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Noncommutative Two Dimensional Gravities

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Abstract

We give formulations of noncommutative two dimensional gravities in terms of noncommutative gauge theories. We survey their classical solutions and show that solutions of the corresponding commutative theories continue to be solutions in the noncommutative theories as well. We argue that the existence of "twisted" diffeomorphisms, recently introduced in [1], is crucial for this conclusion.

Dedicated to Rafael Sorkin on his 60^{th} *Birthday.*

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1 Introduction

Recently progress have been made in formulating gravity theories on noncommutative(NC) spaces. In [1], a new approach has been developed to restore the action of diffeomorphisms as a symmetry of the Groenewold-Moyal type NC spacetimes. . Using this new approach, the authors of [1] were able to construct a noncommutative version of the Einstein-Hilbert action, which is invariant under the deformed or "twisted" diffeomorphisms. However, due to the complicated nature of the resulting expressions, it is not easy to extract the physical predictions of the new theory [However, for recent progress along this line see [2]]. Thus it could be practically advantageous to study the consequences of these developments in the less complicated setting of two dimensional gravities. The latter are well-known in the literature and elegantly formulated as gauge theories [3, 4, 5, 6, 7, 8, 9].

In this article, we construct "twisted" generally covariant noncommutative gauge theories describing noncommutative gravity models on two dimensional noncommutative spaces. We analyze their classical solutions and show that they are the same as the solutions of their corresponding commutative theories. We argue that the presence of "twisted" diffeomorphisms is essential for the latter conclusion.

Our results for the case of the noncommutative version of the standard two dimensional gravity theory bear strong similarities with the earlier work of Cacciatori et. al. [12], our difference being in the class of diffemorphisms the relevant action admits and its implementation.

Our results in this article are also consistent with the results of work in progress of Balachandran et. al. [10], where another approach is being developed to understand the action of "twisted" symmetries. It has as one of its predictions that gravity theories without sources on the Groenewold-Moyal plane have the solutions of the corresponding commutative theories.

The need for "twisted" symmteries in formulating NC two dimensional gravities have also been recently argued in [11].

This paper is organized as follows. In section 2 we discuss NC gauge theory formulation of two dimensional gravity with zero cosmological constant and show that it has trivial solutions. In section 3 we formulate the NC gauge theories for two dimensional gravities with nonzero cosmological constant. Here we give the appropriate action and demonstrate its invariances under NC gauge transformations and "twisted" diffeomorphisms. In section 4, we study the classical solutions of the theories formulated in section 3 and arrive at the result that they are same as that of their corresponding commutative theories.

2 Noncommutative Gauge Theory for Two Dimensional Gravity with Zero Cosmological Constant

Let us formulate two dimensional noncommutative gravity using the noncommutative version of the SO(1,1) gauge group. In the commutative case, we can think of the $SO(1,1)$ gauge group as generated by a single Pauli matrix, say σ_2. It is well-known that gravity based on this group is completely trivial. We will see below that the same holds for gravity based on the noncommutative version of the $SO(1,1)$ gauge group. In this case, the gauge group has two commuting generators, σ_2 and the 2×2 identity matrix I, the latter arising due to the noncommutativity of space-time.

Consider the connection one form

$$A = \omega\sigma_2 + fI \tag{1}$$

where ω is the spin connection and f is an additional one form. The corresponding curvature two form is

$$\begin{aligned} F &= dA + A \wedge_* A \\ &= (d\omega + f \wedge_* \omega + \omega \wedge_* f)\, \sigma_2 + (df + \omega \wedge_* \omega + f \wedge_* f)\, I \\ &\equiv F^1 \sigma_2 + F^2 I\,, \end{aligned} \tag{2}$$

where $\wedge_*$ is understood to be the ordinary wedge product, except that the components of differential forms are now being multiplied with the Groenewold-Moyal $*$-product[1].

Let us also introduce a two component scalar field $\phi = (\phi_1 \sigma_2 + \phi_2 I)$. Using ϕ and F we can form the gauge invariant action

$$\begin{aligned} S &= \frac{1}{2} \int Tr\,(\phi * F) \\ &= \frac{1}{2} \int \phi_1 * F^1 + \phi_2 * F^2\,. \end{aligned} \tag{3}$$

S is invariant under infinitesimal gauge transformations

$$F \to F + i[\upsilon\,, F]_*\,, \quad \phi \to \phi + i[\upsilon\,, \phi]_*\,, \tag{4}$$

where $\upsilon = \upsilon_1 \sigma_2 + \upsilon_2 I$ is the gauge transformation parameter.

