^A, ξ^A have nontrivial θ-dependence. A three dimensional plot of the action density for the $m = 1$, $n = 4$ self-dual instanton is presented in Figure 1.

The $m > 1$ configurations satisfy only the second order Euler-lagrange field equations and are not self-dual. The function a_θ does not exhibit a strong angular dependence, while χ_1 and ξ_1 have rather similar shapes. The package FIDISOL provides an error estimate for each unknown function. The typical numerical error is estimated to be on the order of 10^{-3}, except for the $n = 1$ solutions which are somehow special.

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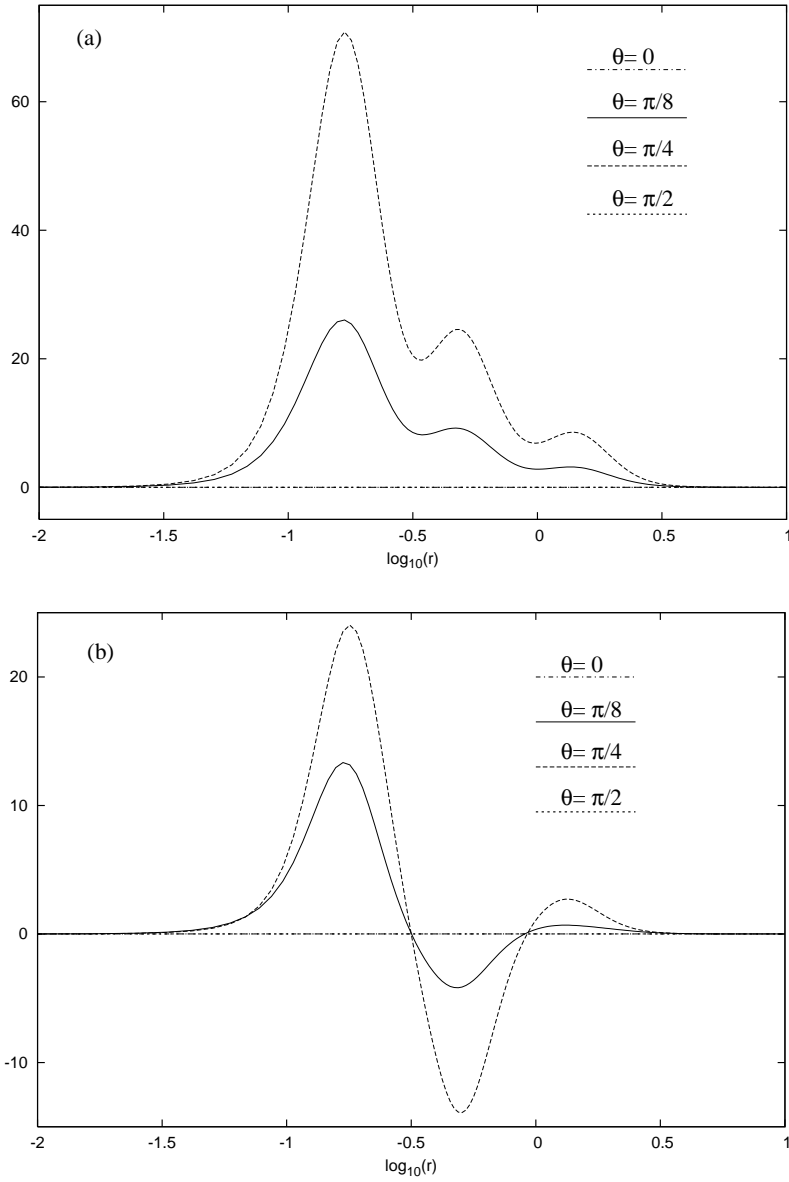
Figure 4. Same as Figure 2 for the $m = 3$, $n = 2$ solution.

In this case, although the numerical iteration still converges, the error is larger. This seems to originate in the behaviour of the function a_r which has a rather small $\theta-$dependence (as opposed to the $n > 1$ configurations) and takes very small values. Its maximal error is around 4% and comes from the vicinity of the origin. Thus our numerical results in this case are less conclusive, the existence and the properties of the $m > 1$ solutions with $n = 1$ requiring further work.