Using (2) and (3) and the fact that one $*$-product can be removed under the integral sign we see the nonlinear terms in the integrand of (3) either become zero or reduce to total derivatives. Thus, we can rewrite S as

$$\begin{aligned} S &= \frac{1}{2} \int \phi_1 * d\omega + \phi_2 * df \\ &= \frac{1}{2} \int \phi_1 \; d\omega + \phi_2 \; df\,. \end{aligned} \tag{5}$$

From (5) we infer that $d\omega = df = 0$. They give trivial solutions for gravity.

It will become clear after the discussion in section 3 that the action (5) is indeed invariant under "twisted" diffemorphisms.

Hereafter we focus on theories with nonzero cosmological constant.

3 Noncommutative Gauge Theories for Two Dimensional Gravity with Nonzero Cosmological Constant

3.1 Generalities

Let us now direct our attention to the formulation of possible gauge theories with nonzero cosmological constant and with or without a dilaton. They are based on the gauge group $U(1,1) \approx SO(2,1) \times U(1)$ and its contractions. The presence of the extra $U(1)$ factor is due to the noncommutativity of the theory.

[1] For two functions $f\,, g \in \mathcal{A}_\theta(\mathbb{R}^2)$ the Groenewold-Moyal $*$-product is defined as:

$$f * g(x) \;=\; f(x) e^{\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} g(x)\,,$$

where $\theta^{\mu\nu} = \theta \varepsilon^{\mu\nu}$, ($\varepsilon^{01} = 1$) and θ is the noncommutativity parameter.

The associated Lie algebra $so(2,1) \oplus u(1)$ is generated by P_a, J and I $(a = 0,1)$. The commutation relations among these generators are given by

$$[P_a, P_b] = -\frac{1}{2}\frac{\Lambda}{s}\varepsilon_{ab}(2J - sI)\,, \quad [P_a\,, J] = \varepsilon_a{}^b P_b\,, \quad [P_a\,, I] = [J\,, I] = 0\,, \quad (\varepsilon^{01} = 1)\,. \tag{6}$$

Thus I is a central element. Here Λ is the cosmological constant and s is a dimensionless parameter whose role will be explained below. For the generators we will sometimes use the notation

$$(Q_a, Q_2, Q_3) \equiv (P_a, J, I)\,, \quad a \in \{0,1\} \tag{7}$$

in the text.

For finite values of Λ and s, the commutation relations in (6) are those of the standard $so(2,1) \oplus u(1)$ Lie algebra as can be seen by making the substitutions $\frac{\Lambda}{s} \to \Lambda$ and $(J - \frac{1}{2}sI) \to J$. Keeping Λ finite and letting $s \to \infty$ results in what is known as the centrally extended Poincaré algebra [9]. We also recall that the gauge theory formulation of the two dimensional gravity model is based on $so(2,1)$, while the centrally extended Poincaré algebra is required for the formulation "string-inspired" gravity [9].

Throughout the paper we work with the fundamental representation of (6). It is given by:

$$P_0 = \frac{1}{2}\sqrt{\frac{\Lambda}{s}}i\sigma_3\,, \quad P_1 = \frac{1}{2}\sqrt{\frac{\Lambda}{s}}\sigma_1\,, \quad J = \frac{1}{2}(\sigma_2 + sI)\,, \tag{8}$$

where $\sigma_i\,, (i = 1,2,3)$ as usual denote the Pauli matrices. In this representation the following relations hold:

$$\begin{aligned} J^2 = sJ + \frac{1-s^2}{4}I\,, \quad \{J, P_a\} = sP_a\,, \quad \{P_a, P_b\} = -\frac{\Lambda}{2s}h_{ab}I \\ TrP_aP_b = -\frac{\Lambda}{2s}h_{ab}\,, \quad TrJ^2 = \frac{1+s^2}{2}\,, \quad TrI = 2\,, \end{aligned} \tag{9}$$

where $h_{ab} = diag(1,-1)$ and $\{.,.\}$ denote anticommutators.

Let us consider the connection one form A. It is composed of the *zweibein*'s $e^a (a = 0,1)$, the spin connection ω and the additional one form k. Expanding in the Lie algebra basis, A reads

$$A := A^\alpha Q_\alpha = e^a P_a + \omega J + \frac{\Lambda}{2}kI\,, \quad (\alpha = 0,1,2,3)\,. \tag{10}$$

We compute the curvature associated to A in a straightforward fashion using

$$F = dA + A \wedge_* A\,. \tag{11}$$

We find

$$\begin{aligned} F = \Big[de^a + \frac{1}{2}\varepsilon_b{}^a(e^b \wedge_* \omega - \omega \wedge_* e^b) + \frac{1}{2}\Lambda(k \wedge_* e^a + e^a \wedge_* k) + \frac{s}{2}(e^a \wedge_* \omega + \omega \wedge_* e^a)\Big]P_a + \\ \Big[d\omega + s\omega \wedge_* \omega - \frac{\Lambda}{2s}\varepsilon_{ab}e^a \wedge_* e^b + \frac{\Lambda}{2}(k \wedge_* \omega + \omega \wedge_* k)\Big]J + \\ \Big[\frac{\Lambda}{2}dk + \frac{\Lambda^2}{4}k \wedge_* k - \frac{\Lambda}{4s}h_{ab}e^a \wedge_* e^b + \frac{\Lambda}{4}\varepsilon_{ab}e^a \wedge_* e^b + \frac{1-s^2}{4}\omega \wedge_* \omega\Big]I\,. \end{aligned} \tag{12}$$

Under the infinitesimal gauge transformations generated by $\upsilon = \upsilon^a P_a + \upsilon^2 J + \upsilon^3 I$, we have

$$\begin{aligned} A \longrightarrow A' &= A + iD^*\upsilon\,, \quad D^*\upsilon = d\upsilon + i[\upsilon\,, A]_* \\ F \longrightarrow F' &= F + i[\upsilon\,, F]_*\,, \end{aligned} \tag{13}$$

while under finite gauge transformations

$$\begin{aligned} A \longrightarrow A' &= e^{i\upsilon} * (A + d) * (e^{i\upsilon})_*^{-1}\,, \\ F \longrightarrow F' &= e^{i\upsilon} * F * (e^{i\upsilon})_*^{-1}\,. \end{aligned} \tag{14}$$

Note that in above $(e^{i\upsilon})_*^{-1}$ is the $*$-inverse of $(e^{i\upsilon})$.

3.2 The Action

We now give the gauge theory action describing NC gravity theories with nonzero cosmological constant, in two dimensions. Generalizing from the commutative theory we write,

$$S = \int Tr(\xi * F)\,, \tag{15}$$

Here we introduced the 4-component scalar field

$$\xi = -\frac{2s}{\Lambda}\eta^a P_a + \frac{2}{1+s^2}\eta^2 J + \frac{1}{\Lambda}\eta^3 I\,, \tag{16}$$

and the trace is taken over the Lie algebra basis. We note the peculiar factors in front of the component fields in (16); in the action S, they cancel with the factors coming from the traces.

In terms of the component fields, the action S reads

$$\begin{aligned} S = \int \eta_a * \Big[de^a + \frac{1}{2}\varepsilon_b{}^a(e^b \wedge_* \omega - \omega \wedge_* e^b) + \frac{\Lambda}{2}(k \wedge_* e^a + e^a \wedge_* k) + \frac{s}{2}(e^a \wedge_* \omega + \omega \wedge_* e^a)\Big] + \\ \int \eta_2 * \Big[d\omega + s\omega \wedge_* \omega - \frac{\Lambda}{2s}\varepsilon_{ab}e^a \wedge_* e^b + \frac{\Lambda}{2}(k \wedge_* \omega + \omega \wedge_* k)\Big] + \\ \int \eta_3 * \Big[dk + \frac{\Lambda}{2}k \wedge_* k - \frac{1}{2}h_{ab}e^a \wedge_* e^b + \frac{1}{2}\varepsilon_{ab}e^a \wedge_* e^b + \frac{1-s^2}{2\Lambda}w \wedge_* \omega\Big]\,. \end{aligned} \tag{17}$$