A general feature of the solutions we found is that the action density $\mathcal{L}$ (or the Lagrangean L) posseses m maxima on the $\theta = \pi/4$ axis. Thus, in all cases studied it is possible to distinguish m individual concentrations of action, the relative distance between them being fixed by the location of the maximum values of the Lagrangean L (see Figures 2a and 4a).

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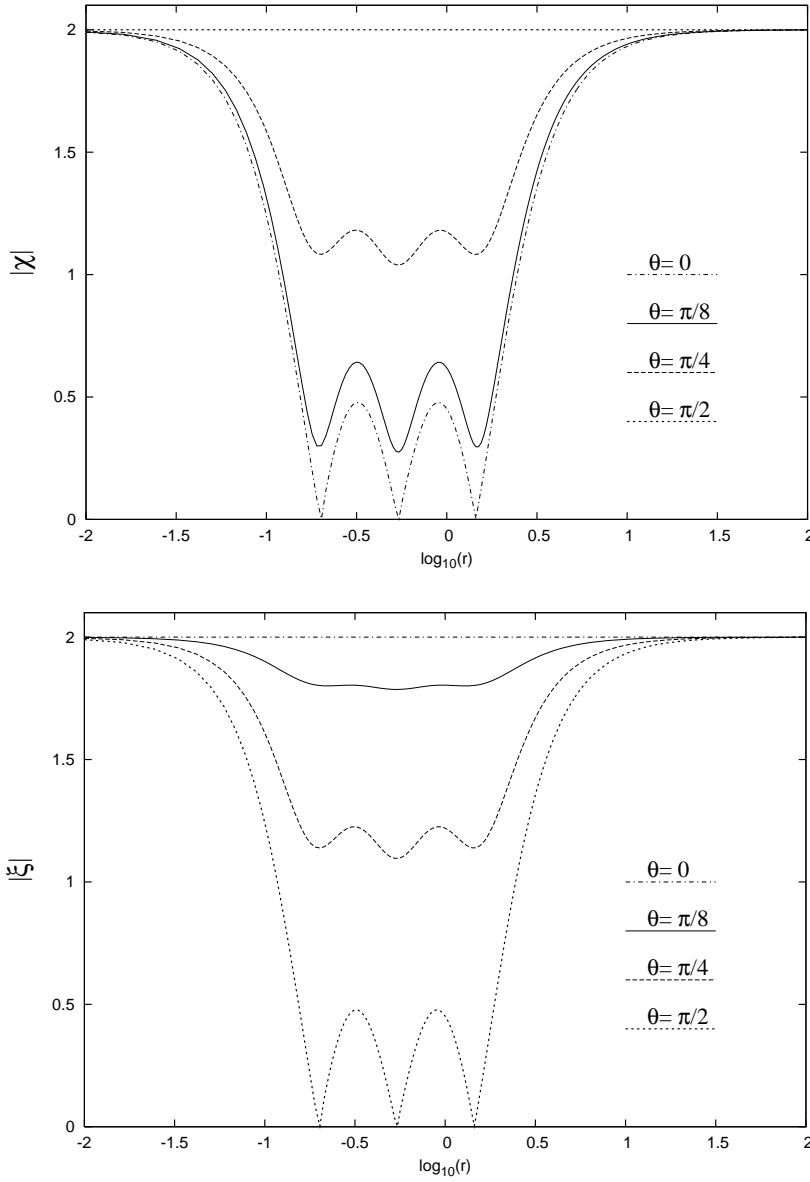
Figure 5. Same as Figure 3 for the $m = 3$, $n = 2$ solution.

Increasing the value of n causes these maxima to become sharper but does not significantly affect the distances between the locations of these pseudoparticles. It is also interesting to note that, as shown in Figures 3 and 5 for the $m = 2$, 3 configurations with $n = 2$, the moduli of the effective Higgs fields $|\chi| = (\chi^A \chi^A)^{1/2}$ and $|\xi| = (\xi^A \xi^A)^{1/2}$ possess m nodes on the ρ and σ axis, respectively. The positions of these nodes coincides, within the numerical accuracy, with the locations of individual pseudoparticles.