Let us note that several terms in this action do in fact vanish. Recalling that one $*$-product can be removed under the integral and removing the $*$ in the $\wedge_*$, we find that the 3^{rd} and 4^{th} terms of the first integral, the 2^{nd} and 4^{th} terms of the second integral and the 2^{nd} and 3^{rd} and 5^{th} terms of the third integral are zero. Thus we have

$$\begin{aligned} S = \int \eta_a * \Big(de^a + \frac{1}{2}\varepsilon_b{}^a(e^b \wedge_* \omega - \omega \wedge_* e^b)\Big) + \eta_2 * \Big(d\omega - \frac{\Lambda}{2s}\varepsilon_{ab}e^a \wedge_* e^b\Big) + \\ \eta_3 * \Big(dk + \frac{1}{2}\varepsilon_{ab}e^a \wedge_* e^b\Big)\,. \end{aligned} \tag{18}$$

S is invariant under both the infinitesimal NC gauge transformations and "twisted" diffeomorphisms. We now explicitly demonstrate these invariances.

3.3 Symmetries

Gauge Invariance:

The action given in (15) is gauge invariant under the infinitesimal gauge transformation given in (13) for the curvature F and the standard infinitesimal transformation law of scalar fields in the NC gauge theories:

$$\xi \to \xi + i[\upsilon\,,\xi]_* \tag{19}$$

Explicitly, we have

$$\begin{aligned}
&S(\xi+\delta\xi, F+\delta F)\\
&= \int Tr(\xi + i[\upsilon\,,\xi]_*) * (F + i[\upsilon\,,F]_*)\\
&= \int Tr\left(\xi * F + i\xi * [\upsilon\,,F]_* + i[\upsilon\,,\xi]_* * F + \mathcal{O}(\upsilon^2)\right)\\
&= \int Tr(\xi * F) + i\int Tr[\upsilon\,,\xi * F]_* + \mathcal{O}(\upsilon^2)\\
&= \int Tr(\xi * F) + i\int \upsilon^\alpha * \xi^\beta * F^\gamma\, Tr[Q^\alpha\,,Q^\beta Q^\gamma] + i\int\, TrQ^\alpha Q^\beta Q^\gamma\,[\upsilon^\alpha\,,\xi^\beta * F^\gamma] + \mathcal{O}(\upsilon^2)\\
&= \int Tr(\xi * F) + \mathcal{O}(\upsilon^2)\,,
\end{aligned} \tag{20}$$

as was to be shown.

Twisted Diffemorphisms:

Implementation of space-time symmetries on noncommutative spaces was a long standing problem until very recently. It is well-known that on a d-dimensional noncommutative space $\mathbb{R}^d_\theta$ generated by the coordinates $x_\mu \in \mathcal{A}_\theta(\mathbb{R}^d)$, the Poincaré and diffeomorphism symmetries are explicitly broken due to the noncommutativity

$$[x_\mu\,,x_\nu]_* = i\theta_{\mu\nu}\,, \tag{21}$$

if they are naively implemented. Very recently it has been reported by Chaichian et. al.[13] and Aschieri et. al. [1] that these symmetries can be restored by twisting their coproduct (See also the earlier work of Oeckl [15], Such a twist in a general context is due to Drinfel'd [16]). A clear way to understand these developments is as follows [1, 14].

Let $\mathcal{A}$ be an algebra. $\mathcal{A}$ comes with a rule for multiplying its elements. For $f, g \in \mathcal{A}$ there exists the multiplication map μ such that

$$\begin{aligned}
&\mu : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\,,\\
&f\otimes g \to \mu(f\otimes g)\,.
\end{aligned} \tag{22}$$

Now let $\mathcal{G}$ be the group of symmetries acting on $\mathcal{A}$ by a given representation $D : g \to D(g)$ for $g \in \mathcal{G}$. We can denote this action by

$$f \longrightarrow D(g)f\,. \tag{23}$$

The action of $\mathcal{G}$ on $\mathcal{A}\otimes\mathcal{A}$ is formally implemented by the coproduct $\Delta(g)$. The action is compatible with μ only if a certain compatibility condition between $\Delta(g)$ and μ is satisfied. This action is

$$f \otimes g \longrightarrow (D \otimes D)\Delta(g) f \otimes g \,, \tag{24}$$

and the compatibility condition requires that

$$\mu\left((D \otimes D)\Delta(g) f \otimes g\right) = D(g)\,\mu(f \otimes g)\,. \tag{25}$$

The latter can be expressed neatly in terms of the following commutative diagram :

$$\begin{tikzcd} f \otimes g \arrow[r, "\Delta"] \arrow[d, "\mu"] & (D \otimes D)\Delta(g) f \otimes g \arrow[d, "\mu"] \\ \mu(f \otimes g) \arrow[r] & D(g)\mu(f \otimes g) \end{tikzcd}$$

If a Δ satisfying the above compatibility condition exists, then $\mathcal{G}$ is an automorphism of $\mathcal{A}$. If such a Δ cannot be found, then $\mathcal{G}$ does not act on $\mathcal{A}$.

We can now specialize to the algebra $\mathcal{A}_\theta(\mathbb{R}^d)$. The multiplication law on $\mathcal{A}_\theta(\mathbb{R}^d)$ is nothing but the Groenewold-Moyal $*$-product

$$\mu_\theta(f \otimes g) = \mu_{\theta=0}\left(\mathcal{F}\, f \otimes g\right) = f * g \,, \tag{26}$$

where $\mu_{\theta=0}(f \otimes g) = fg$ is the pointwise product and we have introduced

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} \,. \tag{27}$$

The twisted coproduct is given by

$$\Delta_\theta(g) = \mathcal{F}^{-1}\Delta_{\theta=0}(g)\mathcal{F} \,. \tag{28}$$

$\Delta_\theta(g)$ satisfies the compatibility condition (25) with μ_θ as can be easily checked.

For infinitesimal symmetries, it is sufficient to consider only the Lie algebra G of $\mathcal{G}$ and the coproduct associated with its universal enveloping algebra.

Infinitesimal diffeomorphisms are generated by vector fields of the form $\zeta = \zeta^\mu(x)\partial_\mu$ and they form a Lie algebra. The twisted coproduct of ζ is given by

$$\begin{aligned} \Delta_\theta(\zeta) &= \mathcal{F}^{-1}\Delta_0(\zeta)\mathcal{F} \\ &= \mathcal{F}^{-1}(\zeta \otimes 1 + 1 \otimes \zeta)\mathcal{F} \,. \end{aligned} \tag{29}$$

It is compatible with the multiplication map μ_θ on $\mathcal{A}_\theta(\mathbb{R}^d)$.

Variation of tensor fields under twisted diffeomorphisms can be suitably represented by defining the operator X^*_ζ acting on $\mathcal{A}_\theta(\mathbb{R}^d)$ as

$$\hat{\delta}_\zeta f \equiv -X^*_\zeta(f) := -\zeta^\alpha \partial_\alpha f \,, \tag{30}$$

on scalars f. Its action on tensor fields follows by the standard transformation rules of tensors. For example, on a contravariant vector field V^μ we have

$$\hat{\delta}_\zeta V^\mu \equiv -X^*_\zeta(V^\mu) + X^*_{(\partial_\rho \zeta^\mu)}(V^\rho) := -\zeta^\rho(\partial_\rho V^\mu) + (\partial_\rho \zeta^\mu)V^\rho \,. \tag{31}$$

We observe that the Leibniz rule for the "twisted" vector fields is a deformed one. It is given by

$$X^*_\zeta(f * g) = \mu_\theta \left\{ \mathcal{F}^{-1}(X^*_\zeta \otimes 1 + 1 \otimes X^*_\zeta)\mathcal{F}\,(f \otimes g) \right\} . \tag{32}$$

This deformation of the Leibniz rule ensures that the product of two tensor fields of rank n and m transforms as a tensor field of rank $n + m$.

This much of information on "twisted" diffeomorphisms is sufficient for our purposes. For further details on the subject we refer to [1].

We are now ready to demonstrate the invariance of the action S in (15) under the "twisted" diffemorphisms. We have ξ transforming as a scalar, and $\varepsilon^{\mu\nu}F_{\mu\nu}$ transforming as a tensor density of weight -1:

$$\begin{aligned} \delta_{\hat\zeta}\xi &= -X^*_\zeta(\xi) := -\zeta^\alpha\partial_\alpha\xi \\ \delta_{\hat\zeta}(\varepsilon^{\mu\nu}F_{\mu\nu}) &= -X^*_\zeta(\varepsilon^{\mu\nu}F_{\mu\nu}) - X^*_{(\partial_\alpha\zeta^\alpha)}(\varepsilon^{\mu\nu}F_{\mu\nu})\,. \end{aligned} \tag{33}$$