The topological charge density, namely the integrand of (22), also presents m local extrema on the $\theta = \pi/4$ axis, whose locations always coincide with the action density extrema. However, as shown in Figures 2b and 4b, the signs of the charges alternate between the locations of the successive lumps.

Thus the $m = 1$ configurations describe self-dual instanton solutions, $m = 2$ corresponds to an instanton-antiinstanton pair, while the $m > 2$ solutions are instanton-antiinstanton bound states composed of several pseudoparticles.

The numerical results indicate that the action $S(m, n)$ of a composite $(m, n)-$solution (with $m > 1$) is smaller than the action $S_0(m, n) = mn^2$, of m single infinitely separated self-dual instantons each with vorticity n. We found e.g. $S(2, 2) = 7.64$, $S(2, 3) = 15.96$ while $S(3, 2) = 11.38$.

Also, we verified numerically by integrating (22) that, within the numerical accuracy, the solutions with $m = 3$ carry topological charge n^2, while the topological charge of $m = 2$ solutions vanishes for all values of n.

4   Summary and discussion

To summarise, we have constructed instantons of the four dimensional $SU(2)$ YM system by numerically integrating the second order Euler–Lagrange equations of the residual two dimensional subsystem resulting from the imposition of azimuthal symmetries in both the $x - y$ and the $z - t$ planes. The residual system is a $U(1)$ Higgs like model featuring two complex scalar functions. The instantons are labeled by a triple of integers (m, n_1, n_2) and have topological charges $q = \frac{1}{2}[1 - (-1)^m]n_1 n_2$. Concrete constructions were presented for the cases $m = 1, 2, 3$ and several values of $n_1 = n_2 = n$, although nontrivial solutions are likely to exist for any positive (m, n_1, n_2).

Clearly, all solutions with even m carry vanishing topological charge, while all those with odd m carry charge $n_1 n_2 = n^2$, in our case. Thus our solutions describe instanton-antiinstanton lumps, rather analogous to the monopole-antimonopole chains of the Yang-Mills-Higgs model [13].

With the exception of the instantons with $m = 1$, all the rest (with $m \geq 2$) do not saturate the topological bound and are only solutions to the second order field equations. They are non self-dual solutions with vorticity n, whose actions are always smaller than the action of m infinitely separated charge-n^2, $m = 1$ self-dual instantons. This is supported by our numerical results. Instantons with $m = 1$ by contrast, do saturate the topological lower bound and satisfy the first order self-duality equations. Of these, the $n = 1$ solution is the charge-1 spherically symmetric BPST instanton, while $m = 1$ instantons of charges n^2 are not spherically symmetric. These features are also borne out by our numerical results.

The instantons and antiinstantons are located in alternating order on the $\rho = \sigma$ symmetry axis, at the nodes of the moduli $|\chi^A|$ and $|\xi^A|$ of effective Higgs fields. A detailed study of these solutions will be presented in a future study.

Concerning the relation of our results with those of [9, 10] proving existence of non self-dual instantons, it may be interesting to speculate as follows: In the latter [9, 10], it is stated that the proof of existence for non self-dual instantons of charge $|q| = 1$ is absent. In our case, while the self-dual solution with $m = 1$, $n = 1$ is evaluated in closed form, the $m \geq 2$, $n = 1$ non self-dual instantons of charge $q = n^2 = 1$ are the only solutions for which the numerical process had some difficulties, discussed in Section 3. This is the class of non self-dual instantons for

which the existence proofs of $m \geq 2$, $n = 1$ are absent.

On a physical level, we hope that our solutions will prove useful both in the sense that they are labeled by integers (m, n) and describe higher charged instantons, and especially since they present exact non self-dual solutions that can be useful in the construction of instanton gases and liquids. Hopefully, these numerical results will be of help in constructing the solutions analytically.

Acknowledgements

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