Using (33) we find

$$\delta_{\hat\zeta}(\xi * \varepsilon^{\mu\nu}F_{\mu\nu}) = -\partial_\alpha\big(\zeta^\alpha(\xi * \varepsilon^{\mu\nu}F_{\mu\nu})\big)\,, \tag{34}$$

or equivalently

$$\delta_{\hat\zeta}(\xi * \varepsilon^{\mu\nu}F_{\mu\nu}) = -\partial_\alpha\big(X^*_{\zeta^\alpha}(\xi * \varepsilon^{\mu\nu}F_{\mu\nu})\big)\,. \tag{35}$$

Thus under the infinitesimal "twisted" diffemorphisms generated by $\delta_{\hat\zeta}$, the Lagrangian changes by a total derivative:

$$Tr(\xi * \varepsilon^{\mu\nu}F_{\mu\nu}) \longrightarrow Tr(\xi * \varepsilon^{\mu\nu}F_{\mu\nu}) - \partial_\alpha\big(X^*_{\zeta^\alpha}(Tr(\xi * \varepsilon^{\mu\nu}F_{\mu\nu}))\big)\,, \tag{36}$$

and hence the action S is invariant.

4 Classical Solutions

4.1 Equations of Motion

Let us first examine the equations of motion following from the S in (18) when the fields η_a, η_2, η_3 are varied. We find

$$D^*e^a := de^a + \frac{1}{2}\varepsilon^a{}_b(\omega \wedge_* e^b - e^b \wedge_* \omega) = 0\,, \tag{37a}$$

$$d\omega - \frac{\Lambda}{2s}\varepsilon_{ab}e^a \wedge_* e^b = 0\,, \tag{37b}$$

$$dk + \frac{1}{2}\varepsilon_{ab}e^a \wedge_* e^b = 0\,. \tag{37c}$$

The commutative limit of these classical equations can immediately be obtained by replacing the $*$-product with the usual pointwise product. That gives the familiar equations

$$De^a = de^a + \varepsilon^a{}_b\omega \wedge e^b = 0\,, \tag{38a}$$

$$d\omega - \frac{1}{2}\frac{\Lambda}{s}\varepsilon_{ab}e^a \wedge e^b = 0\,, \tag{38b}$$

$$dk + \frac{1}{2}\varepsilon_{ab}e^a \wedge e^b = 0\,. \tag{38c}$$

We now study the solutions of the equations of motion (37) for various values of the parameters Λ and s.

4.2 The AdS_2 solution

In this case we take both Λ and s to be finite and let $\frac{\Lambda}{s} \to \Lambda$.

Let us first study the commutative theory from the equations of motion given in (38). To this end first substitute $k = a - \frac{s}{\Lambda}\omega$ to (38c) and use (38b) to find $da = 0$. Thus the 1-form field a is closed and hence non-dynamical, and can be eliminated. Thus we set $a = 0$ in what follows. The equations of motion then describe standard two dimensional gravity [4, 9]. Making the substitution $k = a - \frac{s}{\Lambda}\omega$ in A as well and letting $(J - \frac{1}{2}sI) \to J$, the gauge field and the algebra can also be put into their conventional form. Thus the connection one form A in (10) can be rewritten as

$$A = e^a P_a + \omega J\,. \tag{39}$$

For $\Lambda < 0$, this commutative model has the well-known AdS_2 solution given by the metric

$$ds^2 = \Lambda r^2 dt^2 - \frac{1}{\Lambda r^2} dr^2\,. \tag{40}$$

It gives the connection

$$A_t = \frac{i\Lambda r}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad A_r = \frac{1}{2r}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{41}$$

The *zweibein's* and the spin connection can be read from (41) as

$$\begin{gathered} e_t^0 = \sqrt{\Lambda} r\,, \quad e_t^1 = 0\,, \quad e_r^0 = 0\,, \quad e_r^1 = \frac{1}{\sqrt{\Lambda} r} \\ \omega_t = -\Lambda r\,, \quad \omega_r = 0\,. \end{gathered} \tag{42}$$

It can be easily verified that (42) satisfies the equations of motion given in (38).

It has been shown in [12] that the above AdS_2 geometry is in fact a solution of the noncommutative version of the standard two dimensional gravity model. Below we also reach this result by virtue of the presence of twisted diffeomorphisms in our theory.

Let us now show that (42) is indeed also a solution for the NC equations of motion given in (37). We reason that due to the invariance of our theory under twisted diffeomorphisms, the commutation relation between coordinates is preserved under a general coordinate transformation. Thus we have

$$[x_0\,, x_1] = i\theta \quad \xrightarrow{\text{(twisted diffeos)}} \quad [t\,, r] = i\theta\,. \tag{43}$$

We observe that the solution given in (42) is time-independent. Hence all the $*$-products in (37) collapse to pointwise products and these equations of motion are satisfied immediately. This indeed shows that the AdS_2 geometry is a solution to our noncommutative gauge theory.

4.3 The Black Hole Solution

In this case we keep Λ finite and take $s \to \infty$. This is the NC version of the "string inspired" gravity model [9].

In order to discuss the solutions to the equations of motion in this case, it is useful to express the action in terms of the light cone coordinates

$$x^{\pm} = x^0 \pm x^1\,. \tag{44}$$

Following Verlinde [5], this task can be carried out rather easily. The action S in (18) takes the form

$$\int dx^+dx^-\Big[\eta^+ * \big(de^+ + \frac{1}{2}(\omega \wedge_* e^- - e^- \wedge_* \omega)\big) + \eta^- * \big(de^- - \frac{1}{2}(\omega \wedge_* e^+ - e^+ \wedge_* \omega)\big) + \chi * d\omega + \eta_3 * dk + \eta_3 * e^+ \wedge_* e^-\Big]\,, \quad (45)$$

where $\chi := \eta_2 = e^{\varphi}$ and φ is the dilaton field.

Let us first note that the variation of (45) with respect to k gives $\eta_3 =$ constant. We set this constant equal to Λ. Variations with respect to the e^- and e^+ give

$$\begin{aligned} e^+ &= -\frac{1}{\Lambda}D^*\eta^+ = -\frac{1}{\Lambda}\big(d\eta^+ - \frac{1}{2}(\eta^+ * \omega + \omega * \eta^+)\big)\,, \\ e^- &= \frac{1}{\Lambda}D^*\eta^- = \frac{1}{\Lambda}\big(d\eta^- + \frac{1}{2}(\eta^- * \omega + \omega * \eta^-)\big) \end{aligned} \quad (46)$$

respectively, and from variation of ω we find

$$d\chi = -\frac{1}{2}\{\eta^-, e^+\}_* + \frac{1}{2}\{\eta^+, e^-\}_*\,. \quad (47)$$

Let us also define

$$M^* = -\Lambda\chi + \frac{1}{2}(\eta^+ * \eta^- + \eta^- * \eta^+)\,. \quad (48)$$

In the commutative limit M^* approaches the black hole mass M. Using the above equations of motion we find

$$dM^* = \frac{1}{4}\big[\omega\,, [\eta^+\,, \eta^-]_*\big]_* + \frac{1}{4}\big[\eta^+\,, [\omega\,, \eta^-]_*\big]_*\,. \quad (49)$$

We recall that in the commutative theory the conformally scaled metric is given by [5]

$$\widetilde{G} = h_{ab}\frac{e^a \otimes e^b}{\chi} \equiv \frac{D\eta^+ \otimes D\eta^-}{-\frac{1}{\Lambda}(M - \eta^+\eta^-)}\,. \quad (50)$$

We take the ansatz below as the natural generalization of (50) to the NC case:

$$\widetilde{G}^*_{\mu\nu} = \frac{1}{8}(D^*_\mu\eta^+ * D^*_\nu\eta^- + D^*_\nu\eta^+ * D^*_\mu\eta^-) * \left(\frac{1}{-\frac{1}{\Lambda}\big(M^* - \frac{1}{2}(\eta^+ * \eta^- + \eta^- * \eta^+)\big)}\right) + \frac{1}{8}\left(\frac{1}{-\frac{1}{\Lambda}\big(M^* - \frac{1}{2}(\eta^+ * \eta^- + \eta^- * \eta^+)\big)}\right) * (D^*_\mu\eta^+ * D^*_\nu\eta^- + D^*_\nu\eta^+ * D^*_\mu\eta^-) + (+ \longleftrightarrow -)\,. \quad (51)$$

We note that $\widetilde{G}_{\mu\nu}$ as given above is symmetric and transforms as a second rank covariant tensor under "twisted" diffeomorphisms. Thus according to the definition given in [1], it qualifies as a metric. In what follows, we proceed by setting $\Lambda = -1$.

In the commutative theory, the fields $\eta^\pm$ are related to light cone coordinates u and v by

$$u(x) = \eta^+(x)e^{-\int^x \omega}\,, \quad v(x) = \eta^-(x)e^{\int^x \omega}\,. \quad (52)$$

In these coordinates the black hole metric and the dilaton are given by

$$ds^2 = \frac{dudv}{1 - uv}\,, \quad \varphi = \ln(1 - uv)\,, \quad (53)$$

where M is set to be equal to 1. u and v are also closely related to the Schwarzchild type of coordinates t and r by

$$u = \sinh r e^{t} \,, v = -\sinh r e^{-t} \,. \tag{54}$$

The metric and the dilaton in these coordinates take the form

$$ds^2 = dr^2 - \tanh^2 r dt^2 \,, \quad \varphi = \ln \cosh^2 r \,. \tag{55}$$

Using the metric (55), it is easy to see that one has

$$e_t^+ e_t^- = -\sinh^2 r \,, \quad e_r^+ e_r^- = \cosh^2 r \,. \tag{56}$$

We see from (52) that $\eta^+ \eta^- = -\sinh^2 r$ and thus is time-independent. Let us make the gauge choice that η^+ and η^- are time-independent and only functions of r. The equations of motion of the commutative theory for e^- and e^+ then become

$$\begin{aligned} e_t^+ = D_t \eta^+ &= -\omega_t \eta^+ \\ e_t^- = D_t \eta^- &= -\omega_t \eta^- \,. \end{aligned} \tag{57}$$

Thus

$$e_t^+ e_t^- = \eta^+ \eta^- \omega_t^2 \,, \tag{58}$$

and hence $\omega_t^2 = 1$ and

$$\omega_t = \pm 1 \,. \tag{59}$$

The scalar curvature can be directly computed from the metric to be

$$R = \frac{4}{\cosh^2 r} \,. \tag{60}$$

It can also be expressed as

$$R = \frac{2}{det\ e} \varepsilon^{\mu\nu} \partial_\mu \omega_\nu \,. \tag{61}$$

Using this and (59) and (60) in (61) and the result $det\ e = \tanh r$, we find

$$\partial_t \omega_r = -2 \frac{\tanh r}{\cosh^2 r} \tag{62}$$

with the obvious solution

$$\omega_r = -2 \frac{\tanh r}{\cosh^2 r} t + h(r) \,. \tag{63}$$

Once more we reason that due to the invariance of our theory under twisted diffeomorphisms, the canonical commutation relation between coordinates are preserved under a general coordinate transformation. Thus we have

$$[x_0 \,, x_1] = i\theta \quad \xrightarrow{\text{(twisted diffeos)}} \quad [t \,, r] = i\theta \,. \tag{64}$$

Now using the above developments we find, expanding in powers of θ:

$$\begin{aligned} D^* \eta^+ &= d\eta^+ - \omega \eta^+ + \frac{1}{2} \theta^2 \varepsilon_{ij} \varepsilon_{kl} \partial_i \partial_k \omega \partial_j \partial_l \eta^+ + \text{higher order terms in } \theta \\ &= d\eta^+ - \omega \eta^+ = D\eta^+ \,. \end{aligned} \tag{65}$$

The result above is exact to all orders in θ since odd powers in θ vanish anyway due to antisymmetry of $\theta_{\mu\nu}$ in its indices and even powers like the θ^2 term above vanish after the differentiatons on ω. Similarly we find $D^*\eta^- = D\eta^-$ to all orders in θ. **Time dependent solutions of the theory can be obtained through NC gauge transformations of $\eta^{\pm}$.**

Thus we conclude that the black hole solution of the commutative theory, given by the metric in (55) is at the same time a solution to our noncommutative theory. For this conclusion the presence of twisted diffeomorphisms is crucial as can be clearly observed from the arguments above.

5 Concluding Remarks

In this article we have constructed noncommutative two dimensional gravities in terms of noncommutative gauge theories. We have shown that the presence of twisted diffemorphisms ensure that solutions of the corresponding commutative gauge theories continue to be solutions to these noncommutative models.

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